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1 **VARIATIONAL ANALYSIS FOR OPTIONS WITH STOCHASTIC**
2 **VOLATILITY AND MULTIPLE FACTORS***

3 J. FRÉDÉRIC BONNANS[†] AND AXEL KRÖNER[‡]

4 **Abstract.** This paper performs a variational analysis for a class of European or American
5 options with stochastic volatility models, including those of Heston and Achdou-Tchou. Taking into
6 account partial correlations and the presence of multiple factors, we obtain the well-posedness of the
7 related partial differential equations, in some weighted Sobolev spaces. This involves a generalization
8 of the commutator analysis introduced by Achdou and Tchou in [3].

9 **Key words.** finance, options, partial differential equations, variational formulation, parabolic
10 variational inequalities

11 **AMS subject classifications.** 35K20, 35K85, 91G80

12 **1. Introduction.** In this paper we consider variational analysis for the par-
13 tial differential equations associated with the pricing of European or American op-
14 tions. For an introduction to these models, see Fouque et al., [11]. We will set up
15 a general framework of variable volatility models, which is in particular applicable
16 on the following standard models which are well established in mathematical finance.
17 The well-posedness of PDE formulations of variable volatility problems was studied in
18 [2, 3, 1, 18], and in the recent work [9, 10].

19 Let the $W_i(t)$ be Brownian motions on a filtered probability space. The variable
20 s denotes a financial asset, and the components of y are factors that influence the
21 volatility:

22 (i) The *Achdou-Tchou model* [3], see also Achdou, Franchi, and Tchou [1]:

23 (1.1)
$$\begin{cases} ds(t) = rs(t)dt + \sigma(y(t))s(t)dW_1(t), \\ dy(t) = \theta(\mu - y(t))dt + \nu dW_2(t), \end{cases}$$

24 with the interest rate r , the volatility coefficient σ function of the factor y
25 whose dynamics involves a parameter $\nu > 0$, and positive constants θ and μ .

26 (ii) The *Heston model* [14]

27 (1.2)
$$\begin{cases} ds(t) = s(t) \left(rdt + \sqrt{y(t)}dW_1(t) \right), \\ dy(t) = \theta(\mu - y(t))dt + \nu \sqrt{y(t)}dW_2(t). \end{cases}$$

28 (iii) The *Double Heston model*, see Christoffersen, Heston and Jacobs [17], and
29 also Gauthier and Possamaï [12]:

30 (1.3)
$$\begin{cases} ds(t) = s(t) \left(rdt + \sqrt{y_1(t)}dW_1(t) + \sqrt{y_2(t)}dW_2(t) \right), \\ dy_1(t) = \theta_1(\mu_1 - y_1(t))dt + \nu_1 \sqrt{y_1(t)}dW_3(t), \\ dy_2(t) = \theta_2(\mu_2 - y_2(t))dt + \nu_2 \sqrt{y_2(t)}dW_4(t). \end{cases}$$

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31 In the last two models we have similar interpretations of the coefficients;
 32 in the double Heston model, denoting by $\langle \cdot, \cdot \rangle$ the correlation coefficients, we
 33 assume that there are correlations only between W_1 and W_3 , and W_2 and W_4 .
 34 Consider now the *general multiple factor model*

$$35 \quad (1.4) \quad \begin{aligned} ds &= rs(t)dt + \sum_{k=1}^N f_k(y_k(t))s^{\beta_k}(t)dW_k(t), \\ dy_k &= \theta_k(\mu_k - y_k(t))dt + g_k(y_k(t))dW_{N+k}(t), \quad k = 1, \dots, N. \end{aligned}$$

36 Here the y_k are volatility factors, $f_k(y_k)$ represents the volatility coefficient due to
 37 y_k , $g_k(y_k)$ is a volatility coefficient in the dynamics of the k th factor with positive
 38 constants θ_k and μ_k . Let us denote the correlation between the i th and j th Brownian
 39 motions by κ_{ij} : this is a measurable function of (s, y, t) with value in $[0, 1]$ (here
 40 $s \in (0, \infty)$ and y_k belongs to either $(0, \infty)$ or \mathbb{R}), see below. We assume that we have
 41 nonzero correlations only between the Brownian motions W_k and W_{N+k} , for $k = 1$
 42 to N , i.e.

$$43 \quad (1.5) \quad \kappa_{ij} = 0 \quad \text{if } i \neq j \text{ and } |j - i| \neq N.$$

44 Note that, in some of the main results, we will assume for the sake of simplicity that
 45 the correlations are constant.

46 We apply the developed analysis to a subclass of stochastic volatility models, obtained
 47 by assuming that κ is constant and

$$48 \quad (1.6) \quad |f_k(y_k)| = |y_k|^{\gamma_k}; \quad |g_k(y_k)| = \nu_k |y_k|^{1-\gamma_k}; \quad \beta_k \in (0, 1]; \quad \nu_k > 0; \quad \gamma_k \in (0, \infty).$$

49 This covers in particular a variant of the Achdou and Tchou model with multiple
 50 factors (VAT), when $\gamma_k = 1$, as well as a generalized multiple factor Heston model
 51 (GMH), when $\gamma_k = 1/2$, i.e., for $k = 1$ to N :

$$52 \quad (1.7) \quad \begin{array}{ll} \text{VAT:} & f_k(y_k) = y_k, \quad g_k(y_k) = \nu_k, \\ \text{GMH:} & f_k(y_k) = \sqrt{y_k}, \quad g_k(y_k) = \nu_k \sqrt{y_k}. \end{array}$$

53 For a general class of stochastic volatility models with correlation we refer to Lions
 54 and Musiela [16].

55 The main contribution of this paper is variational analysis for the pricing equa-
 56 tion corresponding to the above general class in the sense of the Feynman-Kac theory.
 57 This requires in particular to prove continuity and coercivity properties of the corre-
 58 sponding bilinear form in weighted Sobolev spaces H and V , respectively, which have
 59 the Gelfand property and allow the application of the Lions and Magenes theory [15]
 60 recalled in Appendix A and the regularity theory for parabolic variational inequalities
 61 recalled in Appendix B. A special emphasis is given to the continuity analysis of the
 62 rate term in the pricing equation. Two approaches are presented, the standard one
 63 and an extension of the one based on the commutator of first-order differential oper-
 64 ators as in Achdou and Tchou [3], extended to the Heston model setting by Achdou
 65 and Pironneau [18]. Our main result is that the commutator analysis gives stronger
 66 results for the subclass defined by (1.6), generalizing the particular cases of the VAT
 67 and GMH classes, see remarks 6.2 and 6.4. In particular we extend some of the results
 68 by [3].

