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ADDENDUM TO "LOCAL CONTROLLABILITY OF THE TWO-LINK MAGNETO-ELASTIC MICRO-SWIMMER"

LAETITIA GIRALDI PIERRE LISSY
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Abstract. In the above mentioned note (<hal-01145537>, <arXiv:1506.05918>, published in IEEE Trans. Autom. Cont., 2017), the first and fourth authors proved a local controllability result around the straight configuration for a class of magneto-elastic micro-swimmers. That result is weaker than the usual small-time local controllability (STLC), and the authors left the STLC question open. The present addendum closes it by showing that these systems cannot be STLC.

1. Model of the magneto-elastic micro-swimmer

Keeping the same notations as in [1], the planar micro swimmer's dynamics are given by

$$\dot{\mathbf{z}} = \mathbf{F}_0(\mathbf{z}) + H_\parallel \, \mathbf{F}_1(\mathbf{z}) + H_\perp \, \mathbf{F}_2(\mathbf{z}) \tag{1}$$

where (see figure 1 in [1]):

- the state is $\mathbf{z} = (x, y, \theta, \alpha)$ with α an angle describing the swimmer's shape and x, y, θ two coordinates and an angle describing its position,
- the control is $(H_\perp, H_\parallel)$, the coordinate vector of the external magnetic field in a moving frame, the norm on the control space $\mathbb{R}^2$ being the sup-norm:
$$\|(H_\perp, H_\parallel)\| = \max\{\, |H_\perp| \,,\, |H_\parallel| \,\} \,,$$
- the $\mathbf{F}_i$'s may be expressed as follows, with $f_{i,j}$ twelve functions[1] of one variable explicitly derived from [1, Prop. II.1 and (12)-(16)]:
$$\mathbf{F}_i(\mathbf{z}) = \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} f_{i,1}(\alpha) \\ f_{i,2}(\alpha) \\ f_{i,3}(\alpha) \\ f_{i,4}(\alpha) \end{pmatrix} . \tag{2}$$

In [1], the dynamics, hence the functions $f_{i,j}$, depend on: the length ℓ_i of each segment ($i = 1, 2$), its magnetization M_i, its longitudinal and transversal hydrodynamic drag

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[1]The notation $f_{i,j}$ is not present in [1].

constants ξ_i, η_i, and an elastic constant κ. It is assumed that $\kappa > 0$ and that, for each i, $\ell_i > 0$, $\xi_i > 0$, $\eta_i > 0$ and $M_i \neq 0$. In this addendum, we further assume that the two links have the same length and hydrodynamic constants, i.e. we define:

$$\ell = \ell_1 = \ell_2,\ \xi = \xi_1 = \xi_2,\ \eta = \eta_1 = \eta_2. \tag{3}$$

This assumption makes the redaction easier to follow but it does not alter the nature of the proofs.

The equilibria of interest are $((x^e, y^e, \theta^e, 0), (0, 0))$ in the state-control space, with (x^e, y^e, θ^e) arbitrary in $\mathbb{R}^2 \times [0, 2\pi]$. Using invariance by translation and rotation [1], one may, without loss of generality, suppose $(x_e, y_e, \theta_e) = (0, 0, 0)$ and consider only the equilibrium $\mathbf{O} = \big((0, 0, 0, 0), (0, 0)\big)$.

2. Some local controllability concepts

Consider a smooth continuous-time control system

$$\dot{z} = f(z, u) \tag{4}$$

with state z in $\mathbb{R}^n$ and control u in $\mathbb{R}^m$. We endow $\mathbb{R}^m$ with a norm $\|\cdot\|$ and always assume that u is essentially bounded. Let (z_e, u_e) be an equilibrium of (4), i.e. $f(z_e, u_e) = 0$.

The following definition introduces an ad hoc notion of controllability for the sake of clarity.

Definition 1 (STLC(q))**.** *Let q be a non-negative number. The control system* (4) *is* STLC(q) at (z_e, u_e) *if and only if, for every $\varepsilon > 0$, there exists $\eta > 0$ such that, for every z_0, z_1 in the ball centered at z_e with radius η, there exists a solution $(z(\cdot), u(\cdot)) : [0, \varepsilon] \to \mathbb{R}^{n+m}$ of* (4) *such that $z(0) = z_0$, $z(\varepsilon) = z_1$, and, for almost all t in $[0, \varepsilon]$,*

$$\|u(t) - u_e\| \leqslant q + \varepsilon\,.$$

Let us also recall the classical definition of STLC.

