

High order Padé schemes for ODEs Mamadou N'Diaye

▶ To cite this version:

Mamadou N'Diaye. High order Padé schemes for ODEs. Journée de l'ecole doctorale ED211, Jun 2016, Pau, France. hal-01535802

HAL Id: hal-01535802 https://inria.hal.science/hal-01535802

Submitted on 9 Jun2017

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers. L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



High order Padé schemes for ODEs

Mamadou N'DIAYE



Advisors : Hélène BARUCQ and Marc DURUFLÉ

(1)



We consider the following linear test equation

$$y'(t) = i\lambda y(t), \quad y(t_0) = y_0, \quad t \in [t_0, T],$$

which has the exact solution after one step $(h = t_{n+1} - t_n)$

$$y_n = e^{ih\lambda}y_{n-1}$$

Numerical solution is obtained by approximating

 $e^{ih\lambda} \approx R(ih\lambda).$

• The scheme is **stable** if $|R(ih\lambda)| \leq 1$, • non dissipative if $|R(ih\lambda)| = 1$, • non dispersive if $\phi(ih\lambda) = ih\lambda - arg(R(ih\lambda)) = 0$. $\forall z \in \mathbb{C}$, R(z) is called the stability function of the numerical scheme.

Undesirable effects of time schemes



Fig. 2: Stability and instability effects

• Unstable scheme: the numerical solution grows more rapidly than the physical solution.



Main goals

Find R(z) such that the scheme is unconditionally stable, less dissipative and less dispersive.

\mathbf{M} M M

Fig. 3: Dissipation and dispersion effects

• Dissipative scheme: numerical solution has amplitude error.

• Dispersive scheme: numerical solution has phase error.

Diagonal Padé schemes

The
$$(r + s)^{th}$$
-order Padé approximation of e^{z} [2] is $R_{r,s}(z) = \frac{P_{r,s}(z)}{Q_{r,s}(z)}$, with

$$P_{r,s}(z) = \sum_{k=0}^{s} \frac{s!(r+s-k)!}{(r+s)!k!(s-k)!} (z)^k$$
 and $Q_{r,s}(z) = P_{s,r}(-z).$

✓ **Diagonal Padé schemes** correspond to the case where r = s = m:

Good news

- Diagonal Padé schemes are always stable.
- When applied to the test equation (1) they are not dissipative.

Furthermore, dispersion error decreases with high order

Application to Maxwell's equations

We consider

$$\frac{\varepsilon \partial E}{\partial t} - \nabla \times H + \sigma E = F_E,$$
$$\frac{\mu \partial H}{\partial t} + \nabla \times E = F_H,$$

PAYS DE L'ADOUR

• $E(F_E)$ and $H(F_H)$ are respectively the electric and magnetic fields (source term), • ε is the permittivity of the medium, μ is the permeability of the medium and σ is the conductivity.

Efficiency in term of computational time

We performed numerical tests with the C++ code Montjoie [3].

• 10-th order FEM on regular mesh $[-4, 4]^2$.

• Time interval: [0, 10].

Comparison 2,	4, 6, 8-th order Pad	é vs 4, 5-th order Rung	je-Kutta for Maxwell
Schemes	Time steps	Relative errors	Times in s
Padé2	0.003	0.001241	1713.04
Padé4	0.054	0.000988	181.09
Padé6	0.153	0.000871	183.84
Padé8	0.285	0.001051	149.50
SDIRK34 [1]	0.0166	0.001030	985.66
SDIRK55 [1]	0.066	0.001244	455.39



We have compared computational times of different schemes after imposing 0.1% of errors.



Fig. 4: Simulation results of Maxwell's equations in 2D on regular square mesh at t = 1, 2, 3.

Further research and perspectives

- Development of high order implicit time schemes for ODEs, application to Maxwell's equations (conference ICOSAHOM2016 Rio June, 27th- July, 1st),
- CFL optimization for explicit time schemes (Runge-Kutta and Runge-Kutta-Nyström),



development of locally implicit time schemes,

modeling of seismoelectric waves,

• teaching: 64 hours (helping class and TD).

References

- [1] Ernst Hairer and Gerhard Wanner, "Solving ordinary differential equations II Stiff and differentialalgebraic problem", Springer, 2010.
- [2] John C. Butcher, "Numerical methods for ordinary differential equations second edition", John Willey & Sons Ltd, 2008.

[3] Marc Duruflé, "Intégration numérique et éléments finis d'ordre élevé appliqués aux équations de Maxwell en regime harmonique", PhD Thesis of University Paris Dauphine, 2006.