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Quadratic Stabilizability of Homogeneous Systems

Andrey Polyakov*

Abstract

Topological equivalence between an asymptotically stable nonlinear homogeneous system and a quadratically stable one is proven. A necessary and sufficient condition of quadratic-like stability (stabilizability) for nonlinear homogeneous (control) system is obtained.

1 Introduction

Symmetry is one of well-known properties of physical systems [1]. A type of symmetry studied in systems and control theory is known as homogeneity [2], [3], [4], [5], [6]. The standard homogeneity introduced by L. Euler in 17th century is the symmetry of a mathematical object f (e.g. function, vector field, operator, etc) with respect to the uniform *dilation* of the argument $x \rightarrow \lambda x$, namely, $f(\lambda x) = \lambda f(x)$, $\lambda > 0$. Type of homogeneity is basically identified by dilation. For example, in [7], [8], [9] the uniform dilation is utilized, while the paper [2], [10], [11], [12] deal with the so-called weighted dilation. Nonlinear homogeneous differential equations/inclusions form an important class of control systems models [4], [14], [15], [16], [17], [18]. They appear as local approximations [3] or set-valued extensions [12] of nonlinear systems and include models of process control [19], nonholonomic systems [20], mechanical models with frictions [11], etc.

The generalized homogeneity (to be studied in this paper) was introduced originally in [21] for infinite dimensional models such as partial differential equations (PDEs). It considers a strongly continuous group of linear bounded operators generated by a possibly unbounded linear operator as a *dilation* in a Banach space. A lot of well-known PDEs are homogeneous in generalized sense, e.g. heat, wave, Navier-Stocks, Saint-Venant, Korteweg-de Vries, fast diffusion equations. This paper deals with the finite-dimensional models of generalized homogeneous systems represented by ordinary differential equations (ODEs). It is worth stressing that all standard

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and weighted homogeneous systems are particular cases of the generalized homogeneous ones. Geometric homogeneity [13], [5], [4] is more general type of symmetry allowing dilations to be nonlinear.

Stabilization of nonlinear plant is one of classical control problems that is hard to solve constructively in general case, so particular solutions are still of the interest [22], [23]. Stability and stabilizability problems were studied for both standard [2], [9] and weighted homogeneous [10], [24], [25], [6], [26], [27], [28] systems which are the most popular today [11], [12], [15], [16], [18].

This paper deals with quadratic stability (stabilizability) problem for nonlinear generalized homogeneous (control) systems. To the best of author's knowledge, the issue of quadratic stability (existence of quadratic Lyapunov function) for the generalized homogeneous systems have not been treated before. Quadratic stabilizability is useful property for control design, since the control tuning procedures in this case can be formalized as Linear Matrix Inequalities [29] and/or semidefinite programming problems [30], which are supported by MATLAB and other software tools.

First of all, we show that any generalized homogeneous system is diffeomorphic to a standard homogeneous one. Next, a change of coordinate, which transforms asymptotically stable generalized homogeneous system to a quadratically stable one is presented. In both cases the so-called canonical homogeneous norm [31] is utilized for construction of the corresponding coordinate transformation. Combination of these two results allows the necessary and sufficient stability condition to be presented in terms of existence of the quadratic-like Lyapunov function $V(x) = x^\top \Xi(x) P \Xi(x) x$, where $P = P^\top > 0$ and the nonsingular matrix Ξ is constant along any ray from the origin (i.e. $\Xi(e^s x) = \Xi(x)$ for $x \neq 0$, $s \in \mathbb{R}$) and $\frac{\partial \Xi(x)}{\partial x_i} x = 0$ for $x \neq 0$. The properties of matrix Ξ motivate the conjecture about existence of the quadratic Lyapunov function ($\Xi = \text{const}$), which fails in the general case. However, some examples show that the obtained stability condition may simplify finite-time stability/stabilizability analysis and control design in some particular cases.

Notation: \mathbb{R} is the field of real numbers; $e_i = (0, \dots, 1, \dots, 0)^\top$ is the identity vector in \mathbb{R}^n ; $\frac{\partial}{\partial u} = \left(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}, \dots, \frac{\partial}{\partial u_n} \right)$; $C^n(X, Y)$ is the set of continuously differentiable (at least up to the order n) maps $X \rightarrow Y$, where X, Y are open subsets of finite dimensional spaces; $I_n \in \mathbb{R}^{n \times n}$ - the identity matrix; $\mathbf{0}$ denotes zero element, e.g. $\mathbf{0} \in \mathbb{R}^n$ is the zero vector but $\mathbf{0} \in \mathbb{R}^{n \times n}$ is the zero matrix; $\text{diag}\{\lambda_1, \dots, \lambda_n\}$ - diagonal matrix with elements λ_i .

2 Problem Statement

Let us consider the nonlinear control system

$$\dot{x} = g(x, u), \quad (1)$$

where $x \in \mathbb{R}^n$ is the state vector of a plant, $u \in \mathbb{R}^m$ is the control input, $g \in C(\mathbb{R}^{n+m}, \mathbb{R}^n)$, $g(\mathbf{0}, \mathbf{0}) = \mathbf{0}$.

We deal with the stabilization problem of the origin of the system (1) by means of the dynamical state-feedback law

$$\dot{u} = k(x, u). \quad (2)$$

We restrict ourselves with the class of the so-called *generalized homogeneous nonlinear systems* introduced below. The goal of the paper is to derive a necessary and sufficient condition for existence of a generalized homogeneous map $k \in C(\mathbb{R}^{n+m} \setminus \{\mathbf{0}\}, \mathbb{R}^m)$ that makes the origin of the closed-loop system (1), (2) to be globally uniformly asymptotically stable. In particular, we are interested in a (more or less) *constructive* stabilizability criterion, e.g. quadratic stabilizability condition, supported with the control design procedure. It is also worth to point out that the homogeneity guarantees the robustness (in the sense of Input-to-State Stability) of the closed-loop system with respect to a wide class of perturbations [2], [32], [33], [16], [34], e.g. measurement noises, additive and some parametric disturbances.

3 Generalized Homogeneity

Homogeneity is a sort of symmetry of an object (e.g. function or vector field) with respect to a group of transformations. The corresponding group is usually called group of dilations (or simply *dilation*). The generalized homogeneity [21] deals with the groups of linear transformations (*linear dilations*).

3.1 Dilation Group

Let $\|\cdot\|$ be a norm in \mathbb{R}^n and $\|\cdot\|_{\mathbb{A}}$ be the matrix norm induced by $\|\cdot\|$, i.e. $\|A\|_{\mathbb{A}} = \sup_{u \in \mathbb{R}^n} \frac{\|Au\|}{\|u\|}$ if $A \in \mathbb{R}^{n \times n}$.

Definition 1 ([21]) A map $d : \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$ is called *dilation* in \mathbb{R}^n if it satisfies

- **Group property:** $d(0) = I_n$ and $d(t+s) = d(t)d(s)$, $t, s \in \mathbb{R}$;

- **Continuity property:** \mathbf{d} is continuous map, i.e.
 $\forall t > 0, \forall \varepsilon > 0, \exists \delta > 0 : |s - t| < \delta \Rightarrow \|\mathbf{d}(s) - \mathbf{d}(t)\|_{\mathbb{A}} \leq \varepsilon;$
- **Limit property:** $\lim_{s \rightarrow -\infty} \|\mathbf{d}(s)u\| = 0$ and $\lim_{s \rightarrow +\infty} \|\mathbf{d}(s)u\| = +\infty$ uniformly on the unit sphere $S := \{u \in \mathbb{R}^n : \|u\| = 1\}$.

Obviously, the dilation \mathbf{d} is a uniformly continuous group of invertible linear maps $\mathbf{d}(s)$ such that $\mathbf{d}(-s) = [\mathbf{d}(s)]^{-1}$.

The matrix $G_{\mathbf{d}} \in \mathbb{R}^{n \times n}$ defined as $G_{\mathbf{d}} = \lim_{s \rightarrow 0} \frac{\mathbf{d}(s) - I}{s}$ is known (see, e.g. [35, Ch. 1]) as the **generator** of the group $\mathbf{d}(s)$. It satisfies the following properties

$$\frac{d}{ds} \mathbf{d}(s) = G_{\mathbf{d}} \mathbf{d}(s) = \mathbf{d}(s) G_{\mathbf{d}} \quad \text{and} \quad \mathbf{d}(s) = e^{G_{\mathbf{d}} s} := \sum_{i=0}^{+\infty} \frac{s^i G_{\mathbf{d}}^i}{i!}.$$

Denote also $[A]_{\mathbb{A}} = \inf_{u \in \mathbb{R}^n} \frac{\|Au\|}{\|u\|}$. Limit property implies

- $\|\mathbf{d}(s)\|_{\mathbb{A}} \rightarrow 0$ as $s \rightarrow -\infty$;
- $\mathbf{d}(s) \neq I_n$ if $s \neq 0$;
- $[\mathbf{d}(s)]_{\mathbb{A}} \rightarrow +\infty$ as $s \rightarrow +\infty$;
- $[G_{\mathbf{d}}] > 0$ ($\ker G_{\mathbf{d}} = \{\mathbf{0}\}$).

The most popular dilations [14], [16], [15], [18]

- *uniform (or standard) dilation* (L. Euler 17th century) : $\mathbf{d}(s) = e^s$, $s \in \mathbb{R}$;
- *weighted dilation* (Zubov 1958, [2]):

$$\mathbf{d}(s) = \begin{pmatrix} e^{r_1 s} & 0 & \dots & 0 \\ 0 & e^{r_2 s} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & e^{r_n s} \end{pmatrix}, \quad s \in \mathbb{R} \text{ and } r_i > 0, i = 1, \dots, n;$$

satisfy Definition 1 with $G_{\mathbf{d}} = I_n$ and $G_{\mathbf{d}} = \text{diag}\{r_i\}$, respectively. The concept of geometric dilation [4], [5] is more general since it allows the map $\mathbf{d}(s) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ to be nonlinear.