Definition 2 (STLC)**.** *The system* (4) *is* STLC (small-time locally controllable) at (z_e, u_e) *if and only if it is STLC(0) at (z_e, u_e).*

The following necessary condition for STLC will be used.

Lemma 3 (Loop trajectories)**.** *If* (4) *is STLC at (z_e, u_e), then, for any $\varepsilon > 0$, there exists a solution $t \mapsto (z^\varepsilon(t), u^\varepsilon(t))$ of* (4)*, defined for t in $[0, \varepsilon]$, such that*

- $z^\varepsilon(0) = z^\varepsilon(\varepsilon) = z_e$,
- $z^\varepsilon(t) \neq z_e$ *for at least one t in* $[0, \varepsilon]$,
- $\|u^\varepsilon(t) - u_e\| \leqslant \varepsilon$ *for almost all t in* $[0, \varepsilon]$.

Proof. Let $\varepsilon > 0$. There exists $\eta > 0$ such that, for every $z_\star$ in the ball centered at z_e with radius η, there is a solution $(z^\varepsilon(\cdot), u^\varepsilon(\cdot)) : [0, \varepsilon/2] \to \mathbb{R}^{n+m}$ of (4) such that $z^\varepsilon(0) = z_e$, $z^\varepsilon(\varepsilon/2) = z_\star$, and $\|u^\varepsilon(t) - u_e\| \leqslant \varepsilon/2$ for almost all t. Pick one such $z_\star$ diffwrent from z_e. System (4) being autonomous, there also exists a solution $(z^\varepsilon(\cdot), u^\varepsilon(\cdot)) : [\varepsilon/2, \varepsilon] \to \mathbb{R}^{n+m}$ of (4) such that $z(\varepsilon/2) = z_\star$, $z(\varepsilon) = z_e$, and, for almost all t in $[\varepsilon/2, \varepsilon]$, $\|u(t) - u_e\| \leqslant \varepsilon/2$. Then, $(z^\varepsilon(\cdot), u^\varepsilon(\cdot)) : [0, \varepsilon] \to \mathbb{R}^{n+m}$ verifies all the desired properties. $\square$

3. Complements to the original note

The following proposition reformulates the results from [1]. Without assumption (3), $\xi \neq \eta$ would be replaced by $(\xi_1, \xi_2) \neq (\eta_1, \eta_2)$ and $M_1 \neq M_2$ by [1, eqn. (20)].

Proposition 4 ([1], Thm. III.4 and Prop. III.1)**.** *Assume* (3)*. The control system* (1) *is STLC*$\big(2\kappa\,|M_1+M_2|/\,|M_1M_2|\big)$ *at* $\boldsymbol{O}$ *if* $\xi \neq \eta$ *and* $M_1 \neq M_2$*. Otherwise, it is not STLC*(q) *for any* $q \geq 0$*.*

Unless $M_1 + M_2 = 0$, STLC$\big(2\kappa\,|M_1+M_2|/\,|M_1M_2|\big)$ does not imply STLC. The purpose of the present addendum is to prove the following result:

Theorem 5. *Assume* (3)*.*
If $\xi \neq \eta$*,* $M_1 \neq M_2$ *and* $M_1 + M_2 \neq 0$*, system* (1) *is not STLC at* $\boldsymbol{O}$*.*

Proof. From [1, Prop. II.1 and (12)-(16)], one readily verifies that the functions $f_{i,j}$ introduced in (2) have the following expansions around $\alpha = 0$:

$$\begin{aligned}
&f_{2,j}(\alpha) = \mathcal{O}(\alpha), \qquad j \in \{1,2,3,4\}, \\
&f_{0,1}(\alpha) = a_1\alpha^2 + \mathcal{O}(\alpha^3), && f_{1,1}(\alpha) = b_1\alpha + \mathcal{O}(\alpha^2), \\
&f_{0,2}(\alpha) = \mathcal{O}(\alpha^2), && f_{1,2}(\alpha) = b_2 + \mathcal{O}(\alpha), \\
&f_{0,3}(\alpha) = \frac{a}{2}\alpha + \mathcal{O}(\alpha^2), && f_{1,3}(\alpha) = b_3 + \mathcal{O}(\alpha), \\
&f_{0,4}(\alpha) = -a\,\alpha + \mathcal{O}(\alpha^2), && f_{1,4}(\alpha) = b_4 + \mathcal{O}(\alpha),
\end{aligned} \tag{5}$$

