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Abstract

In this paper, we study the homogenization of a thermal diffusion problem in a highly heterogeneous medium formed by two constituents. The main characteristics of the medium are the discontinuity of the thermal conductivity over the domain as we go from one constituent to another and the presence of an imperfect interface between the two constituents, where both the temperature and the flux exhibit jumps. The limit problem, obtained via the periodic unfolding method, captures the influence of the jumps in the limit temperature field, in an additional source term, and in the correctors, as well.

Key words: homogenization, imperfect interface, the periodic unfolding method.

AMS subject classification: 35B27, 80M35, 80M40.

1 Introduction

Our goal in this paper is to analyze the effective thermal transfer in a periodic composite material formed by two constituents occupying a domain Ω in $\mathbb{R}^N (N \geq 2)$, divided in two open subdomains, denoted by Ω_1^ε and Ω_2^ε, and separated by an imperfect interface Γ^ε. We assume that the phase Ω_1^ε is connected and reaches the external fixed boundary $\partial\Omega$ and that Ω_2^ε is disconnected, being the union of domains of size ε, periodically distributed in Ω with period of order ε, where ε is a positive real number less than one. Nevertheless, if $N \geq 3$, our results still hold true if the domain Ω_2^ε is connected, too. The order of magnitude of the thermal conductivity of the material occupying the domain Ω_2^ε is of order ε^2, while the conductivity of the material occupying the domain Ω_1^ε is supposed to be of order one. Our problem presents various sources of singularities: the geometric one related to the interspersed periodic distribution of the components, the material one related to the conductivities and the ones generated by the presence of an imperfect interface between the two materials. All these singularities are described in terms of ε.

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More precisely, we study the asymptotic behavior, as the small parameter ε tends to zero, of the solution $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ of the following problem:

$$\begin{cases} -\text{div } (A^\varepsilon \nabla u_1^\varepsilon) = f \quad \text{in } \Omega_1^\varepsilon, \\ -\text{div } (\varepsilon^2 A^\varepsilon \nabla u_2^\varepsilon) = f \quad \text{in } \Omega_2^\varepsilon, \\ A^\varepsilon \nabla u_1^\varepsilon \cdot n^\varepsilon = \varepsilon h^\varepsilon\,(u_1^\varepsilon - u_2^\varepsilon) - G^\varepsilon \quad \text{on } \Gamma^\varepsilon, \\ \varepsilon^2 A^\varepsilon \nabla u_2^\varepsilon \cdot n^\varepsilon = \varepsilon h^\varepsilon\,(u_1^\varepsilon - u_2^\varepsilon) \quad \text{on } \Gamma^\varepsilon, \\ u_1^\varepsilon = 0 \quad \text{on } \partial\Omega. \end{cases}$$

The case $G^\varepsilon = 0$, which corresponds to a continuous flux, proportional to the jump of the temperature field across the imperfect interface, has attracted, in the last two decades, the interest of a broad category of researchers. In the pioneering work [8], using the asymptotic expansion method, the authors study the homogenization of a thermal problem in a two-component composite with interfacial barrier in the particular case in which the conductivities of the two components are both of order one. For this problem, the convergence results were rigorously justified later by using various mathematical methods: the energy method in [21] and [33], the two-scale convergence method in [23] and the unfolding method for periodic homogenization in [20], [39], [40] and [32], to quote just a few of them. Also, for problems involving jumps in the solution in other contexts, such as heat transfer in polycrystals with interfacial resistance, linear elasticity problems or problems modeling the electrical conduction in biological tissues, see [4], [5], [24], [25], [26], [30] and [41]. The case corresponding to the scaling of the conductivities considered in this paper is addressed, among others, in [6], [36], [35], [34], [22], [28], [1], [2]. After passing to the limit with respect to the small parameter ε, a regularised model of diffusion is obtained, which in fact is a special case of the double-porosity model, introduced in [37] in the frame of the heat transfer and in [9] in the context of the flow in porous media. For a review of such models in various types of fissured porous media, see, for instance, [38] and the references therein.

In this paper, we consider the case $G^\varepsilon \neq 0$, which corresponds to a discontinous flux as well. We study here two representative cases for the jump function G^ε, stated explicitly in Section 2, relations (2.2) and (2.3), which both lead to different modified reguralized models of diffusion. More precisely, in the first case, a new global source term, macroscopically distributed over the entire equivalent domain, appears in the right-hand side of the homogenized equation (3.8). In the second case, the novelty brought by the presence of the flux jump is the emergence of the new non homogeneous Neumann cell problem (3.25) and the presence of its solution in the corrector (3.24). We notice that this jump plays no role in the homogenized problem (3.23). Nevertheless, in Remark 3.11 we mention a case in which the homogenized problem depends on this jump, too. This last result is to be compared with the Neumann problem in perforated domains (see [14] and [18]), where the same cases of G^ε are considered on the boundary of the holes and a similar phenomenon occurs. More recently, this type of functions G^ε is encountered in [12] for the study of a thermal problem with flux jump, involving conductivities of order one and a scaling of the jump in the temperature field of order ε^{-1}. We also point out here the effect of the jump of the solution, which is recovered in the corrector (3.10) and in the weak limit (3.15), via the solution of the local Robin problem (3.13). This phenomenon was already noticed in the case in which $G^\varepsilon = 0$. For transmission problems involving jump in the flux in other contexts, such as linear elasticity, theory of semiconductors, the study of photovoltaic systems, combustion theory or heat transfer problems, see [7], [10], [11], [27], [29], [31].

Let us notice that if in the microscopic problem the temperature and the flux are continuous across Γ^ε and if moreover the thermal conductivities of the two materials are both of order one, we have a standard transmission problem, and, then, the limit process leads to a single-diffusion equation. However, if we assume that the thermal conductivities of the two materials are both of order one, but we keep our jump conditions at the interface, then the limit problem is a system of two coupled equations. The case $G^\varepsilon = 0$ is studied in [23] and the limit process leads to the celebrated Barenblatt system, introduced in [9]. The case $G^\varepsilon \neq 0$, which leads to a modified Barenblatt model, is addressed in [13].

The rest of the paper is organized as follows: in Section 2, we introduce the microscopic problem and we fix the notation. In Section 3, we state and prove the main homogenization results of this paper. Corrector results are given, too. We end our paper with a few concluding remarks, an appendix in which we review the definition and the basic properties of the unfolding operators and their adjoints, and some references.

2 Setting of the problem

Let Ω be a bounded open set in $\mathbb{R}^N$ $(N \geqslant 2)$, with a Lipschitz continuous boundary $\partial\Omega$ and let $Y = (0,1)^N$ be the reference cell in $\mathbb{R}^N$. We suppose that Y_1 and Y_2 are two non-empty disjoint connected open subsets of Y such that $\overline{Y}_2 \subset Y$ and $Y = Y_1 \cup \overline{Y}_2$. We also assume that $\Gamma = \partial Y_2$ is Lipschitz continuous and that Y_2 is connected.

Throughout the paper, the small parameter ε takes values in a positive real sequence tending to zero and C is a positive constant independent of ε, whose value can change from line to line.

For each $k \in \mathbb{Z}^N$, we denote $Y^k = k + Y$ and $Y^k_\alpha = k + Y_\alpha$, for $\alpha = 1, 2$. We also define for each ε, $\mathbb{Z}_\varepsilon = \left\{ k \in \mathbb{Z}^N : \ \varepsilon \overline{Y}^k_2 \subset \Omega \right\}$ and we set $\Omega^\varepsilon_2 = \bigcup_{k \in \mathbb{Z}_\varepsilon} \left(\varepsilon Y^k_2\right)$ and $\Omega^\varepsilon_1 = \Omega \setminus \overline{\Omega}^\varepsilon_2$. The boundary of Ω^ε_2 is denoted by Γ^ε and n^ε is the unit outward normal to Ω^ε_2.

Our goal in this paper is to analyze the asymptotic behavior, as $\varepsilon \to 0$, of the solution

$u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ of the following problem:

$$\begin{cases} -\mathrm{div}\ (A^\varepsilon \nabla u_1^\varepsilon) = f \quad \text{in } \Omega_1^\varepsilon, \\ -\mathrm{div}\ (\varepsilon^2 A^\varepsilon \nabla u_2^\varepsilon) = f \quad \text{in } \Omega_2^\varepsilon, \\ A^\varepsilon \nabla u_1^\varepsilon \cdot n^\varepsilon = \varepsilon h^\varepsilon\, (u_1^\varepsilon - u_2^\varepsilon) - G^\varepsilon \quad \text{on } \Gamma^\varepsilon, \\ \varepsilon^2 A^\varepsilon \nabla u_2^\varepsilon \cdot n^\varepsilon = \varepsilon h^\varepsilon\, (u_1^\varepsilon - u_2^\varepsilon) \quad \text{on } \Gamma^\varepsilon, \\ u_1^\varepsilon = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Remark 2.1 *We remark that the flux of the solution is discontinuous across Γ^ε. Indeed, we have*

$$A^\varepsilon \nabla u_1^\varepsilon \cdot n^\varepsilon - \varepsilon^2 A^\varepsilon \nabla u_2^\varepsilon \cdot n^\varepsilon = -G^\varepsilon.$$

The function $f \in L^2(\Omega)$ is given. Let g be a Y-periodic function that belongs to $L^2(\Gamma)$. We define

$$g^\varepsilon(x) = g\left(\frac{x}{\varepsilon}\right) \text{ a.e. on } \Gamma^\varepsilon.$$

For the given function G^ε in (2.1), we consider the following two relevant situations (see, also, [12], [14] and [18]):

$$\textbf{Case 1} : G^\varepsilon(x) = \varepsilon g\Big(\frac{x}{\varepsilon}\Big), \text{ if } \mathcal{M}_\Gamma(g) \neq 0, \tag{2.2}$$

$$\textbf{Case 2} : G^\varepsilon(x) = g\Big(\frac{x}{\varepsilon}\Big), \text{ if } \mathcal{M}_\Gamma(g) = 0. \tag{2.3}$$

Here, $\mathcal{M}_\Gamma(g) = \dfrac{1}{|\Gamma|}\displaystyle\int_\Gamma g(y)\,\mathrm{d}y$ denotes the mean value of the function g on Γ.