Definition 2 *Dilation \mathbf{d} is monotone if $\|\mathbf{d}(s)\|_{\mathbb{A}} < 1$ as $s < 0$.*

Monotonicity of dilation depends on the norm $\|\cdot\|$. For example, the dilation

$$\mathbf{d}(s) = e^s \begin{pmatrix} \cos(s) & \sin(s) \\ -\sin(s) & \cos(s) \end{pmatrix} \quad \text{with } G_{\mathbf{d}} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

is monotone on \mathbb{R}^2 equipped with weighted norm $\|u\|_P = \sqrt{u^T P u}$ if $P = \begin{pmatrix} 1 & 1/\sqrt{2} \\ 1/\sqrt{2} & 1 \end{pmatrix} > 0$ and it is not monotone if, for example, $P = \begin{pmatrix} 1 & 3/4 \\ 3/4 & 1 \end{pmatrix} > 0$. In the latter case, the curve $\{\mathbf{d}(s)u : s \in \mathbb{R}\}$ may cross the unit sphere S in two different points.

Theorem 1 *The next four conditions are equivalent*

- 1) *the dilation \mathbf{d} is **monotone**;*
- 2) $\lfloor \mathbf{d}(s) \rfloor_{\mathbb{A}} > 1$ for $s > 0$;
- 3) *the continuous function $\|\mathbf{d}(\cdot)u\| : \mathbb{R} \rightarrow \mathbb{R}_+$ is strictly increasing for any $u \in S$;*
- 4) *for any $u \in \mathbb{R}^n \setminus \{0\}$ there exists a **unique pair** $(s_0, u_0) \in \mathbb{R} \times S$ such that $u = \mathbf{d}(s_0)u_0$.*

Proof. **1)** \Rightarrow **2)** For any $u \in S$ we have $1 = \|u\| = \|\mathbf{d}(s)\mathbf{d}(-s)u\| \leq \|\mathbf{d}(s)\|_{\mathbb{A}}\|\mathbf{d}(-s)u\|$. Hence, $1 \leq \|\mathbf{d}(s)\|_{\mathbb{A}}\lfloor \mathbf{d}(-s) \rfloor_{\mathbb{A}}$ for any $s \in \mathbb{R}$. **2)** \Rightarrow **3)** If $u \neq 0$ and $s_1 < s_2$ then $\|\mathbf{d}(s_1)u\| - \|\mathbf{d}(s_2)u\| = \|\mathbf{d}(s_1)u\| - \|\mathbf{d}(s_2 - s_1)\mathbf{d}(s_1)u\| \leq (1 - \lfloor \mathbf{d}(s_2 - s_1) \rfloor_{\mathbb{A}})\|\mathbf{d}(s_1)u\| < 0$. The implication **3)** \Rightarrow **4)** is straightforward since $\|\mathbf{d}(0)u\| = \|u\|$. **4)** \Rightarrow **1)** If $u \in S$ then $\mathbf{d}(s)u \notin S$ for all $s \neq 0$. Indeed, otherwise the pair $(s_0, u_0) \in \mathbb{R} \times S$ such that $u_0 = \mathbf{d}(s)u \in S$ is not unique. Hence, the limit property of the dilation (see, Definition 1) implies $\|\mathbf{d}(s)u\| < 1$ for all $s < 0$ and all $u \in S$. ■

Theorem 1 also proves that the functions $\|\mathbf{d}(\cdot)\|_{\mathbb{A}} : \mathbb{R} \rightarrow \mathbb{R}_+$ and $\lfloor \mathbf{d}(\cdot) \rfloor_{\mathbb{A}} : \mathbb{R} \rightarrow \mathbb{R}_+$ are also continuous and strictly increasing.

Definition 3 *The dilation \mathbf{d} is said to be **strictly monotone** on \mathbb{R}^n if there exists $\beta > 0$ such that $\|\mathbf{d}(s)\|_{\mathbb{A}} \leq e^{\beta s}$ for $s \leq 0$.*

The dilation \mathbf{d} considered in the above example is strictly monotone on \mathbb{R}^2 equipped with the conventional Euclidian norm.

Theorem 2 *Let \mathbf{d} be a dilation in \mathbb{R}^n then*

- *the matrix $-G_{\mathbf{d}}$ is Hurwitz, i.e. all eigenvalues λ_i of $G_{\mathbf{d}}$ are placed in the right complex half-plane;*
- *for any $\beta \in (0, \beta^*]$ there exists a matrix $P \in \mathbb{R}^{n \times n}$:*

$$PG_{\mathbf{d}} + G_{\mathbf{d}}^{\top}P \geq 2\beta P, \quad P = P^{\top} > 0; \quad (3)$$

where $\beta^* = \min \Re(\lambda_i)$

- *the dilation \mathbf{d} is strictly monotone with respect to the weighted Euclidean norm $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$ induced by the inner product $\langle u, v \rangle = u^{\top}Pv$ with P satisfying (3):*

$$\begin{aligned} e^{\alpha s} &\leq \lfloor \mathbf{d}(s) \rfloor_{\mathbb{A}} \leq \|\mathbf{d}(s)\|_{\mathbb{A}} \leq e^{\beta s} \quad \text{if } s \leq 0, \\ e^{\beta s} &\leq \lfloor \mathbf{d}(s) \rfloor_{\mathbb{A}} \leq \|\mathbf{d}(s)\|_{\mathbb{A}} \leq e^{\alpha s} \quad \text{if } s \geq 0, \end{aligned}$$

$$\begin{aligned} \text{where } \alpha &= 0.5\lambda_{\max} \left(P^{\frac{1}{2}}G_{\mathbf{d}}P^{-\frac{1}{2}} + P^{-\frac{1}{2}}G_{\mathbf{d}}^{\top}P^{\frac{1}{2}} \right), \\ \beta &= 0.5\lambda_{\min} \left(P^{\frac{1}{2}}G_{\mathbf{d}}P^{-\frac{1}{2}} + P^{-\frac{1}{2}}G_{\mathbf{d}}^{\top}P^{\frac{1}{2}} \right). \end{aligned}$$

Proof. Since $\frac{d}{ds}\mathbf{d}(s) = G_{\mathbf{d}}\mathbf{d}(s)$, $\mathbf{d}(0) = I$ then $\mathbf{d}(s)$ is the fundamental matrix of the linear system ODEs with the matrix $G_{\mathbf{d}}$. The limit property of the dilation implies that this system of ODEs is globally asymptotically stable in the inverse time, i.e. the matrix $-G_{\mathbf{d}}$ is Hurwitz. Hence, there exists a symmetric positive definite matrix such that (3) holds and for any $u \in S$ one has $\frac{d}{ds}\|\mathbf{d}(s)u\|^2 = u^{\top}\mathbf{d}(s)^{\top}(G_{\mathbf{d}}^{\top}P + PG_{\mathbf{d}})\mathbf{d}(s)u \geq 2\beta\|\mathbf{d}(s)u\|^2$. Similarly we derive $\frac{d}{ds}\|\mathbf{d}(s)u\|^2 \leq 2\alpha\|\mathbf{d}(s)u\|^2$, i.e. the inequalities for matrix norms also hold. ■

Therefore, any dilation \mathbf{d} is strictly monotone if \mathbb{R}^n is equipped with the weighted Euclidian norm $\|u\| = \sqrt{u^{\top}Pu}$ provided that the matrix $P > 0$ satisfies (3) for some $\beta > 0$.

3.2 Canonical Homogeneous Norm

Dilation \mathbf{d} introduces a new topology in \mathbb{R}^n (spheres and balls [4], [17], [16]) by means of the so-called "homogeneous norm".

Definition 4 A continuous function $p : \mathbb{R}^n \rightarrow \mathbb{R}_+$ is said to be \mathbf{d} -homogeneous norm if $p(u) \rightarrow 0$ as $u \rightarrow \mathbf{0}$ and $p(\mathbf{d}(s)u) = e^s p(u) > 0$ for $u \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and $s \in \mathbb{R}$.

For monotone dilations the **canonical homogeneous norm** $\|\cdot\|_{\mathbf{d}} : \mathbb{R}^n \rightarrow \mathbb{R}_+$ can be defined as follows:

$$\|u\|_{\mathbf{d}} = e^{s_u} \quad \text{where } s_u \in \mathbb{R} \quad \text{such that} \quad \|\mathbf{d}(-s_u)u\| = 1. \quad (4)$$

In [31] such a homogeneous norm was called canonical since it is induced by the canonical norm $\|\cdot\|$ in \mathbb{R}^n and $\|x\|_{\mathbf{d}} = \|x\| = 1$ on the unit sphere S . Obviously,

$$\lfloor \mathbf{d}(\ln \|u\|_{\mathbf{d}}) \rfloor_{\mathbb{A}} \leq \|u\| \leq \|\mathbf{d}(\ln \|u\|_{\mathbf{d}})\|_{\mathbb{A}}.$$

Note also that $\|\cdot\|_{\mathbf{d}} = \|\cdot\|$ if $\mathbf{d}(s) = e^s$.

