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► **To cite this version:**

| Jf Aujol, Ch Dossal. An analysis of on ODE associated to FISTA. 2017. hal-01547251v1

HAL Id: hal-01547251

<https://inria.hal.science/hal-01547251v1>

Preprint submitted on 26 Jun 2017 (v1), last revised 17 Jul 2017 (v2)

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An analysis of on ODE associated to FISTA

JF Aujol and Ch. Dossal

June 2017

Abstract This paper is devoted to the study of an ODE associated to FISTA. New vonvergence results are presented, even in the case of non convex functions. The asymptotic rate are proven to be optimal with the study of particular instances. Robustness to noise is also investigated.

1 Introduction

We study the following equation

$$\ddot{x}(t) + \frac{b}{t}\dot{x}(t) + \nabla F(x(t)) = 0 \quad (1)$$

where F is a differentiable function and $b > 0$, having at least one minimizer. We denote by x^* such a minimizer and $W(t) = F(x(t)) - F(x^*)$.

This ODE is associated to FISTA [5] or Accelerated Gradient Method [9] :

$$x_{n+1} = y_n - \gamma \nabla F(y_n) \text{ and } y_n = x_n + \frac{n}{n+b}(x_n - x_{n-1}) \quad (2)$$

with γ and β positive parameter. This equation, including or not a perturbation term, has been widely studied in the literature [3, 10, 6, 4, 8]. It was proved that if F is convex with Lipschitz gradient and if $b > 0$, the trajectory $F(x(t))$, where x is solution of (1), converges to the minimum $F^* := F(x^*)$ of F . It is also known that for $b \geq 3$ and F convex we have

$$W(t) := F(x(t)) - F(x^*) = o\left(\frac{1}{t^2}\right) \quad (3)$$

Extending to the continuous setting the proof of Chambolle et al. [7] of the convergence of iterates of FISTA, Attouch et al. [1] proved that for $b > 3$ the trajectory x weakly converges to a minimizer \tilde{x} of F .

Su et al. [10] proposed some new results, proving the integrability of $t \mapsto tW(t)$

if $b > 3$, and they gave more accurate bounds on $W(t)$ in the case of strong convexity and if $g = (F - F^*)^\beta$ is convex. More recently several studies including a perturbation term [1] have been proposed but always in the setting when $b \geq 3$.

The main contribution of this article is to study the decay of $W(t)$ for any $b > 0$ and to give more precise results on the convergence of the trajectory of x . We also study the robustness to perturbations. The results are available for any function F such that $g = (F - F^*)^\beta$ is convex. These results extend the known results about solutions of (1) to a class of non-convex functions and complete the known results on convex functions. Such hypotheses with $\beta > 1$ were also used by Su et al. [10] to show that the $\frac{1}{t^2}$ rate may hold for such functions even for some $b < 3$.

Moreover we prove that the given bounds on the decay of $W(t)$ are optimal, by building examples where they are achieved.

The plan of the paper is the following. In Section 2, we present the main results of the paper, and we give some comments on their implications. Section 3 details the proof of Theorem 1 which states the main result of the paper. We then study a particular case in Section 4 to show that the asymptotic rate of point 1. of Theorem 1 is optimal. Robustness to noise is investigated in Section 5. The proofs of technical lemmas is postponed to the appendix.

2 Decay of $F - F^*$ and convergence of the trajectory x for $b > 0$.

In this paper, we consider a function F with values in \mathbb{R} , and defined on some finite dimensional Hilbert space H (e.g. $H = \mathbb{R}^n$). Notice that in this case, if F is continuous, then there exists locally a solution x for (1) (thanks to the Cauchy-Peano-Arzela theorem, see e.g p. 137 in J-P Demailly: Analyse numerique et equations aux drivees partielles). Notice also that this existence theorem only holds since H is of finite dimension.

The main contribution of this paper is the following Theorem :

Theorem 1. *Let F be a lower semi-continuous function (lsc) from a (finite dimension) Hilbert space H to \mathbb{R} having a minimizer $x^* \in H$ and $\beta > 0$ such that the function $g(x) = (F(x) - F(x^*))^\beta$ is convex. Then*

1. *If $b \in (0, 1 + 2\beta)$ then for any solution $x(t)$ of (1)*

$$W(t) = O\left(\frac{1}{t^{\frac{2b}{2\beta+1}}}\right) \quad (4)$$

2. if $b \geq 2\beta + 1$ then for any solution $x(t)$ of (1)

$$W(t) = O\left(\frac{1}{t^2}\right) \quad (5)$$

3. If $b > 2\beta + 1$ then for any solution $x(t)$ of (1)

(a) The trajectory $x(t)$ converges to a minimizer \tilde{x} of F .

(b)

$$W(t) = o\left(\frac{1}{t^2}\right) \quad (6)$$

Several comments about this theorem:

- Points 2. and 3. are known for the convex case, ie when $\beta = 1$, see for example [8, 1].
- The two last points generalize what is known on convex functions. If the function F is convex and if β can be chosen smaller than 1, the proposition ensures that the optimal decay $\frac{1}{t^2}$ is achieved even for some $b \leq 3$. If F is not convex but there exists $\beta > 1$ such that $g = (F - F(x^*))^\beta$ is convex, the solution of (1) may converge and ensure good decay if b is chosen sufficiently large.
- The first point ensures that if b is too small, for example $b < 3$ for the convex case, one also has some decay of W , which is not as fast as what can be expected for large b .
This decay was proved by Su et al. [10] for strongly convex functions.
- The decay in $O\left(\frac{1}{t^{\frac{2b}{2\beta+1}}}\right)$ is optimal in the sens that we show in Proposition 6 that it can be achieved for some explicit functions F for any $\beta > 1$. As a consequence one cannot expect a $o\left(\frac{1}{t^{\frac{2b}{2\beta+1}}}\right)$ decay if $b \in]0, 1 + 2\beta]$
- For $\beta = 1$ and $b = 3$, we cannot exhibit such a function F as differential inclusion are beyond the scope of this paper. But the same analysis shows that this optimal rate is reached for the differential inclusion associated to (1) with $F(x) = |x|$. Hence for this function we cannot have $W(t) = o\left(\frac{1}{t^2}\right)$ with $b = 3$.

It is worth rewriting Theorem 1 in the particular case when $\beta = 1$, ie when F is a convex function.