Moreover, we make the following assumptions on the data:

(H1) h is a Y–periodic function such that $h \in L^\infty(\Gamma)$ and there exists $h_0 \in \mathbb{R}$ with $0 < h_0 < h(y)$ a.e. on Γ. We set

$$h^\varepsilon(x) = h\left(\frac{x}{\varepsilon}\right) \qquad \text{a.e. on } \Gamma^\varepsilon.$$

(H2) For $\lambda, \mu \in \mathbb{R}$, with $0 < \lambda \leq \mu$, let $\mathcal{M}(\lambda, \mu, Y)$ be the set of all the matrices $A \in (L^\infty(Y))^{N\times N}$ such that for any $\xi \in \mathbb{R}^N$, $\lambda|\xi|^2 \leq (A(y)\xi, \xi) \leq \mu|\xi|^2$, almost everywhere in Y. For a Y-periodic symmetric matrix $A \in \mathcal{M}(\lambda, \mu, Y)$, we set

$$A^\varepsilon(x) = A\left(\frac{x}{\varepsilon}\right) \text{ a.e. in } \Omega.$$

In order to write the variational formulation of problem (2.1), we introduce, for every positive $\varepsilon < 1$, the Hilbert space

$$H^\varepsilon = V^\varepsilon \times H^1(\Omega_2^\varepsilon).$$

The space $V^\varepsilon = \{v \in H^1(\Omega_1^\varepsilon),\ v = 0 \text{ on } \partial\Omega\}$ is endowed with the norm $\|v\|_{V^\varepsilon} = \|\nabla v\|_{L^2(\Omega_1^\varepsilon)}$, for any $v \in V^\varepsilon$, and the space $H^1(\Omega_2^\varepsilon)$ is equipped with the standard norm. On the space H^ε, we consider the scalar product

$$(u, v)_{H^\varepsilon} = \int_{\Omega_1^\varepsilon} \nabla u_1 \nabla v_1 \,\mathrm{d}x + \int_{\Omega_2^\varepsilon} \varepsilon^2 \nabla u_2 \nabla v_2 \,\mathrm{d}x + \varepsilon \int_{\Gamma^\varepsilon} (u_1 - u_2)(v_1 - v_2) \,\mathrm{d}\sigma_x \tag{2.4}$$

where $u = (u_1, u_2)$ and $v = (v_1, v_2)$ belong to H^ε. The norm generated by the scalar product (2.4) is given by

$$\|v\|_{H^\varepsilon}^2 = \|\nabla v_1\|_{L^2(\Omega_1^\varepsilon)}^2 + \varepsilon^2 \|\nabla v_2\|_{L^2(\Omega_2^\varepsilon)}^2 + \varepsilon \|v_1 - v_2\|_{L^2(\Gamma^\varepsilon)}^2. \tag{2.5}$$

The variational formulation of problem (2.1) is the following one: find $u^\varepsilon \in H^\varepsilon$ such that

$$a(u^\varepsilon, v) = l(v), \quad \forall v \in H^\varepsilon, \tag{2.6}$$

where the bilinear form $a : H^\varepsilon \times H^\varepsilon \to \mathbb{R}$ and the linear form $l : H^\varepsilon \to \mathbb{R}$ are given by

$$a(u, v) = \int_{\Omega_1^\varepsilon} A^\varepsilon \nabla u_1 \nabla v_1 \,\mathrm{d}x + \varepsilon^2 \int_{\Omega_2^\varepsilon} A^\varepsilon \nabla u_2 \nabla v_2 \,\mathrm{d}x + \varepsilon \int_{\Gamma^\varepsilon} h^\varepsilon (u_1 - u_2)(v_1 - v_2) \,\mathrm{d}\sigma_x$$

and

$$l(v) = \int_{\Omega_1^\varepsilon} f v_1 \,\mathrm{d}x + \int_{\Omega_2^\varepsilon} f v_2 \,\mathrm{d}x + \int_{\Gamma^\varepsilon} G^\varepsilon v_1 \,\mathrm{d}\sigma_x,$$

respectively.

We recall in the next lemma a result from [22], which is a key argument allowing us to prove an existence and uniqueness result and *a priori* estimates for the solution of the variational problem (2.6). In the sequel, unless otherwise mentioned, by C we denote a positive constant which is independent of ε and whose value can change from line to line.

Lemma 2.2 *For every v given in the space H^ε, the following inequalities hold true:*

$$\|v_1\|_{L^2(\Omega_1^\varepsilon)} \le C \|v\|_{H^\varepsilon}$$

and

$$\|v_2\|_{L^2(\Omega_2^\varepsilon)} \le C \|v\|_{H^\varepsilon}.$$

Proof. The first inequality is a direct consequence of the definition (2.5), together with the Poincaré inequality applied to functions from the space V^ε, namely

$$\|v_1\|_{L^2(\Omega_1^\varepsilon)} \le C \|\nabla v_1\|_{L^2(\Omega_1^\varepsilon)}. \tag{2.7}$$

In order to prove the second inequality, we need the following inequalities (see [22]):

$$\|v_2\|_{L^2(\Omega_2^\varepsilon)} \leq C(\varepsilon\|\nabla v_2\|_{L^2(\Omega_2^\varepsilon)} + \varepsilon^{\frac{1}{2}}\|v_2\|_{L^2(\Gamma^\varepsilon)}) \tag{2.8}$$

and

$$\varepsilon^{\frac{1}{2}}\|v_1\|_{L^2(\Gamma^\varepsilon)} \leq C(\varepsilon\|\nabla v_1\|_{L^2(\Omega_1^\varepsilon)} + \|v_1\|_{L^2(\Omega_1^\varepsilon)}). \tag{2.9}$$

The triangular inequality applied in (2.8), together with (2.9) and (2.7), imply

$$\begin{aligned}\|v_2\|_{L^2(\Omega_2^\varepsilon)} &\leq C(\varepsilon\|\nabla v_2\|_{L^2(\Omega_2^\varepsilon)} + \varepsilon^{\frac{1}{2}}\|v_2 - v_1\|_{L^2(\Gamma^\varepsilon)} + \varepsilon^{\frac{1}{2}}\|v_1\|_{L^2(\Gamma^\varepsilon)})\\ &\leq C(\varepsilon\|\nabla v_1\|_{L^2(\Omega_1^\varepsilon)} + \varepsilon\|\nabla v_2\|_{L^2(\Omega_2^\varepsilon)} + \varepsilon^{\frac{1}{2}}\|v_1 - v_2\|_{L^2(\Gamma^\varepsilon)} + \|v_1\|_{L^2(\Omega_1^\varepsilon)})\\ &\leq C(\|\nabla v_1\|_{L^2(\Omega_1^\varepsilon)} + \varepsilon\|\nabla v_2\|_{L^2(\Omega_2^\varepsilon)} + \varepsilon^{\frac{1}{2}}\|v_1 - v_2\|_{L^2(\Gamma^\varepsilon)}),\end{aligned}$$

and the second inequality then follows, by using the definition (2.5). □

Theorem 2.3 *For any $\varepsilon \in (0,1)$, the variational problem (2.6) has a unique solution $u^\varepsilon \in H^\varepsilon$. Moreover, there exists a constant $C > 0$, independent of ε, such that*

$$\|u_1^\varepsilon\|_{L^2(\Omega_1^\varepsilon)} \leq C, \quad \|u_2^\varepsilon\|_{L^2(\Omega_2^\varepsilon)} \leq C \tag{2.10}$$

and

$$\|\nabla u_1^\varepsilon\|_{L^2(\Omega_1^\varepsilon)} \leq C, \quad \varepsilon\|\nabla u_2^\varepsilon\|_{L^2(\Omega_2^\varepsilon)} \leq C, \quad \varepsilon^{1/2}\|u_1^\varepsilon - u_2^\varepsilon\|_{L^2(\Gamma^\varepsilon)} \leq C. \tag{2.11}$$

Proof. In order to prove the existence and the uniqueness of the solution for problem (2.6), we apply the Lax-Milgram theorem for the space H^ε endowed with the norm (2.5). Due to the hypotheses $(H1)$ and $(H2)$, we easily get that the bilinear form a is coercive and continuous. Indeed, we have

$$a(v,v) \geq C\|v\|_{H^\varepsilon}^2, \quad \forall v \in H^\varepsilon,$$

and

$$a(u,v) \leq C\|u\|_{H^\varepsilon}\|v\|_{H^\varepsilon}, \quad \forall u, v \in H^\varepsilon.$$