with

$$\begin{aligned}
&a_1 = \frac{3\kappa}{\ell^2\eta},\ b_1 = \frac{3}{2}\frac{M_2 - M_1}{\ell^2\eta} - \frac{3}{8}\frac{M_1+M_2}{\ell^2\xi},\ b_2 = \frac{3}{4}\frac{M_1+M_2}{\ell^2\eta}, \\
&a_2 = \frac{24\,\kappa}{\ell^3\eta},\ b_3 = \frac{3(5M_2 - 3M_1)}{2\,\ell^3\eta},\ b_4 = \frac{12(M_1 - M_2)}{\ell^3\eta}.
\end{aligned} \tag{6}$$

The assumptions before (3) and these of the theorem imply $b_4 \neq 0$, $a_2 \neq 0$, $M_1 + M_2 \neq 0$ and $M_1 - M_2 \neq 0$, hence

$$z_4 = \frac{1}{b_4}\,\alpha,\ z_3 = \frac{8\,(M_1 - M_2)}{a_2\,(M_1 + M_2)}\Big(b_4\,\theta - b_3\,\alpha\Big) \tag{7}$$

defines a change of coordinates[2] $(x, y, \theta, \alpha) \mapsto (x, y, z_3, z_4)$. Since $8(M_1 - M_2)/(M_1 + M_2) = 1/(1/2 + b_3/b_4)$, one deduces from (1), (2), (5), and (7) the following expressions of $\dot{z}_3$ and $\dot{z}_4$, where $r_{i,j}$ $(i = 0, 1, 2,\ j = 3, 4)$ are smooth functions of one variable:

$$\begin{aligned}
\dot{z}_3 &= z_4\,\big(1 + z_4\,r_{0,3}(z_4) + H_\perp\,r_{1,3}(z_4) + H_\parallel\,r_{2,3}(z_4)\big)\,, \\
\dot{z}_4 &= H_\perp - z_4\big(a_2 + z_4\,r_{0,4}(z_4) + H_\perp\,r_{1,4}(z_4) + H_\parallel\,r_{2,4}(z_4)\big).
\end{aligned} \tag{8}$$

Substituting $\alpha = b_4\,z_4$ and $\theta = b_3\,z_3 + \dfrac{a_2\,(M_1 + M_2)}{8b_4\,(M_1 - M_2)}\,z_4$ in (2), expanding $\sin(\theta)$ and $\cos(\theta)$ around 0 and using (5) one gets, with c_1, c_2, c_3 three constants that may easily be

[2]For the reader's information: the linear approximation of (1) is in (non-controllable) Brunovsky form in coordinates $(x,\ y - b_2 z_3\,,\ z_3, z_4)$.

computed from $a_1, a_2, b_1, b_2, b_3, b_4$, the expression:

$$\dot{x} = c_3\, z_4^2 + (c_1 z_3 + c_2 z_4) H_\perp + z_4^2 R_1 + z_3 z_4\, R_2 + z_3^2 R_3 \tag{9}$$

$$\begin{aligned} \text{with } R_1 &= z_4\, \rho_1(z_3, z_4) + \rho_2(z_3, z_4)\, H_\perp + \rho_3(z_3, z_4)\, H_\parallel, \\ R_2 &= \rho_4(z_3, z_4) + \rho_5(z_3, z_4)\, H_\perp + \rho_6(z_3, z_4)\, H_\parallel, \\ R_3 &= \rho_7(z_3, z_4) + \rho_8(z_3, z_4)\, H_\perp + \rho_9(z_3, z_4)\, H_\parallel, \end{aligned} \tag{10}$$

and ρ_i, $i = 1 \ldots 9$, nine smooth functions of two variables. Then, defining $\zeta = x - c_1 z_3 z_4 - \frac{1}{2} c_2 z_4^2$, one has

