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# Linear Solution to Attitude Determination from Vector Observations 

Jin Wu, Member, IEEE, Zebo Zhou, Jinling Wang and Hassen Fourati


#### Abstract

This paper deals with the attitude determination problem where the effect of weights is ignored. In this way, the developed method is named as the Sub-Optimal Linear Estimator of Quaternion (SOLEQ). The theory is established based on a recent publication and the two-vector matrix multiplications are decomposed with the Eigenvalue Factorization. Some analytical results are proved and given which provides the audience with a brand new viewpoint of the attitude determination and its evolution inside. With the derivations of two-vector case, the $n$ vector cased is then naturally formed. Simulations are carried out showing the accuracy and time consumption of the proposed SOLEQ, compared with representative ones. The algorithm is then implemented using $\mathrm{C}++$ programming language on the designed hardware with 3-axis accelerometer, 3-axis magnetometer, giving the effectiveness validation of the SOLEQ in real-world applications.


Index Terms-Attitude Determination, Wahba's Problem, Quaternion Transformation, Vector Observations.

## I. Introduction

ATTITUDE determination (or estimation) from vector observation pairs is a significant technology in aerospace engineering and related applications [1], [2]. For instance, the inertial navigation, as an important role in modern military applications, has a high demand on attitude determination accuracy for initial alignment [3], [4], [5].

Representative methods are mainly about the solution to the famous Wahba's problem [6], which aims to obtain the optimal attitude determination using weighted least squares. Among these algorithms, the QUEST (By M. D. Shuster [7] and improved by Y. Cheng [8] for extreme cases), SVD (By F. L. Markley, [9]) and ESOQ (By D. Mortari, [10]) are the most frequently used ones. Some other interesting approaches are proposed as well, investigating the other internals of this problem [11], [12].

There are still some weight-less algorithms for multisensor attitude determination. For example, using the nonlinear special orthogonal group [13], it is able to obtain the

[^0]attitude determination results. Via optimization approaches like gradient-decent algorithm (GDA, [14]), Gauss-Newton algorithm (GNA, [15]), we may also compute stable solutions. Apart from these, a famous analytical method was proposed by Arun et al. where the Singular Value Decomposition (SVD) is employed to calculate the attitude matrix [16].

In this paper, regardless of the weights of sensors, a novel quaternion attitude determination algorithm from arbitrary pair(s) of vector measurements is proposed. The theory is based on our previous research contribution and is extended to any vector measurements linearly. We prove that based on our model, the steady-state result is not influenced by specific orders of quaternion transformations. Using the eigenvalue decompositions of specific matrices and their simplified computations, the algorithm is arranged analytically. The proposed algorithm is named as the Sub-Optimal Linear Estimator of Quaternion (SOLEQ). Several numerical experiments are carried out showing the effectiveness, correspondence and advantages of our proposed method compared with classical solutions to Wahba's problem.

This paper is structured as follows:

1) Section II contains the problem formulation and presents that quaternion estimation from a single sensor observation. The two-vector attitude determination theory along with the $n$-vector one are given accordingly.
2) Section III involves the numerical examples and simulations where comparisons between the proposed SOLEQ and other representative methods are given.
3) Section IV consists of concluding remarks and discussion about the future work.

## II. Problem Formulation

The conventional Wahba's problem aims to find the optimal attitude matrix from vector observation pairs in the sense of least squares, such that

$$
\begin{equation*}
L(\mathbf{C})=\sum_{i=1}^{n} a_{i}\left\|\mathbf{D}_{i}^{b}-\mathbf{C D}_{i}^{r}\right\|^{2}, n=2,3, \cdots \tag{1}
\end{equation*}
$$

in which $\mathbf{C}$ denotes the optimal direction cosine ma$\operatorname{trix}(\mathrm{DCM}) ; \mathbf{D}_{i}^{b}=\left(D_{x, i}^{b}, D_{y, i}^{b}, D_{z, i}^{b}\right)^{T}$ and $\mathbf{D}_{i}^{r}=$ $\left(D_{x, i}^{r}, D_{y, i}^{r}, D_{z, i}^{r}\right)^{T}$ are the $i$-th pair of normalized vector observations from the body frame $b$ and the reference frame $r$ respectively; $a_{i}$ is the weight of the $i$-th sensor output. Wahba's problem is solved via many representative methods [17]. Many of these algorithms solve the problem via eigenvalue decompositions analytically or numerically [18], [11]. When there are only one pair of vector observations, the Wahba's
solutions fail to obtain the optimal quaternion since there will be ambiguous quaternions corresponding to the maximum eigenvalue 1 [19], [20]. In our previous contribution [21], the continuous stable quaternion solution to an accelerometerbased attitude determination system is derived.

## A. Quaternion from A Single Sensor Observation

Considering an attitude determination model from a pair of vector observations

$$
\begin{equation*}
\mathbf{D}^{b}=\mathbf{C D}^{r} \tag{2}
\end{equation*}
$$

, this section deals with the attitude determination from a single pair of sensor observations. Note that we have developed a method using the decomposition of the DCM, which can be given with quaternions $\mathbf{q}=\left(q_{0}, q_{1}, q_{2}, q_{3}\right)^{T}$ [21]:

$$
\begin{equation*}
\mathbf{C}=\left(\mathbf{P}_{1} \mathbf{q}, \mathbf{P}_{2} \mathbf{q}, \mathbf{P}_{3} \mathbf{q}\right) \tag{3}
\end{equation*}
$$

in which

$$
\begin{align*}
& \mathbf{P}_{1}=\left(\begin{array}{cccc}
q_{0} & q_{1} & -q_{2} & -q_{3} \\
-q_{3} & q_{2} & q_{1} & -q_{0} \\
q_{2} & q_{3} & q_{0} & q_{1} \\
q_{3} & q_{2} & q_{1} & q_{0} \\
\mathbf{P}_{2} & -q_{1} & q_{2} & -q_{3} \\
-q_{1} & -q_{0} & q_{3} & q_{2} \\
-q_{2} & q_{3} & -q_{0} & q_{1} \\
q_{1} & q_{0} & q_{3} & q_{2} \\
q_{0} & -q_{1} & -q_{2} & q_{3}
\end{array}\right), ~
\end{align*} \mathbf{P}_{3}=\left(\begin{array}{ll}
\end{array}\right)
$$