Let us prove now that the linear form l is continuous, i.e.

$$l(v) \leq C\|v\|_{H^\varepsilon}, \quad \forall v \in H^\varepsilon.$$

One obviously has

$$|l(v)| \leq \|f\|_{L^2(\Omega_1^\varepsilon)}\|v_1\|_{L^2(\Omega_1^\varepsilon)} + \|f\|_{L^2(\Omega_1^\varepsilon)}\|v_2\|_{L^2(\Omega_2^\varepsilon)} + \left|\int_{\Gamma^\varepsilon} G^\varepsilon(x)v_1(x)\,\mathrm{d}\sigma_x\right|. \tag{2.12}$$

According to Proposition 3.8 in [18], we obtain the estimate of the last term in (2.12) as follows:

(i) if G^ε satisfies (2.2), then

$$\left|\int_{\Gamma^\varepsilon} G^\varepsilon(x)v_1(x)\,\mathrm{d}\sigma_x\right| = \left|\int_{\Gamma^\varepsilon} \varepsilon g\left(\frac{x}{\varepsilon}\right)v_1(x)\,\mathrm{d}\sigma_x\right| \leq \varepsilon\frac{C}{\varepsilon}\left(|\mathcal{M}_\Gamma(g)| + \varepsilon\right)\|\nabla v_1\|_{L^2(\Omega_1^\varepsilon)} \leq C\|\nabla v_1\|_{L^2(\Omega_1^\varepsilon)};$$

(ii) if G^ε satisfies (2.3), then

$$\left|\int_{\Gamma^\varepsilon} G^\varepsilon(x) v_1(x)\,\mathrm{d}\sigma_x\right| = \left|\int_{\Gamma^\varepsilon} g\left(\frac{x}{\varepsilon}\right) v_1(x)\,\mathrm{d}\sigma_x\right| \le \frac{C}{\varepsilon}\left(|\mathcal{M}_\Gamma(g)| + \varepsilon\right)\|\nabla v_1\|_{L^2(\Omega_1^\varepsilon)} \le C\|\nabla v_1\|_{L^2(\Omega_1^\varepsilon)},$$

since $\mathcal{M}_\Gamma(g) = 0$.

Coming back to (2.12), we obtain:

$$|l(v)| \le C(\|v_1\|_{L^2(\Omega_1^\varepsilon)} + \|v_2\|_{L^2(\Omega_2^\varepsilon)} + \|\nabla v_1\|_{L^2(\Omega_1^\varepsilon)}).$$

By using Lemma 2.2 and the definition (2.5), we get the continuity of l. Thus, the Lax-Milgram theorem applies.

In order to obtain the *a priori* estimates (2.10) and (2.11), we take $v = u^\varepsilon$ in the variational formulation (2.6). By using the coerciveness of a and the continuity of l, we obtain

$$\|u^\varepsilon\|_{H^\varepsilon} \le C,$$

which obviously imply (2.11). Estimates (2.10) are then obtained by applying Lemma 2.2. □

3 Homogenization results

Our goal in this section is to pass to the limit, with $\varepsilon \to 0$, in the variational formulation (2.6) of the problem (2.1). To this end, we make use of the periodic unfolding method and the general compactness results given in the appendix of this paper.

More precisely, using the *a priori* estimates (2.10)-(2.11) and the general compactness results from Proposition 5.6, it follows that there exist $u_1 \in H_0^1(\Omega)$, $\widehat{u}_1 \in L^2(\Omega, H^1_{\mathrm{per}}(Y_1))$, $\widehat{u}_2 \in L^2(\Omega, H^1(Y_2))$ such that $\mathcal{M}_\Gamma(\widehat{u}_1) = 0$ and up to a subsequence, for $\varepsilon \to 0$, we get:

$$\begin{aligned}
&\mathcal{T}_1^\varepsilon(u_1^\varepsilon) \to u_1 \quad \text{strongly in } L^2(\Omega, H^1(Y_1)),\\
&\mathcal{T}_1^\varepsilon(\nabla u_1^\varepsilon) \rightharpoonup \nabla u_1 + \nabla_y \widehat{u}_1 \quad \text{weakly in } L^2(\Omega \times Y_1),\\
&\mathcal{T}_2^\varepsilon(u_2^\varepsilon) \rightharpoonup \widehat{u}_2 \quad \text{weakly in } L^2(\Omega, H^1(Y_2)),\\
&\varepsilon\mathcal{T}_2^\varepsilon(\nabla u_2^\varepsilon) \rightharpoonup \nabla_y \widehat{u}_2 \quad \text{weakly in } L^2(\Omega \times Y_2),\\
&\widetilde{u_1^\varepsilon} \rightharpoonup |Y_1| u_1 \quad \text{weakly in } L^2(\Omega),\\
&\widetilde{u_2^\varepsilon} \rightharpoonup \int_{Y_2} \widehat{u}_2(x,y)\,\mathrm{d}y \quad \text{weakly in } L^2(\Omega),
\end{aligned} \tag{3.1}$$

where the space $H^1_{\mathrm{per}}(Y_1)$ is defined by

$$H^1_{\mathrm{per}}(Y_1) = \{v \in H^1(Y_1) \,|\, v \text{ is Y-periodic}\}.$$

Remark 3.1 *We notice that in* (3.1) *we omitted to write* $|Y|$. *Indeed, since* Y *is the unit cube, we have* $|Y| = 1$ *and, in order to simplify the presentation, in the sequel we shall not write it.*

Let $W_{\text{per}}(Y_1) = \{v \in H^1_{\text{per}}(Y_1) \,|\, \mathcal{M}_\Gamma(v) = 0\}$. We introduce the space

$$\mathcal{V} = H^1_0(\Omega) \times L^2\left(\Omega; W_{\text{per}}(Y_1)\right) \times L^2\left(\Omega, H^1(Y_2)\right),$$

equipped with the norm

$$\|V\|^2_{\mathcal{V}} = \|\nabla v + \nabla_y \widehat{v}_1\|^2_{L^2(\Omega\times Y_1)} + \|\nabla_y \widehat{v}_2\|^2_{L^2(\Omega\times Y_2)} + \|v - \widehat{v}_2\|^2_{L^2(\Omega\times\Gamma)},$$

for all $V = (v, \widehat{v}_1, \widehat{v}_2) \in \mathcal{V}$.

In order to pass to the limit in (2.6), we need to distinguish between two cases, depending on the form of the function G^ε.

Case 1: $G^\varepsilon = \varepsilon g\left(\dfrac{x}{\varepsilon}\right)$, if $\mathcal{M}_\Gamma(g) \neq 0$.

Theorem 3.2 *The unique solution $u^\varepsilon = (u^\varepsilon_1, u^\varepsilon_2)$ of the variational problem (2.6) converges, in the sense of (3.1), to the unique solution $(u_1, \widehat{u}_1, \widehat{u}_2) \in \mathcal{V}$ of the following unfolded limit problem:*

$$\int_{\Omega\times Y_1} A(y)(\nabla u_1 + \nabla_y \widehat{u}_1)(\nabla\varphi + \nabla_y \Phi_1)\,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega\times Y_2} A(y)\nabla_y \widehat{u}_2 \nabla_y \Phi_2\,\mathrm{d}x\,\mathrm{d}y +$$

$$\int_{\Omega\times\Gamma} h(y)(u_1 - \widehat{u}_2)(\varphi - \Phi_2)\,\mathrm{d}x\,\mathrm{d}\sigma_y = \int_{\Omega\times Y_1} f(x)\varphi(x)\,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega\times Y_2} f(x)\Phi_2(x,y)\,\mathrm{d}x\,\mathrm{d}y +$$

$$|\Gamma|\mathcal{M}_\Gamma(g)\int_\Omega \varphi(x)\,\mathrm{d}x, \tag{3.2}$$

for all $\varphi \in H^1_0(\Omega)$, $\Phi_1 \in L^2(\Omega, H^1_{per}(Y_1))$ and $\Phi_2 \in L^2(\Omega, H^1(Y_2))$.

