

A-stable high-order implicit time schemes

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Abstract

We address the problem of constructing high-order implicit schemes for wave equations. We considered two classes of one-step schemes adapted to linear Ordinary Differential Equations, one based upon Pad   approximant of exponential, other one requiring the inversion of a unique linear system such as SDIRK (Singly Diagonally Implicit Runge Kutta) schemes.

Keywords: forward problem, implicit time integration, Runge-Kutta schemes, Pad   schemes

Construction of diagonal Pad   schemes

Let us consider the following linear ODE:

$$y'(t) = Ay(t) + F(t), \quad t \in (0, T] \quad (1)$$

where A is a given matrix, coming from spatial discretization, and F the term source. After each step $\Delta t = t_{n+1} - t_n$ the analytical solution to (1) reads

$$y(t_{n+1}) = e^{\Delta t A} \left(y(t_n) + \int_0^{\Delta t} e^{-uA} F(n\Delta t + u) du \right) \quad (2)$$

The diagonal Pad   approximant [1] of the exponential function e^z is a rational function

$$R_m(z) = \frac{P_m(z)}{Q_m(z)}, \quad z \in \mathbb{C},$$

where $P_m(z) = \sum_{i=0}^m \frac{m! (2m-i)!}{(2m)! i! (m-i)!} (z)^i$ and $Q_m(z) = P_m(-z)$. R_m is the stability function and the stability region is defined as

$$S = \{z \in \mathbb{C} \text{ such that } |R_m(z)| \leq 1\}.$$

R_m satisfies the A-stability condition which requires that $S \supset \mathbb{C}^- = \{z \in \mathbb{C}, \operatorname{Re}(z) \leq 0\}$ [2].

Herein, we assume that $Q_m(\Delta t A)^{-1}$ is well defined. Solution (2) can then be written as

$$Q_m(\Delta t A)y(t_{n+1}) = P_m(\Delta t A)y(t_n) + \phi, \quad (3)$$

where

$$\phi = P_m(\Delta t A) \int_0^{\Delta t} e^{-uA} F(n\Delta t + u) du + O(\Delta t^{2m+1}). \quad (4)$$

By using the Taylor expansion of $y(t_n)$ and $y(t_{n+1})$ around $t_n + \frac{\Delta t}{2}$ in (3) we obtain the approximation ϕ_n of ϕ :

$$\phi_n = \sum_{r=1}^m A^{r-1} \Delta t^r \sum_{i=0}^{2m-r} \omega_i^r F(t_n + \Delta t c_i)$$

where c_i are quadrature points and ω_i^r are the weights. We then end up with the following numerical scheme:

$$Q_m(\Delta t A)y_{n+1} = P_m(\Delta t A)y_n + \phi_n. \quad (5)$$

The obtained scheme is the diagonal Pad   scheme of order $2m$. It is implicit and A-stable. Its stability function is the same as that of the Gauss-Runge-Kutta (Gauss RK) schemes [1]. However, Gauss RK schemes which are used also to solve non-linear ODE, are too expensive in practice. The Pad   schemes presented here can be seen as a simplification of Gauss RK schemes in the case of linear ODE. In fact with these diagonal Pad   schemes we need to solve $m/2$ complex linear system if m is even, $(m-1)/2$ complex and one real linear system if m is odd.

Numerical dispersion and dissipation

To study the dispersion and dissipation properties of the scheme, we consider the following linear test equation

$$y'(t) = i\lambda y(t), \quad y(t_0) = y_0 \text{ and } \lambda \in \mathbb{R},$$

which analytical solution is approximated by

$$y(t_{n+1}) = e^{iz} y(t_n) \approx R_m(iz) y_n = y_{n+1}.$$

The dispersion and dissipation errors can be measured by considering the ratio between e^{iz} and $R_m(iz)$. We then define the dissipation error as $\psi(z) = |R_m(iz)| - 1$ and the dispersion error as $\Phi(z) = z - \arg[R_m(iz)]$. Since $|P_m(iz)| = |Q_m(iz)|$, the diagonal Pad   schemes are not dissipative: $\psi(z) = 0$. Furthermore, we show in Figure 1 the relative dispersion errors, for $m = 1, \dots, 5$, which is much smaller with high-order schemes. We choose z/m in the x -axis because m represents the computational complexity of the schemes.