69 This paper is organized as follows. In section 2 we give the expression of the bi-
 70 linear form associated with the original PDE, and check the hypotheses of continuity
 71 and semi-coercivity of this bilinear form. In section 3 we show how to refine this anal-
 72 ysis by taking into account the commutators of the first-order differential operators

73 associated with the variational formulation. In section 4 we show how to compute
 74 the weighting function involved in the bilinear form. In section 5 we develop the re-
 75 sults for a general class introduced in the next section. In section 6 we specialize the
 76 results to stochastic volatility models. The appendix recalls the main results of the
 77 variational theory for parabolic equations, with a discussion on the characterization
 78 of the V functional spaces in the case of one dimensional problems.

79 **Notation.** We assume that the domain Ω of the PDEs to be considered in the
 80 sequel of this paper has the following structure. Let (I, J) be a partition of $\{0, \dots, N\}$,
 81 with $0 \in J$, and

$$82 \quad (1.8) \quad \Omega := \prod_{k=0}^N \Omega_k; \quad \text{with} \quad \Omega_k := \begin{cases} \mathbb{R} & \text{when } k \in I, \\ (0, \infty) & \text{when } k \in J. \end{cases}$$

83 Let $L^0(\Omega)$ denote the space of measurable functions over Ω . For a given weighting
 84 function $\rho : \Omega \rightarrow \mathbb{R}$ of class C^1 , with positive values, we define the *weighted space*

$$85 \quad (1.9) \quad L^{2,\rho}(\Omega) := \{v \in L^0(\Omega); \int_{\Omega} v(x)^2 \rho(x) dx < \infty\},$$

86 which is a Hilbert space endowed with the norm

$$87 \quad (1.10) \quad \|v\|_{\rho} := \left(\int_{\Omega} v(x)^2 \rho(x) dx \right)^{1/2}.$$

88 By $\mathcal{D}(\Omega)$ we denote the space of C^∞ functions with compact support in Ω . By
 89 $H_{loc}^2(\Omega)$ we denote the space of functions over Ω whose product with an element of
 90 $\mathcal{D}(\Omega)$ belongs to the Sobolev space $H^2(\Omega)$.

91 Besides, let Φ be a vector field over Ω (i.e., a mapping $\Omega \rightarrow \mathbb{R}^n$). The *first-order*
 92 *differential operator* associated with Φ is, for $u : \Omega \rightarrow \mathbb{R}$ the function over Ω defined
 93 by

$$94 \quad (1.11) \quad \Phi[u](x) := \sum_{i=0}^n \Phi_i(x) \frac{\partial u}{\partial x_i}(x), \quad \text{for all } x \in \Omega.$$

95 **2. General setting.** Here we give compute the bilinear form associated with
 96 the original PDE, in the setting of the general multiple factor model (1.4). Then we
 97 will check the hypotheses of continuity and semi-coercivity of this bilinear form.

98 **2.1. Variational formulation.** We compute the bilinear form of the variational
 99 setting, taking into account a general weight function. We will see how to choose the
 100 functional spaces for a given ρ , and then how to choose the weight itself.

101 **2.1.1. The elliptic operator.** In financial models the underlying is solution of
 102 stochastic differential equations of the form

$$103 \quad (2.1) \quad dX(t) = b(t, X(t))dt + \sum_{i=1}^{n_\sigma} \sigma_i(t, X(t))dW_i.$$

104 Here $X(t)$ takes values in Ω , defined in (1.8). That is, X_1 corresponds to the s variable,
 105 and X_{k+1} , for $k = 1$ to N , corresponds to y_k . We have that $n_\sigma = 2N$.

106 So, b and σ_i , for $i = 1$ to n_σ , are mappings $(0, T) \times \Omega \rightarrow \mathbb{R}^n$, and the W_i , for
 107 $i = 1$ to n_σ , are standard Brownian processes with correlation $\kappa_{ij} : (0, T) \times \Omega \rightarrow \mathbb{R}$

109 between W_i and W_j for $i, j \in \{1, \dots, n_\sigma\}$. The $n_\sigma \times n_\sigma$ symmetric *correlation matrix*
 110 $\kappa(\cdot, \cdot)$ is nonnegative with unit diagonal:

$$111 \quad (2.2) \quad \kappa(t, x) \succeq 0; \quad \kappa_{ii} = 1, \quad i = 1, \dots, n_\sigma, \quad \text{for a.a. } (t, x) \in (0, T) \times \Omega.$$

112 Here, for symmetric matrices B and C of same size, by " $C \succeq B$ " we mean that $C - B$
 113 is positive semidefinite. The expression of the second order differential operator A
 114 corresponding to the dynamics (2.1) is, skipping the time and space arguments, for
 115 $u: (0, T) \times \Omega \rightarrow \mathbb{R}$:

$$116 \quad (2.3) \quad Au := ru - b \cdot \nabla u - \frac{1}{2} \sum_{i,j=1}^{n_\sigma} \kappa_{ij} \sigma_j^\top u_{xx} \sigma_i,$$

117 where

$$118 \quad (2.4) \quad \sigma_j^\top u_{xx} \sigma_i := \sum_{k,\ell=1}^{n_\sigma} \sigma_{kj} \frac{\partial^2 u}{\partial x_k \partial x_\ell} \sigma_{\ell i},$$

119 $r(x, t)$ represents an interest rate, and u_{xx} is the matrix of second derivatives in space
 120 of u . The associated backward PDE for a European option is of the form

$$121 \quad (2.5) \quad \begin{cases} -\dot{u}(t, x) + A(t, x)u(t, x) = f(t, x), & (t, x) \in (0, T) \times \Omega; \\ u(x, T) = u_T(x), & x \in \Omega, \end{cases}$$

122 with \dot{u} the notation for the time derivative of u , $u_T(x)$ payoff at final time (horizon)
 123 T and the r.h.s. $f(t, x)$ represents dividends (often equal to zero).