In [21], we mainly focus on getting the attitude solution to the accelerometer case. In this section, the theory is extended to arbitrary sensor with exactly the similar approach. Inserting (3) into (2) gives

$$
\begin{equation*}
\mathbf{D}^{b}=\left(D_{x}^{r} \mathbf{P}_{1}+D_{y}^{r} \mathbf{P}_{2}+D_{z}^{r} \mathbf{P}_{3}\right) \mathbf{q} \tag{5}
\end{equation*}
$$

It has been proved in [21] that

$$
\begin{equation*}
\mathbf{P}_{1}^{T}=\mathbf{P}_{1}^{\dagger}, \mathbf{P}_{2}^{T}=\mathbf{P}_{2}^{\dagger}, \mathbf{P}_{3}^{T}=\mathbf{P}_{3}^{\dagger} \tag{6}
\end{equation*}
$$

where $\dagger$ stands for the Moore-Penrose pseudo-inverse. In fact, another property can be shown in the following theorem

Theorem 1. $\mathbf{P}_{1} \mathbf{P}_{2}^{T}+\mathbf{P}_{2} \mathbf{P}_{1}^{T}=\mathbf{P}_{1} \mathbf{P}_{3}^{T}+\mathbf{P}_{3} \mathbf{P}_{1}^{T}=\mathbf{P}_{2} \mathbf{P}_{3}^{T}+$ $\mathbf{P}_{3} \mathbf{P}_{2}^{T}=\mathbf{0}_{3 \times 3}$.

Proof. This can be proved via brute-force calculation:

$$
\begin{aligned}
& \mathbf{P}_{1} \mathbf{P}_{2}^{T}= \\
& \left(\begin{array}{ccc}
0 & -q_{0}^{1}+q_{1}^{2}+q_{2}^{2}-q_{3}^{2} & 2 q_{0} q_{1}+2 q_{2} q_{3} \\
q_{0}^{1}-q_{1}^{2}-q_{2}^{2}+q_{3}^{2} & 0 & 2 q_{0} q_{2}-2 q_{1} q_{3} \\
-2 q_{0} q_{1}-2 q_{2} q_{3} & -2 q_{0} q_{2}+2 q_{1} q_{3} & 0
\end{array}\right) \\
& \mathbf{P}_{2} \mathbf{P}_{1}^{T}= \\
& \left(\begin{array}{ccc}
0 & q_{0}^{1}-q_{1}^{2}-q_{2}^{2}+q_{3}^{2} & -2 q_{0} q_{1}-2 q_{2} q_{3} \\
-q_{0}^{1}+q_{1}^{2}+q_{2}^{2}-q_{3}^{2} & 0 & -2 q_{0} q_{2}+2 q_{1} q_{3} \\
2 q_{0} q_{1}+2 q_{2} q_{3} & 2 q_{0} q_{2}-2 q_{1} q_{3} & 0
\end{array}\right) \\
& \mathbf{P}_{1} \mathbf{P}_{3}^{T}= \\
& \left(\begin{array}{ccc}
0 & 2 q_{0} q_{1}-2 q_{2} q_{3} & q_{0}^{1}-q_{1}^{2}+q_{2}^{2}-q_{3}^{2} \\
-2 q_{0} q_{1}+2 q_{2} q_{3} & 0 & -2 q_{1} q_{2}-2 q_{0} q_{3} \\
-q_{0}^{1}+q_{1}^{2}-q_{2}^{2}+q_{3}^{2} & 2 q_{1} q_{2}+2 q_{0} q_{3} & 0
\end{array}\right) \\
& \left(\begin{array}{ccc}
\mathbf{P}_{3} \mathbf{P}_{1}^{T}= & -2 q_{0} q_{1}+2 q_{2} q_{3} & -q_{0}^{1}+q_{1}^{2}-q_{2}^{2}+q_{3}^{2} \\
0 & 0 & 2 q_{1} q_{2}+2 q_{0} q_{3} \\
2 q_{0} q_{1}-2 q_{2} q_{3} & 0 \\
q_{0}^{1}-q_{1}^{2}+q_{2}^{2}-q_{3}^{2} & -2 q_{1} q_{2}-2 q_{0} q_{3} & 0
\end{array}\right)
\end{aligned}
$$

$$
\begin{aligned}
& \mathbf{P}_{2} \mathbf{P}_{3}^{T}= \\
& \left(\begin{array}{ccc}
0 & 2 q_{0} q_{2}+2 q_{1} q_{3} & -2 q_{1} q_{2}+2 q_{0} q_{3} \\
-2 q_{0} q_{2}-2 q_{1} q_{3} & 0 & q_{0}^{1}+q_{1}^{2}-q_{2}^{2}-q_{3}^{2} \\
2 q_{1} q_{2}-2 q_{0} q_{3} & -q_{0}^{1}-q_{1}^{2}+q_{2}^{2}+q_{3}^{2} & 0
\end{array}\right) \\
& \mathbf{P}_{3} \mathbf{P}_{2}^{T}= \\
& 0
\end{aligned} \begin{array}{ccc}
2 q_{0} q_{2}+2 q_{1} q_{3} & -2 q_{0} q_{2}-2 q_{1} q_{3} & 2 q_{1} q_{2}-2 q_{0} q_{3} \\
-2 q_{1} q_{2}+2 q_{0} q_{3} & q_{0}^{1}+q_{1}^{2}-q_{2}^{2}-q_{3}^{2} & -q_{0}^{1}-q_{1}^{2}+q_{2}^{2}+q_{3}^{2} \\
\left(\begin{array}{l}
\text { a }
\end{array}\right)
\end{array}
$$