Corollary 1. *Let F be a convex lower semi-continuous function (lsc) function from a (finite dimension) Hilbert space H to \mathbb{R} having a minimizer $x^* \in H$. Then*

1. *If $b \in (0, 3)$ then for any solution $x(t)$ of (1)*

$$W(t) = O\left(\frac{1}{t^{\frac{2b}{3}}}\right) \quad (7)$$

2. *if $b \geq 3$ then for any solution $x(t)$ of (1)*

$$W(t) = O\left(\frac{1}{t^2}\right) \quad (8)$$

3. *If $b > 3$ then for any solution $x(t)$ of (1)*

(a) *The trajectory $x(t)$ converges to a minimizer \tilde{x} of F .*

(b)

$$W(t) = o\left(\frac{1}{t^2}\right) \quad (9)$$

The two first points of the corollary are well known. But so far, nothing was reported in the literature for the case $b \in (0, 3)$, even in the convex case. Our results show that the same decay for W cannot be expected, but that some control of the W nevertheless remains.

3 Proof of Theorem 1

This section is devoted to the proof of Theorem 1. We split the proof in 3 points.

The points 2. and 3. of Theorem 1 are generalizations of classical results to $\beta > 0$. The main contribution is the point 1.

3.1 Preliminary material and proof of point 2. of Theorem 1

The proofs of all the following results rely on the next lemma, bounding the value of $F(x)$ using $\nabla F(x)$, if g is convex.

Lemma 1. *Let $\beta \in]0, 1[$, if $g(x) = (F(x) - F(x^*))^\beta$ is convex then*

$$\frac{1}{\beta}(F(x(t)) - F(x^*)) \leq \langle \nabla F(x(t)), x(t) - x^* \rangle \quad (10)$$

Proof Since g is convex we have

$$g(x(t)) \leq \langle \nabla g(x(t)), x(t) - x^* \rangle \quad (11)$$

and $\nabla g(x(t)) = \beta(F(x(t)) - F(x^*))^{\beta-1} \nabla F(x(t))$. Replacing $g(x(t))$ by $(F(x(t)) - F(x^*))^\beta$ we get the result. \square

Notations: In the following we use the notation :

$$W(t) := F(x(t)) - F(x^*) \quad (12)$$

where x^* is a minimizer of F . If x^* is a fixed minimizer of F , $\lambda \in \mathbb{R}$ and x a solution of (1) we denote :

$$u(t) := tW(t) + \frac{1}{2t} \|\lambda(x(t) - x^*) + t\dot{x}(t)\|^2 \quad (13)$$

and

$$v(t) := \frac{1}{2t} \|x(t) - x^*\|^2 \quad (14)$$

The analysis is based on Lyapunov functions. Following Su, Boyd and Candès, for any $(\lambda, \xi) \in \mathbb{R}^2$ we define

$$\mathcal{E}_{\lambda, \xi}(t) = t^2 W(t) + \frac{1}{2} \|\lambda(x(t) - x^*) + t\dot{x}(t)\|^2 + \frac{\xi}{2} \|x(t) - x^*\|^2. \quad (15)$$

One can observe that this energy can be defined using functions u and v

$$\mathcal{E}(t) = tu(t) + \xi tv(t). \quad (16)$$

As usual, we need to compute the derivative $\mathcal{E}'_{\lambda, \xi}$. We will need two different expressions of the derivative.

Lemma 2.

$$\mathcal{E}'_{\lambda, \xi}(t) = 2tW(t) + t(\lambda + 1 - b) \|\dot{x}(t)\|^2 \quad (17)$$

$$+ \lambda t \langle -\nabla F(x(t)), x(t) - x^* \rangle + (\lambda(\lambda + 1) - b\lambda + \xi) \langle \dot{x}(t), x(t) - x^* \rangle \quad (18)$$

The proof of Lemma 2 is given in Appendix and is only a sequence of calculations.

Under the hypotheses of Proposition 1 we have the following Lemma :

Lemma 3. *If g is convex, then:*

$$\mathcal{E}'_{\lambda, \xi}(t) \leq (2 - \frac{\lambda}{\beta})tW(t) + (\xi + \lambda^2 + \lambda - b\lambda) \langle \dot{x}(t), x(t) - x^* \rangle + (\lambda + 1 - b)t \|\dot{x}(t)\|^2 \quad (19)$$

Proof. To prove Lemma 3 we only apply the inequality of Lemma 1 in the equality of Lemma 2. \square

The following result will give the proof of point 2. of Theorem 1.

Proposition 1. *If $b \geq 2\beta + 1$, if $\lambda = 2\beta$ and $\xi = \lambda(b - \lambda - 1) \geq 0$, then*

$$\mathcal{E}'_{\lambda,\xi}(t) \leq 0 \quad (20)$$

Proof. Let us suppose $b \geq 2\beta + 1$. If we apply Lemma 3 with $\lambda = 2\beta$ and $\xi = \lambda(b - \lambda - 1) \geq 0$ we have

$$\mathcal{E}'_{\lambda,\xi}(t) \leq (2\beta + 1 - b) \|\dot{x}(t)\|^2 \leq 0 \quad (21)$$

\square

Proof of point 2. of Theorem 1: It follows from Proposition 1 that the function $\mathcal{E}_{\lambda,\xi}$ is non increasing and thus bounded. As a consequence, we see that the second point of Theorem 1 holds true. \square

3.2 Proof of point 1. of Theorem 1.

In this section suppose that $b \in (0, 2\beta + 1)$. To prove this point we use another Lyapunov energy. We need another expression of $\mathcal{E}_{\lambda,\xi}(t)$:

Lemma 4.

$$\mathcal{E}'_{\lambda,\xi}(t) = 2tW(t) + \lambda t \langle -\nabla F(x(t)), x(t) - x^* \rangle \quad (22)$$

$$+ (\xi - \lambda(\lambda + 1 - b)) \langle \dot{x}(t), x(t) - x^* \rangle \quad (23)$$

$$+ (\lambda + 1 - b) \frac{1}{t} \|\lambda(x(t) - x^*) + t\dot{x}(t)\|^2 - \frac{\lambda^2(\lambda + 1 - b)}{t} \|x(t) - x^*\|^2 \quad (24)$$

The proof of Lemma 4 is given in Appendix.