124 In case of an American option we obtain a variational inequality; for details we
 125 refer to Appendix D.

126 **2.1.2. The bilinear form.** In the sequel we assume that

$$127 \quad (2.6) \quad b, \sigma, \kappa \text{ are } C^1 \text{ mappings over } [0, T] \times \Omega.$$

128 Multiplying (2.3) by the test function $v \in \mathcal{D}(\Omega)$ and the continuously differentiable
 129 weight function $\rho: \Omega \rightarrow \mathbb{R}$ and integrating over the domain we can integrate by parts;
 130 since $v \in \mathcal{D}(\Omega)$ there will be no contribution from the boundary. We obtain

$$131 \quad (2.7) \quad -\frac{1}{2} \int_{\Omega} \sigma_j^\top u_{xx} \sigma_i v \kappa_{ij} \rho = \sum_{p=0}^3 a_{ij}^p(u, v),$$

132 with

$$133 \quad (2.8) \quad a_{ij}^0(u, v) := \frac{1}{2} \int_{\Omega} \sum_{k,\ell=1}^n \sigma_{kj} \sigma_{\ell i} \frac{\partial u}{\partial x_k} \frac{\partial v}{\partial x_\ell} \kappa_{ij} \rho = \frac{1}{2} \int_{\Omega} \sigma_j [u] \sigma_i [v] \kappa_{ij} \rho,$$

134

$$135 \quad (2.9) \quad a_{ij}^1(u, v) := \frac{1}{2} \int_{\Omega} \sum_{k,\ell=1}^n \sigma_{kj} \sigma_{\ell i} \frac{\partial u}{\partial x_k} \frac{\partial(\kappa_{ij} \rho)}{\partial x_\ell} v = \frac{1}{2} \int_{\Omega} \sigma_j [u] \sigma_i [\kappa_{ij} \rho] \frac{v}{\rho},$$

136

$$137 \quad (2.10) \quad a_{ij}^2(u, v) := \frac{1}{2} \int_{\Omega} \sum_{k,\ell=1}^n \sigma_{kj} \frac{\partial(\sigma_{\ell i})}{\partial x_\ell} \frac{\partial u}{\partial x_k} v \kappa_{ij} \rho = \frac{1}{2} \int_{\Omega} \sigma_j [u] (\operatorname{div} \sigma_i) v \kappa_{ij} \rho,$$

138

$$139 \quad (2.11) \quad a_{ij}^3(u, v) := \frac{1}{2} \int_{\Omega} \sum_{k, \ell=1}^n \frac{\partial(\sigma_{kj})}{\partial x_{\ell}} \sigma_{\ell i} \frac{\partial u}{\partial x_k} v \kappa_{ij} \rho = \frac{1}{2} \int_{\Omega} \sum_{k=1}^n \sigma_i[\sigma_{kj}] \frac{\partial u}{\partial x_k} v \kappa_{ij} \rho.$$

140 Also, for the contributions of the first and zero order terms resp. we get

$$141 \quad (2.12) \quad a^4(u, v) := - \int_{\Omega} b[u] v \rho; \quad a^5(u, v) := \int_{\Omega} r u v \rho.$$

142 Set

$$143 \quad (2.13) \quad a^p := \sum_{i, j=1}^{n_{\sigma}} a_{ij}^p, \quad p = 0, \dots, 3.$$

144 The *bilinear* form associated with the above PDE is

$$145 \quad (2.14) \quad a(u, v) := \sum_{p=0}^5 a^p(u, v).$$

146 From the previous discussion we deduce that

147 LEMMA 2.1. *Let $u \in H_{loc}^2(\Omega)$ and $v \in \mathcal{D}(\Omega)$. Then we have that*

$$148 \quad (2.15) \quad a(u, v) = \int_{\Omega} A(t, x) u(x) v(x) \rho(x) dx.$$

149 **2.1.3. The Gelfand triple.** We can view a^0 as the principal term of the bilinear
150 form $a(u, v)$. Let σ denote the $n \times n_{\sigma}$ matrix whose σ_j are the columns. Then

$$151 \quad (2.16) \quad a^0(u, v) = \sum_{i, j=1}^{n_{\sigma}} \int_{\Omega} \sigma_j[u] \sigma_i[v] \kappa_{ij} \rho = \int_{\Omega} \nabla u^{\top} \sigma \kappa \sigma^{\top} \nabla v \rho.$$

152 Since $\kappa \succeq 0$, the above integrand is nonnegative when $u = v$; therefore, $a^0(u, u) \geq 0$.
153 When κ is the identity we have that $a^0(u, u)$ is equal to the seminorm $a^{00}(u, u)$, where

$$154 \quad (2.17) \quad a^{00}(u, u) := \int_{\Omega} |\sigma^{\top} \nabla u|^2 \rho.$$

155 In the presence of correlations it is natural to assume that we have a coercivity of the
156 same order. That is, we assume that

$$157 \quad (2.18) \quad \text{For some } \gamma \in (0, 1]: \quad \sigma \kappa \sigma^{\top} \succeq \gamma \sigma \sigma^{\top}, \quad \text{for all } (t, x) \in (0, T) \times \Omega.$$

158 Therefore, we have

$$159 \quad (2.19) \quad a^0(u, u) \geq \gamma a^{00}(u, u).$$

160 REMARK 2.2. *Condition (2.18) holds in particular if*

$$161 \quad (2.20) \quad \kappa \succeq \gamma I,$$

162 *but may also hold in other situations, e.g., when $n = 1$, $n_{\sigma} = 2$, $\kappa_{12} = 1$, and*
163 *$\sigma_1 = \sigma_2 = 1$. Yet when the σ_i are linearly independent, (2.19) is equivalent to (2.20).*

164 We need to choose a pair (V, H) of Hilbert spaces satisfying the Gelfand condi-
 165 tions for the variational setting of Appendix A, namely V densely and continuously
 166 embedded in H , $a(\cdot, \cdot)$ continuous and semi-coercive over V . Additionally, the r.h.s.
 167 and final condition of (2.5) should belong to $L^2(0, T; V^*)$ and H resp. (and for the
 168 second parabolic estimate, to $L^2(0, T; H)$ and V resp.).