Proposition 1 If \mathbf{d} is a strictly monotone dilation then

- $\left| \|u_1\|_{\mathbf{d}}^{\beta} - \|u_2\|_{\mathbf{d}}^{\beta} \right| \leq \|u_1 - u_2\|$ for $u_1, u_2 \in \mathbb{R}^n \setminus B_{\mathbf{d}}(1)$,

- the homogeneous norm $\|\cdot\|_{\mathbf{d}}$ is Lipschitz continuous outside the origin;
- if the norm $\|\cdot\|$ is smooth outside the origin then the homogeneous norm $\|\cdot\|_{\mathbf{d}}$ is also smooth outside the origin, $\frac{d\|\mathbf{d}(-s)u\|}{ds} < 0$ if $s \in \mathbb{R}$, $u \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and

$$\frac{\partial\|u\|_{\mathbf{d}}}{\partial u} = \frac{\|u\|_{\mathbf{d}} \frac{\partial\|z\|}{\partial z} \Big|_{z=\mathbf{d}(-s)u}}{\frac{\partial\|z\|}{\partial z} \Big|_{z=\mathbf{d}(-s)u} G_{\mathbf{d}}\mathbf{d}(-s)u} \Big|_{s=\ln\|u\|_{\mathbf{d}}} \quad (5)$$

Proof. Since for $u_i \in \mathbb{R}^n$ we have $\|u_i\|_{\mathbf{d}} = e^{s_i} : \|\mathbf{d}(-s_i)u\| = 1$ then $1 = \|\mathbf{d}(-s_1)u_1\| = \|\mathbf{d}(-s_1)(u_1 - u_2) + \mathbf{d}(s_2 - s_1)\mathbf{d}(-s_2)u_2\| \leq \|\mathbf{d}(-s_1)\|_{\mathbb{A}}\|(u_1 - u_2)\| + \|\mathbf{d}(s_2 - s_1)\|_{\mathbb{A}}$. For $1 < \|u_2\|_{\mathbf{d}} < \|u_1\|_{\mathbf{d}}$ we have $0 < s_2 < s_1$ and $1 \leq e^{-\beta s_1}\|u_1 - u_2\| + e^{\beta s_2 - \beta s_1}$ or equivalently, $\|u_1\|_{\mathbf{d}} - \|u_2\|_{\mathbf{d}} \leq \|u_1 - u_2\|$. Lipschitz continuity follows from the proven inequality, the identity $\|\mathbf{d}(s)u\|_{\mathbf{d}} = e^s\|u\|_{\mathbf{d}}$ and monotonicity of the dilation. The existence of the unique function $s : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $\|\mathbf{d}(-s(u))u\| = 1$ has been proven in Theorem 1. Since the dilation is strictly monotone then $\frac{d}{ds}\|\mathbf{d}(-s)u\| < 0$ on S (and, on $\mathbb{R}^n \setminus \{0\}$) for all $s \in \mathbb{R}$ (see, Theorem 1). Since the norm $\|\cdot\|$ is smooth them $\frac{d}{ds}\|\mathbf{d}(-s)u\| = -\frac{\partial\|z\|}{\partial z} \Big|_{z=\mathbf{d}(-s)u} G_{\mathbf{d}}\mathbf{d}(-s)u$. Taking into account $\frac{\partial}{\partial u}\|u\|_{\mathbf{d}} = e^s \frac{\partial s}{\partial u} \Big|_{s=\ln\|u\|_{\mathbf{d}}}$ the formula (5) can be derived using Implicit Function Theorem [36] applied to the equality $\|\mathbf{d}(-s)u\| = 1$. ■

3.3 Generalized Homogeneous Functions and Vectors Fields

Vector fields, which are symmetric (in a certain sense) with respect to dilation \mathbf{d} , have a lot of properties useful for control design and state estimation of both linear and nonlinear plants as well as for analysis of convergence rates [5],[25],[37],[15].

Definition 5 A vector field $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ (a function $h : \mathbb{R}^n \rightarrow \mathbb{R}$) is said to be \mathbf{d} -homogeneous of degree $\nu \in \mathbb{R}$ if

$$f(\mathbf{d}(s)u) = e^{\nu s}\mathbf{d}(s)f(u), \quad \forall u \in \mathbb{R}^n \setminus \{\mathbf{0}\}, \quad \forall s \in \mathbb{R}. \quad (6)$$

$$(resp. h(\mathbf{d}(s)u) = e^{\nu s}h(u), \quad \forall u \in \mathbb{R}^n \setminus \{\mathbf{0}\}, \quad \forall s \in \mathbb{R}.)$$

Example 1 Let us consider the dilation

$$\mathbf{d}(s) = e^s \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(s) & \sin(s) \\ 0 & -\cos(s) & \sin(s) \end{pmatrix}$$

that is strictly monotone with respect to the Euclidean norm $\|x\| = \sqrt{x^T x}$ and $G_{\mathbf{d}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$. The vector field $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined as

$$f(x) = \begin{pmatrix} x_2^2 + x_3^2 \\ x_1^2 (\cos(\ln|x_1|) + \sin(\ln|x_1|)) \\ x_1^2 (\cos(\ln|x_1|) - \sin(\ln|x_1|)) \end{pmatrix}$$

and the function $h : \mathbb{R}^3 \rightarrow \mathbb{R}$ given by $h = x_1^3 + (x_2^2 + x_3^2)^{\frac{3}{2}}$ are \mathbf{d} -homogeneous of degree 1 and 3, respectively.

Example 2 (Homogeneous vector field of arbitrary degree) *The vector field may have different degrees of homogeneity dependently of the dilation group. Indeed, the linear vector field $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $f(x) = Ax$ defined by a chain of integrators $A = \begin{pmatrix} 0 & I_{n-1} \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{n \times n}$ is \mathbf{d}_1 -homogeneous of degree $\mu \in [0, 1]$ with $\mathbf{d}_1(s) = \text{diag}\{e^{(n+(i-1)\mu)s}\}_{i=1}^n$ and \mathbf{d}_2 -homogeneous of degree $\mu \in [-1, 0]$ with $\mathbf{d}_2(s) = \text{diag}\{e^{(n+(n-i)\mu)s}\}_{i=1}^n$.*

Let $\mathbb{F}_{\mathbf{d}}(\mathbb{R}^n)$ (resp. $\mathbb{H}_{\mathbf{d}}(\mathbb{R}^n)$) be the set of \mathbf{d} -homogeneous vector fields $\mathbb{R}^n \rightarrow \mathbb{R}^n$ (resp. functions $\mathbb{R}^n \rightarrow \mathbb{R}$), which are continuous on $\mathbb{R}^n \setminus \{\mathbf{0}\}$. Let $\text{deg}_{\mathbf{d}}(h)$ (resp. $\text{deg}_{\mathbf{d}}(f)$) denote the homogeneity degree of $h \in \mathbb{H}_{\mathbf{d}}(\mathbb{R}^n)$ (resp. $f \in \mathbb{F}_{\mathbf{d}}(\mathbb{R}^n)$).

The homogeneity allows local properties (e.g. smoothness) of vector fields (functions) to be extended globally [2], [3].

Corollary 1 *The vector field $f \in \mathbb{F}_{\mathbf{d}}(\mathbb{R}^n)$ is Lipschitz continuous (smooth) on $\mathbb{R}^n \setminus \{\mathbf{0}\}$ **if and only if** it satisfies Lipschitz condition (it is smooth) on the unit sphere S provided that \mathbf{d} is strictly monotone on \mathbb{R}^n equipped with a (smooth) norm $\|\cdot\|$.*

Proof. The necessity is straightforward since any locally Lipschitz function on a compact set satisfies Lipschitz condition on it. Let us prove sufficiency. Let $u_i \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, $i = 1, 2$ then $u_i = \mathbf{d}(\ln \|u_i\|_{\mathbf{d}})z_i$ for some $z_i \in S$,

$$\begin{aligned} f(u_1) - f(u_2) &= f(\mathbf{d}(\ln \|u_1\|_{\mathbf{d}})z_1) - f(\mathbf{d}(\ln \|u_2\|_{\mathbf{d}})z_2) \\ &= \|u_1\|_{\mathbf{d}}^{\nu} \mathbf{d}(\ln \|u_1\|_{\mathbf{d}}) f(z_1) - \|u_2\|_{\mathbf{d}}^{\nu} \mathbf{d}(\ln \|u_2\|_{\mathbf{d}}) f(z_2) \\ &= \|u_1\|_{\mathbf{d}}^{\nu} \mathbf{d}(\ln \|u_1\|_{\mathbf{d}}) (f(z_1) - f(z_2)) + (\|u_1\|_{\mathbf{d}}^{\nu} \mathbf{d}(\ln \|u_1\|_{\mathbf{d}}) \\ &\quad - \|u_2\|_{\mathbf{d}}^{\nu} \mathbf{d}(\ln \|u_1\|_{\mathbf{d}})) f(z_2) + \|u_2\|_{\mathbf{d}}^{\nu} (\mathbf{d}(\ln \|u_1\|_{\mathbf{d}}) - \mathbf{d}(\ln \|u_2\|_{\mathbf{d}})) f(z_2). \end{aligned}$$

If $L > 0$ is a Lipschitz constant on S then

$$\begin{aligned} \|f(u_1) - f(u_2)\| &\leq L \|u_1\|_{\mathbf{d}}^{\nu} \mathbf{d}(\ln \|u_1\|_{\mathbf{d}}) \|z_1 - z_2\| \\ &\quad + \|\mathbf{d}(\ln \|u_1\|_{\mathbf{d}}) f(z_2)\| (\|u_1\|_{\mathbf{d}}^{\nu} - \|u_2\|_{\mathbf{d}}^{\nu}) \\ &\quad + \|f(z_2)\| \|u_2\|^{\nu} \|\mathbf{d}(\ln \|u_1\|_{\mathbf{d}}) - \mathbf{d}(\ln \|u_2\|_{\mathbf{d}})\|_{\mathbb{A}}. \end{aligned}$$

Since $\mathbf{d}(s_1) - \mathbf{d}(s_2) = G_{\mathbf{d}} \int_{s_2}^{s_1} \mathbf{d}(s) ds$ and the function $\|\mathbf{d}(\cdot)\|_{\mathbb{A}}$ is strictly monotone increasing then $\|\mathbf{d}(\ln \|u_1\|_{\mathbf{d}}) - \mathbf{d}(\ln \|u_2\|_{\mathbf{d}})\|_{\mathbb{A}} \leq M \|G_{\mathbf{d}}\| |\ln \|u_1\|_{\mathbf{d}} - \ln \|u_2\|_{\mathbf{d}}|$, where $M = \max\{\|\mathbf{d}(\ln \|u_1\|_{\mathbf{d}})\|_{\mathbb{A}}, \|\mathbf{d}(\ln \|u_2\|_{\mathbf{d}})\|_{\mathbb{A}}\}$. Since the homogeneous norm is Lipschitz continuous (see Proposition 1) on $\mathbb{R}^n \setminus \{\mathbf{0}\}$ but power and logarithm functions are Lipschitz continuous outside zero then f is Lipschitz continuous outside the origin. (Differentiability of homogeneous vector field (function) f on $\mathbb{R}^n \setminus \{\mathbf{0}\}$ can be proven by means of the formula (5), the identity $\frac{d}{ds} \mathbf{d}(s) = G_{\mathbf{d}} \mathbf{d}(s)$ and the representation $f(u) = \|u\|_{\mathbf{d}}^{\nu} \mathbf{d}(\ln \|u\|_{\mathbf{d}}) f(z)$ with $z = \mathbf{d}(-\ln \|u\|) u \in S$.) ■

If a function (or a vector field) is smooth then homogeneity is inherited by its derivatives in a certain way.