Under the hypotheses of Proposition 1 we have the following Lemma :

Lemma 5. *If g is convex, then:*

$$\mathcal{E}'_{\lambda,\xi}(t) \leq (2 - \frac{\lambda}{\beta})tW(t) + (\xi - \lambda(\lambda + 1 - b)) \langle \dot{x}(t), x(t) - x^* \rangle \quad (25)$$

$$+ (\lambda + 1 - b) \frac{1}{t} \|\lambda(x(t) - x^*) + t\dot{x}(t)\|^2 - \frac{\lambda^2(\lambda + 1 - b)}{t} \|x(t) - x^*\|^2 \quad (26)$$

Proof. To prove Lemma 5 we only apply the inequality of Lemma 1 in the equality of Lemma 4. \square

Notations: Now, we choose $\lambda = \frac{2b\beta}{2\beta+1}$ and $\xi = \lambda(\lambda + 1 - b)$ and we denote $\mathcal{E}_{\lambda,\xi} = \mathcal{E}$.

For these parameters one can observe that $\lambda + 1 - b = 1 - \frac{b}{2\beta+1} \geq 0$. We can define

$$c := 2 - \frac{2b}{2\beta+1} \quad (27)$$

Notice that $c > 0$. We introduce the new Lyapunov energy:

$$\mathcal{H}(t) := t^{-c}\mathcal{E}(t). \quad (28)$$

We can rewrite Lemma 5 in a more compact way:

Lemma 6. *If g is convex, then:*

$$\mathcal{E}'(t) \leq cu(t) - dv(t). \quad (29)$$

where $d > 0$ is defined as:

$$d := - \left(\frac{2b\beta}{2\beta+1} \right)^2 \left(1 - \frac{b}{2\beta+1} \right) \quad (30)$$

Proof.

$$\mathcal{E}'(t) \leq \left(2 - \frac{2b}{2\beta+1} \right) tW(t) + \left(1 - \frac{b}{2\beta+1} \right) \frac{1}{t} \left\| \frac{2b\beta}{2\beta+1} (x(t) - x^*) + t\dot{x}(t) \right\|^2 \quad (31)$$

$$- \left(\frac{2b\beta}{2\beta+1} \right)^2 \left(1 - \frac{b}{2\beta+1} \right) \frac{1}{t} \|x(t) - x^*\|^2 \quad (32)$$

Using

$$u(t) = tW(t) + \frac{1}{2t} \left\| \frac{2b\beta}{2\beta+1} (x(t) - x^*) + t\dot{x}(t) \right\|^2, \quad (33)$$

the definition of $v(t)$, c and d , we get the result. \square

The following result will give the proof of point 1. of Theorem 1.

Proposition 2. *If $b \in (0, 2\beta + 1)$, if $\lambda = \frac{2b\beta}{2\beta+1}$ and $\xi = \lambda(\lambda + 1 - b)$, then*

$$\mathcal{H}'(t) \leq 0. \quad (34)$$

Proof. We have

$$\mathcal{H}'(t) = t^{-c-1}(t\mathcal{E}'(t) - c\mathcal{E}(t)). \quad (35)$$

Using Lemma 6, we deduce that:

$$\mathcal{H}'(t) \leq - \left(2 - \frac{2b(1-\beta)}{2\beta+1} \right) t^{-c}v(t) \leq 0. \quad (36)$$

\square

Proof of point 1. of Theorem 1: It follows from Proposition 2 that the function \mathcal{H}' is non increasing and thus bounded. We thus get that for all $t \geq t_0$, we have

$$t^2 W(t) \leq \mathcal{E}(t) \leq \frac{t^c}{t_0^c} \mathcal{E}(t_0) \quad (37)$$

ie for any $t \geq t_0$

$$W(t) \leq \frac{\mathcal{E}(t_0)}{t_0^{2-\frac{2b}{2\beta+1}}} \frac{1}{t^{\frac{2b}{2\beta+1}}}. \quad (38)$$

As a consequence, we see that point 1. of Theorem 1 holds true. \square

3.3 Convergence of the trajectory and proof of point 3. of Theorem 1

We start with some classical energy estimates.

Proposition 3. *If $b > 1 + 2\beta$, then we have:*

$$\int_{t_0}^{+\infty} t W(t) dt < +\infty \quad (39)$$

and

$$\int_{t_0}^{+\infty} t \|\dot{x}(t)\|^2 dt < +\infty \quad (40)$$

Proof. if $T > t_0$ and $b > 1 + 2\beta$, choosing $\lambda = b - 1$ and $\xi = \lambda(b - \lambda - 1) = 0$, by integrating (25) in $[t_0, T]$, we obtain (notice that we thus have $\lambda > 2\beta$, i.e. $2 - \lambda/\beta < 0$):

$$\left(\frac{\lambda}{\beta} - 2\right) \int_{t_0}^T t W(t) dt \leq \mathcal{E}_{b-1,0}(t_0) - \mathcal{E}_{b-1,0}(T) \leq \mathcal{E}_{b-1,0}(t_0) \quad (41)$$

In a similar way if we choose $\lambda = 2\beta$ and $\xi = \lambda(b - \lambda - 1) > 0$, by integrating (25) in $[t_0, T]$, we obtain :

$$(b - 1 - 2\beta) \int_{t_0}^T t \|\dot{x}(t)\|^2 dt \leq \mathcal{E}_{2\beta,\xi}(t_0) - \mathcal{E}_{2\beta,\xi}(T) \leq \mathcal{E}_{2\beta,\xi}(t_0) \quad (42)$$

Since (41) and (42) hold for every $T > t_0$, we can conclude. \square

Thanks to the energy estimates of Proposition 3, we can establish the convergence of the trajectory.

Proposition 4. *If $b > 1 + 2\beta$, then the trajectory $x(t)$ converges to x^* .*

For the proof of Proposition 4 we use the continuous version of Opial's Lemma for which we omit the proof

Lemma 7. *Let $S \subset \mathcal{H}$ be a non-empty set and $x : [t_0, +\infty)$ such that the following conditions hold:*

1. $\lim_{t \rightarrow +\infty} \|x(t) - x^*\| \in \mathbb{R}$, for all $x^* \in S$
2. Every weak-cluster point of $x(t)$ belongs to S

Then we have that $x(t)$ converges weakly to a point of S as $t \rightarrow +\infty$.