169 We do as follows: for some measurable function $h : \Omega \rightarrow \mathbb{R}_+$ to be specified later
 170 we define

$$171 \quad (2.21) \quad \begin{cases} H := \{v \in L^0(\Omega); hv \in L^{2,\rho}(\Omega)\}, \\ \mathcal{V} := \{v \in H; \sigma_i[v] \in L^{2,\rho}(\Omega), i = 1, \dots, n_\sigma\}, \\ V := \{\text{closure of } \mathcal{D}(\Omega) \text{ in } \mathcal{V}\}, \end{cases}$$

172 endowed with the natural norms,

$$173 \quad (2.22) \quad \|v\|_H := \|hv\|_\rho; \quad \|u\|_V^2 := a^{00}(u, u) + \|u\|_H^2.$$

174 We do not try to characterize the space V since this is problem dependent.

175 Obviously, $a^0(u, v)$ is a bilinear continuous form over \mathcal{V} . We next need to choose
 176 h so that $a(u, v)$ is a bilinear and semi-coercive continuous form, and $u_T \in H$.

177 **2.2. Continuity and semi-coercivity of the bilinear form over \mathcal{V} .** We will
 178 see that the analysis of a^0 to a^2 is relatively easy. It is less obvious to analyze the
 179 term

$$180 \quad (2.23) \quad a^{34}(u, v) := a^3(u, v) + a^4(u, v).$$

181 Let $\bar{q}_{ij}(t, x) \in \mathbb{R}^n$ be the vector with k th component equal to

$$182 \quad (2.24) \quad \bar{q}_{ijk} := \kappa_{ij} \sigma_i[\sigma_{kj}].$$

183 Set

$$184 \quad (2.25) \quad \hat{q} := \sum_{i,j=1}^{n_\sigma} \bar{q}_{ij}, \quad q := \hat{q} - b.$$

185 Then by (2.11)-(2.12), we have that

$$186 \quad (2.26) \quad a^{34}(u, v) = \int_\Omega q[u]v\rho.$$

187 We next need to assume that it is possible to choose η_k in $L^0((0, T) \times \Omega)$, for $k = 1$
 188 to n_σ , such that

$$189 \quad (2.27) \quad q = \sum_{k=1}^{n_\sigma} \eta_k \sigma_k.$$

190 Often the $n \times n_\sigma$ matrix $\sigma(t, x)$ has a.e. rank n . Then the above decomposition is
 191 possible. However, the choice for η is not necessarily unique. We will see in examples
 192 how to do it. Consider the following hypotheses:

$$193 \quad (2.28) \quad h_\sigma \leq c_\sigma h, \quad \text{where } h_\sigma := \sum_{i,j=1}^{n_\sigma} |\sigma_i[\kappa_{ij}\rho]/\rho + \kappa_{ij} \operatorname{div} \sigma_i|, \quad \text{a.e., for some } c_\sigma > 0,$$

$$194 \quad (2.29) \quad h_r \leq c_r h, \quad \text{where } h_r := |r|^{1/2}, \quad \text{a.e., for some } c_r > 0,$$

$$195 \quad (2.30) \quad h_\eta \leq c_\eta h, \quad \text{where } h_\eta := |\eta|, \quad \text{a.e., for some } c_\eta > 0.$$

197 REMARK 2.3. Let us set for any differentiable vector field $Z: \Omega \rightarrow \mathbb{R}^n$

$$198 \quad (2.31) \quad G_\rho(Z) := \operatorname{div} Z + \frac{Z[\rho]}{\rho}.$$

199 Since $\kappa_{ii} = 1$, (2.28) implies that

$$200 \quad (2.32) \quad |G_\rho(\sigma_i)| \leq c_\sigma h, \quad i = 1; \dots, n_\sigma.$$

201 REMARK 2.4. Since

$$202 \quad (2.33) \quad \sigma_i[\kappa_{ij}\rho] = \sigma_i[\kappa_{ij}]\rho + \sigma_i[\rho]\kappa_{ij},$$

203 and $|\kappa_{ij}| \leq 1$ a.e., a sufficient condition for (2.28) is that there exist a positive con-
204 stants c'_σ such that

$$205 \quad (2.34) \quad h'_\sigma \leq c'_\sigma h; \quad h'_\sigma := \sum_{i,j=1}^{n_\sigma} |\sigma_i[\kappa_{ij}]| + \sum_{i=1}^{n_\sigma} (|\operatorname{div} \sigma_i| + |\sigma_i[\rho]/\rho|).$$

206 We will see in section 4 how to choose the weight ρ so that $|\sigma_i[\rho]/\rho|$ can be easily
207 estimated as a function of σ .

208 LEMMA 2.5. Let (2.28)-(2.30) hold. Then the bilinear form $a(u, v)$ is both (i)
209 continuous over V , and (ii) semi-coercive, in the sense of (A.5).