Proof. In order to get the limit problem (3.2), we unfold the variational formulation (2.6). Then, by using Proposition 5.3 and Lemma 5.4, we obtain

$$\lim_{\varepsilon\to 0}\left(\int_{\Omega\times Y_1} \mathcal{T}^\varepsilon_1(A^\varepsilon)\mathcal{T}^\varepsilon_1(\nabla u^\varepsilon_1)\mathcal{T}^\varepsilon_1(\nabla v_1)\,\mathrm{d}x + \int_{\Omega\times Y_2} \mathcal{T}^\varepsilon_2(A^\varepsilon)\mathcal{T}^\varepsilon_2(\varepsilon\nabla u^\varepsilon_2)\mathcal{T}^\varepsilon_2(\varepsilon\nabla v_2)\,\mathrm{d}x + \right.$$

$$\left.\int_{\Omega\times\Gamma} h(y)(\mathcal{T}^\varepsilon_1(u^\varepsilon_1) - \mathcal{T}^\varepsilon_2(u^\varepsilon_2))(\mathcal{T}^\varepsilon_1(v_1) - \mathcal{T}^\varepsilon_2(v_2))\,\mathrm{d}\sigma_x\right) =$$

$$\lim_{\varepsilon\to 0}\left(\int_{\Omega\times Y_1} \mathcal{T}^\varepsilon_1(f)\mathcal{T}^\varepsilon_1(v_1)\,\mathrm{d}x + \int_{\Omega\times Y_2} \mathcal{T}^\varepsilon_2(f)\mathcal{T}^\varepsilon_2(v_2)\,\mathrm{d}x + \frac{1}{\varepsilon}\int_{\Omega\times\Gamma} \mathcal{T}^\varepsilon_b(G^\varepsilon)\mathcal{T}^\varepsilon_b(v_1)\,\mathrm{d}\sigma_x\right).$$

In this unfolded problem, we choose the admissible test functions

$$v_1 = \varphi(x) + \varepsilon\omega_1(x)\psi_1\left(\frac{x}{\varepsilon}\right), \qquad v_2 = \omega_2(x)\psi_2\left(\frac{x}{\varepsilon}\right), \tag{3.3}$$

with $\varphi, \omega_1, \omega_2 \in \mathcal{D}(\Omega)$, $\psi_1 \in H^1_{\text{per}}(Y_1)$, $\psi_2 \in H^1(Y_2)$. It is not difficult to see that we have the following convergences:

$$\mathcal{T}^\varepsilon_1(v_1) \to \varphi(x) \quad \text{strongly in } L^2(\Omega\times Y_1), \tag{3.4}$$

$$\mathcal{T}_1^\varepsilon(\nabla v_1) \to \nabla\varphi(x) + \nabla_y\Phi_1 \quad \text{strongly in } L^2(\Omega\times Y_1), \tag{3.5}$$

$$\mathcal{T}_2^\varepsilon(v_2) \to \Phi_2(x,y) \quad \text{strongly in } L^2(\Omega\times Y_2), \tag{3.6}$$

and

$$\mathcal{T}_2^\varepsilon(\varepsilon\nabla v_2) \to \nabla_y\Phi_2 \quad \text{strongly in } L^2(\Omega\times Y_2), \tag{3.7}$$

where $\Phi_1(x,y) = \omega_1(x)\psi_1(y)$ and $\Phi_2(x,y) = \omega_2(x)\psi_2(y)$.

The passage to the limit with $\varepsilon \to 0$ is standard, by using convergences (3.1) and (3.4)-(3.7). The only term which needs more attention is the one involving the function G^ε. For this term, we get:

$$\frac{1}{\varepsilon}\int_{\Omega\times\Gamma} \mathcal{T}_b^\varepsilon(G^\varepsilon)\mathcal{T}_b^\varepsilon(v_1)\,\mathrm{d}\sigma_x = \int_{\Omega\times\Gamma} \mathcal{T}_b^\varepsilon\left(g\left(\frac{x}{\varepsilon}\right)\right)\mathcal{T}_b^\varepsilon\left(\varphi(x) + \varepsilon\omega_1(x)\psi_1\left(\frac{x}{\varepsilon}\right)\right)\mathrm{d}\sigma_x =$$

$$\int_{\Omega\times\Gamma} g(y)\mathcal{T}_b^\varepsilon(\varphi)(x,y)\,\mathrm{d}x\,\mathrm{d}\sigma_y + \varepsilon\int_{\Omega\times\Gamma} g(y)\mathcal{T}_b^\varepsilon(\omega_1)(x,y)\mathcal{T}_b^\varepsilon(\psi_1)(x,y)\,\mathrm{d}x\,\mathrm{d}\sigma_y \to$$

$$|\Gamma|\mathcal{M}_\Gamma(g)\int_\Omega \varphi(x)\,\mathrm{d}x.$$

By the density of $\mathcal{D}(\Omega)\otimes H^1_{\text{per}}(Y_1)$ in $L^2(\Omega, H^1_{\text{per}}(Y_1))$ and of $\mathcal{D}(\Omega)\otimes H^1(Y_2)$ in $L^2(\Omega, H^1(Y_2))$, we obtain (3.2).

We notice that our limit problem (3.2) is similar with the one obtained in [34], the only difference being the right-hand side, in which an extra constant term involving the function g arises. Indeed, our right-hand side actually writes

$$\int_\Omega F(x)\varphi(x)\,\mathrm{d}x + \int_{\Omega\times Y_2} f(x)\Phi_2(x,y)\,\mathrm{d}x\,\mathrm{d}y,$$

with

$$F(x) = |Y_1|f(x) + |\Gamma|\mathcal{M}_\Gamma(g).$$

The existence and the uniqueness for the solution of problem (3.2) is a consequence of the Lax-Milgram theorem. Due to the uniqueness of $(u_1, \widehat{u}_1, \widehat{u}_2) \in \mathcal{V}$, all the above convergences hold true for the whole sequence. □

Theorem 3.3 *The unique solution $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ of the variational problem (2.6) converges, in the sense of (3.1), to $(u_1, \widehat{u}_1, \widehat{u}_2) \in \mathcal{V}$, where u_1 is the unique solution of the homogenized problem*

$$\begin{cases} -div\,(A^{hom}\nabla u_1(x)) = f(x) + |\Gamma|\mathcal{M}_\Gamma(g) & in\ \Omega, \\ u_1 = 0 \quad on\ \partial\Omega \end{cases} \tag{3.8}$$

and

$$\widehat{u}_1(x,y) = -\sum_{j=1}^N \frac{\partial u_1}{\partial x_j}(x)\chi_1^j(y) \quad in\ \Omega\times Y_1, \tag{3.9}$$

$$\widehat{u}_2(x,y) = u_1(x) + f(x)\chi_2(y) \quad \textit{in } \Omega \times Y_2. \tag{3.10}$$

Here, A^{hom} is the constant homogenized matrix whose entries are defined, for $i, j = 1, \ldots, N$ by

$$A_{ij}^{hom} = \int_{Y_1} \left(a_{ij} - \sum_{k=1}^{N} a_{ik} \frac{\partial \chi_1^j}{\partial y_k} \right) \mathrm{d}y. \tag{3.11}$$

The vectorial function $\chi_1^j \in H_{per}^1(Y_1)$ ($j = 1, \ldots, N$) and the scalar function $\chi_2 \in H^1(Y_2)$ are the weak solutions of the following cell problems:

$$\begin{cases} -div_y(A(y)(\nabla_y \chi_1^j - e_j)) = 0 & \textit{in } Y_1, \\ (A(y)(\nabla_y \chi_1^j - e_j)) \cdot n = 0 & \textit{on } \Gamma, \\ \mathcal{M}_\Gamma(\chi_1^j) = 0 \end{cases} \tag{3.12}$$

and

$$\begin{cases} -div_y(A(y)\nabla_y \chi_2) = 1 & \textit{in } Y_2, \\ A(y)\nabla_y \chi_2 \cdot n + h\chi_2 = 0 & \textit{on } \Gamma, \end{cases} \tag{3.13}$$

where n denotes the unit outward normal to Y_2.

Moreover, the weak solution $(u_1^\varepsilon, u_2^\varepsilon)$ of problem (2.6) verifies:

$$\widetilde{u}_1^\varepsilon \rightharpoonup |Y_1| u_1 \quad \textit{weakly in } L^2(\Omega) \tag{3.14}$$

and

$$\widetilde{u}_2^\varepsilon \rightharpoonup |Y_2| u_1 + \left(\int_{Y_2} \chi_2(y)\,\mathrm{d}y \right) f \quad \textit{weakly in } L^2(\Omega). \tag{3.15}$$

Proof. By choosing $\varphi = 0$ in the unfolded limit problem (3.2), we obtain:

$$\int_{\Omega \times Y_1} A(y)(\nabla u_1 + \nabla_y \widehat{u}_1)\nabla_y \Phi_1 \,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega \times Y_2} A(y)(\nabla_y \widehat{u}_2)\nabla_y \Phi_2 \,\mathrm{d}x\,\mathrm{d}y -$$

$$\int_{\Omega \times \Gamma} h(y)(u_1 - \widehat{u}_2)\Phi_2 \,\mathrm{d}x\,\mathrm{d}\sigma_y = \int_{\Omega \times Y_2} f \Phi_2 \,\mathrm{d}x\,\mathrm{d}y. \tag{3.16}$$

Then, taking $\Phi_2 = 0$ in (3.16), we get:

$$\int_{\Omega \times Y_1} A(y)(\nabla u_1 + \nabla_y \widehat{u}_1)\nabla_y \Phi_1 \,\mathrm{d}x\,\mathrm{d}y = 0,$$

which implies

$$-\mathrm{div}_y(A(y)\nabla_y \widehat{u}_1) = \mathrm{div}_y(A(y)\nabla u_1) \quad \text{in } \Omega \times Y_1$$

and

$$A(y)(\nabla u_1 + \nabla_y \widehat{u}_1) \cdot n = 0 \quad \text{on } \Omega \times \Gamma.$$

Classical results from the theory of homogenization then imply (3.9) and (3.12).