Remark: We will use the previous Lemma with $S = \operatorname{argmin} F$. In fact Opial's Lemma holds true for a general separable Hilbert space \mathcal{H} , but in our case as \mathcal{H} has a finite dimension, we also deduce strong convergence of $x(t)$ to a point of S .

Proof of Proposition 4: In order to apply Opial's Lemma, we define $\psi(t) = \frac{1}{2} \|x(t) - x^*\|^2$.

We have

$$\dot{\psi}(t) = \langle \dot{x}(t), x(t) - x^* \rangle \quad \text{and} \quad \ddot{\psi}(t) = \|\dot{x}(t)\|^2 + \langle \ddot{x}(t), x(t) - x^* \rangle \quad \text{a.e.}$$

By using (1), we obtain

$$\begin{aligned} \ddot{\psi}(t) + \frac{b}{t} \dot{\psi}(t) &= \|\dot{x}(t)\|^2 + \langle \ddot{x}(t) + \frac{b}{t} \dot{x}(t), x(t) - x^* \rangle \\ &= \|\dot{x}(t)\|^2 - \langle z(t), x(t) - x^* \rangle \\ &\leq \|\dot{x}(t)\|^2 - W(t) \leq \|\dot{x}(t)\|^2 \end{aligned} \tag{43}$$

Hence by multiplying both sides by t^b we obtain

$$t^b \ddot{\psi}(t) + b t^{b-1} \dot{\psi}(t) \leq t^b \|\dot{x}(t)\|^2 \tag{44}$$

By integrating over $[t_0, s]$ we find

$$\dot{\psi}(s) \leq \frac{t_0^b \dot{\psi}(t_0)}{s^b} + \frac{1}{s^b} \int_{t_0}^s t^b \|\dot{x}(t)\|^2 dt \leq \frac{C_0}{s^b} + \frac{1}{s^b} \int_{t_0}^s t^b \|\dot{x}(t)\|^2 dt \tag{45}$$

where C_0 is a positive constant. If we consider the positive part of $\dot{\psi}$, i.e. $[\dot{\psi}]_+(t) = \max\{0, \dot{\psi}(t)\} \quad \forall t \geq t_0$, we have

$$[\dot{\psi}]_+(s) \leq \frac{C_0}{s^b} + \frac{1}{s^b} \int_{t_0}^s t^b \|\dot{x}(t)\|^2 dt \tag{46}$$

Hence, by applying Fubini's Theorem, for $T > t_0$, we have that :

$$\begin{aligned}
\int_{t_0}^T [\dot{\psi}]_+(s) ds &\leq C_0 \int_{t_0}^T \frac{1}{s^b} + \int_{t_0}^T \frac{1}{s^b} \int_{t_0}^s t^b \|\dot{x}(t)\|^2 dt ds \\
&= (b-1)C_0(t_0^{1-b} - T^{1-b}) + \int_{t_0}^T t^b \|\dot{x}(t)\|^2 \left(\int_t^T s^{-b} ds \right) dt \quad (47) \\
&\leq C_0 + (b-1) \int_{t_0}^T t \|\dot{x}(t)\|^2 dt
\end{aligned}$$

Since, by Proposition 3, for $b > 1 + 2\beta$, the right-hand side of this inequality is finite for every $T > t_0$, we deduce that :

$$\int_{t_0}^{+\infty} [\dot{\psi}]_+(s) ds < +\infty \quad (48)$$

Hence if we consider the function $\theta(t) = \psi(t) - \int_{t_0}^t [\dot{\psi}]_+(s) ds \quad \forall t \in [t_0, +\infty)$, we have that θ is non-increasing and bounded from below on $[t_0, +\infty)$, so it converges to its infimum $\theta_\infty = \inf_{t \geq t_0} \{\theta(t)\}$.

As a consequence we obtain that :

$$\lim_{t \rightarrow \infty} \psi(t) = \theta_\infty + \int_{t_0}^{+\infty} [\dot{\psi}]_+(s) ds \in \mathbb{R} \quad (49)$$

This shows that the first condition of Opial's Lemma is satisfied.

For the second condition, let \tilde{x} be a weak-cluster point of the trajectory $x(t)$, when $t \rightarrow +\infty$. By lower semi-continuity of F , we have that :

$$F(\tilde{x}) \leq \liminf_{t \rightarrow \infty} F(x(t)) \quad (50)$$

From point 2. of Theorem 1, we have that $\lim_{t \rightarrow \infty} F(x(t)) = F(x^*)$, where x^* is a minimizer, so that $\tilde{x} \in \operatorname{argmin} F$, which shows that the second condition of Opial's Lemma is satisfied, therefore we can conclude the proof by applying Opial's Lemma. \square

It is now classical to establish that a better convergence rate can be reached [1].

Proposition 5. *Let $f b > 1 + 2\beta$, x a solution of (1) and x^* a minimizer of F . Then*

$$\lim_{t \rightarrow \infty} t^2 W(t) = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} t \|\dot{x}(t)\| = 0 \quad (51)$$

In other words : $W(t) = o(t^{-2})$ and $\|\dot{x}(t)\| = o(t^{-1})$

Proof. First of all, we consider the following energy function

$$U(t) = t^2W(t) + \frac{t^2}{2} \|\dot{x}(t)\|^2 \geq 0 \quad , \forall t \in [t_0, +\infty) \quad (52)$$

Let $T > t_0$. We have

$$\frac{d}{dt}U(t) = t^2\langle z, \dot{x}(t) \rangle + t^2\langle \ddot{x}, \dot{x}(t) \rangle + 2tW(t) + t \|\dot{x}(t)\|^2 \quad (53)$$

By using (1) and $b > 1 + 2\beta$, we find

$$\frac{d}{dt}U(t) = 2tW(t) + (1 - b)t \|\dot{x}(t)\|^2 \leq 2tW(t) \quad (54)$$

Hence if we consider the positive part of $\frac{d}{dt}U(t)$, i.e.

$[\frac{d}{dt}U(t)]_+(t) = \max\{0, \frac{d}{dt}U(t)(t)\} \quad \forall t \geq t_0$, we obtain :

$$\left[\frac{d}{dt}U(t) \right]_+ \leq 2tW(t) \quad (55)$$

By Proposition 3 for $b > 1 + 2\beta$ the term $2tW(t)$ is integrable on $[t_0, +\infty)$, and so is $\left[\frac{d}{dt}U(t) \right]_+$.