210 *Proof.* (i) We have that $a^1 + a^2$ is continuous, since by (2.9)-(2.10), (2.28) and
211 the Cauchy-Schwarz inequality:

$$\begin{aligned} |a^1(u, v) + a^2(u, v)| &\leq \sum_{i,j=1}^{n_\sigma} |a_{ij}^1(u, v) + a_{ij}^2(u, v)| \\ 212 \quad (2.35) \quad &\leq \sum_{j=1}^{n_\sigma} \|\sigma_j[u]\|_\rho \sum_{i=1}^{n_\sigma} \|(\sigma_i[\kappa_{ij}\rho]/\rho + \kappa_{ij} \operatorname{div} \sigma_i) v\|_\rho \\ &\leq c_\sigma n_\sigma \|v\|_H \sum_{j=1}^{n_\sigma} \|\sigma_j[u]\|_\rho. \end{aligned}$$

213

214 (ii) Also, a^{34} is continuous, since by (2.27) and (2.30):

$$215 \quad (2.36) \quad |a^{34}(u, v)| \leq \sum_{k=1}^{n_\sigma} \|\sigma_k[u]\|_\rho \|\eta_k v\|_\rho \leq c_\eta \|v\|_H \sum_{k=1}^{n_\sigma} \|\sigma_k[u]\|_\rho.$$

216 Set $c := c_\sigma n_\sigma + c_\eta^2$. By (2.35)–(2.36), we have that

$$217 \quad (2.37) \quad \begin{cases} |a^5(u, v)| \leq \| |r|^{1/2} u \|_{2,\rho} \| |r|^{1/2} v \|_{2,\rho} \leq c_r^2 \|u\|_H \|v\|_H, \\ |a^1(u, v) + a^2(u, v) + a^{34}(u, v)| \leq c a^{00}(u)^{1/2} \|v\|_H. \end{cases}$$

218 Since a^0 is obviously continuous, the continuity of $a(u, v)$ follows.

219 (iii) Semi-coercivity. Using (2.37) and Young's inequality, we get that

$$\begin{aligned} 220 \quad (2.38) \quad a(u, u) &\geq a_0(u, u) - |a^1(u, u) + a^2(u, u) + a^{34}(u, u)| - |a^5(u, u)| \\ &\geq \gamma a^{00}(u) - c a^{00}(u)^{1/2} \|u\|_H - c_r \|u\|_H^2 \\ &\geq \frac{1}{2} \gamma a^{00}(u) - \left(\frac{1}{2} \frac{c^2}{\gamma} + c_r \right) \|u\|_H^2, \end{aligned}$$

221 which means that a is semi-coercive. \square

222 The above consideration allow to derive well-posedness results for parabolic equations
223 and parabolic variational inequalities.

224 **THEOREM 2.6.** (i) *Let (V, H) be given by (2.21), with h satisfying (2.28)-(2.30),*
225 *$(f, u_T) \in L^2(0, T; V^*) \times H$. Then equation (2.5) has a unique solution u in $L^2(0, T; V)$*
226 *with $\dot{u} \in L^2(0, T; V^*)$, and the mapping $(f, u_T) \mapsto u$ is nondecreasing. (ii) *If in**
227 *addition the semi-symmetry condition (A.8) holds, then u in $L^\infty(0, T; V)$ and $\dot{u} \in$*
228 *$L^2(0, T; H)$.*

229 *Proof.* This is a direct consequence of Propositions A.1, A.2 and C.1. □

230 We next consider the case of parabolic variational inequalities associated with the set

$$231 \quad (2.39) \quad K := \{\psi \in V : \psi(x) \geq \Psi(x) \quad \text{a.e. in } \Omega\},$$

232 where $\Psi \in V$. The strong and weak formulations of the parabolic variational ine-
233 equality are defined in (B.2) and (B.5) resp. The abstract notion of monotonicity is
234 discussed in appendix B. We denote by \mathcal{K} the closure of K in V .

235 **THEOREM 2.7.** (i) *Let the assumptions of theorem 2.6 hold, with $u_T \in \mathcal{K}$. Then*
236 *the weak formulation (B.5) has a unique solution u in $L^2(0, T; K) \cap C(0, T; H)$, and*
237 *the mapping $(f, u_T) \mapsto u$ is nondecreasing.*

238 (ii) *Let in addition the semi-symmetry condition (A.8) be satisfied. Then u is the*
239 *unique solution of the strong formulation (B.2), belongs to $L^\infty(0, T; V)$, and \dot{u} belongs*
240 *to $L^2(0, T; H)$.*

241 *Proof.* This follows from Propositions B.1 and C.2. □

242 **3. Variational analysis using the commutator analysis.** In the following a
243 commutator for first order differential operators is introduced, and calculus rules are
244 derived.

245 **3.1. Commutators.** Let $u : \Omega \rightarrow \mathbb{R}$ be of class C^2 . Let Φ and Ψ be two vector
246 fields over Ω , both of class C^1 . Recalling (1.11), we may define the *commutator* of
247 the first-order differential operators associated with Φ and Ψ as

$$248 \quad (3.1) \quad [\Phi, \Psi][u] := \Phi[\Psi[u]] - \Psi[\Phi[u]].$$

249 Note that

$$250 \quad (3.2) \quad \Phi[\Psi[u]] = \sum_{i=1}^n \Phi_i \frac{\partial(\Psi u)}{\partial x_i} = \sum_{i=1}^n \Phi_i \left(\sum_{k=1}^n \frac{\partial \Psi_k}{\partial x_i} \frac{\partial u}{\partial x_k} + \Psi_k \frac{\partial^2 u}{\partial x_k \partial x_i} \right).$$

251 So, the expression of the commutator is

$$252 \quad (3.3) \quad \begin{aligned} [\Phi, \Psi][u] &= \sum_{i=1}^n \left(\Phi_i \sum_{k=1}^n \frac{\partial \Psi_k}{\partial x_i} \frac{\partial u}{\partial x_k} - \Psi_i \sum_{k=1}^n \frac{\partial \Phi_k}{\partial x_i} \frac{\partial u}{\partial x_k} \right) \\ &= \sum_{k=1}^n \left(\sum_{i=1}^n \Phi_i \frac{\partial \Psi_k}{\partial x_i} - \Psi_i \frac{\partial \Phi_k}{\partial x_i} \right) \frac{\partial u}{\partial x_k}. \end{aligned}$$

253 It is another first-order differential operator associated with a vector field (which
254 happens to be the Lie bracket of Φ and Ψ , see e.g.[4]).