By choosing now $\Phi_1 = 0$ in (3.16), we get:

$$\int_{\Omega\times Y_2} A(y)(\nabla_y \widehat{u}_2)\nabla_y \Phi_2 \,\mathrm{d}x\,\mathrm{d}y - \int_{\Omega\times\Gamma} h(y)(u_1 - \widehat{u}_2)\Phi_2 \,\mathrm{d}x\,\mathrm{d}\sigma_y = \int_{\Omega\times Y_2} f\Phi_2 \,\mathrm{d}x\,\mathrm{d}y,$$

which implies

$$-\mathrm{div}_y(A(y)\nabla_y \widehat{u}_2) = f \quad \text{in } \Omega\times Y_2$$

and

$$A(y)\nabla_y \widehat{u}_2 \cdot n = h(y)(u_1 - \widehat{u}_2) \quad \text{on } \Omega\times\Gamma.$$

This suggests us to search the function $\widehat{u}_2$ of the form

$$\widehat{u}_2(x,y) = u_1(x) + f(x)\chi_2(y) \quad \text{in } \Omega\times Y_2.$$

By replacing this form of $\widehat{u}_2$ in the two previous equations, we obtain

$$-\mathrm{div}_y\left(A(y)(f(x)\nabla_y\chi_2)\right) = f(x) \quad \text{in } \Omega\times Y_2,$$

and

$$A(y)\left(f(x)\nabla_y\chi_2\right)\cdot n = -h(y)f(x)\chi_2(y) \quad \text{on } \Omega\times\Gamma,$$

which imply that the scalar function χ_2 is the solution of the Robin cell problem (3.13).

By choosing now $\Phi_1 = \Phi_2 = 0$ in (3.2), we obtain:

$$\int_{\Omega\times Y_1} A(y)(\nabla u_1 + \nabla_y\widehat{u}_1)\nabla\varphi \,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega\times\Gamma} h(y)(u_1 - \widehat{u}_2)\varphi \,\mathrm{d}x\,\mathrm{d}\sigma_y =$$
$$|Y_1|\int_\Omega f(x)\varphi(x)\,\mathrm{d}x + |\Gamma|\mathcal{M}_\Gamma(g)\int_\Omega \varphi(x)\,\mathrm{d}x. \tag{3.17}$$

We have the equality

$$\int_{\Omega\times\Gamma} h(y)(u_1-\widehat{u}_2)\varphi\,\mathrm{d}x\,\mathrm{d}\sigma_y = -\int_{\Omega\times\Gamma} h(y)f(x)\chi_2(y)\varphi(x)\,\mathrm{d}x\,\mathrm{d}\sigma_y =$$
$$\left(-\int_\Gamma h(y)\chi_2(y)\,\mathrm{d}\sigma_y\right)\int_\Omega f(x)\varphi(x)\,\mathrm{d}x = \left(\int_\Gamma A(y)\nabla_y\chi_2(y)\cdot n\,\mathrm{d}\sigma_y\right)\int_\Omega f(x)\varphi(x)\,\mathrm{d}x =$$
$$\left(\int_{Y_2}\mathrm{div}_y(A(y)\nabla_y\chi_2(y))\,\mathrm{d}y\right)\int_\Omega f(x)\varphi(x)\,\mathrm{d}x =$$
$$\int_{Y_2}(-1)\,\mathrm{d}y\int_\Omega f(x)\varphi(x)\,\mathrm{d}x = -|Y_2|\int_\Omega f(x)\varphi(x)\,\mathrm{d}x,$$

and then relation (3.17) becomes:

$$\int_{\Omega\times Y_1} A(y)(\nabla u_1 + \nabla_y\widehat{u}_1)\nabla\varphi\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega f(x)\varphi(x)\,\mathrm{d}x + |\Gamma|\mathcal{M}_\Gamma(g)\int_\Omega\varphi(x)\,\mathrm{d}x.$$

We integrate this last equality by parts with respect to x and, by using (3.9) and (3.12), we are led to the homogenized problem (3.8).

□

Remark 3.4 *Due to the right scaling ε in front of the function g^ε given at the interface Γ^ε, we obtain at the limit a new source term distributed all over the domain Ω. Our initial problem (2.1) can be also studied for a nonzero function g with mean-value $\mathcal{M}_\Gamma(g)$ equal to zero. In this situation, there is no contribution of g in the right-hand side of the homogenized equation and, thus, the limit problem is the same as in the case with no g at all in the microscopic problem.*

Remark 3.5 *The solution $u_1(x)$ of problem (3.8) represents the contribution coming from the first material distributed in Ω_1^ε and the solution $\widehat{u}_2(x,y)$, verifying relation (3.10), is an additional contribution coming from the second material distributed in Ω_2^ε. This shows that the diffusion within the domain $\Omega \times Y_2$ has more than a local character. The limit problems keep information from the two different materials, but on two different scales, and this is an important particularity of such models.(see e.g. [3]) We remark that the homogenized matrix A^{hom} and, so, the solution u_1, are independent of the function h, while the limit $\widehat{u}_2$ depends on h, via the function χ_2.*

Remark 3.6 *All the above results remain true for the case in which the set Y_2 is not connected, but consists on a finite number of connected components, as in [20].*

We can also state the convergence of the energy and corrector results for the solution $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ of problem (2.1). They are obtained in a classical way, by adapting to our case the proof of Proposition 4.7 in [20]. We have the following result:

Theorem 3.7 *Under the assumptions of Theorem 3.1, if $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ is the unique solution of problem (2.1), then*

$$\lim_{\varepsilon\to 0}\left(\int_{\Omega_1^\varepsilon} A^\varepsilon \nabla u_1^\varepsilon \nabla u_1^\varepsilon \,\mathrm{d}x + \int_{\Omega_2^\varepsilon} \varepsilon^2 A^\varepsilon \nabla u_2^\varepsilon \nabla u_2^\varepsilon \,\mathrm{d}x\right) =$$

$$\int_{\Omega\times Y_1} A(y)(\nabla u_1 + \nabla_y \widehat{u}_1)(\nabla u_1 + \nabla_y \widehat{u}_1)\,\mathrm{d}x\,\mathrm{d}y +$$

$$\int_{\Omega\times Y_2} A(y)\nabla_y \widehat{u}_2 \nabla_y \widehat{u}_2 \,\mathrm{d}x\,\mathrm{d}y, \tag{3.18}$$

$$\lim_{\varepsilon\to 0}\left(\int_{\Lambda_1^\varepsilon} |\nabla u_1^\varepsilon|^2 \,\mathrm{d}x + \int_{\Lambda_2^\varepsilon} |\nabla u_2^\varepsilon|^2 \,\mathrm{d}x\right) = 0, \tag{3.19}$$

$$\mathcal{T}_1^\varepsilon(\nabla u_1^\varepsilon) \to \nabla\, u_1 + \nabla_y \widehat{u}_1 \quad \textit{strongly in } L^2(\Omega\times Y_1) \tag{3.20}$$

and

$$\mathcal{T}_2^\varepsilon(\varepsilon\nabla u_2^\varepsilon) \to \nabla_y \widehat{u}_2 \quad \textit{strongly in } L^2(\Omega\times Y_2). \tag{3.21}$$

Moreover, the following corrector result holds true:

$$\left\| \nabla u_1^\varepsilon - \nabla u_1 + \sum_{j=1}^{N} \mathcal{U}_1^\varepsilon \left(\frac{\partial u_1}{\partial x_j} \right) \mathcal{U}_1^\varepsilon \left(\nabla_y \chi_1^j \right) \right\|_{L^2(\Omega_1^\varepsilon)} \longrightarrow 0$$

and

$$\| \nabla u_2^\varepsilon - \nabla u_1 - f(x) \mathcal{U}_2^\varepsilon \left(\nabla_y \chi_2 \right) \|_{L^2(\Omega_2^\varepsilon)} \longrightarrow 0.$$

Let us analyse now the second relevant situation for the jump function G^ε.

Case 2: $G^\varepsilon(x) = g\left(\dfrac{x}{\varepsilon}\right)$, if $\mathcal{M}_\Gamma(g) = 0$.

Theorem 3.8 *The unique solution $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ of the variational problem (2.6) converges, in the sense of (3.1), to the unique solution $(u_1, \widehat{u}_1, \widehat{u}_2) \in \mathcal{V}$ of the following unfolded limit problem:*

$$\begin{aligned}
&\int_{\Omega \times Y_1} A(y)(\nabla u_1 + \nabla_y \widehat{u}_1)(\nabla \varphi + \nabla_y \Phi_1) \, \mathrm{d}x \, \mathrm{d}y + \int_{\Omega \times Y_2} A(y) \nabla_y \widehat{u}_2 \nabla_y \Phi_2 \, \mathrm{d}x \, \mathrm{d}y + \\
&\int_{\Omega \times \Gamma} h(y)(u_1 - \widehat{u}_2)(\varphi - \Phi_2) \, \mathrm{d}x \, \mathrm{d}\sigma_y = \int_{\Omega \times Y_1} f(x) \varphi(x) \, \mathrm{d}x \, \mathrm{d}y + \\
&\int_{\Omega \times Y_2} f(x) \Phi_2(x, y) \, \mathrm{d}x \, \mathrm{d}y + \int_{\Omega \times \Gamma} g(y) \Phi_1(x, y) \, \mathrm{d}x \, \mathrm{d}\sigma_y,
\end{aligned} \tag{3.22}$$

for all $\varphi \in H_0^1(\Omega)$, $\Phi_1 \in L^2(\Omega, H^1_{per}(Y_1))$, $\Phi_2 \in L^2(\Omega, H^1(Y_2))$.