Lemma 1. Let $\mathbf{K}(\mathbf{q})=\left(D_{x}^{r} \mathbf{P}_{1}+D_{y}^{r} \mathbf{P}_{2}+D_{z}^{r} \mathbf{P}_{3}\right), \mathbf{K}^{T}(\mathbf{q})=$ $\mathbf{K}^{\dagger}(\mathbf{q})$.
Proof. We have

$$
\begin{align*}
& \mathbf{K}(\mathbf{q}) \mathbf{K}^{T}(\mathbf{q}) \\
& =\left(D_{x}^{r} \mathbf{P}_{1}+D_{y}^{r} \mathbf{P}_{2}+D_{z}^{r} \mathbf{P}_{3}\right)\left(D_{x}^{r} \mathbf{P}_{1}+D_{y}^{r} \mathbf{P}_{2}+D_{z}^{r} \mathbf{P}_{3}\right)^{T} \\
& =\left(D_{x}^{r}\right)^{2} \mathbf{P}_{1} \mathbf{P}_{1}^{T}+\left(D_{y}^{r}\right)^{2} \mathbf{P}_{2} \mathbf{P}_{2}^{T}+\left(D_{z}^{r}\right)^{2} \mathbf{P}_{3} \mathbf{P}_{3}^{T} \\
& \quad+D_{x}^{r} D_{y}^{r}\left(\mathbf{P}_{1} \mathbf{P}_{2}^{T}+\mathbf{P}_{2} \mathbf{P}_{1}^{T}\right)  \tag{10}\\
& \quad+D_{x}^{r} D_{z}^{r}\left(\mathbf{P}_{1} \mathbf{P}_{3}^{T}+\mathbf{P}_{3} \mathbf{P}_{1}^{T}\right) \\
& \quad+D_{y}^{r} D_{z}^{r}\left(\mathbf{P}_{2} \mathbf{P}_{3}^{T}+\mathbf{P}_{3} \mathbf{P}_{2}^{T}\right) \\
& =\left[\left(D_{x}^{r}\right)^{2}+\left(D_{y}^{r}\right)^{2}+\left(D_{z}^{r}\right)^{2}\right] \mathbf{I}_{3 \times 3}=\mathbf{I}_{3 \times 3}
\end{align*}
$$

which gives Lemma 1.
Hence (5) can be transformed into

$$
\begin{equation*}
\mathbf{D}^{b}=\mathbf{K}(\mathbf{q}) \mathbf{q} \Rightarrow \mathbf{K}^{\dagger}(\mathbf{q}) \mathbf{D}^{b}=\mathbf{q} \Rightarrow \mathbf{K}^{T}(\mathbf{q}) \mathbf{D}^{b}=\mathbf{q} \tag{11}
\end{equation*}
$$

Since $\mathbf{K}(\mathbf{q})$ is in the form of quaternion, $\mathbf{K}^{T}(\mathbf{q}) \mathbf{D}^{b}$ can be expanded by

$$
\begin{equation*}
\mathbf{K}^{T}(\mathbf{q}) \mathbf{D}^{b}=D_{x}^{r} \mathbf{P}_{1}^{T} \mathbf{D}^{b}+D_{y}^{r} \mathbf{P}_{2}^{T} \mathbf{D}^{b}+D_{z}^{r} \mathbf{P}_{3}^{T} \mathbf{D}^{b} \tag{12}
\end{equation*}
$$

Also it can be obtained that

$$
\left.\begin{array}{l}
\mathbf{P}_{1}^{T} \mathbf{D}^{b}=\left(\begin{array}{c}
D_{x}^{b} q_{0}+D_{z}^{b} q_{2}-D_{y}^{b} q_{3} \\
D_{x}^{b} q_{1}+D_{y}^{b} q_{2}+D_{z}^{b} q_{3} \\
D_{z}^{b} q_{0}+D_{y}^{b} q_{1}-D_{x}^{b} q_{2} \\
-D_{y}^{b} q_{0}+D_{z}^{b} q_{1}-D_{x}^{b} q_{3}
\end{array}\right) \\
=\left(\begin{array}{ccc}
D_{x}^{b} & 0 & D_{z}^{b} \\
0 & -D_{y}^{b} \\
D_{x}^{b} & D_{y}^{b} & D_{z}^{b} \\
D_{z}^{b} & D_{y}^{b} & -D_{x}^{b} \\
-D_{y}^{b} & D_{z}^{b} & 0 \\
\hline
\end{array}\right) \mathbf{- D _ { x } ^ { b }}
\end{array}\right) \mathbf{q}=\mathbf{M}_{1} \mathbf{q} .
$$

Then we have

$$
\begin{equation*}
\mathbf{K}^{T}(\mathbf{q}) \mathbf{D}^{b}=D_{x}^{r} \mathbf{M}_{1} \mathbf{q}+D_{y}^{r} \mathbf{M}_{2} \mathbf{q}+D_{z}^{r} \mathbf{M}_{3} \mathbf{q}=\mathbf{W} \mathbf{q} \tag{16}
\end{equation*}
$$

where $\mathbf{W}$ is given by

$$
\begin{equation*}
\mathbf{W}=D_{x}^{r} \mathbf{M}_{1}+D_{y}^{r} \mathbf{M}_{2}+D_{z}^{r} \mathbf{M}_{3} \tag{17}
\end{equation*}
$$

Therefore the attitude determination problem is shifted to

$$
\begin{equation*}
\mathbf{W} \mathbf{q}=\mathbf{q} \tag{18}
\end{equation*}
$$

Theorem 2. For any normalized vector observation pair $\left\{\mathbf{D}^{r}, \mathbf{D}^{b}\right\}, \mathbf{W}^{2}=\mathbf{I}_{4 \times 4}$.
Proof. We have

$$
\begin{equation*}
(\mathbf{W} \mathbf{q})^{T} \mathbf{W} \mathbf{q}=\mathbf{q}^{T} \mathbf{W}^{T} \mathbf{W} \mathbf{q}=\mathbf{q}^{T} \mathbf{q} \tag{19}
\end{equation*}
$$

and this gives

$$
\begin{equation*}
\mathbf{W}^{T} \mathbf{W}=\mathbf{I}_{4 \times 4} \tag{20}
\end{equation*}
$$

Since $\mathbf{W}$ is symmetric, we also have

$$
\begin{equation*}
\mathbf{W}^{2}=\mathbf{W}^{T} \mathbf{W}=\mathbf{I}_{4 \times 4} \tag{21}
\end{equation*}
$$

In [21], we showed that (18) can be seen as an iterative dynamical system

$$
\begin{equation*}
\mathbf{q}(n)=\mathbf{W} \mathbf{q}(n-1) \tag{22}
\end{equation*}
$$

where $\mathbf{q}(n)$ denotes the quaternion for the $n$th iteration. Also, as has been proved, the discrete system can be converted to a corresponding continuous system. The stable solution to the continuous system is calculated as

$$
\begin{equation*}
\mathbf{q}=\frac{\mathbf{W}+\mathbf{I}_{4 \times 4}}{2} \mathbf{q}_{\text {rand }} \tag{23}
\end{equation*}
$$

if $\mathbf{W}^{2}=\mathbf{I}_{4 \times 4}$. Where $\mathbf{q}_{\text {rand }}$ denotes an randomly-chosen unit quaternion. This provides us with a new approach to obtaining the measurement quaternion from strapdown sensors.