Corollary 2 *If $h \in \mathbb{H}_{\mathbf{d}}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n \setminus \{\mathbf{0}\}, \mathbb{R})$ then*

$$e^{\deg(h)s} \frac{\partial h(u)}{\partial u} = \left. \frac{\partial h(z)}{\partial z} \right|_{z=\mathbf{d}(s)u} \mathbf{d}(s), \quad (7)$$

$$\frac{\partial h(u)}{\partial u} G_{\mathbf{d}} u = \deg_{\mathbf{d}}(h) h(u), \quad (8)$$

for $u \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and $s \in \mathbb{R}$.

Proof. The formula (7) can be obtained using the definition of the (Frechét) derivative, which coincides with $\frac{\partial h}{\partial u}$ if h is smooth. Namely,

$$\lim_{\|\Delta\| \rightarrow 0} \frac{|h(u+\Delta) - h(u) - \frac{\partial h(u)}{\partial u} \Delta|}{\|\Delta\|} = 0$$

and

$$\lim_{\|\Delta\| \rightarrow 0} \frac{|h(\mathbf{d}(s)u+\Delta) - h(\mathbf{d}(s)u) - \left. \frac{\partial h(z)}{\partial z} \right|_{z=\mathbf{d}(s)u} \Delta|}{\|\Delta\|} = 0,$$

where $\Delta \in \mathbb{R}^n$. Since $h \in \mathbb{H}_{\mathbf{d}}$ then

$$\frac{|h(\mathbf{d}(s)u+\Delta) - h(\mathbf{d}(s)u) - \left. \frac{\partial h(z)}{\partial z} \right|_{z=\mathbf{d}(s)u} \Delta|}{\|\Delta\|}$$

$$\begin{aligned}
&= e^{\nu s} \frac{\left| h(u+\tilde{\Delta})-h(u)-e^{-\nu s} \frac{\partial h(z)}{\partial z} \Big|_{z=\mathbf{d}(s)u} \mathbf{d}(s)\tilde{\Delta} \right|}{\|\mathbf{d}(s)\tilde{\Delta}\|} \\
&\leq \frac{e^{\nu s}}{\lfloor \mathbf{d}(s) \rfloor_{\mathbb{A}}} \times \frac{\left| h(u+\tilde{\Delta})-h(u)-e^{-\nu s} \frac{\partial h(z)}{\partial z} \Big|_{z=\mathbf{d}(s)u} \mathbf{d}(s)\tilde{\Delta} \right|}{\|\tilde{\Delta}\|},
\end{aligned}$$

where $\nu = \deg(h)$ and $\tilde{\Delta} = \mathbf{d}(-s)\Delta$ such that $\|\tilde{\Delta}\| \rightarrow 0$ implies $\|\Delta\| \rightarrow 0$. Therefore the identity (7) holds.

To prove (8) let us consider a homogeneous norm $\|\cdot\|_{\mathbf{d}}$ induced by a weighted Euclidean norm $\|u\| = \sqrt{u^{\top} P u}$ in $u \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, $P = P^{\top} > 0$. In this case, due to (5) we have $\frac{\partial \|u\|_{\mathbf{d}}}{\partial u} u > 0$ if $u \neq 0$. From $h(u) = \|u\|_{\mathbf{d}}^{\nu} h(\mathbf{d}(-\ln \|u\|_{\mathbf{d}})u)$ we derive

$$\begin{aligned}
\frac{\partial h(u)}{\partial u} &= \nu h(\mathbf{d}(-\ln \|u\|_{\mathbf{d}})u) \|u\|_{\mathbf{d}}^{\nu-1} \frac{\partial \|u\|_{\mathbf{d}}}{\partial u} + \|u\|_{\mathbf{d}}^{\nu} \frac{\partial h(\mathbf{d}(-\ln \|u\|_{\mathbf{d}})u)}{\partial u} \\
&= \nu h(\mathbf{d}(-\ln \|u\|_{\mathbf{d}})u) \|u\|_{\mathbf{d}}^{\nu-1} \frac{\partial \|u\|_{\mathbf{d}}}{\partial u} + \|u\|_{\mathbf{d}}^{\nu} \frac{\partial h(z)}{\partial z} \Big|_{z=\mathbf{d}(-\ln \|u\|_{\mathbf{d}})u} \frac{\partial}{\partial u} (\mathbf{d}(-\ln \|u\|_{\mathbf{d}})u) \\
&= \nu h(\mathbf{d}(-\ln \|u\|_{\mathbf{d}})u) \|u\|_{\mathbf{d}}^{\nu-1} \frac{\partial \|u\|_{\mathbf{d}}}{\partial u} + \|u\|_{\mathbf{d}}^{\nu} \frac{\partial h(z)}{\partial z} \Big|_{z=\mathbf{d}(-\ln \|u\|_{\mathbf{d}})u} \\
&\quad \times \left(\mathbf{d}(-\ln \|u\|_{\mathbf{d}}) - \frac{G_{\mathbf{d}} \mathbf{d}(-\ln \|u\|_{\mathbf{d}})u \frac{\partial \|u\|_{\mathbf{d}}}{\partial u}}{\|u\|_{\mathbf{d}}} \right).
\end{aligned}$$

Therefore, for $\|u\| = 1$ we have $\frac{\partial h(u)}{\partial u} G_{\mathbf{d}} u \frac{\partial \|u\|_{\mathbf{d}}}{\partial u} = \nu h(u) \frac{\partial \|u\|_{\mathbf{d}}}{\partial u}$. Hence, multiplying by u we obtain that (8) holds for $\|u\| = 1$. Since $\|\cdot\|_{new} = \gamma \|\cdot\|$ with $\gamma > 0$ is again a weighted Euclidean norm in \mathbb{R}^n then the obtained identity holds on $\mathbb{R}^n \setminus \{\mathbf{0}\}$. ■

Remark 1 If \mathbf{d} is a dilation with the generator $G_{\mathbf{d}}$ then for any fixed $\alpha > 0$ the group \mathbf{d}^{α} defined as $\mathbf{d}^{\alpha}(s) := \mathbf{d}(\alpha s)$, $s \in \mathbb{R}^n$ is the dilation with the generator $G_{\mathbf{d}^{\alpha}} = \alpha G_{\mathbf{d}}$. Moreover, if $f \in \mathbb{F}_{\mathbf{d}}(\mathbb{R}^n)$ then $f \in \mathbb{F}_{\mathbf{d}^{\alpha}}(\mathbb{R}^n)$ and $\deg_{\mathbf{d}^{\alpha}}(f) = \alpha \deg_{\mathbf{d}}(f)$.

4 Quadratic Stability of Homogeneous Systems

Homogeneity may simplify the analysis of differential equations, e.g. to prove existence and uniqueness of solution on $\mathbb{R}^n \setminus \{\mathbf{0}\}$ it is sufficient to prove that the right hand side is Lipschitz continuous (or differentiable) on a sphere (see, Corollary 2. The most important property of \mathbf{d} -homogeneous systems is scalability of the solutions [2], [4], [14], [37], [38], [21].

Theorem 3 If $\varphi_{\xi_0} : [0, T) \rightarrow \mathbb{R}^n$ is a solution to

$$\dot{\xi} = f(\xi), \quad f \in \mathbb{F}_{\mathbf{d}}(\mathbb{R}^n) \tag{9}$$

with the initial condition $\xi(0) = \xi_0 \in \mathbb{R}^n$ then $\varphi_{\mathbf{d}(s)\xi_0} : [0, e^{-\nu s}T) \rightarrow \mathbb{R}^n$ defined as $\varphi_{\mathbf{d}(s)\xi_0}(t) := \mathbf{d}(s)\varphi_{\xi_0}(te^{\nu s})$ with $s \in \mathbb{R}$ is a solution to (9) with the initial condition $\xi(0) = \mathbf{d}(s)\xi_0$, where $\nu = \deg_{\mathbf{d}}(f)$.

Proof. The scheme of the proof is standard (see [2]) for any type of homogeneity. Since $\frac{d}{dt}\varphi_{\xi_0}(t) = f(\varphi_{\xi_0}(t))$ then

$$\mathbf{d}(s)\frac{d}{dt}\varphi_{\xi_0}(t) = \frac{d}{dt}\mathbf{d}(s)\varphi_{\xi_0}(t) = \mathbf{d}(s)f(\varphi_{\xi_0}(t)) = e^{-\nu s}f(\mathbf{d}(s)\varphi_{\xi_0}(t)).$$

Making the change of time $t = e^{\nu s}t^{new}$ we complete the proof. ■

This theorem has a lot of corollaries, which are very useful for qualitative analysis of homogeneous systems. For example, local stability always implies the global one or the existence of strictly positively invariant compact set implies asymptic stability [4], [14], [2], etc. Moreover, using the classical result [2], [14] about existence of homogeneous Lyapunov function for asymptotically stable homogeneous system we can prove topological equivalence of such system to a quadratically stable one.