Proof. To obtain the problem (3.22), we pass to the limit in the unfolded form of the variational formulation (2.6) with the test functions (3.3), which satisfy (3.4)-(3.7). The only difference with respect to the proof of Theorem 3.2 is the passage to the limit in the term involving the function G^ε. More precisely, we have now:

$$\begin{aligned}
&\frac{1}{\varepsilon} \int_{\Omega \times \Gamma} \mathcal{T}_b^\varepsilon(G^\varepsilon) \mathcal{T}_b^\varepsilon(v_1) \, \mathrm{d}\sigma_x = \frac{1}{\varepsilon} \int_{\Omega \times \Gamma} \mathcal{T}_b^\varepsilon \left(g\left(\frac{x}{\varepsilon}\right) \right) \mathcal{T}_b^\varepsilon \left(\varphi(x) + \varepsilon \omega_1(x) \psi_1 \left(\frac{x}{\varepsilon}\right) \right) \mathrm{d}\sigma_x = \\
&\frac{1}{\varepsilon} \int_{\Omega \times \Gamma} g(y) \mathcal{T}_b^\varepsilon(\varphi)(x, y) \, \mathrm{d}x \, \mathrm{d}\sigma_y + \int_{\Omega \times \Gamma} g(y) \mathcal{T}_b^\varepsilon(\omega_1)(x, y) \mathcal{T}_b^\varepsilon(\psi_1)(x, y) \, \mathrm{d}x \, \mathrm{d}\sigma_y \to \\
&\int_{\Omega \times \Gamma} g(y) \omega_1(x) \psi_1(y) \, \mathrm{d}x \, \mathrm{d}\sigma_y.
\end{aligned}$$

from the fact that $\mathcal{M}_\Gamma(g) = 0$. By using the density of $\mathcal{D}(\Omega) \otimes H^1_{\mathrm{per}}(Y_1)$ in $L^2(\Omega, H^1_{\mathrm{per}}(Y_1))$ and of $\mathcal{D}(\Omega) \otimes H^1(Y_2)$ in $L^2(\Omega, H^1(Y_2))$, we are led to the unfolded limit problem (3.22).

Due to the uniqueness of $(u_1, \widehat{u}_1, \widehat{u}_2) \in \mathcal{V}$, which can be proven by the Lax-Milgram theorem, all the above convergences hold true for the whole sequence and our theorem is proven.

□

Remark 3.9 *Let us notice that the term* $\int_{\Omega\times\Gamma} g(y)\Phi_1(x,y)\,\mathrm{d}x\,\mathrm{d}\sigma_y$ *in (3.22) constitutes the main difference with respect to the unfolded equation (3.2), where the term involving* g *is a nonzero constant, appearing explicitly in the right-hand side of the homogenized equation (3.8). This is not the case here, since this term involves now explicitly both variables* x *and* y*. Our task now is to understand the contribution in the homogenized problem of this non standard term generated by the discontinuity of the flux in the microscopic problem. As we shall see in Theorem 3.10, apart from the cell problems (3.12) and (3.13), an additional non homogeneous Neumann cell problem needs to be introduced.*

Theorem 3.10 *The unique solution* $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ *of the variational problem (2.6) converges, in the sense of (3.1), to* $(u_1, \widehat{u}_1, \widehat{u}_2) \in \mathcal{V}$*, where* u_1 *is the unique solution of the homogenized problem*

$$\begin{cases} -div\,(A^{hom}\nabla u_1(x)) = f(x) \quad in\ \Omega, \\ u_1 = 0 \quad on\ \partial\Omega \end{cases} \tag{3.23}$$

and

$$\widehat{u}_1(x,y) = -\sum_{j=1}^{N} \frac{\partial u_1}{\partial x_j}(x)\chi_1^j(y) + \eta(y), \tag{3.24}$$

$$\widehat{u}_2(x,y) = u_1(x) + f(x)\chi_2(y).$$

Here, A^{hom} *is the homogenized matrix whose entries are given by (3.11) and the functions* χ_1^j *and* χ_2 *are defined by (3.12) and (3.13). The* Y*-periodic function* η *is the unique solution of the following non homogeneous Neumann cell problem:*

$$\begin{cases} -div_y(A(y)\nabla_y\eta) = 0 \quad in\ Y_1, \\ A(y)\nabla_y\eta\cdot n = -g(y) \quad on\ \Gamma, \\ \mathcal{M}_\Gamma(\eta) = 0. \end{cases} \tag{3.25}$$

Convergences (3.14) and (3.15) still hold true, with u_1 *solution of (3.23).*

Proof. By taking $\varphi = 0$ in the unfolded limit problem (3.22), we obtain:

$$\int_{\Omega\times Y_1} A(y)(\nabla u_1 + \nabla_y\widehat{u}_1)\nabla_y\Phi_1\,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega\times Y_2} A(y)\nabla_y\widehat{u}_2\nabla_y\Phi_2\,\mathrm{d}x\,\mathrm{d}y -$$
$$\int_{\Omega\times\Gamma} h(y)(u_1 - \widehat{u}_2)\Phi_2\,\mathrm{d}x\,\mathrm{d}\sigma_y =$$
$$\int_{\Omega\times Y_2} f(x)\Phi_2(x,y)\,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega\times\Gamma} g(y)\Phi_1(x,y)\,\mathrm{d}x\,\mathrm{d}\sigma_y. \tag{3.26}$$

By choosing $\Phi_1 = 0$ in (3.26), we have

$$\int_{\Omega\times Y_2} A(y)\nabla_y\widehat{u}_2\nabla_y\Phi_2\,\mathrm{d}x\,\mathrm{d}y - \int_{\Omega\times\Gamma} h(y)(u_1 - \widehat{u}_2)\Phi_2\,\mathrm{d}x\,\mathrm{d}\sigma_y = \int_{\Omega\times Y_2} f(x)\Phi_2\,\mathrm{d}x\,\mathrm{d}y.$$

By taking now suitable test functions Φ_2, we obtain

$$-\mathrm{div}_y(A(y)\nabla_y\widehat{u}_2) = f \quad \text{ in } \Omega \times Y_2$$

and

$$A(y)\nabla_y\widehat{u}_2 \cdot n = h(y)(u_1 - \widehat{u}_2) \quad \text{ on } \Omega \times \Gamma. \tag{3.27}$$

We then find the functions $\widehat{u}_2$ and χ_2 exactly like in the proof of Theorem 3.3.

Now, let us take $\Phi_2 = 0$ in (3.26). We obtain

$$\int_{\Omega\times Y_1} A(y)(\nabla u_1 + \nabla_y\widehat{u}_1)\nabla_y\Phi_1 \,\mathrm{d}x\,\mathrm{d}y = \int_{\Omega\times\Gamma} g(y)\Phi_1 \,\mathrm{d}x\,\mathrm{d}\sigma_y.$$

By taking suitable test functions Φ_1, we obtain

$$-\mathrm{div}_y(A(y)\nabla_y\widehat{u}_1) = \mathrm{div}_y(A(y)\nabla u_1) \quad \text{ in } \Omega \times Y_1, \tag{3.28}$$

$$A(y)(\nabla_x u_1 + \nabla_y\widehat{u}_1) \cdot n = -g(y) \quad \text{ on } \Omega \times \Gamma. \tag{3.29}$$

We remark that (3.27) and (3.29) imply that we also have a discontinuity type condition:

$$A(y)(\nabla_x u_1 + \nabla_y\widehat{u}_1) \cdot n - A(y)\nabla_y\widehat{u}_2 \cdot n = -h(y)(u_1 - \widehat{u}_2) - g(y) \quad \text{ on } \Omega \times \Gamma.$$

The presence of the function g in relation (3.29) suggests us to search $\widehat{u}_1$ in the following non standard form:

$$\widehat{u}_1(x,y) = -\sum_{j=1}^{N} \frac{\partial u_1}{\partial x_j}(x)\chi_1^j(y) + \eta(y), \tag{3.30}$$

where the functions χ_1^j are defined by (3.12) and the function η remains to be found. To this end, we replace $\widehat{u}_1$ given by (3.30) in (3.28)-(3.29). We obtain:

$$\begin{cases} -\mathrm{div}_y\left(A(y)(-\nabla_x u_1 \nabla_y \chi_1 + \nabla_y \eta(y)\right) = \mathrm{div}_y\left(A(y)\nabla_x u_1\right) & \text{in } \Omega \times Y_1, \\ A(y)\left(\nabla_x u_1 - \nabla_x u_1 \nabla_y \chi_1 + \nabla_y \eta\right) \cdot n = -g(y) & \text{on } \Omega \times \Gamma, \\ \mathcal{M}_\Gamma(\eta) = 0. \end{cases} \tag{3.31}$$

By using (3.12), we deduce that the scalar function η is the unique Y-periodic solution of the cell problem

$$\begin{cases} -\mathrm{div}_y(A(y)\nabla_y\eta) = 0 & \text{in } Y_1, \\ A(y)\nabla_y\eta \cdot n = -g(y) & \text{on } \Gamma, \\ \mathcal{M}_\Gamma(\eta) = 0. \end{cases} \tag{3.32}$$

We observe that (3.32) is a non homogeneous Neumann problem. The compatibility condition

$$\int_\Gamma \eta(y)\,\mathrm{d}y = 0.$$

is satisfied, thanks to the hypothesis (2.3) imposed on the function g.