255 **3.2. Adjoint.** Remembering that H was defined in (2.21), given two vector fields
 256 Φ and Ψ over Ω , we define the spaces

$$257 \quad (3.4) \quad \mathcal{V}(\Phi, \Psi) := \{v \in H; \Phi[v], \Psi[v] \in H\},$$

$$258 \quad (3.5) \quad V(\Phi, \Psi) := \{\text{closure of } \mathcal{D}(\Omega) \text{ in } \mathcal{V}(\Phi, \Psi)\}.$$

260 We define the adjoint Φ^\top of Φ (view as an operator over say $C^\infty(\Omega, \mathbb{R})$, the latter
 261 being endowed with the scalar product of $L^{2,\rho}(\Omega)$), by

$$262 \quad (3.6) \quad \langle \Phi^\top[u], v \rangle_\rho = \langle u, \Phi[v] \rangle_\rho \quad \text{for all } u, v \in \mathcal{D}(\Omega),$$

263 where $\langle \cdot, \cdot \rangle_\rho$ denotes the scalar product in $L^{2,\rho}(\Omega)$. Thus, there holds the identity

$$264 \quad (3.7) \quad \int_\Omega \Phi^\top[u](x)v(x)\rho(x)dx = \int_\Omega u(x)\Phi[v](x)\rho(x)dx \quad \text{for all } u, v \in \mathcal{D}(\Omega).$$

265 Furthermore,

$$266 \quad (3.8) \quad \begin{aligned} \int_\Omega u \sum_{i=1}^n \Phi_i \frac{\partial v}{\partial x_i} \rho dx &= - \sum_{i=1}^n \int_\Omega v \frac{\partial}{\partial x_i} (u \rho \Phi_i) dx \\ &= - \sum_{i=1}^n \int_\Omega v \left(\frac{\partial}{\partial x_i} (u \Phi_i) + \frac{u}{\rho} \Phi_i \frac{\partial \rho}{\partial x_i} \right) \rho dx. \end{aligned}$$

267 Hence,

$$268 \quad (3.9) \quad \Phi^\top[u] = - \sum_{i=1}^n \frac{\partial}{\partial x_i} (u \Phi_i) - u \Phi_i \frac{\partial \rho}{\partial x_i} / \rho = -u \operatorname{div} \Phi - \Phi[u] - u \Phi[\rho] / \rho.$$

269 Remembering the definition of $G_\rho(\Phi)$ in (2.31), we obtain that

$$270 \quad (3.10) \quad \Phi[u] + \Phi^\top[u] + G_\rho(\Phi)u = 0.$$

271 **3.3. Continuity of the bilinear form associated with the commutator.**

272 Setting, for v and w in $V(\Phi, \Psi)$:

$$273 \quad (3.11) \quad \Delta(u, v) := \int_\Omega [\Phi, \Psi][u](x)v(x)\rho(x)dx,$$

274 we have

$$275 \quad (3.12) \quad \begin{aligned} \Delta(u, v) &= \int_\Omega (\Phi[\Psi[u]]v - \Psi[\Phi[u]]v)\rho dx = \int_\Omega \Psi[u]\Phi^\top[v] - \Phi[u]\Psi^\top[v]\rho dx \\ &= \int_\Omega (\Phi[u]\Psi[v] - \Psi[u]\Phi[v])\rho dx + \int_\Omega (\Phi[u]G_\rho(\Psi)v - \Psi[u]G_\rho(\Phi)v)\rho dx. \end{aligned}$$

276 **LEMMA 3.1.** For $\Delta(\cdot, \cdot)$ to be a continuous bilinear form on $V(\Phi, \Psi)$, it suffices
 277 that, for some $c_\Delta > 0$:

$$278 \quad (3.13) \quad |G_\rho(\Phi)| + |G_\rho(\Psi)| \leq c_\Delta h \quad \text{a.e.,}$$

279 and we have then:

$$280 \quad (3.14) \quad |\Delta(u, v)| \leq \|\Psi[u]\|_\rho \left(\|\Phi[v]\|_\rho + c_\Delta \|v\|_H \right) + \|\Phi[u]\|_\rho \left(\|\Psi[v]\|_\rho + c_\Delta \|v\|_H \right).$$

281 *Proof.* Apply the Cauchy Schwarz inequality to (3.12), and use (3.13) combined
282 with the definition of the space H . \square

283 We apply the previous results with $\Phi := \sigma_i$, $\Psi := \sigma_j$. Set for v, w in V :

$$284 \quad (3.15) \quad \Delta_{ij}(u, v) := \int_{\Omega} [\sigma_i, \sigma_j][u](x)v(x)\rho(x)dx, \quad i, j = 1, \dots, n_{\sigma}.$$

285 We recall that V was defined in (2.21).

286 **COROLLARY 3.2.** *Let (2.28) hold. Then the $\Delta_{ij}(u, v)$, $i, j = 1, \dots, n_{\sigma}$, are con-*
287 *tinuous bilinear forms over V .*

288 *Proof.* Use remark 2.3 and conclude with lemma 3.1. \square

289 **3.4. Redefining the space H .** In section 2.2 we have obtained the continuity
290 and semi-coercivity of a by decomposing q , defined in (2.26), as a linear combination
291 (2.27) of the σ_i . We now take advantage of the previous computation of commutators
292 and assume that, more generally, instead of (2.27), we can decompose q in the form

$$293 \quad (3.16) \quad q = \sum_{k=1}^{n_{\sigma}} \eta_k'' \sigma_k + \sum_{1 \leq i < j \leq n_{\sigma}} \eta'_{ij} [\sigma_i, \sigma_j] \quad \text{a.e.}$$