Theorem 4 *The next five claims are equivalent*

- 1) *The origin of the system (9) is asymptotically stable.*
- 2) *There exists a Lyapunov function $V \in \mathbb{H}_{\mathbf{d}}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n)$;*
- 3) *The origin of the system*

$$\dot{z} = \|z\|^{1+\deg_{\mathbf{d}}(f)} \left(\frac{(I_n - G_{\mathbf{d}})z^\top z P}{z^\top P G_{\mathbf{d}} z} + I_n \right) f\left(\frac{z}{\|z\|}\right) \quad (10)$$

is asymptotically stable, where $\|z\| = \sqrt{z^\top P z}$ with

$$P G_{\mathbf{d}} + G_{\mathbf{d}}^\top P > 0, \quad 0 < P = P^\top \in \mathbb{R}^{n \times n}. \quad (11)$$

- 4) *For any matrix $P \in \mathbb{R}^{n \times n}$ satisfying (11) there exists a map $\Psi \in \mathbb{F}_{\mathbf{d}}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n \setminus \{\mathbf{0}\})$, $\deg_{\mathbf{d}}(\Psi) = 0$ such that Ψ is diffeomorphism on $\mathbb{R}^n \setminus \{\mathbf{0}\}$, homeomorphism on \mathbb{R}^n , $\Psi(\mathbf{0}) = \mathbf{0}$ and*

$$\frac{\partial(\Psi^\top(\xi)P\Psi(\xi))}{\partial \xi} f(\xi) < 0 \quad \text{if} \quad \Psi^\top(\xi)P\Psi(\xi) = 1. \quad (12)$$

Moreover, $\|\Psi\|_{\mathbf{d}} \in \mathbb{H}_{\mathbf{d}}(\mathbb{R}^n) \cap C^\infty(\mathbb{R}^n \setminus \{\mathbf{0}\})$ is Lyapunov function to the system (9), where $\|\cdot\|_{\mathbf{d}}$ is the canonical homogeneous norm induced by $\|\xi\| = \sqrt{\xi^\top P \xi}$.

5) For any matrix $P \in \mathbb{R}^{n \times n}$ satisfying (11) there exists a map $\Xi \in C^\infty(\mathbb{R}^n \setminus \{\mathbf{0}\}, \mathbb{R}^{n \times n})$ such that

$$\det(\Xi(z)) \neq 0, \quad \frac{\partial \Xi(z)}{\partial z_i} z = 0, \quad \Xi(e^s z) = \Xi(z) \quad \text{for } z \in \mathbb{R}^n \setminus \{\mathbf{0}\}, s \in \mathbb{R}$$

and

$$z^\top \Xi^\top(z) P \Xi(z) \left(\frac{(I_n - G_{\mathbf{d}}) z z^\top P}{z^\top P G_{\mathbf{d}} z} + I_n \right) f \left(\frac{z}{\sqrt{z^\top P z}} \right) < 0. \quad (13)$$

Proof. Without loss of generality we assume that f is continuous at zero and $f(\mathbf{0}) = \mathbf{0}$, since asymptotic stability of (10) is equivalent to asymptotic stability of $\dot{\xi} = \tilde{f}(\xi) := \|\xi\|_{\mathbf{d}}^{-\text{deg}_{\mathbf{d}}(f)} f(\xi)$, which is always continuous at the origin.

1) \Leftrightarrow 2) We use the scheme developed in [14]. The Converse Lyapunov Theorem (see, e.g. [39]) implies that there exists a smooth Lyapunov function $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$. Let the smooth function $a : \mathbb{R} \rightarrow \mathbb{R}_+$ be defined as $a(\rho) = e^{\frac{1}{1-\rho}}$ if $\rho > 0$ and $a(\rho) = 0$ if $\rho \leq 1$. Obviously, $a'(\rho) > 0$ if $\rho > 1$. Then the function $V_h : \mathbb{R}^n \rightarrow \mathbb{R}_+$ defined as $V_h(x) = \int_{-\infty}^{+\infty} e^{-s} a(V(\mathbf{d}(s)x)) ds$ is \mathbf{d} -homogeneous Lyapunov function to the system (9). Indeed, it is well-defined (due to cut-off function a), smooth, positive definite and radially unbounded. Finally, V_h is \mathbf{d} -homogeneous $V_h(\mathbf{d}(q)x) = \int_{-\infty}^{+\infty} e^{-s} a(V(\mathbf{d}(-s+q)x)) ds = e^q V_h(x)$ and

$$\dot{V}_h(x) = \int_{-\infty}^{+\infty} e^{-s} a'(V(\mathbf{d}(-s)x)) \frac{\partial V(z)}{\partial z} \Big|_{z=\mathbf{d}(s)x} \mathbf{d}(s) f(x) ds < 0$$

since $f \in \mathbb{F}(\mathbb{R}^n) : \frac{\partial V(z)}{\partial z} \Big|_{z=\mathbf{d}(s)x} \mathbf{d}(s) f(x) = \frac{1}{e^{\nu s}} \frac{\partial V(z)}{\partial z} f(z) \Big|_{z=\mathbf{d}(s)x} < 0$.

1) \Leftrightarrow 3) Since P satisfies (3) then the dilation \mathbf{d} is strictly monotone on \mathbb{R}^n equipped with the norm $\|\xi\| = \sqrt{\xi^\top P \xi}$. The change of coordinates $z = \|\xi\|_{\mathbf{d}} \mathbf{d}(-\ln \|\xi\|_{\mathbf{d}}) \xi$ gives $\|z\| = \|\xi\|_{\mathbf{d}}$ and $\xi = \mathbf{d}(\ln \|z\|) \frac{z}{\|z\|}$,

$$\begin{aligned} \dot{z} &= (I_n - G_{\mathbf{d}}) \mathbf{d}(-\ln \|\xi\|_{\mathbf{d}}) \xi \frac{d\|\xi\|_{\mathbf{d}}}{dt} + \|\xi\|_{\mathbf{d}} \mathbf{d}(-\ln \|\xi\|_{\mathbf{d}}) f(\xi) = \\ &\|\xi\|_{\mathbf{d}} (I_n - G_{\mathbf{d}}) \mathbf{d}(-\ln \|\xi\|_{\mathbf{d}}) \xi \frac{\xi^\top \mathbf{d}^\top(-\ln \|\xi\|_{\mathbf{d}}) P \mathbf{d}(-\ln \|\xi\|_{\mathbf{d}}) f(\xi)}{\xi^\top \mathbf{d}^\top(-\ln \|\xi\|_{\mathbf{d}}) P G_{\mathbf{d}} \mathbf{d}(-\ln \|\xi\|_{\mathbf{d}}) \xi} + \\ &\quad + \|\xi\|_{\mathbf{d}} \mathbf{d}(-\ln \|\xi\|_{\mathbf{d}}) f(\xi) = \\ &\|\xi\|_{\mathbf{d}} \left(\frac{(I - G_{\mathbf{d}}) \mathbf{d}(-\ln \|\xi\|_{\mathbf{d}}) \xi \xi^\top \mathbf{d}^\top(-\ln \|\xi\|_{\mathbf{d}}) P}{\xi^\top \mathbf{d}^\top(-\ln \|\xi\|_{\mathbf{d}}) P G_{\mathbf{d}} \mathbf{d}(-\ln \|\xi\|_{\mathbf{d}}) \xi} + I_n \right) \mathbf{d}(-\ln \|\xi\|_{\mathbf{d}}) f(\xi). \end{aligned}$$

Taking into account $f \in \mathbb{F}_{\mathbf{d}}(\mathbb{R}^n)$ we derive (10).

4) \Rightarrow 2) Since $\Psi(\mathbf{d}(s)\xi) = \mathbf{d}(s)\Psi(\xi)$ then $\Psi(\mathbf{0}) = \mathbf{0}$ due to continuity at zero. Note also that $\Psi(\xi) \neq \mathbf{0}$ for $\xi \neq \mathbf{0}$, otherwise (i.e. $\exists \xi^* \neq \mathbf{0} : \Psi(\xi^*) = \mathbf{0}$), due to homogeneity we derive that $\Psi(\xi) = \mathbf{0}$ on a smooth

curve $\{\mathbf{d}(s)\xi^*, s \in \mathbb{R}\}$, which starts at the origin goes to ∞ . The latter contradicts the assumption that Ψ is diffeomorphism (continuously differentiable invertible map with continuously differentiable inverse) on $\mathbb{R}^n \setminus \{0\}$. Since $\deg_{\mathbf{d}}(\Psi) = 0$ then $\|\Psi(\mathbf{d}(s)\xi)\|_{\mathbf{d}} = \|\mathbf{d}(s)\Psi(\xi)\|_{\mathbf{d}} = e^s \|\Psi(\xi)\|_{\mathbf{d}}$ and the function $\|\Psi(\cdot)\|_{\mathbf{d}}$ is \mathbf{d} -homogeneous of degree 1, radially unbounded, continuous at the origin and continuously differentiable outside the origin. Due to (5) the inequality $\frac{\partial \|\Psi(\xi)\|_{\mathbf{d}}}{\partial \xi} f(\xi) \Big|_{\|\Psi(\xi)\|_{\mathbf{d}}=1} < 0$ implies $\frac{\partial \|\Psi(\xi)\|_{\mathbf{d}}}{\partial \xi} f(\xi) \Big|_{\|\Psi(\xi)\|_{\mathbf{d}}=1} < 0$.

Applying homogeneity we derive $\frac{\partial \|\Psi(\xi)\|_{\mathbf{d}}}{\partial \xi} f(\xi) < 0$ if $\xi \in \mathbb{R}^n \setminus \{0\}$, i.e. $\|\Psi\|_{\mathbf{d}}$ is a Lyapunov function to the system (9).

2) \Rightarrow 4) Since the origin of the system (9) is asymptotically stable then there exists a smooth \mathbf{d} -homogeneous Lyapunov function $\tilde{V} : \mathbb{R}^n \rightarrow \mathbb{R}_+$ of degree $\mu > 0$. The function $V = \tilde{V}^{1/\mu}$ is also Lyapunov function to (9) that is \mathbf{d} -homogeneous of degree 1, continuous at the origin and smooth outside the origin. Let us consider the map $\Psi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined as

$$\Psi(\xi) = \mathbf{d} \left(\ln \left(\frac{V(\xi)}{\|\xi\|_{\mathbf{d}}} \right) \right) \xi.$$

Since $\frac{V(\xi)}{\|\xi\|_{\mathbf{d}}} \in \mathbb{H}_{\mathbf{d}}(\mathbb{R}^n)$, $\deg_{\mathbf{d}} \left(\frac{V(\xi)}{\|\xi\|_{\mathbf{d}}} \right) = 0$ then $\liminf_{\xi \rightarrow 0} \frac{V(\xi)}{\|\xi\|_{\mathbf{d}}} = \inf_{\xi \in S} \frac{V(\xi)}{\|\xi\|_{\mathbf{d}}} > 0$, so Ψ is continuous at $\mathbf{0}$ and $\Psi(\mathbf{0}) = \mathbf{0}$. Obviously, $\Psi(\mathbf{d}(s)\xi) = \mathbf{d}(s)\Psi(\xi)$ and $\|\Psi(\xi)\|_{\mathbf{d}} = V(\xi)$. Using (5) we derive (12). The map Ψ is bijection. The inverse map $\Psi^{-1} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is given by

$$\Psi^{-1}(x) = \mathbf{d} \left(-\ln \left(\frac{V(x)}{\|x\|_{\mathbf{d}}} \right) \right) x.$$

Indeed, $\Psi^{-1}(\Psi(\xi)) = \xi$ and $\Psi(\Psi^{-1}(x)) = x$ for all $\xi, x \in \mathbb{R}^n$. Since Ψ and Ψ^{-1} are continuous at the origin, smooth outside the origin then Ψ is diffeomorphism on $\mathbb{R}^n \setminus \{0\}$.