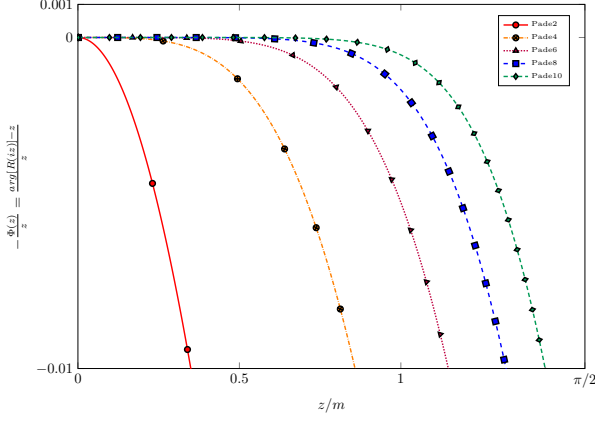


Figure 1: Relative dispersion error

Efficiency in terms of computational time

We consider the acoustic wave equation

$$\begin{cases} \rho \partial_t u - \operatorname{div} v = 0, & \forall (x, t) \in \Omega \times \mathbb{R}^+ \\ \mu^{-1} \partial_t v - \nabla u = 0, & \forall (x, t) \in \Omega \times \mathbb{R}^+ \\ u = f_D, & x \in \Gamma_D \\ \mu \partial_n u = f_N, & x \in \Gamma_N \end{cases} \quad (6)$$

Equation (6) is solved in 1-D with $\rho = \mu = 1$ in $\Omega = [0, 500]$ using the finite element solver Montjoie. We choose $[0, 1000]$ for the time interval. We used Dirichlet condition on the left

$$u(x=0, t) = e^{-i\omega t} \exp\left(-\frac{1}{2} \left(\frac{t-T}{\tau}\right)^2\right).$$

where

$$\omega = 2\pi, \quad \tau = \frac{20}{2\sqrt{2}\log 2}, \quad T = 100$$

and a homogeneous Neumann condition is set on the right extremity. The analytical solution reads

$$u(x, t) = e^{i\omega(x-t)} \exp\left(-\frac{1}{2} \left(\frac{t-T-x}{\tau}\right)^2\right).$$

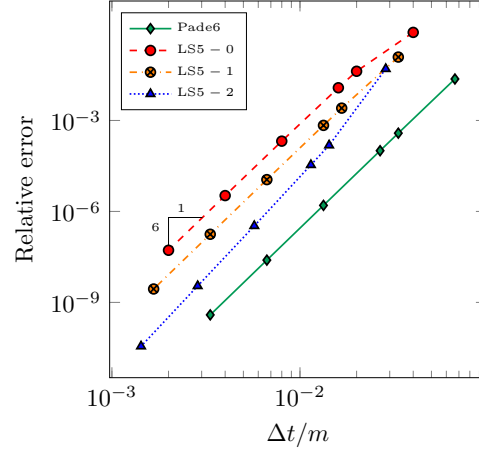
Since Padé schemes can be used only for linear ODE, their comparison with classical SDIRK [1] which solve also non-linear ODE may not be fair. So, to get a comparable scheme, we consider the case:

$$R_s^l(z) = \frac{N(z)}{(1 + \gamma z)^{s+l}}, \quad s \in \mathbb{N}, l \in \mathbb{N}. \quad (7)$$

The numerator $N(z)$ and γ are computed such that R_s^l satisfies the A-stability condition and

represents an approximation of order $s+1$ to the exponential function. The obtained schemes are new and we called them Linear-SDIRK schemes ($LSs-l$) since they share the same property as SDIRK.

In Figure 2 we show the convergence curves of Linear-SDIRK and Padé schemes of order 6. We choose $\Delta t/m$ to take into account the complexity of each scheme ($m = s+l$ for $LSs-l$).


 Figure 2: Convergence curves at $t = 200$

In Table 1, we present the computational times needed for Linear-SDIRK and Padé schemes to reach 1% of relative L^2 error computed at $t = 1000$ between the numerical solution and the analytical solution. The results show that the Padé schemes are more efficient than the Linear-SDIRK schemes. The obtained results in 1-D

Padé	Padé4	Padé6	Padé8
Nb time step	33333	8360	3875
Times	1mn36	37s	24s
Linear-SDIRK	LS3-1	LS5-2	LS7-3
Nb time step	25960	7355	3700
Times	1mn48	53s	38s

 Table 1: Computational time after imposing 1% of relative errors at $t = 1000$.

and 2-D confirmed the efficiency of Padé schemes. The next step is to combine Padé schemes with optimized high-order explicit schemes we are developing to construct locally implicit schemes.

References

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