By choosing now $\Phi_1 = \Phi_2 = 0$ in (3.22), we get:

$$\int_{\Omega\times Y_1} A(y)(\nabla u_1 + \nabla_y \widehat{u}_1)\nabla\varphi \,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega\times\Gamma} h(y)(u_1 - \widehat{u}_2)\varphi \,\mathrm{d}x\,\mathrm{d}\sigma_y =$$

$$\int_{\Omega\times Y_1} f(x)\varphi(x)\,\mathrm{d}x\,\mathrm{d}y. \tag{3.33}$$

Also, since

$$u_1(x) - \widehat{u}_2(x,y) = -f_2(x)\chi_2(y) \quad \text{in } \Omega\times Y_2,$$

we have, as in the proof of Theorem 3.3, the equality

$$\int_{\Omega\times\Gamma} h(y)(u_1 - \widehat{u}_2)\varphi \,\mathrm{d}x\,\mathrm{d}\sigma_y = -|Y_2| \int_\Omega f(x)\varphi(x)\,\mathrm{d}x$$

and relation (3.33) then becomes:

$$\int_{\Omega\times Y_1} A(y)(\nabla u_1 + \nabla_y \widehat{u}_1)\nabla\varphi \,\mathrm{d}x\,\mathrm{d}y = \int_\Omega f(x)\varphi(x)\,\mathrm{d}x.$$

We integrate this last equality by parts with respect to x and, using (3.30) and the definition (3.11) of the matrix A^{hom}, we obtain

$$-\mathrm{div}_x\,(A^{\mathrm{hom}}\nabla u_1) = f + \mathrm{div}_x\left(\int_{Y_1} A(y)\nabla\eta(y)\,\mathrm{d}y\right) \quad \text{in } \Omega,$$

which leads immediately to the homogenized problem (3.23). We notice that this problem does not involve the function g. Nevertheless, the influence of the flux jump g appears in the corrector function $\widehat{u}_1$, via the cell problem (3.25).

□

Remark 3.11 *The above results can be generalized to the case in which A^ε is a sequence of matrices in $\mathcal{M}(\lambda,\mu,\Omega)$ such that*

$$\mathcal{T}^\varepsilon_\alpha(A^\varepsilon) \to A \text{ strongly in } L^1(\Omega\times Y),$$

for some matrix $A = A(x,y)$ in $\mathcal{M}(\lambda,\mu,\Omega\times Y)$. The heterogeneity of the medium described by such a matrix generates different effects in our limit problems (3.2) and (3.22), respectively. In both situations, due to the fact that the correctors χ_1^j depend also on the variable x, the new homogenized matrix A_x^{hom} is no longer constant, but it depends on x. A more interesting effect occurs in the second case. As proven in Theorem 3.10, if the matrix A depends only on the variable y, the function η is independent of x and there is no contribution of the term containing g in the decoupled form of the limit problem. So, the limit equation is the same as that corresponding to the case with no jump of the flux in the microscopic problem. Now, the *dependence of A on x prevents this phenomenon to occur, and, as a consequence, the function g gives an explicit contribution in the homogenized problem, which becomes*

$$-div_x\,(A_x^{hom}\nabla u) = f + div_x\left(\int_{Y_1} A(x,y)\nabla\eta(x,y)\,\mathrm{d}y\right) \quad in\ \Omega.$$

A similar effect was observed in the homogenization of the Neumann problem in perforated domains (see [14]).

In this second case too, a corrector result similar to the one stated in Theorem 3.7 holds true. The main difference now is that the function η, solution of the cell problem (3.25), appears in the correctors of the solution $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ of problem (2.1), as well.

Theorem 3.12 *Under the assumptions of Theorem 3.8, if $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ is the unique solution of problem (2.1), then (3.18)-(3.21) hold true. Moreover, we have the following corrector result:*

$$\left\|\nabla u_1^\varepsilon - \nabla u_1 + \sum_{j=1}^N \mathcal{U}_1^\varepsilon\left(\frac{\partial u_1}{\partial x_j}\right)\mathcal{U}_1^\varepsilon\left(\nabla_y \chi_1^j\right) - \mathcal{U}_1^\varepsilon\left(\nabla_y \eta\right)\right\|_{L^2(\Omega_1^\varepsilon)} \longrightarrow 0$$

and

$$\|\nabla u_2^\varepsilon - \nabla u_1 - f(x)\,\mathcal{U}_2^\varepsilon\left(\nabla_y \chi_2\right)\|_{L^2(\Omega_2^\varepsilon)} \longrightarrow 0.$$

Remark 3.13 *Let us point out that similar corrector results can be stated in the case in which the matrix A depends both on x and y, as in Remark 3.11.*

4 Conclusions

Using the periodic unfolding method, the effective thermal transfer in a periodic composite material formed by two constituents, with different thermal properties, was analyzed. The main features of the considered composite material were the discontinuity of the thermal conductivity over the domain as we go from one constituent to another and the presence of an imperfect interface between the two constituents, where both the temperature and the flux exhibit jumps. The limit problem captures the influence of the jumps in the limit temperature field, in an additional source term, and in the correctors, as well.

5 Appendix

We briefly recall here the definitions and the main properties of the unfolding operators $\mathcal{T}_1^\varepsilon$ and $\mathcal{T}_2^\varepsilon$, introduced, for a two-component domain, by P. Donato et al. in [20] (see, also, [14], [15], [16] and [19]) and of the boundary unfolding operator $\mathcal{T}_b^\varepsilon$, introduced in [16] and [17]. The main particularity of these operators is that they map functions defined on the oscillating domains Ω_1^ε, Ω_2^ε and, respectively, Γ^ε, into functions defined on the fixed domains $\Omega \times Y_1$, $\Omega \times Y_2$ and $\Omega \times \Gamma$, respectively.

For $x \in \mathbb{R}^N$, we denote by $[x]_Y$ its integer part $k \in \mathbb{Z}^N$, such that $x - [x]_Y \in Y$ and we set $\{x\}_Y = x - [x]_Y$ for $x \in \mathbb{R}^N$. So, for every $x \in \mathbb{R}^N$, we have $x = \varepsilon\left(\left[\frac{x}{\varepsilon}\right]_Y + \left\{\frac{x}{\varepsilon}\right\}_Y\right)$. For defining the above mentioned periodic unfolding operators, we consider the following sets (see [20]):

$$\widehat{\mathbb{Z}}_\varepsilon = \left\{k \in \mathbb{Z}^N \mid \varepsilon Y^k \subset \Omega\right\}, \quad \widehat{\Omega}^\varepsilon = \operatorname{int} \bigcup_{k \in \widehat{\mathbb{Z}}_\varepsilon} \left(\varepsilon \overline{Y}^k\right), \quad \Lambda^\varepsilon = \Omega \setminus \widehat{\Omega}^\varepsilon,$$

$$\widehat{\Omega}_\alpha^\varepsilon = \bigcup_{k \in \widehat{\mathbb{Z}}_\varepsilon} \left(\varepsilon Y_\alpha^k\right), \quad \Lambda_\alpha^\varepsilon = \Omega_\alpha^\varepsilon \setminus \widehat{\Omega}_\alpha^\varepsilon, \quad \widehat{\Gamma}^\varepsilon = \partial \widehat{\Omega}_2^\varepsilon.$$

Definition 5.1 *For any Lebesgue measurable function φ on $\Omega_\alpha^\varepsilon$, $\alpha \in \{1,2\}$, we define the periodic unfolding operators by the formula*

$$\mathcal{T}_\alpha^\varepsilon(\varphi)(x,y) = \begin{cases} \varphi\left(\varepsilon\left[\dfrac{x}{\varepsilon}\right]_Y + \varepsilon y\right) & \text{for a.e. } (x,y) \in \widehat{\Omega}^\varepsilon \times Y_\alpha \\ 0 & \text{for a.e. } (x,y) \in \Lambda^\varepsilon \times Y_\alpha \end{cases}$$

If φ is a function defined in Ω, for simplicity, we write $\mathcal{T}_\alpha^\varepsilon(\varphi)$ instead of $\mathcal{T}_\alpha^\varepsilon(\varphi|_{\Omega_\alpha^\varepsilon})$.

For any function φ which is Lebesgue-measurable on Γ^ε, the periodic boundary unfolding operator $\mathcal{T}_b^\varepsilon$ is defined by

$$\mathcal{T}_b^\varepsilon(\varphi)(x,y) = \begin{cases} \varphi\left(\varepsilon\left[\dfrac{x}{\varepsilon}\right]_Y + \varepsilon y\right) & \text{for a.e. } (x,y) \in \widehat{\Omega}^\varepsilon \times \Gamma \\ 0 & \text{for a.e. } (x,y) \in \Lambda^\varepsilon \times \Gamma \end{cases}$$

Remark 5.2 *We notice that if $\varphi \in H^1(\Omega_\alpha^\varepsilon)$, then $\mathcal{T}_b^\varepsilon(\varphi) = \mathcal{T}_\alpha^\varepsilon(\varphi)|_{\widehat{\Omega}^\varepsilon \times \Gamma}$.*

We give now a few useful properties of these operators (see, e.g., [14], [19] and [20]).