3) \Rightarrow 5) The system (10) is homogeneous of degree ν with respect to the dilation $\mathbf{d}_0(s) = e^s$ with $G_{\mathbf{d}_0} = I_n$. Note that $P_0 G_{\mathbf{d}_0} + G_{\mathbf{d}_0}^\top P_0 = 2P_0 > 0$ holds for an arbitrary symmetric positive definite matrix P_0 . Taking into account $\|z\|_{\mathbf{d}_0} = \|z\| = \sqrt{z^\top P_0 z}$ we use the claim 4) to obtain the homogeneous Lyapunov function V defined as $V(z) = \|\Psi_0(z)\|_{\mathbf{d}_0}^2 = \Psi_0^\top(z) P_0 \Psi_0(z)$ for $z \in \mathbb{R}^n$. Since Ψ_0 is diffeomorphism then $\det \left(\frac{\partial \Psi_0(z)}{\partial z} \right) \neq 0$ for $z \in \mathbb{R}^n \setminus \{0\}$. Using (8) we derive $\frac{\partial \Psi_0(z)}{\partial z} z = \Psi_0(z)$, $\frac{\partial V}{\partial z} = 2z^\top \Xi^\top(z) P_0 \Xi(z)$ and $\frac{\partial \Xi(z)}{\partial z_i} z = 0$, where $\Xi(z) = \frac{\partial \Psi_0(z)}{\partial z}$. Finally, $\Xi(e^s z) = \Xi(z)$ for $z \in \mathbb{R}^n \setminus \{0\}$ and $s \in \mathbb{R}$ due to (7) and $\deg_{\mathbf{d}_0}(\Psi_0) = 1$.

5) \Rightarrow 3) Let us consider the function $V(z) = z^\top \Xi^\top P \Xi(z) z$, $V \in C(\mathbb{R}^n, \mathbb{R}) \cap C^\infty(\mathbb{R}^n \setminus \{\mathbf{0}\}, \mathbb{R})$. Since $\frac{\partial \Xi(z)}{z} = 0$ then $\frac{\partial V(z)}{\partial z} = 2z^\top \Xi^\top P \Xi(z)$ and the condition (13) implies that V is the Lyapunov function to (10). \blacksquare

This theorem proves two important facts.

- Any generalized homogeneous system is diffeomorphic to standard homogeneous one (see the formula (10)). Moreover, it is diffeomorphic to a **quadratically stable** system. Indeed, making the change of variables $z = \Psi(\xi)$ we derive $\dot{z} = \tilde{f}(z)$, where $\tilde{f}(z) = \left. \frac{\partial \Psi(\xi)}{\partial \xi} f(\xi) \right|_{\xi = \Psi^{-1}(z)}$, but the criterion (12) implies that $z^\top P \dot{z} < 0$ if $z^\top P z = 1$, so the homogeneous norm $\|\cdot\|_{\mathbf{d}}$ is the Lyapunov function to the latter system. Finally, the change of variable $x = \|z\|_{\mathbf{d}} \mathbf{d}(-\ln \|z\|_{\mathbf{d}}) z$ gives $\|z\|_{\mathbf{d}} = \|x\|$, so the transformed system $\dot{x} = \tilde{f}(x)$ is quadratically stable.
- The formula (13) gives a more constructive stability criterion. *Since $\Xi(z) = \Xi(e^s z)$ for $z \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, $s \in \mathbb{R}$ then the map Ξ is constant along any straight line $\{e^s z : s \in \mathbb{R}\}$ if $z \neq \mathbf{0}$. In addition, the property $\frac{\partial \Xi(z)}{\partial z_i} z = 0$ motivates the conjecture that a quadratic Lyapunov function ($\Xi \equiv \text{const}$) always exists for asymptotically stable standard homogeneous system. Such assumption was also given in [9]. However, in the view of [39, Proposition 5.2] this conjecture seems to be wrong. Therefore, the condition*

$$\begin{aligned} z^\top Q_0 \left(\frac{(I_n - G_{\mathbf{d}}) z z^\top P}{z^\top P G_{\mathbf{d}} z} + I_n \right) f \left(\frac{z}{\sqrt{z^\top P z}} \right) < 0 \text{ if } z \neq \mathbf{0}, \\ P G_{\mathbf{d}} + G_{\mathbf{d}} P > 0, \quad P > 0, \quad Q_0 > 0 \end{aligned} \quad (14)$$

derived from (13) is just sufficient stability condition.

Recall [37, 11, 12], if the standard (or weighted) homogeneous system (9) is asymptotically stable and $\deg_{\mathbf{d}}(f) < 0$ then it is globally uniformly *finite-time stable*, i.e there exists a locally bounded function $T : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $\varphi_{x_0}(t) = 0$, $\forall t > T(x_0)$ and $\forall x_0 \in \mathbb{R}^n$. The proof of finite-time stability in the case of the generalized homogeneous system is similar, e.g. it is given in [21] for infinite dimensional systems.

Example 3 (Quadratic Finite-time Stability) *The system*

$$\dot{x} = Ax + Bu(x), \quad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$u(x) = \frac{k_1 x_1 + k_2 x_2 \sqrt{\sigma(x)}}{\sigma(x)}, \quad \sigma(x) := q_1 |x_1|^{\frac{2}{3}} + q_2 |x_2|,$$

is \mathbf{d} -homogeneous of negative degree -1 if the dilation \mathbf{d} is generated by $G_{\mathbf{d}} = \text{diag}\{3, 2\}$. Let us find conditions to

$$k_1, k_2 \in \mathbb{R}, \quad q_1, q_2 \in \mathbb{R}_+$$

allowing the finite-time stability of the system. Selecting $Q_0 = P$ in (14) we derive the stability condition

$$x^\top P \tilde{f}(x) < 0, \quad \tilde{f}(x) = \begin{pmatrix} x_2 \\ \frac{k_1 x_1}{\tilde{N}(x)} + \frac{k_2 x_2}{\sqrt{\tilde{N}(x)}} \end{pmatrix},$$

where $\tilde{N}(x) = \frac{q_1 |x_1|^{\frac{2}{3}} \|x\|^{\frac{1}{3}} + q_2 |x_2|}{\|x\|}$, $\|x\| = \sqrt{x^\top P x}$, $0 < \delta_{\min} \leq \tilde{N}(x) \leq \delta_{\max}$,

$$\delta_{\min} = \min_{x^\top P x = 1} q_1 |x_1|^{\frac{2}{3}} + q_2 |x_2| \quad \text{and} \quad \delta_{\max} = \min_{x^\top P x = 1} q_1 |x_1|^{\frac{2}{3}} + q_2 |x_2|.$$

Since

$$x^\top P \tilde{f}(x) = x^\top P \begin{pmatrix} 0 & 1 \\ k_1 & k_2 \end{pmatrix} x + \frac{1 - \sqrt{\tilde{N}(x)}}{\sqrt{\tilde{N}(x)}} x^\top P \begin{pmatrix} 0 & 0 \\ k_1 & k_2 \end{pmatrix} x + \frac{1 - \sqrt{\tilde{N}(x)}}{\tilde{N}(x)} x^\top P \begin{pmatrix} 0 & 0 \\ k_1 & 0 \end{pmatrix} x,$$

then the Cauchy-Schwarz inequality implies

$$\begin{aligned} x^\top P \tilde{f}(x) &\leq x^\top P \begin{pmatrix} 0 & 1 \\ k_1 & k_2 \end{pmatrix} x + \frac{|1 - \sqrt{\tilde{N}(x)}|}{\sqrt{\tilde{N}(x)}} x^\top P x + \frac{|1 - \sqrt{\tilde{N}(x)}|}{\tilde{N}(x)} x^\top P x \\ &= x^\top P \begin{pmatrix} 0 & 1 \\ k_1 & k_2 \end{pmatrix} x + \frac{|1 - \tilde{N}(x)|}{\tilde{N}(x)} x^\top P x \\ &\leq x^\top P \begin{pmatrix} 0 & 1 \\ k_1 & k_2 \end{pmatrix} x + \max \left\{ \frac{1}{\delta_{\min}} - 1, 1 - \frac{1}{\delta_{\max}} \right\} x^\top P x \end{aligned}$$

provided that

$$\begin{pmatrix} 0 & 0 \\ k_1 & k_2 \end{pmatrix}^\top P \begin{pmatrix} 0 & 0 \\ k_1 & k_2 \end{pmatrix} \leq P, \quad \begin{pmatrix} 0 & 0 \\ k_1 & 0 \end{pmatrix}^\top P \begin{pmatrix} 0 & 0 \\ k_1 & 0 \end{pmatrix} \leq P.$$

Hence, if the system of LMIs

$$AX + XA^\top + BY + Y^\top B^\top + \tau X < 0, \quad G_{\mathbf{d}} X + X G_{\mathbf{d}}^\top > 0, \quad X > 0,$$

$$\begin{pmatrix} \gamma X & Y^\top B^\top \\ BY & X \end{pmatrix} > 0, \quad \begin{pmatrix} \gamma^{-1} X & E_1 X \\ X E_1 & X \end{pmatrix} > 0, \quad E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

is feasible with respect to $X \in \mathbb{R}^{2 \times 2}$, $Y \in \mathbb{R}^{1 \times 2}$ for some fixed $\gamma, \tau \in (0, 1)$ then the considered homogeneous ODE is finite stable with $(k_1 \ k_2) = YX^{-1}$, and the parameters $q_1, q_2 \in \mathbb{R}_+$ selected such that $\delta_{\min} \geq (1 - \tau)^{-1}$, e.g.

$$q_1 = \frac{1}{(1-\tau)\sqrt[3]{e_1^\top P e_1}}, \quad q_2 = \frac{1}{(1-\tau)\sqrt{e_1^\top P e_1}}, \quad e_1 = (1, 0)^\top, \quad e_2 = (0, 1)^\top, \quad P = X^{-1}.$$

In particular, MATLAB gives the following solution to the LMIs with $\tau = \gamma = 0.5$:

$$X = \begin{pmatrix} 22.1245 & -6.8466 \\ -6.8466 & 9.1080 \end{pmatrix}, \quad Y = \begin{pmatrix} -3.8858 & -3.0105 \end{pmatrix}.$$

5 Stabilization of Homogeneous Control Systems via Dynamical Feedback

Based on the scheme of universal stabilizing control design given [40] we derive the following corollary of Theorem 4.