## B. Two-Vector Case

When there are two pairs of vector observations, regardless of the weights of respective sensors, it is natural that the attitude quaternion may be computed via

$$
\begin{equation*}
\mathbf{q}=\frac{1}{4}\left(\mathbf{W}_{1}+\mathbf{I}_{4 \times 4}\right)\left(\mathbf{W}_{2}+\mathbf{I}_{4 \times 4}\right) \mathbf{q}_{\text {rand }} \tag{24}
\end{equation*}
$$

where $\mathbf{W}_{1}$ and $\mathbf{W}_{2}$ are for the 1st and 2nd sensors. (23) is reasonable since we have

$$
\begin{align*}
& {\left[\frac{1}{2}\left(\mathbf{W}+\mathbf{I}_{4 \times 4}\right)\right]^{2}=\frac{1}{4}\left(\mathbf{W}^{2}+2 \mathbf{W}+\mathbf{I}_{4 \times 4}\right)}  \tag{25}\\
& =\frac{1}{4}\left(2 \mathbf{W}+2 \mathbf{I}_{4 \times 4}\right)=\frac{1}{2}\left(\mathbf{W}+\mathbf{I}_{4 \times 4}\right)
\end{align*}
$$

However, for the two-vector case, one can write

$$
\begin{align*}
& {\left[\frac{1}{4}\left(\mathbf{W}_{1}+\mathbf{I}_{4 \times 4}\right)\left(\mathbf{W}_{2}+\mathbf{I}_{4 \times 4}\right)\right]^{2}} \\
& =\left[\frac{1}{4}\left(\mathbf{W}_{1} \mathbf{W}_{2}+\mathbf{W}_{1}+\mathbf{W}_{2}+\mathbf{I}_{4 \times 4}\right)\right]^{2}  \tag{26}\\
& \neq \frac{1}{4}\left(\mathbf{W}_{1}+\mathbf{I}_{4 \times 4}\right)\left(\mathbf{W}_{2}+\mathbf{I}_{4 \times 4}\right)
\end{align*}
$$

This reflects that the accurate quaternion may be recursively computed with

$$
\left\{\begin{array}{l}
\mathbf{q}_{k}=\frac{1}{4}\left(\mathbf{W}_{1}+\mathbf{I}_{4 \times 4}\right)\left(\mathbf{W}_{2}+\mathbf{I}_{4 \times 4}\right) \mathbf{q}_{k-1}  \tag{27}\\
\mathbf{q}_{k}=\frac{\mathbf{q}_{k}}{\left\|\mathbf{q}_{k}\right\|}
\end{array}\right.
$$

which exits when the Euler distance $\left\|\mathbf{q}_{k}-\mathbf{q}_{k-1}\right\|$ is less than one predetermined threshold $\eta$.

Note that the above initialization procedure is equivalent to the following process

$$
\left\{\begin{array}{l}
\mathbf{q}^{n}=\left[\frac{1}{4}\left(\mathbf{W}_{1}+\mathbf{I}_{4 \times 4}\right)\left(\mathbf{W}_{2}+\mathbf{I}_{4 \times 4}\right)\right]^{n} \mathbf{q}_{\text {rand }}  \tag{28}\\
\mathbf{q}^{n}=\frac{\mathbf{q}^{n}}{\left\|\mathbf{q}^{n}\right\|}
\end{array}\right.
$$

where $n$ is chosen to make sure $\left\|\mathbf{q}^{n}-\mathbf{q}^{n-1}\right\|<\eta$. Let us define some notations i.e. transformation operators

$$
\begin{equation*}
\mathcal{A}=\frac{1}{4}\left(\mathbf{W}_{1}+\mathbf{I}_{4 \times 4}\right)\left(\mathbf{W}_{2}+\mathbf{I}_{4 \times 4}\right) \tag{29}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathcal{B}=\mathcal{A}^{T}=\frac{1}{4}\left(\mathbf{W}_{2}+\mathbf{I}_{4 \times 4}\right)\left(\mathbf{W}_{1}+\mathbf{I}_{4 \times 4}\right) \tag{30}
\end{equation*}
$$

Theorem 3. For the two-vector attitude determination case, the steady-state evolution in (28) is not affected by transformation operators' order, such that

$$
\left\{\begin{array} { l } 
{ \mathbf { q } ^ { n } = \mathcal { A } ^ { n } \mathbf { q } _ { \text { rand } } }  \tag{31}\\
{ \mathbf { q } ^ { n } = \frac { \mathbf { q } ^ { n } } { \| \mathbf { q } ^ { n } \| } }
\end{array} \Leftrightarrow \left\{\begin{array}{l}
\mathbf{q}^{n}=\mathcal{B}^{n} \mathbf{q}_{\text {rand }} \\
\mathbf{q}^{n}=\frac{\mathbf{q}^{n}}{\left\|\mathbf{q}^{n}\right\|}
\end{array}, n \rightarrow+\infty\right.\right.
$$

Proof. The integrated transformation can be computed by

$$
\begin{align*}
& \mathcal{A B}=\frac{1}{16}\left(\mathbf{W}_{1}+\mathbf{I}_{4 \times 4}\right)\left(\mathbf{W}_{2}+\mathbf{I}_{4 \times 4}\right)^{2}\left(\mathbf{W}_{1}+\mathbf{I}_{4 \times 4}\right) \\
& =\frac{1}{8}\left(\mathbf{W}_{1}+\mathbf{I}_{4 \times 4}\right)\left(\mathbf{W}_{2}+\mathbf{I}_{4 \times 4}\right)\left(\mathbf{W}_{1}+\mathbf{I}_{4 \times 4}\right)  \tag{32}\\
& =\frac{1}{2} \mathcal{A}\left(\mathbf{W}_{1}+\mathbf{I}_{4 \times 4}\right)=\frac{1}{2}\left(\mathbf{W}_{1}+\mathbf{I}_{4 \times 4}\right) \mathcal{B}
\end{align*}
$$

and accordingly

$$
\begin{equation*}
\mathcal{B A}=\frac{1}{2} \mathcal{B}\left(\mathbf{W}_{2}+\mathbf{I}_{4 \times 4}\right)=\frac{1}{2}\left(\mathbf{W}_{2}+\mathbf{I}_{4 \times 4}\right) \mathcal{A} \tag{33}
\end{equation*}
$$