We define the function $\Theta(t) = U(t) - \int_{t_0}^t \left[\frac{d}{ds}U(s) \right]_+ ds$. By definition, Θ has negative derivative on $(t_0, +\infty)$; hence it is non-increasing in $(t_0, +\infty)$. This together with the fact that the function Θ is bounded from below on $[t_0, +\infty)$, implies that when $t \rightarrow +\infty$, $\Theta(t)$ converges to its infimum $\Theta_\infty = \inf_{t \geq t_0} \{\Theta(t)\}$

As a consequence we have that $U(t)$ also converges when $t \rightarrow +\infty$ with :

$$\lim_{t \rightarrow +\infty} U(t) = \lim_{t \rightarrow +\infty} \Theta(t) + \int_{t_0}^{+\infty} [\dot{U}(t)]_+ dt \in \mathbb{R} \quad (56)$$

Finally since $b > 1 + 2\beta$, by Proposition 3, we have:

$$\int_{t_0}^{+\infty} \frac{1}{t} U(t) dt = \int_{t_0}^{+\infty} tW(t) dt + \frac{1}{2} \int_{t_0}^{+\infty} t \|\dot{x}(t)\|^2 dt < +\infty \quad (57)$$

As $\int_{t_0}^{+\infty} \frac{1}{t} dt = +\infty$ and $U(t)$ converges when $t \rightarrow +\infty$, we deduce that $U(t)$ converges to zero, when $t \rightarrow +\infty$. This together with the positivity of $t^2W(t)$ and $\frac{t^2}{2} \|\dot{x}(t)\|^2$ allow us to conclude the proof of the proposition. \square

Proof of point 3. of Theorem 1: Propositions 4 and 5 give the results of point 3. of Theorem 1. \square

4 ODE associated to $F(x) = |x|^\alpha$.

The goal of this section is to show that the asymptotic rate of point 1. of Theorem 1 is optimal, in the sense that it can be obtained as a limit case for class of functions F .

If $F(x) = |x|^\alpha$ with $\alpha > 1$ we can consider the following ordinary differential equation (ODE):

$$\ddot{x}(t) + \frac{b}{t}\dot{x}(t) + \alpha \frac{|x|^\alpha}{x} = 0 \quad (58)$$

Of course, since $\alpha > 1$, we have $\lim_{x \rightarrow 0} \frac{|x|^\alpha}{x} = 0$ and the ODE is well defined.

Notations: As before, we choose $\lambda = \frac{2b\beta}{2\beta+1}$ and $\xi = \lambda(\lambda + 1 - b)$ and we denote $\mathcal{E}_{\lambda,\xi} = \mathcal{E}$. For these parameters one can observe that $\lambda + 1 - b = 1 - \frac{b}{2\beta+1} \geq 0$. We recall that

$$c = 2 - \frac{2b}{2\beta + 1} \quad (59)$$

Notice that $c \geq 0$. We introduce the new Lyapunov energy:

$$\mathcal{H}_\varepsilon(t) := t^{-c+\varepsilon}\mathcal{E}(t) = t^\varepsilon\mathcal{H}(t). \quad (60)$$

Proposition 6. *Let $\beta \in (0, 1)$ and $b \leq 1 + 2\beta$, if $x_0 \neq 0$, for any solution x of (58) with $\beta = \frac{1}{\alpha}$,*

1. *if $b = 2\beta + 1$, \mathcal{E} is constant,*
2. *if $b \in (0, 2\beta+1)$, the Lyapunov energy satisfies $\mathcal{E}(t) \sim t^{2-\frac{2b}{2\beta+1}}$ when $t \rightarrow +\infty$.*
3. *There exists $K > 0$ such that for any $T > 0$, there exists $t > T$ such that*

$$W(t) \geq \frac{K}{t^{\frac{2b}{2\beta+1}}} \quad (61)$$

Remarks :

- We can observe that for the function $F(x) = |x|^{\frac{1}{\beta}}$, $g(x) = (F(x) - F(x^*))^\beta$ is convex and thus we can apply Theorem 1. This Proposition ensures that the decay stated in point 1. of Theorem 1 is actually optimal and is reached for an explicit function.
- For the function $F(x) = |x|^{\frac{1}{\beta}}$ the inequality (10) is actually an equality for any $x \in \mathbb{R}$, ie:

$$W(t) := F(x(t)) - F(x^*) = |x(t)|^{\frac{1}{\beta}} = \beta \langle \nabla F(x(t)), x(t) - x^* \rangle \quad (62)$$

Proof of Proposition 6: To prove the first point of Proposition 6, using equation (62) we have that for this specific ODE (58) the inequality (25) is actually an equality :

$$\mathcal{E}'_{\lambda,\xi}(t) = (2 - \frac{\lambda}{\beta})tW(t) + (\xi - \lambda(\lambda + 1 - b))\langle \dot{x}(t), x(t) - x^* \rangle \quad (63)$$

$$+ (\lambda + 1 - b)\frac{1}{t} \|\lambda(x(t) - x^*) + t\dot{x}(t)\|^2 - \frac{\lambda^2(\lambda + 1 - b)}{t} \|x(t) - x^*\|^2 \quad (64)$$

and thus choosing $\lambda = 2\beta$ and $\xi = 0$ this derivative is zero.

To prove the two last points of Proposition 6 we need the following lemmas whose proofs are more technical and are given in Appendix.

Lemma 8. *If $b \in (0, 2\beta + 1)$ for any $\varepsilon > 0$, there exists $K > 0$ and $T_\varepsilon > t_0$ such that for any $t \geq T_\varepsilon$*

$$\mathcal{H}(t) \geq \frac{K}{t^\varepsilon}. \quad (65)$$

when $t \rightarrow +\infty$.

Lemma 9. *If $b \in (0, 2\beta + 1)$ there exists $A > 0$ such that $\mathcal{H}(t) \rightarrow A$ when $t \rightarrow +\infty$.*

The second point of Proposition 6 is a direct consequence of Lemmas 9, 8, point 1. of Theorem 1, and the definition of \mathcal{H} .