294 We assume that η' and η'' are measurable functions over $[0, T] \times \Omega$, that η' is weakly
295 differentiable, and that for some $c'_{\eta} > 0$:

$$296 \quad (3.17) \quad h'_{\eta} \leq c'_{\eta} h, \quad \text{where } h'_{\eta} := |\eta''| + \sum_{i,j=1}^N |\sigma_i[\eta'_{ij}]| \quad \text{a.e., } \eta' \in L^{\infty}(\Omega).$$

297 **LEMMA 3.3.** *Let (2.28), (2.29), and (3.17) hold. Then the bilinear form $a(u, v)$*
298 *defined in (2.14) is both (i) continuous and (ii) semi-coercive over V .*

299 *Proof.* (i) We only have to analyze the contribution of a^{34} (defined in (2.23)),
300 since the other contributions to $a(\cdot, \cdot)$ do not change. For the terms in the first sum
301 in (3.16) we have, as was done in (2.36):

$$302 \quad (3.18) \quad \left| \int_{\Omega} \sigma_k[u] \eta_k'' v \rho \right| \leq \|\sigma_k[u]\|_{\rho} \|\sigma_k[u] \eta_k'' v\|_{\rho} \leq \|\sigma_k[u]\|_{\rho} \|v\|_H.$$

303 (ii) Setting $w := \eta'_{ij} v$ and taking here $(\Phi, \Psi) = (\sigma_i, \sigma_j)$, we get that

$$304 \quad (3.19) \quad \int_{\Omega} \eta'_{ij} [\sigma_i, \sigma_j][u] v \rho = \Delta(u, w),$$

305 where $\Delta(\cdot, \cdot)$ was defined in (3.11). Combining with lemma 3.1, we obtain

$$306 \quad (3.20) \quad |\Delta_{ij}(u, v)| \leq \|\sigma_j[u]\|_{\rho} \left(\|\sigma_i[w]\|_{\rho} + c_{\sigma} \|\eta'_{ij}\|_{\infty} \|v\|_H \right) \\ + \|\sigma_i[u]\|_{\rho} \left(\|\sigma_j[w]\|_{\rho} + c_{\sigma} \|\eta'_{ij}\|_{\infty} \|v\|_H \right).$$

307 Since

$$308 \quad (3.21) \quad \sigma_i[\eta'_{ij} v] = \eta'_{ij} \sigma_i[v] + \sigma_i[\eta'_{ij}] v,$$

309 by (3.17):

$$310 \quad (3.22) \quad \|\sigma_i[w]\|_\rho \leq \|\eta'_{ij}\|_\infty \|\sigma_i[v]\|_\rho + \|\sigma_i[\eta'_{ij}v]\|_\rho \leq \|\eta'_{ij}\|_\infty \|\sigma_i[v]\|_\rho + c_\eta \|v\|_H.$$

311 Combining these inequalities, point (i) follows.

312 (ii) Use $u = v$ in (3.21) and (3.12). We find after cancellation in (3.12) that

$$313 \quad (3.23) \quad \begin{aligned} \Delta_{ij}(u, \eta'_{ij}u) &= \int_\Omega u(\sigma_i[u]\sigma_j[\eta'_{ij}] - \sigma_j[u]\sigma_i(\eta'_{ij}))\rho \\ &\quad + \int_\Omega (\sigma_i[u]G_\rho(\sigma_j) - \sigma_j[u]G_\rho(\sigma_i))\eta'_{ij}u\rho. \end{aligned}$$

314 By (3.17), an upper bound for the absolute value of the first integral is

$$315 \quad (3.24) \quad (\|\sigma_i[u]\|_\rho + \|\sigma_j[u]\|_\rho) \|hu\|_\rho \leq 2 \|u\|_V \|u\|_H.$$

316 With (2.28), we get an upper bound for the absolute value of the second integral in
317 the same way, so, for any $\varepsilon > 0$:

$$318 \quad (3.25) \quad |\Delta_{ij}(u, \eta'_{ij}u)| \leq 4 \|u\|_V \|u\|_H.$$

319 We finally have that for some $c > 0$

$$320 \quad (3.26) \quad \begin{aligned} a(u, u) &\geq a_0(u, u) - c \|u\|_V \|u\|_H, \\ &\geq a_0(u, u) - \frac{1}{2} \|u\|_V^2 - \frac{1}{2} c^2 \|u\|_H^2, \\ &= \frac{1}{2} \|u\|_V^2 - \frac{1}{2} (c^2 + 1) \|u\|_H^2. \end{aligned}$$

321 The conclusion follows. \square

322 **REMARK 3.4.** *The statements analogous to theorems 2.6 and 2.7 hold, assuming*
323 *now that h satisfies (2.28), (2.29), and (3.17) (instead of (2.28)-(2.30)).*

324 **4. The weight ρ .** Classes of weighting functions characterized by their growth
325 are introduced. A major result is the independence of the growth order of the function
326 h on the choice of the weighting function ρ in the class under consideration.

327 **4.1. Classes of functions with given growth.** In financial models we usually
328 have nonnegative variables and the related functions have polynomial growth. Yet,
329 after a logarithmic transformation, we get real variables whose related functions have
330 exponential growth. This motivates the following definitions.