$$\begin{aligned} &\dot{\zeta} = z_4^2 \big(c_0 + \widetilde{R}_1\big) + z_3 z_4\, \widetilde{R}_2 + z_3^2\ \widetilde{R}_3 \\ &\text{with } c_0 = c_3 + a_2 c_2 - c_1 \end{aligned} \tag{11}$$

with $\widetilde{R}_1, \widetilde{R}_2, \widetilde{R}_3$ three functions of z_3, z_4, $H_\perp, H_\parallel$ that can be expended similarly to R_1, R_2 and R_3 in (10). Computing c_0 from the expressions of c_1, c_2, c_3, one finds that it is nonzero from the assumptions of Theorem 5:

$$c_0 = \frac{108\kappa}{\ell^8 \eta^3 \xi} (M_2^2 - M_1^2)(\eta - \xi) \neq 0. \tag{12}$$

From Lemma 3, for each $\varepsilon > 0$, there exists a "loop"

$$t \mapsto (x^\varepsilon(t), y^\varepsilon(t), \theta^\varepsilon(t), \alpha^\varepsilon(t), H_\perp^\varepsilon(t), H_\parallel^\varepsilon(t)\,)$$

defined on $[0, \varepsilon]$, solution of (1), and such that

$$|H_\perp^\varepsilon(t)| \leqslant \varepsilon \text{ and } |H_\parallel^\varepsilon(t)| \leqslant \varepsilon \text{ for all } t \text{ in } [0, \varepsilon], \tag{13}$$

$$\begin{aligned} (x^\varepsilon(0), y^\varepsilon(0), z_3^\varepsilon(0), z_4^\varepsilon(0)) &= (0,0,0,0), \\ (x^\varepsilon(\varepsilon), y^\varepsilon(\varepsilon), z_3^\varepsilon(\varepsilon), z_4^\varepsilon(\varepsilon)) &= (0,0,0,0), \end{aligned} \tag{14}$$

$$(x^\varepsilon(t), y^\varepsilon(t), z_3^\varepsilon(t), z_4^\varepsilon(t)) \neq (0,0,0,0) \text{ for one } t \text{ in } [0, \varepsilon], \tag{15}$$

where $z_3^\varepsilon(t), z_4^\varepsilon(t)$ are defined from $(\theta^\varepsilon(t), \alpha^\varepsilon(t))$ as in (7). Along these solutions, the functions $r_{i,j}$ and ρ_i are bounded uniformly with respect to t in $[0, \varepsilon]$ and ε in $(0, \varepsilon_0]$ for some small enough $\varepsilon_0 > 0$. In particular, using (7) and (11), we deduce that for each $\varepsilon \in (0, \varepsilon_0]$, there are three functions $u^\varepsilon(.)$, $v^\varepsilon(.)$, $w^\varepsilon(.)$ such that

$$\dot{z}_3^\varepsilon(t) = z_4^\varepsilon(t)\, u^\varepsilon(t)\,, \tag{16}$$

$$\dot{\zeta}^\varepsilon(t) = z_4^\varepsilon(t)^2 \big(c_0 + \varepsilon\, w^\varepsilon(t)\big) + z_3^\varepsilon(t) z_4^\varepsilon(t)\ v^\varepsilon(t) + z_3^\varepsilon(t)^2 s^\varepsilon(t)\,, \tag{17}$$

$$|u^\varepsilon(t)| \leqslant K,\ |v^\varepsilon(t)| \leqslant K,\ |w^\varepsilon(t)| \leqslant K,\ |s^\varepsilon(t)| \leqslant K\,. \tag{18}$$

Here and hereafter, $K > 0$ denotes a constant independent of ε and t that may vary from line to line. One has:

Lemma 6. *Equations* (16) *and* (18) *imply, for* ε *in* $(0, \varepsilon_0]$,

$$\int_0^\varepsilon |z_3^\varepsilon(t) z_4^\varepsilon(t)| \mathrm{d}t \leqslant K\varepsilon \int_0^\varepsilon z_4^\varepsilon(t)^2 \mathrm{d}t \quad \textit{and} \quad \int_0^\varepsilon z_3^\varepsilon(t)^2 \mathrm{d}t \leqslant K^2 \varepsilon^2 \int_0^\varepsilon z_4^\varepsilon(t)^2 \mathrm{d}t\,.$$