These generate

$$
\begin{align*}
& \mathcal{A B A}=\frac{1}{2}\left(\mathbf{W}_{1}+\mathbf{I}_{4 \times 4}\right) \mathcal{B A} \\
& =\frac{1}{4}\left(\mathbf{W}_{1}+\mathbf{I}_{4 \times 4}\right)\left(\mathbf{W}_{2}+\mathbf{I}_{4 \times 4}\right) \mathcal{A}=\mathcal{A}^{2} \tag{34}
\end{align*}
$$

and

$$
\begin{equation*}
\mathcal{A B A B}=\mathcal{A}^{2} \mathcal{B}=\mathcal{A}(\mathcal{A B})=\frac{1}{2} \mathcal{A}^{2}\left(\mathbf{W}_{1}+\mathbf{I}_{4 \times 4}\right) \tag{35}
\end{equation*}
$$

Finally, we have

$$
\begin{equation*}
(\mathcal{A B})^{n}=\frac{1}{2} \mathcal{A}^{n}\left(\mathbf{W}_{1}+\mathbf{I}_{4 \times 4}\right)=\frac{1}{2}\left(\mathbf{W}_{1}+\mathbf{I}_{4 \times 4}\right) \mathcal{B}^{n} \tag{36}
\end{equation*}
$$

This proves that the mixed steady-state transformation $(\mathcal{A B})^{n}$ can be achieved by pure transformations from $\mathcal{A}$ or $\mathcal{B}$, which also reflects that the transformation order does not take any effect.

Following this theorem, the confronted problem is to compute the power $\mathcal{A}^{n}$. In fact, $\mathcal{A}$ is formed by $\frac{1}{2}\left(\mathbf{W}_{1}+\mathbf{I}_{4 \times 4}\right)$ and $\frac{1}{2}\left(\mathbf{W}_{2}+\mathbf{I}_{4 \times 4}\right)$. Their respective eigenvalue decomposition can be given by

$$
\left\{\begin{array}{l}
\frac{1}{2}\left(\mathbf{W}_{1}+\mathbf{I}_{4 \times 4}\right)=\mathbf{V}_{1} \mathbf{S}_{1} \mathbf{V}_{1}^{-1}=\mathbf{V}_{1} \mathbf{S}_{1} \mathbf{V}_{1}^{T}  \tag{37}\\
\frac{1}{2}\left(\mathbf{W}_{2}+\mathbf{I}_{4 \times 4}\right)=\mathbf{V}_{2} \mathbf{S}_{2} \mathbf{V}_{2}^{-1}=\mathbf{V}_{2} \mathbf{S}_{2} \mathbf{V}_{2}^{T}
\end{array}\right.
$$

where $\mathbf{V}$ and $\mathbf{D}$ are constituted by eigenvectors and eigenvalues respectively as $\left(\mathbf{W}_{i}+\mathbf{I}_{4 \times 4}\right)$ is real symmetric [22]. Since $\mathbf{W}_{1}$ and $\mathbf{W}_{2}$ are in the same form, the eigenvalue matrices are equal to each other, i.e.

$$
\begin{equation*}
\mathbf{S}_{1}=\mathbf{S}_{2}=\mathbf{S} \tag{38}
\end{equation*}
$$

Then $\mathcal{A}$ can be rewritten as

$$
\begin{equation*}
\mathcal{A}=\mathbf{V}_{1} \mathbf{S} \mathbf{V}_{1}^{T} \mathbf{V}_{2} \mathbf{S} \mathbf{V}_{2}^{T} \tag{39}
\end{equation*}
$$

Identically, we have

$$
\begin{equation*}
\mathcal{B}=\mathbf{V}_{2} \mathbf{S} \mathbf{V}_{2}^{T} \mathbf{V}_{1} \mathbf{S} \mathbf{V}_{1}^{T} \tag{40}
\end{equation*}
$$

Combining (39) and (40), it is obtained that

$$
\begin{equation*}
\mathcal{A B}=\mathbf{V}_{1} \mathbf{S} \mathbf{V}_{1}^{T} \mathbf{V}_{2} \mathbf{S} \mathbf{V}_{2}^{T} \mathbf{V}_{2} \mathbf{S} \mathbf{V}_{2}^{T} \mathbf{V}_{1} \mathbf{S} \mathbf{V}_{1}^{T} \tag{41}
\end{equation*}
$$

Note that

$$
\begin{equation*}
\mathbf{V}_{1} \mathbf{V}_{1}^{T}=\mathbf{V}_{1}^{T} \mathbf{V}_{1}=\mathbf{V}_{2} \mathbf{V}_{2}^{T}=\mathbf{V}_{2}^{T} \mathbf{V}_{2}=\mathbf{I}_{4 \times 4} \tag{42}
\end{equation*}
$$

(41) is simplified as

$$
\begin{equation*}
\mathcal{A B}=\mathbf{V}_{1} \mathbf{S} \mathbf{V}_{1}^{T} \mathbf{V}_{2} \mathbf{S S} \mathbf{V}_{2}^{T} \mathbf{V}_{1} \mathbf{S} \mathbf{V}_{1}^{T} \tag{43}
\end{equation*}
$$

Here we define

$$
\begin{equation*}
\mathbf{U}=\mathbf{S} \mathbf{V}_{1}^{T} \mathbf{V}_{2} \mathbf{S}^{2} \mathbf{V}_{2}^{T} \mathbf{V}_{1} \mathbf{S} \tag{44}
\end{equation*}
$$

Actually it is decomposed by

$$
\begin{align*}
& \mathbf{U}=\mathbf{H H}^{T} \\
& \mathbf{H}=\mathbf{S V}_{1}^{T} \mathbf{V}_{2} \mathbf{S} \tag{45}
\end{align*}
$$