Theorem 5 (On Homogeneous Dynamical Feedback) *Let d_x and d_u be dilations \mathbb{R}^n and \mathbb{R}^m , respectively,*

$$d := \begin{pmatrix} d_x(s) & \mathbf{0} \\ \mathbf{0} & d_u(s) \end{pmatrix} \text{ and } \tilde{f} = \begin{pmatrix} g \\ \mathbf{0} \end{pmatrix} \in \mathbb{F}_d(\mathbb{R}^{n+m}).$$

The origin of the system (1) is globally asymptotically stabilizable by means of the homogeneous dynamical feedback (2) with

$$\begin{pmatrix} g \\ k \end{pmatrix} \in \mathbb{F}_d(\mathbb{R}^{n+m})$$

if and only if there exist a number $\gamma \geq 0$, a symmetric matrix $P \in \mathbb{R}^{(n+m) \times (n+m)}$ satisfying (11) and a map $\Xi \in C^\infty(\mathbb{R}^n \setminus \{\mathbf{0}\}, \mathbb{R}^{n \times n})$ such that

$$\det(\Xi(z)) \neq 0, \quad \frac{\partial \Xi(z)}{\partial z_i} z = 0, \quad \Xi(e^s z) = \Xi(z) \quad \text{for } z \in \mathbb{R}^n \setminus \{\mathbf{0}\}, s \in \mathbb{R}$$

and

$$a(z) < \gamma \sqrt{b^\top(z)b(z)} \quad \text{for } z \in S, \quad (15)$$

where

$$a(z) = z^\top W(z) \tilde{f}(z), \quad b^\top(z) = z^\top W(z) \begin{pmatrix} \mathbf{0} \\ I_m \end{pmatrix}, \\ W(z) = \Xi^\top(z) P \Xi(z) \left(\frac{(I_n - G_d) z^\top z P}{z^\top P G_d z} + I_n \right)$$

and S is the unit sphere in \mathbb{R}^{n+m} with $\|z\| = \sqrt{z^\top P z}$. Moreover, the corresponding stabilizing homogeneous feedback law can be designed as follows

$$k(\xi) = \|\xi\|_{\mathbf{d}}^{\deg_a(\tilde{f})} \mathbf{d}_u (\ln \|\xi\|_{\mathbf{d}}) k_0(\mathbf{d}(-\ln \|\xi\|_{\mathbf{d}})\xi), \quad \xi = \begin{pmatrix} x \\ u \end{pmatrix}, \quad (16)$$

where

$$k_0(z) = \begin{cases} -\frac{a(z) + \sqrt{a^2(z) + (b^\top(z)b(z))^2}}{b^\top(z)b(z)} b(z) & \text{if } b(z) \neq \mathbf{0} \\ 0 & \text{if } b(z) = \mathbf{0} \end{cases}$$

and $\|\cdot\|_{\mathbf{d}}$ is the canonical homogeneous norm induced by $\|\xi\| = \sqrt{\xi^\top P \xi}$.

Proof. Sufficiency. Let us show that the function k_0 is continuous on S . Indeed, $k_0(z) = \frac{a(z) + \sqrt{a^2(z) + (b^\top(z)b(z))^2}}{\sqrt{b^\top(z)b(z)}} \frac{b(z)}{\sqrt{b^\top(z)b(z)}}$, the first fraction is continuous and equals zero if $b(z) = \mathbf{0}$ (see, [40]), but the norm of the second fraction is globally bounded and continuous for $z \in \mathbb{R}^n : b(z) \neq \mathbf{0}$. Let us consider the closed-loop system (1), (2), (16): $\dot{\xi} = f(\xi)$, where $f = \begin{pmatrix} g \\ \tilde{k} \end{pmatrix} \in \mathbb{F}_{\mathbf{d}}(\mathbb{R}^n)$. Under this notation, the inequality (13) becomes $a(z) + b^\top(z)k_0(z) < 0$. The latter holds for all $z \in S$. Indeed, the inequality (15) implies that $a(z) < 0$ if $b(z) = \mathbf{0}$, but for $b(z) \neq \mathbf{0}$ we have $a(z) + b^\top(z)k_0(z) = a(z) - b^\top(z) \frac{a(z) + \sqrt{a^2(z) + (b^\top(z)b(z))^2}}{b^\top(z)b(z)} b(z) = -\sqrt{a^2(z) + (b^\top(z)b(z))^2} < 0$.

Necessity. Let us assume that there exists a map $\tilde{k} \in C(\mathbb{R}^{n+m}, \mathbb{R}^m)$ such that the closed-loop system (1), (2) is globally asymptotically stable and $f = \begin{pmatrix} g \\ \tilde{k} \end{pmatrix} \in \mathbb{F}_{\mathbf{d}}(\mathbb{R}^{n+m})$. Let $\Xi \in C^\infty(\mathbb{R}^{n+m} \setminus \{\mathbf{0}\}, \mathbb{R}^{(n+m) \times (n+m)})$ be derived according to Theorem 4 such that (13) holds. To show that in this case (15) also holds for some $\gamma \geq 0$, we rewrite the inequality (13) as $a(z) < -b^\top(z)\tilde{k}(z)$. Since $\tilde{k} \in C(\mathbb{R}^{n+m}, \mathbb{R}^m)$ then $\gamma = \max_{z \in S} \sqrt{\tilde{k}^\top(z)\tilde{k}(z)} < +\infty$ and using Cauchy-Schwarz inequality we derive

$$-b^\top(z)\tilde{k}(z) \leq \sqrt{\tilde{k}^\top(z)\tilde{k}(z)} \sqrt{b^\top(z)b(z)} \leq \gamma \sqrt{b^\top(z)b(z)}.$$

■

For $\Xi(z) = I_n$ we derive from (15) the following sufficient condition of quadratic stabilizability:

$$\begin{aligned} z^\top P \tilde{f} \left(\frac{z}{\sqrt{z^\top P z}} \right) &\leq \gamma \sqrt{z^\top P \begin{pmatrix} 0 & 0 \\ 0 & I_m \end{pmatrix} P z}, \quad z \neq \mathbf{0} \\ P G_{\mathbf{d}} + G_{\mathbf{d}}^\top P &> 0, \quad P > 0. \end{aligned} \quad (17)$$

Example 4 (Quadratic Dynamical Stabilization in a Finite Time)

Let us consider the stabilization problem for

$$\dot{x}_1 = |x_1|^{\frac{1}{3}} u + x_2, \quad \dot{x}_2 = u,$$

where $x_1, x_2, u \in \mathbb{R}$. The vector field $\tilde{f} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ defined by

$$\tilde{f}(z) = \begin{pmatrix} |z_1|^{\frac{1}{3}}z_3 + z_2 \\ z_3 \\ 0 \end{pmatrix},$$

$z = (z_1, z_2, z_3)^\top = (x_1, x_2, u)$ is \mathbf{d} -homogeneous of degree -1 with respect to the dilation \mathbf{d} in \mathbb{R}^3 generated by $G_{\mathbf{d}} = \text{diag}\{3, 2, 1\}$. Hence, the sufficient stabilizability condition (17) becomes

$$z^\top PA(z)z < \gamma \sqrt{z^\top Pz} \sqrt{z^\top Pe_3e_3^\top Pz},$$

$P > 0$, $PG_{\mathbf{d}} + G_{\mathbf{d}}^\top P > 0$, where

$$A(z) = \begin{pmatrix} 0 & 1 & \left(\frac{|z_1|}{\sqrt{z^\top Pz}}\right)^{\frac{1}{3}} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $X \in \mathbb{R}^{3 \times 3}$ and $\gamma_0 > 0$ satisfy the linear matrix inequalities

$$XA_i + A_i^\top X - 2\gamma_0 e_3 e_3^\top < 0, \quad XG_{\mathbf{d}} + G_{\mathbf{d}}X > 0, \quad X > 0,$$

$A_i = \begin{pmatrix} 0 & 1 & i \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $i = 0, 1$. Since $A(z) = \alpha(z)A_1 + (1 - \alpha(z))A_0$, where $\alpha(z) = \left(\frac{|z_1|}{\sqrt{z^\top Pz}}\right)^{\frac{1}{3}}$ and $\alpha(z) \in [0, 1]$ provided that $p_{11} = 1$, then for $P = \tilde{P}/(\tilde{p}_{11})$, $\tilde{P} = X^{-1}$ the stabilizability condition (17) holds. So, the considered system can be stabilized in a finite time (due to $\deg_{\mathbf{d}}(\tilde{f}) = -1$) to zero by means of the \mathbf{d} -homogeneous dynamical feedback

$$\dot{u} = k_0 \left(\frac{z}{\|z\|_{\mathbf{d}}} \right),$$

where

$$k_0(z) = \begin{cases} -\frac{a(z) + \sqrt{a^2(z) + b^4(z)}}{b(z)} & \text{if } b(z) \neq 0, \\ 0 & \text{if } b(z) = 0, \end{cases}$$

$a(z) = z^\top P \tilde{f}(z)$ and $b(z) = p_{13}z_1 + p_{23}z_2 + p_{33}z_3$.