To prove the last point of Proposition 6 we observe that in this specific case

$$\mathcal{E}(t) = t^2W(t) + \frac{1}{2}|\lambda x(t) + t\dot{x}(t)|^2 + \frac{\xi}{2}|x(t)|^2. \quad (66)$$

Let $K_1 > 0$ such that for any $t > T$, $\mathcal{E}(t) \geq K_1 t^{2 - \frac{2b}{2\beta+1}}$. We can notice here that $2 - \frac{2b}{2\beta+1} \geq 0$. There are some cases where the conclusion holds directly.

1. If for $t_2 > T$, $\frac{1}{2}|\lambda x(t_2) + t\dot{x}(t_2)|^2 + \frac{\xi}{2}|x(t_2)|^2 \leq \frac{K_1}{2}t_0^{2 - \frac{2b}{2\beta+1}}$ we have $t_2^2|x(t_2)|^{\frac{2}{\beta}} \geq K_1 t_2^{2 - \frac{2b}{2\beta+1}} - \frac{K_1}{2}t_0^{2 - \frac{2b}{2\beta+1}} \geq \frac{K_1}{2}t_2^{2 - \frac{2b}{2\beta+1}}$ and thus we can conclude.
2. If there exists $t > T$ such that $\dot{x}(t) = 0$, using the fact that $\mathcal{E}(t) = t^2|x(t)|^{\frac{2}{\beta}} + \frac{\lambda^2 + \xi}{2}|x(t)|^2$ and the fact that $\lim_{t \rightarrow \infty} \frac{\lambda^2 + \xi}{2}|x(t)|^2 = 0$, we can conclude using the previous point.
3. If there exists $t > T$ such that $x(t) = 0$, since $\lim_{u \rightarrow \infty} |x(u)| = 0$, there exists $t_1 \geq t$ such that $\dot{x}(t_1) = 0$ and we can use the previous point.

We now suppose that $x(T) > 0$ and the sign of \dot{x} is constant on $[T, +\infty)$. Since $\lim_{t \rightarrow \infty} x(t) = 0$ we deduce that $\forall t \in (T, +\infty)$, $\dot{x}(t) < 0$.

For any $t_1 > T$ we have

$$x(t_1) - x(T) = \int_{u=T}^{t_1} \dot{x}(t) dt \quad (67)$$

Since $x(t_1)$ converges to 0, we deduce that for any $\varepsilon' > 0$, there exists $t_2 > T$ such that $|t_2 \dot{x}(t_2)| \leq \varepsilon'$. Hence for any $\varepsilon > 0$, there exists $t_2 > T$ such that $\frac{1}{2} |\lambda x(t_2) + t \dot{x}(t_2)|^2 + \frac{\xi}{2} |x(t_2)|^2 \leq \varepsilon$ and we can conclude since we are back to case 1. This concludes the proof of the Proposition. \square

5 Robustness to noise

We focus in this section on the perturbed ODE :

$$\ddot{x}(t) + \frac{b}{t} \dot{x}(t) + \nabla F(x(t)) = g(t) \quad (68)$$

where $g(t)$ can be seen as a perturbation term.

In [2], Attouch et al. proved that $W(t) = 0(\frac{1}{t^2})$ when F is convex and $b \geq 3$ and

$$\int_{t_1}^{+\infty} t \|g(t)\|_2 dt < +\infty.$$

Moreover under weaker integrability conditions on g (Theorem 4.1) authors proved that $F(x(t))$ tends to 0 when t goes to infinity.

In this section we propose an extension of these results when $b < 2\beta + 1$:

Theorem 2. *Let x be any solution of (68) with $b \in (0, 2\beta + 1)$ and*

$$\int_{t_0}^{+\infty} t^{\frac{b}{2\beta+1}} \|g(t)\| < +\infty. \text{ Then}$$

$$W(t) = 0 \left(\frac{1}{t^{\frac{2b}{2\beta+1}}} \right) \quad (69)$$

Proof. The proof relies on Lyapunov functions.

We define

$$p(t) = \int_{t_0}^t u^{\frac{2b}{2\beta+1}-1} \langle g(u), \lambda(x(u) - x^*) + u\dot{x}(u) \rangle du \quad (70)$$

and

$$\mathcal{H}(t) = t^{-c} \mathcal{E}(t) + p(t) \quad (71)$$

where $\mathcal{E} = \mathcal{E}_{\lambda, \xi}$ with $\lambda = \frac{2b\beta}{2\beta+1}$ and $\xi = \lambda(\lambda + 1 - b)$.

We first prove the following lemma :

Lemma 10. $\mathcal{H}'(t) \leq 0$

and then we conclude using the following classical Gronwall-Bellman Lemma which can be found in May (Asymptotic for the perturbed heavy ball system), Lemma 2.1

Lemma 11. *Let $f \in L^1$ and $c \geq 0$. Suppose that w is a continue function from $[a, b]$ to \mathbb{R} that satisfies : for all $t \in [a, b]$*

$$\frac{1}{2}w^2(t) \leq \frac{1}{2}c^2 + \int_a^t f(s)w(s)ds \quad (72)$$

Then, for all $t \in [a, b]$

$$w(t) \leq c + \int_a^t f(s)ds. \quad (73)$$

More precisely for $t \geq t_0$, $\mathcal{H}(t) \leq \mathcal{H}(t_0)$ and denoting $w(t) := \left\| t^{1-\frac{c}{2}} (\lambda(x(t) - x^*) + t\dot{x}(t)) \right\|_2$ with $c = 2 - \frac{2b}{2\beta+1}$, that is $1 - \frac{c}{2} = \frac{b}{2\beta+1}$ we get

$$w(t) \leq \mathcal{H}(t_0) + \int_{t_0}^t w(u) \left\| u^{1-\frac{c}{2}} g(u) \right\|_2 du \quad (74)$$

and we apply the Gronwall-Bellman Lemma to conclude. \square

A Proofs of lemmas

A.1 Lemmas used in Section 3

Proof of Lemma 2: We differentiate :

$$\mathcal{E}'_{\lambda,\xi}(t) = 2tW(t) + t^2 \langle \nabla F(x(t)), \dot{x}(t) \rangle \quad (75)$$