An interesting fact is that the eigenvalue matrix $\mathbf{S}$ can be analytically calculated and is given by

$$
\begin{equation*}
\mathbf{S}=\operatorname{diag}(0,0,1,1) \tag{46}
\end{equation*}
$$

where $\operatorname{diag}(\cdot)$ represents the diagonal matrix. This further yields $\mathbf{H}$ to be a matrix with the form of

$$
\mathbf{H}=\left(\begin{array}{cccc}
0 & 0 & 0 & 0  \tag{47}\\
0 & 0 & 0 & 0 \\
0 & 0 & h_{1} & h_{2} \\
0 & 0 & h_{3} & h_{4}
\end{array}\right)
$$

Then $\mathbf{U}$ is computed by

$$
\begin{align*}
& \mathbf{U}=\mathbf{H H}^{T} \\
& =\left(\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & u_{1} & u_{2} \\
0 & 0 & u_{3} & u_{4}
\end{array}\right)  \tag{48}\\
& =\left(\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & h_{1}^{2}+h_{2}^{2} & h_{1} h_{3}+h_{2} h_{4} \\
0 & 0 & h_{1} h_{3}+h_{2} h_{4} & h_{3}^{2}+h_{4}^{2}
\end{array}\right)
\end{align*}
$$

where $u_{2}=u_{3}$. Using this, we have

$$
\begin{equation*}
(\mathcal{A B})^{n}=\mathbf{V}_{1} \mathbf{U} \mathbf{V}_{1}^{T} \mathbf{V}_{1} \mathbf{U} \mathbf{V}_{1}^{T} \cdots \mathbf{V}_{1} \mathbf{U} \mathbf{V}_{1}^{T}=\mathbf{V}_{1} \mathbf{U}^{n} \mathbf{V}_{1}^{T} \tag{49}
\end{equation*}
$$

$\mathbf{U}$ can be decomposed with eigenvalue decomposition as well, such that

$$
\begin{equation*}
\mathbf{U}=\mathbf{V}_{\mathbf{U}} \mathbf{S}_{\mathbf{U}} \mathbf{V}_{\mathbf{U}}^{T} \tag{50}
\end{equation*}
$$

which can be analytically given by

$$
\begin{align*}
& \mathbf{S}_{\mathbf{U}}=\operatorname{diag}\left(\begin{array}{l}
0,0, \\
-\frac{1}{2} \sqrt{\Delta}+\frac{u_{1}+u_{4}}{2}, \\
+\frac{1}{2} \sqrt{\Delta}+\frac{u_{1}+u_{4}}{2}
\end{array}\right) \\
& \mathbf{V}_{\mathbf{U}}=\left(\begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \frac{-u_{1}+u_{4}+\sqrt{\Delta}}{2 u_{2}} & \frac{u_{1}-u_{4}+\sqrt{\Delta}}{2 u_{2}} \\
0 & 0 & 1 & 1
\end{array}\right) \tag{51}
\end{align*}
$$

with

$$
\begin{equation*}
\Delta=u_{1}^{2}-2 u_{1} u_{4}+4 u_{2}^{2}+u_{4}^{2} \tag{52}
\end{equation*}
$$

Letting

$$
\begin{align*}
& \lambda_{\mathbf{U}, 1}=-\frac{1}{2} \sqrt{\Delta}+\frac{u_{1}+u_{4}}{2}  \tag{53}\\
& \lambda_{\mathbf{U}, 2}=+\frac{1}{2} \sqrt{\Delta}+\frac{u_{1}+u_{4}}{2}
\end{align*}
$$

, $\mathbf{U}^{n}$ is finally computed by

$$
\begin{equation*}
\mathbf{U}^{n}=\mathbf{V}_{\mathbf{U}} \mathbf{S}^{n} \mathbf{V}_{\mathbf{U}}^{T}=\mathbf{V}_{\mathbf{U}} \operatorname{diag}\left(0,0, \lambda_{\mathbf{U}, 1}^{n}, \lambda_{\mathbf{U}, 2}^{n}\right) \mathbf{V}_{\mathbf{U}}^{T} \tag{54}
\end{equation*}
$$

Therefore

$$
\begin{equation*}
(\mathcal{A B})^{n}=\mathbf{V}_{1} \mathbf{U}^{n} \mathbf{V}_{1}^{T}=\mathbf{V}_{1} \mathbf{V}_{\mathbf{U}} \operatorname{diag}\left(0,0, \lambda_{\mathbf{U}, 1}^{n}, \lambda_{\mathbf{U}, 2}^{n}\right) \mathbf{V}_{\mathbf{U}}^{T} \mathbf{V}_{1}^{T} \tag{55}
\end{equation*}
$$