Proposition 5.3 *For $p \in [1,\infty)$ and $\alpha \in \{1,2\}$, the operators $\mathcal{T}_\alpha^\varepsilon$ are linear and continuous from $L^p(\Omega_\alpha^\varepsilon)$ to $L^p(\Omega \times Y_\alpha)$ and*

(i) if φ and ψ are two Lebesgue measurable functions on $\Omega_\alpha^\varepsilon$, one has $\mathcal{T}_\alpha^\varepsilon(\varphi\psi) = \mathcal{T}_\alpha^\varepsilon(\varphi)\mathcal{T}_\alpha^\varepsilon(\psi)$;

(ii) for every $\varphi \in L^1(\Omega_\alpha^\varepsilon)$, one has

$$\frac{1}{|Y|}\int_{\Omega \times Y_\alpha} \mathcal{T}_\alpha^\varepsilon(\varphi)(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_{\widehat{\Omega}_\alpha^\varepsilon} \varphi(x)\,\mathrm{d}x = \int_{\Omega_\alpha^\varepsilon} \varphi(x)\,\mathrm{d}x - \int_{\Lambda_\varepsilon} \varphi(x)\,\mathrm{d}x;$$

(iii) if $\{\varphi^\varepsilon\}_\varepsilon \subset L^p(\Omega)$ is a sequence such that $\varphi^\varepsilon \longrightarrow \varphi$ strongly in $L^p(\Omega)$, then $\mathcal{T}_\alpha^\varepsilon(\varphi^\varepsilon) \longrightarrow \varphi$ strongly in $L^p(\Omega \times Y_\alpha)$;

(iv) if $\varphi \in L^p(Y_\alpha)$ is Y-periodic and $\varphi^\varepsilon(x) = \varphi(x/\varepsilon)$, then $\mathcal{T}_\alpha^\varepsilon(\varphi^\varepsilon) \longrightarrow \varphi$ strongly in $L^p(\Omega \times Y_\alpha)$;

(v) if $\varphi \in W^{1,p}(\Omega_\alpha^\varepsilon)$, then $\nabla_y\left(\mathcal{T}_\alpha^\varepsilon(\varphi)\right) = \varepsilon \mathcal{T}_\alpha^\varepsilon(\nabla\varphi)$ and $\mathcal{T}_\alpha^\varepsilon(\varphi)$ belongs to $L^2\left(\Omega; W^{1,p}(Y_\alpha)\right)$. Moreover, for every $\varphi \in L^1(\Gamma^\varepsilon)$, one has

$$\int_{\widehat{\Gamma}^\varepsilon} \varphi(x)\,\mathrm{d}\sigma_x = \frac{1}{\varepsilon|Y|}\int_{\Omega\times\Gamma} \mathcal{T}_b^\varepsilon(\varphi)(x,y)\,\mathrm{d}x\,\mathrm{d}\sigma_y.$$

The following result was proven, for our geometry, in [20].

Lemma 5.4 *If $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ is a sequence in H^ε and $\varphi \in \mathcal{D}(\Omega)$, then, for ε small enough and $\alpha \in \{1,2\}$ we have*

$$\varepsilon \int_{\Gamma^\varepsilon} h^\varepsilon (u_1^\varepsilon - u_2^\varepsilon)\varphi\,\mathrm{d}\sigma_x = \int_{\Omega\times\Gamma} h(y)\left(\mathcal{T}_1^\varepsilon(u_1^\varepsilon) - \mathcal{T}_2^\varepsilon(u_2^\varepsilon)\right)\mathcal{T}_\alpha^\varepsilon(\varphi)\,\mathrm{d}x\,\mathrm{d}\sigma_y.$$

We remind some general compactness results obtained in [22] for bounded sequences in H^ε.

Lemma 5.5 *Let $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ be a bounded sequence in H^ε. Then, there exists a constant $C > 0$, independent of ε, such that*

$$\|\mathcal{T}_1^\varepsilon(u_1^\varepsilon)\|_{L^2(\Omega\times Y_1)} \le C,$$

$$\|\mathcal{T}_2^\varepsilon(u_2^\varepsilon)\|_{L^2(\Omega\times Y_2)} \le C,$$

$$\|\mathcal{T}_1^\varepsilon(\nabla u_1^\varepsilon)\|_{L^2(\Omega\times Y_1)} \le C,$$

$$\varepsilon\|\mathcal{T}_2^\varepsilon(\nabla u_2^\varepsilon)\|_{L^2(\Omega\times Y_2)} \le C,$$

$$\|\mathcal{T}_2^\varepsilon(u_1^\varepsilon) - \mathcal{T}_1^\varepsilon(u_2^\varepsilon)\|_{L^2(\Omega\times\Gamma)} \le C.$$

Proposition 5.6 *Let $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon)$ be a bounded sequence in H^ε. Then, up to a subsequence, still denoted by ε, there exist $u_1 \in H_0^1(\Omega)$, $\widehat{u}_1 \in L^2\left(\Omega, H^1_{per}(Y_1)\right)$ and $\widehat{u}_2 \in L^2\left(\Omega, H^1(Y_2)\right)$ such that*

$$\mathcal{T}_1^\varepsilon(u_1^\varepsilon) \longrightarrow u_1 \text{ strongly in } L^2\left(\Omega, H^1(Y_1)\right),$$

$$\mathcal{T}_1^\varepsilon(\nabla u_1^\varepsilon) \rightharpoonup \nabla u_1 + \nabla_y \widehat{u}_1 \text{ weakly in } L^2(\Omega\times Y_1),$$

$$\mathcal{T}_2^\varepsilon(u_2^\varepsilon) \rightharpoonup \widehat{u}_2 \text{ weakly in } L^2(\Omega, H^1(Y_2)),$$

$$\varepsilon\mathcal{T}_2^\varepsilon(\nabla u_2^\varepsilon) \rightharpoonup \nabla_y \widehat{u}_2 \text{ weakly in } L^2\left(\Omega\times Y_2\right),$$

$$\widetilde{u_1^\varepsilon} \rightharpoonup \frac{|Y_1|}{|Y|}u_1 \quad \text{weakly in } L^2(\Omega),$$
$$\widetilde{u_2^\varepsilon} \rightharpoonup \frac{1}{|Y|}\int_{Y_2} \widehat{u}_2(x,y)\,\mathrm{d}y \quad \text{weakly in } L^2(\Omega).$$

where $\mathcal{M}_\Gamma(\widehat{u}_1) = 0$ for almost every $x \in \Omega$ and $\widetilde{\cdot}$ denotes the extension by zero of a function to the whole of the domain Ω.

Finally, we give for $\alpha \in \{1,2\}$ the definition of the adjoints $\mathcal{U}_\alpha$ of the unfolding operators and we state some useful properties for them (see [14] and [20]).

Definition 5.7 *For $p \in [1,\infty]$, the averaging operators $\mathcal{U}_\alpha^\varepsilon : L^p(\Omega \times Y_\alpha) \to L^p(\Omega_\alpha^\varepsilon)$, are given by*

$$\mathcal{U}_\alpha^\varepsilon(\phi)(x) = \begin{cases} \dfrac{1}{|Y|}\displaystyle\int_Y \phi\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon z, \left\{\frac{x}{\varepsilon}\right\}_Y\right) \mathrm{d}z & \textit{for a.e. } x \in \widehat{\Omega}_\alpha^\varepsilon, \\ 0 & \textit{for a.e. } x \in \Lambda_\alpha^\varepsilon. \end{cases}$$

It is not difficult to see that these averaging operators are almost left-inverses of the corresponding unfolding operators $\mathcal{T}_\alpha^\varepsilon$, i.e., for any $\varphi \in L^p(\Omega_\alpha^\varepsilon)$, we have

$$\mathcal{U}_\alpha^\varepsilon(\mathcal{T}_\alpha^\varepsilon(\varphi))(x) = \begin{cases} \varphi(x) & \text{for a.e. } x \in \widehat{\Omega}_\alpha^\varepsilon, \\ 0 & \text{for a.e. } x \in \Lambda_\alpha^\varepsilon. \end{cases}$$

Proposition 5.8 *For $p \in [1,\infty)$, the operators $\mathcal{U}_\alpha^\varepsilon$ are linear and continuous from $L^p(\Omega \times Y_\alpha)$ to $L^p(\Omega_\alpha^\varepsilon)$ and*
(i) $\|\mathcal{U}_\alpha^\varepsilon(\phi) - \phi\|_{L^p(\Omega_\alpha^\varepsilon)} \to 0$ *for every $\phi \in L^p(\Omega)$;*
(ii) if $\varphi_\varepsilon \in L^p(\Omega_\alpha^\varepsilon)$, then the following statements are equivalent:

- $\mathcal{T}_\alpha^\varepsilon(\varphi_\varepsilon) \to \widehat{\varphi}$ *strongly in $L^p(\Omega \times Y_\alpha)$ and* $\displaystyle\int_{\Lambda_\alpha^\varepsilon} |\varphi_\varepsilon|^p \, \mathrm{d}x \to 0$*;*
- $\|\varphi_\varepsilon - \mathcal{U}_\alpha^\varepsilon(\widehat{\varphi})\|_{L^p(\Omega_\alpha^\varepsilon)} \to 0.$

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