$$+ \langle \lambda \dot{x}(t) + t\ddot{x}(t) + \dot{x}(t), \lambda(x(t) - x^*) + t\dot{x}(t) \rangle + \xi \langle \dot{x}(t), x(t) - x^* \rangle \quad (76)$$

$$\mathcal{E}'_{\lambda,\xi}(t) = 2tW(t) + t^2 \langle \nabla F(x(t)) + \ddot{x}(t), \dot{x}(t) \rangle \quad (77)$$

$$+ (\lambda + 1)t \|\dot{x}(t)\|^2 + \lambda t \langle \ddot{x}(t), x(t) - x^* \rangle + (\lambda(\lambda + 1) + \xi) \langle \dot{x}(t), x(t) - x^* \rangle \quad (78)$$

$$\mathcal{E}'_{\lambda,\xi}(t) = 2tW(t) + t^2 \left\langle -\frac{b}{t} \dot{x}(t), \dot{x}(t) \right\rangle \quad (79)$$

$$+ (\lambda + 1)t \|\dot{x}(t)\|^2 + \lambda t \langle \ddot{x}(t), x(t) - x^* \rangle + (\lambda(\lambda + 1) + \xi) \langle \dot{x}(t), x(t) - x^* \rangle \quad (80)$$

$$\mathcal{E}'_{\lambda,\xi}(t) = 2tW(t) + t(\lambda + 1 - b) \|\dot{x}(t)\|^2 \quad (81)$$

$$+ \lambda t \langle \ddot{x}(t), x(t) - x^* \rangle + (\lambda(\lambda + 1) + \xi) \langle \dot{x}(t), x(t) - x^* \rangle \quad (82)$$

$$\mathcal{E}'_{\lambda,\xi}(t) = 2tW(t) + t(\lambda + 1 - b) \|\dot{x}(t)\|^2 \quad (83)$$

$$+ \lambda t \langle -\nabla F(x(t)) - \frac{b}{t} \dot{x}(t), x(t) - x^* \rangle + (\lambda(\lambda + 1) + \xi) \langle \dot{x}(t), x(t) - x^* \rangle \quad (84)$$

$$\mathcal{E}'_{\lambda,\xi}(t) = 2tW(t) + t(\lambda + 1 - b) \|\dot{x}(t)\|^2 \quad (85)$$

$$+ \lambda t \langle -\nabla F(x(t)), x(t) - x^* \rangle + (\lambda(\lambda + 1) - b\lambda + \xi) \langle \dot{x}(t), x(t) - x^* \rangle \quad (86)$$

□

Proof of Lemma 4: We start from the result of Lemma 2. Observing that

$$\frac{1}{t} \|\lambda(x(t) - x^*) + t\dot{x}(t)\|^2 = t \|\dot{x}(t)\|^2 + 2\lambda \langle \dot{x}(t), x(t) - x^* \rangle + \frac{\lambda^2}{t} \|x(t) - x^*\|^2 \quad (87)$$

we can write

$$\mathcal{E}'_{\lambda,\xi}(t) = 2tW(t) + \lambda t \langle -\nabla F(x(t)), x(t) - x^* \rangle \quad (88)$$

$$+ (\xi - \lambda(\lambda + 1 - b)) \langle \dot{x}(t), x(t) - x^* \rangle \quad (89)$$

$$+ (\lambda + 1 - b) \frac{1}{t} \|\lambda(x(t) - x^*) + t\dot{x}(t)\|^2 - \frac{\lambda^2(\lambda + 1 - b)}{t} \|x(t) - x^*\|^2 \quad (90)$$

□

A.2 Lemmas used in Section 4

Proof of Lemma 8: Using equation (62) we have that for this specific ODE (58) the inequality (25) is actually an equality :

$$\mathcal{E}'_{\lambda,\xi}(t) = (2 - \frac{\lambda}{\beta})tW(t) + (\xi - \lambda(\lambda + 1 - b)) \langle \dot{x}(t), x(t) - x^* \rangle \quad (91)$$

$$+ (\lambda + 1 - b) \frac{1}{t} \|\lambda(x(t) - x^*) + t\dot{x}(t)\|^2 - \frac{\lambda^2(\lambda + 1 - b)}{t} \|x(t) - x^*\|^2 \quad (92)$$

and thus choosing $\lambda = \frac{2b\beta}{2\beta+1}$ and $\xi = \lambda(\lambda + 1 - b)$ and denoting $c := 2 - \frac{2b}{2\beta+1} > 0$ and $d := -(\frac{2b\beta}{2\beta+1})^2(1 - \frac{b}{2\beta+1}) > 0$, we have

$$\mathcal{E}'(t) = cu(t) - dv(t). \quad (93)$$

which leads to

$$\mathcal{H}'(t) = t^{-c-1}(t\mathcal{E}'(t) - c\mathcal{E}(t)) = -rt^{-c}v(t) \leq 0. \quad (94)$$

where $r := (2 - \frac{2b(1-\beta)}{2\beta+1}) > 0$. Hence \mathcal{H} is a non increasing function, and thus there exists $K > 0$ such that

$$\mathcal{H}(t) \leq \frac{K}{t^c} \quad (95)$$

Let us now compute the derivative of $\mathcal{H}_\varepsilon(t) = t^\varepsilon\mathcal{H}(t) = t^{-c+\varepsilon}\mathcal{E}(t)$:

$$\mathcal{H}'_\varepsilon(t) = t^\varepsilon\mathcal{H}'(t) + \varepsilon t^{\varepsilon-1}\mathcal{H}(t) \quad (96)$$

$$= -rt^{-c+\varepsilon}v(t) + \varepsilon t^{-c+\varepsilon-1}\mathcal{E}(t) \quad (97)$$

Since $\mathcal{E}(t) = tu(t) + \xi tv(t)$, we deduce that:

$$\mathcal{H}'_\varepsilon(t) = -rt^{-c+\varepsilon}v(t) + \varepsilon t^{-c+\varepsilon-1}(tu(t) + \xi tv(t)) \quad (98)$$

ie:

$$\mathcal{H}'_\varepsilon(t) = t^{-c+\varepsilon}(\varepsilon u(t) + v(t)(\xi\varepsilon - r)) \quad (99)$$

We now define:

$$T_\varepsilon := \sup_{t \geq t_0} \{u(t) \leq \delta v(t)\}. \quad (100)$$

with $\delta = \frac{r}{\varepsilon} - \xi$.