Required computation of $\mathbf{V}_{i}$ is given by

$$
\begin{align*}
& \frac{1}{2}\left(\mathbf{W}_{i}+\mathbf{I}_{4 \times 4}\right)=\tilde{\mathbf{V}}_{i} \mathbf{S} \tilde{\mathbf{V}}_{i}^{-1}, \\
& \tilde{\mathbf{V}}_{i}(1,1)=+\frac{1}{V}\left(D_{x, i}^{b}-D_{x, i}^{r}\right)\left(D_{z, i}^{b}-D_{z, i}^{r}\right) \\
& \tilde{\mathbf{V}}_{i}(1,2)=-\frac{1}{V}\left(D_{x, i}^{b}-D_{x, i}^{r}\right)\left(D_{y, i}^{b}-D_{y, i}^{r}\right) \\
& \tilde{\mathbf{V}}_{i}(1,3)=+\frac{1}{V}\left(D_{x, i}^{b}+D_{x, i}^{r}\right)\left(D_{z, i}^{b}+D_{z, i}^{r}\right) \\
& \tilde{\mathbf{V}}_{i}(1,4)=-\frac{1}{V}\left(D_{x, i}^{b}+D_{x, i}^{r}\right)\left(D_{y, i}^{b}+D_{y, i}^{r}\right) \\
& \tilde{\mathbf{V}}_{i}(2,1)=+\frac{1}{V}\left(D_{x, i}^{b}-D_{x, i}^{r}\right)\left(D_{y, i}^{b}+D_{y, i}^{r}\right) \\
& \tilde{\mathbf{V}}_{i}(2,2)=+\frac{1}{V}\left(D_{x, i}^{b}-D_{x, i}^{r}\right)\left(D_{z, i}^{b}+D_{z, i}^{r}\right) \\
& \tilde{\mathbf{V}}_{i}(2,3)=+\frac{1}{V}\left(D_{x, i}^{b}+D_{x, i}^{r}\right)\left(D_{y, i}^{b}-D_{y, i}^{r}\right) \\
& \tilde{\mathbf{V}}_{i}(2,4)=+\frac{1}{V}\left(D_{x, i}^{b}+D_{x, i}^{r}\right)\left(D_{z, i}^{b}-D_{z, i}^{r}\right)^{b} \\
& \tilde{\mathbf{V}}_{i}(3,1)=1, \tilde{\mathbf{V}}_{i}(3,2)=0, \tilde{\mathbf{V}}_{i}(3,3)=1, \tilde{\mathbf{V}}_{i}(3,4)=0 \\
& \tilde{\mathbf{V}}_{i}(4,1)=0, \tilde{\mathbf{V}}_{i}(4,2)=1, \tilde{\mathbf{V}}_{i}(4,3)=0, \tilde{\mathbf{V}}_{i}(4,4)=1 \\
& V=\left(D_{y, i}^{b}\right)^{2}+\left(D_{z, i}^{b}\right)^{2}-\left(D_{y, i}^{r}\right)^{2}-\left(D_{z, i}^{r}\right)^{2} \tag{56}
\end{align*}
$$

where $\tilde{\mathbf{V}}_{i}(k, j)$ stands for the element of $\tilde{\mathbf{V}}_{i}$ in $k$-th row and $j$-th column. Related information can also be acquired from [23]. It should be noted that

$$
\begin{equation*}
\tilde{\mathbf{V}}_{i} \tilde{\mathbf{V}}_{i}^{T} \neq \tilde{\mathbf{V}}_{i}^{T} \tilde{\mathbf{V}}_{i} \neq \mathbf{I}_{4 \times 4} \tag{57}
\end{equation*}
$$

Thus the Gram-Schmidt orthogonalization should be applied to $\tilde{\mathbf{V}}_{i}$, enabling $\mathbf{V}_{i} \mathbf{V}_{i}^{T}=\mathbf{V}_{i}^{T} \mathbf{V}_{i}=\mathbf{I}_{4 \times 4}$ [22]. A typical commitment to achieve this is to compute the QR decomposition [24], such that

$$
\begin{equation*}
\mathbf{V}_{i} \mathbf{R}=\tilde{\mathbf{V}}_{i} \tag{58}
\end{equation*}
$$

where $\mathbf{R}$ denotes an invertible upper triangular matrix. If $\mathbf{q}_{\text {rand }}=(1,0,0,0)$, the suboptimal quaternion is equal to the normalized first column of $(\mathcal{A B})^{n}$.

## C. $n$-Vector Case

Corresponding to the above notations and derivations, the n -vector case's transformation operators are defined by

$$
\begin{align*}
\mathcal{A} & =\mathbf{V}_{1} \mathbf{S} \mathbf{V}_{1}^{T} \cdots \mathbf{V}_{i} \mathbf{S} \mathbf{V}_{i}^{T} \cdots \mathbf{V}_{n} \mathbf{S} \mathbf{V}_{n}^{T} \\
\mathcal{B} & =\mathbf{V}_{n} \mathbf{S V}_{n}^{T} \cdots \mathbf{V}_{i} \mathbf{S} \mathbf{V}_{i}^{T} \cdots \mathbf{V}_{1} \mathbf{S} \mathbf{V}_{1}^{T} \tag{59}
\end{align*}
$$

Defining

$$
\begin{align*}
\mathbf{H}=\mathbf{S} \mathbf{V}_{1}^{T} & \left(\prod_{i=2}^{n-1} \mathbf{V}_{i} \mathbf{S} \mathbf{V}_{i}^{T}\right) \mathbf{V}_{n} \mathbf{S}  \tag{60}\\
& =\frac{1}{2^{n-2}} \mathbf{S V}_{1}^{T}\left[\prod_{i=2}^{n-1}\left(\mathbf{W}_{i}+\mathbf{I}_{4 \times 4}\right)\right] \mathbf{V}_{n} \mathbf{S}
\end{align*}
$$

, we have
$\mathbf{U}=\mathbf{S} \mathbf{V}_{1}^{T} \cdots \mathbf{V}_{i} \mathbf{S} \mathbf{V}_{i}^{T} \cdots \mathbf{V}_{n} \mathbf{S}^{2} \mathbf{V}_{n}^{T} \cdots \mathbf{V}_{i} \mathbf{S} \mathbf{V}_{i}^{T} \cdots \mathbf{V}_{1} \mathbf{S}=\mathbf{H} \mathbf{H}^{T}$
Then

$$
\begin{align*}
& \mathcal{A B}=\mathbf{V}_{1} \mathbf{U} \mathbf{V}_{1}^{T} \Rightarrow(\mathcal{A B})^{n}=\mathbf{V}_{1} \mathbf{U}^{n} \mathbf{V}_{1}^{T} \\
& =\mathbf{V}_{1} \mathbf{V}_{\mathbf{U}} \operatorname{diag}\left(0,0, \lambda_{\mathbf{U}, 1}^{n}, \lambda_{\mathbf{U}, 2}^{n}\right) \mathbf{V}_{\mathbf{U}}^{T} \mathbf{V}_{1}^{T} \tag{62}
\end{align*}
$$

Accordingly, the normalized first column of $(\mathcal{A B})^{n}$ constitutes the attitude quaternion.