6 Discussions and Conclusions

In this paper it is proved that any asymptotically stable generalized homogeneous system is equivalent (diffeomorphic) to a quadratically stable one. The necessary and sufficient condition of quadratic-like stability (stabilizability) of homogeneous (control) system is derived. This condition may simplify, in some particular cases (see, Examples 3 and 4), design of finite-time stabilizing feedback laws and formulate an algorithm for control parameters tuning in terms of LMIs. In general case, an appropriate computational procedure is needed to be developed for quadratic-like Lyapunov function design (as well as for control application) based on (13) (resp. (15), (16)). This issue is considered as an open problem for future research.

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The present paper had been submitted to IEEE Transactions on Automatic Control (January 17, 2017) as Technical Note. It did not achieve a high enough ranking to be accepted and was rejected (May 15, 2017) based on the reviews given below. However, the author thinks that it is worth to publish the paper in HAL, since, to the best of the author’s knowledge, the obtained results (basically Theorem 4 (claims 3,4 and 5) and Theorem 5) are new and they may simplify (in some cases) control design and analysis of nonlinear homogeneous control system using LMI technique. The version of the paper presented above has been revised according the minor remarks of the reviewers.

Review 124787

This paper deals with quadratic stability problem for nonlinear generalized homogeneous control systems. The author shows the importance of the problem using many references.

The paper shows a necessary and sufficient stability condition in terms of existence of the quadratic-like Lyapunov function for a transformed standard homogeneous system in Theorem 4. Since the properties of the coordinate transformation using homogeneity is well-written, the results have sufficient quality for this journal.

I have checked correctness of Theorem 3 and Corollary 2. Due to the lack of my time and skill, other correctness is not checked.

Basically, the author can trust because the notation is very accurate. I like the well summarized basic properties of homogeneity in the paper :) But, some small mistakes are remained.

The structure of the paper is clear. It is well written in the way of mathematicians, and it is easy to understand even by the researchers outside the field by taking times. The motivation is also well presented in the sixth paragraph of the introduction; diversion possibility of the results of linear control

Due to the above view points, the paper should be published.

Minor comments

I want to enjoy mathematical flow of your proof. So, please refine the shapes of equations in your proof, e.g., Proof of Corollary 1, which has a too long equation in line.

P.1 Right L. 27

control input, $g \in C(\mathbb{R}^{n+1}, \mathbb{R}^n)$, $g(0, 0) = 0$.

->

control input, $g \in C(\mathbb{R}^{n+m}, \mathbb{R}^n)$, $g(0, 0) = 0$.

P.3

I could not understand why the equation (7) holds.

Since $h \in C^1$, I guess that the Frechet differential is not needed.

(We need only the elemental definition of the differential).

Why do you use the Frechet differential?

Let $o(\Delta) := h(x+\Delta) - h(x) - \frac{\partial h}{\partial u}(x)\Delta$.

Note that $o(\Delta) \rightarrow 0$ with $\Delta \rightarrow 0$.

By using the elemental calculation and $h(d(s)\Delta) = e^{-\nu s}h(\Delta)$,

$o(d(s)\Delta) = e^{-\nu s}o(\Delta)$ implies the equation (7) and vice versa.

(If the equation (7) holds, then the remained term of homogeneous function h is also homogeneous.)

But I could not show that $o(d(s)\Delta) = e^{-\nu s}o(\Delta)$.

P.3 Left L. 48

the homogeneity degree of $h \in \mathbb{H}_d(\mathbb{R}^n)$

->

the homogeneity degree of $h \in \mathbb{H}_{\{d\}}(\mathbb{R}^n)$

p.4

in Theorem 3, we can not take t_0 in the whole of \mathbb{R}

because the domain of φ_{x_0} is \mathbb{R}_+ (not \mathbb{R}).

What is the domain of the new solution?

p.4 Left L.10

$\frac{d}{dt}\text{bf}\{d\}(s)\varphi_{x_0}(t) = \text{bf}\{d\}(s)\varphi_{x_0}(t_0 + t)$
 $^{\{deg_{\text{bf}\{d\}}(f)s\}}$ with t_0, s

->

$\frac{d}{dt}\text{bf}\{d\}(s)\varphi_{x_0}(t) = \text{bf}\{d\}(s)\varphi_{x_0}$

$x(t_0 + te^{\nu s})$ with t_0 ,

*Refining usage of ν and $\deg_{\textbf{d}}(f)$ may improve readability.

Review 124945

Review of "Note on Quadratic-like Stabilizability of nonlinear generalized homogeneous systems" by A. Polyakov.

The paper proposes a generalized notion of dilation and, correspondingly, homogeneity with respect to this dilation (generalized homogeneity). It is proved that any generalized homogeneous system is diffeomorphic to a standard homogeneous system. In particular any generalized homogeneous system is diffeomorphic to a quadratically stable one. A stability criterion is given (formula (13)). Finally, the universal stabilizing control scheme of [33] inspires the last section, where a dynamical state-feedback (2) is obtained.

The main criticism about this paper is its practical usefulness in comparison with previous existing results. The author should convince the reader that this transformation into a quadratically stabilizable system to obtain a quadratic "control" Lyapunov function simplifies indeed the control design. Example 4 (and 2) are claimed to prove that this methodology simplifies the design of finite-time stabilizing feedback laws (which is, for systems with negative homogeneity degree). First of all, for completeness the author should provide also some example of stabilization of system with positive homogeneity degree. Secondly, example 4 is not suitable to make a comparison: once the system of example 4 is dynamically extended as pointed out in (2), it becomes a chain of integrators plus a nonlinear vector field. As claimed in the example, this vector field has homogeneity degree with respect to the standard dilation d generated by $G_d = \text{diag}(3, 2, 1)$. Therefore, I believe that in this case it is possible to apply theorem 4.1 of [2] (particularized to homogeneity at 0) to construct a finite-time stabilizing feedback law.

Minor points:

- the notation for induced norm $\| \cdot \|_A$ is a bit confusing with the use of the subscript A
- in formula (4) you mean s_u as the value for which the equality after the semicolon is true?
- in formula (5) the two partial derivatives in the numerator and denominator are equal. Should they simplify or not?
- before corollary 2: the statement (resp. vector field) has to be removed
- In theorem 4: Is (9) assumed to be d -homogeneous? If yes, please include in the statement of the theorem.
- remark before formula (13): please clarify.
- before formula (15): the partial derivative is applied to Σ (not to ξ).

Review 124997

This technical note studies the stabilizability of generalized homogeneous systems by dynamic state feedback, and presents a necessary and sufficient condition for quadratic-like stabilizability. The results presented in the paper seem to be correct but many of them are already known. The authors should make effort to explain clearly what is the motivation for using dynamic v.s. static state feedback, what is the originality of the paper, and most importantly, what is new of the present work compared to the vast majority of the feedback stabilization results in the literature?

Detailed Comments

1. Presentation of the paper

The paper was not written with a care nor clearly presented. As an evidence, there are many grammatical errors that are impossible to list one-by-one. Starting from Abstract, each paragraph of Introduction, and every section, one can find the syntax errors. Please find a

professional English writer to correct/edit the manuscript.

2. Section I Introduction

The section is lack of a through and complete review on the feedback stabilization of homogeneouslike systems and affine nonlinear systems. In particular, many representative and related work by, e.g., Dayawansa, Martin, Mazenc, Kawski, Lin, Qian, Tsiniias, Tzamtzi, Celikovsky, Aranda-Bricaire,Ryan, etc, for homogeneous or homogeneous-like systems, and Byrnes, Isidori, Sussman, Kokotovic, Khalil and more....for the affine system

$$\dot{x} = g(x, u), \quad \dot{u} = v \quad (1)$$

are missing. Without a comparison and discussion on the difference/connection between the current paper and the listed papers, it is difficult to judge the new contribution of the paper, if any.

3. Section II Problem Statement

The motivation of using dynamic v.s. static state feedback is not clearly explained. Although the author begins with a general nonlinear system (1), the stabilization problem under consideration reduces to the stabilization of the affine system (1) if a dynamic instead of static state feedback is utilized. Note that the affine system (1) can be viewed as a nonlinear system with relative degree 1, whose stabilization problem has been extensively studied by various researchers including those listed above, and many fruitful results were obtained.

To sum up, the contribution of this paper is questionable if the author only considers dynamic state feedback control as done in the paper. The paper would be interesting and nontrivial if a static state feedback control scheme $u(x)$ were studied.

4. Section III Generalized Homogeneity

This section is way too long. Most of the results reviewed in this section is well-known for homogeneous systems community and they can be found in a series of papers by Kawski, Hermes and Dayawansa,.... and the classical books by, e.g., Hahn (1967) or Zubov (1964). It can be reduced to less than a half-page by only reviewing what are used in this paper, with appropriate citations and without giving detailed re-proofs.

I suggest that Section II and III be condensed into a section Problem Statement and Preliminary.

5. Section III Quadratic Stability of Homogeneous Systems

Again, Theorem 3 is known and can be put into the section of Preliminary, by stating as one of the properties of homogeneous systems with appropriate references [14], Hahn (1967) and Zubov (1964). Reference [35] is better replaced by Zubov's book in 1964 (English version). The proof of 1) equivalent to 2) can be omitted as it was already in [29].

6. Section IV Stabilization of homogeneous control via dynamic feedback

I am troubling by the result presented in this section. As explained in my previous comment, the stabilization problem reduces to the trivial one for the affine system (1) if a dynamic state feedback strategy is used. For the affine system (1), Theorem 5 is a straightforward consequence of Sontag's formula in [33] and Theorem 4, with the homogeneous vector fields \tilde{f} and $G = [0 \ 1]^T$.

Therefore, I must reiterate my viewpoint: the paper is not particularly exciting and adds little new to what already known if only dynamic state feedback is considered. A more interesting problem is the stabilization problem by static state feedback.

7. References

Many important stabilization results by static state feedback for homogeneous, homogeneous-like and affine systems (1) have not been cited/discussed and should be included. My list above is by no means completed and it is the authors responsibility to do a careful job in this regard. The author should make a convincing case and justify what is new in the present paper, why dynamic state feedback is not trivial, and how this work is related/connected to and differentiated from the missing papers by the researchers listed in my previous comments.