If for any $t \geq t_0$, $\delta v(t) < u(t)$ we define $T_\varepsilon := t_0$.

We have $u(t) \geq tW(t)$, and $W(t) = F(x(t) - F(x^*)) = |x(t)|$. Hence $u(t) \geq t|x(t)|$. Since

$$v(t) = \frac{1}{2t} \|x(t) - x^*\|^2 = \frac{1}{2t} |x(t)|^2 \quad (101)$$

we deduce that for $t > 0$:

$$0 \leq v(t) \leq \frac{1}{2t^2} |x(t)|u(t) \quad (102)$$

Since $|x(t)| = W(t)^\beta$ tends to 0, we deduce that $T_\varepsilon < +\infty$.

Using the continuity of u and v and inequality (102) we deduce that $v(T_\varepsilon) > 0$.

Hence for $t \geq T_\varepsilon$

$$0 \leq v(t) \leq \frac{1}{\delta} u(t) \quad (103)$$

which implies that for any $t \geq T_\varepsilon$,

$$\mathcal{H}'_\varepsilon(t) = \varepsilon t^{-c+\varepsilon} (u(t) - \delta v(t)) \geq 0 \quad (104)$$

We deduce that for all $t \geq T_\varepsilon$

$$t^{-c+\varepsilon} \mathcal{E}(t) \geq T_\varepsilon^{-c+\varepsilon} \mathcal{E}(T_\varepsilon) = K > 0 \quad (105)$$

From the definition of \mathcal{H} we deduce that for all $t \geq T_\varepsilon$,

$$\mathcal{H}(t) \geq \frac{K}{t^\varepsilon}. \quad (106)$$

Proof of Lemma 9: We proved that

$$\mathcal{H}'(t) = -rt^{-c-1}v(t) \leq 0. \quad (107)$$

with $r = (2 - \frac{2b(1-\beta)}{2\beta+1}) > 0$. Since $W(t)$ tends to 0, the function $t \mapsto v(t)$ is bounded by a real number M and we deduce that for all pair $(t_1, t_2) \in \mathbb{R}^2$ such that $t_1 < t_2$

$$\mathcal{H}(t_2) \geq \mathcal{H}(t_1) - \frac{Mr}{c}(t_1^{-c} - t_2^{-c}). \quad (108)$$

Let us consider $\varepsilon < c$, we proved that there exists $K > 0$ and T_ε such that for all $t \geq T_\varepsilon$,

$$\mathcal{H}(t) \geq \frac{K}{t^\varepsilon}. \quad (109)$$

We deduce that for all (t_1, t_2) such that $T_\varepsilon \leq t_1 \leq t_2$

$$\mathcal{H}(t_2) - \frac{Mr}{c}t_2^{-c} \geq \frac{K}{t_1^\varepsilon} - \frac{Mr}{c}t_1^{-c} = t_1^{-\varepsilon} \left(K - \frac{Mr}{c}t_1^{-c+\varepsilon} \right) \quad (110)$$

Since $\varepsilon < c$, we can choose t_1 such that the right hand side of the inequality is strictly positive which ensures that H has a strictly positive limit and concludes the proof of the Lemma. □

Proof of Lemma 10 To prove that $\mathcal{H}'(t) \leq 0$ in the perturbed case as in the unperturbed case we differentiate $\mathcal{E}_{\lambda, \xi}$

$$\mathcal{E}'_{\lambda, \xi}(t) = 2tW(t) + t^2 \langle \nabla F(x(t)), \dot{x}(t) \rangle \quad (111)$$

$$+ \langle \lambda \dot{x}(t) + t\ddot{x}(t) + \dot{x}(t), \lambda(x(t) - x^*) + t\dot{x}(t) \rangle + \xi \langle \dot{x}(t), x(t) - x^* \rangle \quad (112)$$

Previously, when $g = 0$ we used the fact that $\ddot{x}(t) = -\frac{b}{t}\dot{x}(t) - \nabla F(x(t))$. When $g \neq 0$, we replace $\ddot{x}(t)$ using

$$\ddot{x}(t) = -\frac{b}{t}\dot{x}(t) - \nabla F(x(t)) - g(t). \quad (113)$$

This difference makes that the expression of $\mathcal{E}'_{\lambda,\xi}(t)$ is equal to the one without perturbation up to a term equal to $-\langle tg(t), \lambda(x(t) - x^*) + t\dot{x}(t) \rangle$. Using the same calculation that the ones of Lemma 2 we get

$$\mathcal{E}'_{\lambda,\xi}(t) = 2tW(t) + t(\lambda + 1 - b) \|\dot{x}(t)\|^2 - \langle tg(t), \lambda(x(t) - x^*) + t\dot{x}(t) \rangle \quad (114)$$

$$+ \lambda t \langle -\nabla F(x(t)), x(t) - x^* \rangle + (\lambda(\lambda + 1) - b\lambda + \xi) \langle \dot{x}(t), x(t) - x^* \rangle \quad (115)$$

Using the same calculations that in Lemma 4 we have

$$\mathcal{E}'_{\lambda,\xi}(t) = 2tW(t) + \lambda t \langle -\nabla F(x(t)), x(t) - x^* \rangle - \langle tg(t), \lambda(x(t) - x^*) + t\dot{x}(t) \rangle \quad (116)$$

$$+ (\xi - \lambda(\lambda + 1 - b)) \langle \dot{x}(t), x(t) - x^* \rangle \quad (117)$$

$$+ (\lambda + 1 - b) \frac{1}{t} \|\lambda(x(t) - x^*) + t\dot{x}(t)\|^2 - \frac{\lambda^2(\lambda + 1 - b)}{t} \|x(t) - x^*\|^2 \quad (118)$$

It follows that

$$\mathcal{E}'_{\lambda,\xi}(t) \leq cu(t) - dv(t) - \langle tg(t), \lambda(x(t) - x^*) + t\dot{x}(t) \rangle \quad (119)$$

where u , v and c and d are defined as in the unperturbed case (see (13) and (14)). Using the definitions \mathcal{H} and p we get the inequality (36)

$$\mathcal{H}'(t) \leq - \left(2 - \frac{2b(1 - \beta)}{2\beta + 1} \right) t^{-c} v(t) \leq 0.$$

which concludes the proof of the lemma.

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