## III. Numerical Examples, Simulations and EXPERIMENTS

## A. Numerical Example: Common Case

In this sub-section, the vector observation pairs are simulated via

$$
\begin{equation*}
\mathbf{D}^{b}=\mathbf{C}_{t r u e} \mathbf{D}^{r}+\varepsilon \tag{63}
\end{equation*}
$$

where $\varepsilon$ denotes the noise item and the reference DCM $\mathbf{C}_{\text {true }}$ is defined using the Euler parameters of $\left(-10^{\circ}, 65^{\circ},-37^{\circ}\right)^{T}$ in the rotation sequence of $X Y Z$, namely,

$$
\mathbf{C}_{\text {true }}=\left(\begin{array}{ccc}
0.33752 & -0.71836 & -0.60831  \tag{64}\\
0.25434 & 0.69179 & -0.67582 \\
0.90631 & 0.07339 & 0.41620
\end{array}\right)
$$

Three pairs of vector observations are generated:

$$
\left\{\begin{array}{l}
\mathbf{D}_{1}^{b}=(0.8084,0.5500,0.2096)^{T}  \tag{65}\\
\mathbf{D}_{1}^{r}=(0.5992,-0.2011,-0.7750)^{T} \\
\mathbf{D}_{2}^{b}=(0.0175,-0.8691,0.4942)^{T} \\
\mathbf{D}_{2}^{r}=(0.2221,-0.5864,0.7790)^{T}
\end{array} \begin{array}{l}
\mathbf{D}_{3}^{b}=(-0.4870,-0.1006,-0.8676)^{T} \\
\mathbf{D}_{3}^{r}=(-0.9690,0.2341,-0.0787)^{T}
\end{array}\right.
$$

where the reference vectors are chosen randomly and the observation vectors are simulated using the noise standard deviations of $\sigma_{1}=0.01, \sigma_{2}=0.02, \sigma_{3}=0.03$. We run all the algorithms in the MATLAB r2016b software on a MacBook Pro with a CPU of i7-4core 2.4 GHz . The QUEST algorithm is implemented for comparison, where the weights are set to be equal with each other and the accuracy of Newton iteration is set to $1 \times 10^{-8}$. For the proposed SOLEQ, the power order $n$ is set as $1 \times 10^{8}$. The estimated quaternions from QUEST and the proposed SOLEQ are

$$
\begin{align*}
& \mathbf{q}_{\text {QUEST }}=(0.78670,-0.23179,0.47357,-0.32109)^{T} \\
& \mathbf{q}_{\text {SOLEQ }}=(0.77823,-0.24116,0.48459,-0.31837)^{T} \tag{66}
\end{align*}
$$

respectively. The true quaternion is computed as $\mathbf{q}_{\text {true }}=$ $(0.78191,-0.23955,0.48427,-0.31100)^{T}$. The results show that in this case the SOLEQ's estimation result is more accurate than QUEST. However, in the following simulation sub-section, the results prove that the SOLEQ is generally sub-optimal as assumed.

## B. Simulation

In this sub-section, the sensor observations are simulated with random reference vectors and true DCM in (64). The standard deviations are set between 0.005 and 0.025 . Using the simulated samples, the attitude errors in Euler angles are evaluated using QUEST and SOLEQ, which are shown in Fig. 1. It is seen that our sub-optimal method cannot meet the same attitude accuracy with QUEST generally. The standard deviation of the attitude errors of the SOLEQ is about $180 \%$ of that of QUEST.


Fig. 1: Simulated attitude errors from QUEST and proposed SOLEQ.

From another point of view, the time consumption of two algorithms should be investigated. With the simulation samples, each algorithm is tested for 20000 times and the time consumption is averaged, which is plotted in Fig. 2.


Fig. 2: Time consumption of QUEST and proposed SOLEQ.
The results show that the time complexities of the two algorithms are all $O(n)$, where the SOLEQ's slope is larger
denoting higher time consumption as the amount of vector observation pairs increases. However, it is obvious that when there are less than 15 pairs of vector observations, the time consumption of SOLEQ is less than that of QUEST. For common applications, such amount will meet most requirements. Hence, the proposed SOLEQ may be more time-efficient in current engineering practice.

## C. Experiment: Accelerometer-Magnetometer Case

In this sub-section, we conduct an experiment where the accelerometer-magnetometer combination is adopted. Such sensor combination is extensively applied in nowadays lowcost attitude estimation schemes. The accelerometer is precalibrated using the 6 -face bias cancelling while the magnetometer is calibrated using the method proposed by Y. Wu et al. [25].


Fig. 3: Designed hardware with accelerometer and magnetometer for algorithm implementation.

Fig. 4: Compensated accelerometer-magnetometer data from the designed hardware.
 -

The hardware is constituted by a battery, an attitude and heading reference system (AHRS) with high precision internal accelerometer, magnetometer and gyroscope, a telemetry for remote data transmission and an embedded computer for implementation of the algorithm using C++ language on the embedded Linux operating system. With the designed hardware platform shown in Fig. 3, we collect a data set with 10000 samplings (see Fig. 4). The attitude determination is calculated with the proposed SOLEQ (with the same parameters as described in last sub-section) and the results are compared with the reference angles and results from representative methods (see Fig. 5). Note that here the weights between the accelerometer and magnetometer for Wahba's solution are chosen as 0.5 and 0.5 according to their respective noise characteristics. Yet, the local magnetometer's reference vector is calculated as $\mathbf{M}^{r}=(0.60311,0,-0.79766)^{T}$ in Wuxi, Jiangsu Province, China.


Fig. 5: Experiment results using sampled data and different algorithms.

The presented results provides us with the following information:

1) The general attitude determination results of Wahba's solution (QUEST) and the proposed SOLEQ are very similar.
2) Coping with the sub-optimal scheme, the proposed SOLEQ has lower attitude determination accuracy than Wahba's solution mainly on the yaw angle.
Hence the experiment has validated the correctness and effectiveness on embedded platforms for real-world applications.

## IV. Conclusion

This paper revisits the attitude determination from vector observations. A novel linear algorithm is designed to obtain accurate attitude estimates regardless of the effect of weights. Handling in this manner, the computed quaternion is suboptimal with respect to conventional Wahba's solutions. Numerical examples exhibit that the proposed Sub-Optimal Linear Estimator of Quaternion (SOLEQ) owns at least the same
accuracy with conventional Wahba's solutions like QUEST, SVD, LMI, and etc. The computation speed of SOLEQ is comparable with that of QUEST in the sense of absolute time consumption. Moreover, an experiment of accelerometermagnetometer attitude determination is conducted showing the effectiveness of the proposed SOLEQ in real-world embedded applications. The presented approach provides the audience with a brand new viewpoint of attitude evolution and hopefully would benefit related applications.

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