Let us temporarily admit this lemma. Then, equations (14) imply $\int_0^\varepsilon \dot{\zeta}^\varepsilon(t) \mathrm{d}t = 0$. Substituting $\dot{\zeta}^\varepsilon(t)$ from (17) and using (18) and Lemma 6 yields, for any ε in $(0, \varepsilon_0]$,

$$|c_0| \int_0^\varepsilon z_4^\varepsilon(t)^2 \mathrm{d}t \leqslant \big(2K\varepsilon + K^2\varepsilon^2\big) \int_0^\varepsilon z_4^\varepsilon(t)^2 \mathrm{d}t\,. \tag{19}$$

Since $c_0 \neq 0$, this implies that $z_4^\varepsilon(t)$ is identically zero on $[0,\varepsilon]$ for $\varepsilon > 0$ small enough. From (8), this implies that the control $H_\perp^\varepsilon(t)$ is identically zero. Since all the maps $f_{i,j}$ with $i \neq 1$ are zero at zero (see (2) and (5)), all state variables are constant if z_4^ε and $H_\perp^\varepsilon$ are identically zero, meaning that $(x^\varepsilon(t), y^\varepsilon(t), \theta^\varepsilon(t))$ are identically zero on $[0,\varepsilon]$ for $\varepsilon > 0$ small enough. Therefore, any small enough loop with small enough control is trivial, which contradicts (15) and hence contradicts STLC. □

Proof of Lemma 6. From (16) and (18), one gets

$$|z_3^\varepsilon(t)| \leqslant K\int_0^t |z_4^\varepsilon(\tau)|\,\mathrm{d}\tau, \quad \int_0^\varepsilon |z_3^\varepsilon(t)|\mathrm{d}t \leqslant K\varepsilon\int_0^\varepsilon |z_4^\varepsilon(\tau)|\mathrm{d}\tau\,.$$

The following two inequalities follow:

$$\begin{aligned}\int_0^\varepsilon |z_3^\varepsilon(t) z_4^\varepsilon(t)|\,\mathrm{d}t &\leqslant K\int_0^\varepsilon \Big(\int_0^t |z_4^\varepsilon(\tau)|\,\mathrm{d}\tau\Big)|z_4^\varepsilon(t)|\,\mathrm{d}t \\ &\leqslant K\iint_{(t,\tau)\in[0,\varepsilon]^2} |z_4^\varepsilon(t)|\,|z_4^\varepsilon(\tau)|\mathrm{d}t\,\mathrm{d}\tau = K\,\Big(\int_0^\varepsilon |z_4^\varepsilon(t)|\mathrm{d}t\Big)^2,\end{aligned}$$

$$\begin{aligned}\int_0^\varepsilon z_3^\varepsilon(t)^2\mathrm{d}t &\leqslant K\int_0^\varepsilon \Big(\int_0^t |z_4^\varepsilon(\tau)|\,\mathrm{d}\tau\Big)|z_3^\varepsilon(t)|\,\mathrm{d}t \leqslant K\iint_{(t,\tau)\in[0,\varepsilon]^2} |z_3^\varepsilon(t)|\,|z_4^\varepsilon(\tau)|\mathrm{d}t\,\mathrm{d}\tau \\ &= K\Big(\int_0^\varepsilon |z_3^\varepsilon(t)|\mathrm{d}t\Big)\Big(\int_0^\varepsilon |z_4^\varepsilon(t)|\mathrm{d}t\Big) \leqslant K^2\varepsilon\,\Big(\int_0^\varepsilon |z_4^\varepsilon(t)|\mathrm{d}t\Big)^2.\end{aligned}$$

We conclude by applying the Cauchy-Schwartz inequality. □

4. Conclusion

We proved that the local controllability results in [1] are sharp in the sense that STLC occurs only for the values of the parameters for which it was already proved in that note.

On the one hand, from the theoretical point of view of controllability, although it deals with a very specific class of systems, Theorem 5 is a necessary condition for STLC. Conditions for STLC have been much studied in the last decades, see for instance [2] or [3] and references therein. Many sophisticated and powerful sufficient conditions have been stated, but necessary conditions are always specific, see for instance [4, 5]. Theorem 5 is not, to the best of our knowledge, a consequence of known necessary conditions.

On the other hand, the implications for locomotion at low Reynolds number via an external magnetic field are not clear. Comments on that matter are left to further research.

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