331 We remind that (I, J) is a partition of $\{0, \dots, N\}$, with $0 \in J$ and that Ω was
332 defined in (1.8).

333 **DEFINITION 4.1.** *Let γ' and γ'' belong to \mathbb{R}_+^{N+1} , with index from 0 to N . Let*
334 *$\mathcal{G}(\gamma', \gamma'')$ be the class of functions $\varphi : \Omega \rightarrow \mathbb{R}$ such that for some $c > 0$:*

$$335 \quad (4.1) \quad |\varphi(x)| \leq c \left(\prod_{k \in I} (e^{\gamma'_k x_k} + e^{-\gamma''_k x_k}) \right) \left(\prod_{k \in J} (x_k^{\gamma'_k} + x_k^{-\gamma''_k}) \right).$$

336 *We define \mathcal{G} as the union of $\mathcal{G}(\gamma', \gamma'')$ for all nonnegative (γ', γ'') . We call γ'_k and γ''_k*
337 *the growth order of φ , w.r.t. x_k , at $-\infty$ and $+\infty$ (resp. at zero and $+\infty$).*

338 Observe that the class \mathcal{G} is stable by the operations of sum and product, and that
339 if f, g belong to that class, so does $h = fg$, h having growth orders equal to the sum
340 of the growth orders of f and g . For $a \in \mathbb{R}$, we define

$$341 \quad (4.2) \quad a^+ := \max(0, a); \quad a^- := \max(0, -a); \quad N(a) := (a^2 + 1)^{1/2},$$

342 as well as

$$343 \quad (4.3) \quad \rho := \rho_I \rho_J,$$

344 where

$$345 \quad (4.4) \quad \rho_I(x) := \prod_{k \in I} e^{-\alpha'_k N(x_k^+) - \alpha''_k N(x_k^-)},$$

$$346 \quad (4.5) \quad \rho_J(x) := \prod_{k \in J} \frac{x_k^{\alpha'_k}}{1 + x_k^{\alpha'_k + \alpha''_k}},$$

347

348 for some *nonnegative* constants α'_k, α''_k , to be specified later.

349 **LEMMA 4.2.** *Let $\varphi \in \mathcal{G}(\gamma', \gamma'')$. Then $\varphi \in L^{1,p}(\Omega)$ whenever ρ is as above, with α*
 350 *satisfying, for some positive ε' and ε'' , for all $k = 0$ to N :*

$$351 \quad (4.6) \quad \begin{cases} \alpha'_k = \varepsilon' + \gamma'_k, & \alpha''_k = \varepsilon'' + \gamma''_k, & k \in I, \\ \alpha'_k = (\varepsilon' + \gamma''_k - 1)_+, & \alpha''_k = 1 + \varepsilon'' + \gamma'_k, & k \in J. \end{cases}$$

352 *In addition we can choose for $k = 0$ (if element of J):*

$$353 \quad (4.7) \quad \begin{cases} \alpha'_0 := (\varepsilon' + \gamma''_0 - 1)_+; \alpha''_0 := 0 & \text{if } \varphi(s, y) = 0 \text{ when } s \text{ is far from } 0, \\ \alpha'_0 := 0, \alpha''_0 := 1 + \varepsilon'' + \gamma'_0, & \text{if } \varphi(s, y) = 0 \text{ when } s \text{ is close to } 0. \end{cases}$$

354 *Proof.* It is enough to prove (4.6), the proof of (4.7) is similar. We know that φ
 355 satisfy (4.1) for some $c > 0$ and γ . We need to check the finiteness of

$$356 \quad (4.8) \quad \int_{\Omega} \left(\prod_{k \in I} (e^{\gamma'_k y_k} + e^{-\gamma''_k y_k}) \right) \left(\prod_{k \in J} (y_k^{\gamma'_k} + y_k^{-\gamma''_k}) \right) \rho(s, y) d(s, y).$$

357 But the above integral is equal to the product $p_I p_J$ with

$$358 \quad (4.9) \quad p_I := \prod_{k \in I} \int_{\mathbb{R}} (e^{\gamma'_k x_k} + e^{-\gamma''_k x_k}) e^{-\alpha'_k N(x_k^+) - \alpha''_k N(x_k^-)} dx_k,$$

$$359 \quad (4.10) \quad p_J := \prod_{k \in J} \int_{\mathbb{R}_+} \frac{x_k^{\alpha'_k + \gamma'_k} + x_k^{\alpha'_k - \gamma''_k}}{1 + x_k^{\alpha'_k + \alpha''_k}} dx_k.$$

360

361 Using (4.6) we deduce that p_I is finite since for instance

$$362 \quad (4.11) \quad \begin{aligned} & \int_{\mathbb{R}_+} (e^{\gamma'_k x_k} + e^{-\gamma''_k x_k}) e^{-\alpha'_k N(x_k^+) - \alpha''_k N(x_k^-)} dx_k \\ & \leq 2 \int_{\mathbb{R}_+} e^{\gamma'_k x_k} e^{-(1+\gamma'_k)x_k} dx_k = 2 \int_{\mathbb{R}_+} e^{-x_k} dx_k = 2, \end{aligned}$$

363 and p_J is finite since

$$364 \quad (4.12) \quad p_J = \prod_{k \in J} \int_{\mathbb{R}_+} \frac{x_k^{\varepsilon' + \gamma'_k + \gamma''_k} + x_k^{\varepsilon' - 1}}{1 + x_k^{\varepsilon' + \varepsilon'' + \gamma'_k + \gamma''_k}} dx_k < \infty.$$

365 The conclusion follows. □

366 **4.2. On the growth order of h .** Set for all k

$$367 \quad (4.13) \quad \alpha_k := \alpha'_k + \alpha''_k.$$

368 Remember that we take ρ in the form (4.3)-(4.4).

369 **LEMMA 4.3.** *We have that:*

370 (i) *We have that*

$$371 \quad (4.14) \quad \left\| \frac{\rho_{x_k}}{\rho} \right\|_{\infty} \leq \alpha_k, \quad k \in I; \quad \left\| \frac{x}{\rho} \rho_{x_k} \right\|_{\infty} \leq \alpha_k, \quad k \in J.$$

372 (ii) *Let h satisfying either (2.28)-(2.30) or (2.28)-(2.29), and (3.17). Then the growth*
373 *order of h does not depend on the choice of the weighting function ρ .*

374 *Proof.* (i) For $k \in I$ this is an easy consequence of the fact that $N(\cdot)$ is non
375 expansive. For $k \in J$, we have that

$$376 \quad (4.15) \quad \frac{x}{\rho} \rho_{x_k} = \frac{x}{\rho} \frac{\alpha'_k x^{\alpha'_k - 1} (1 + x^{\alpha_k}) - x^{\alpha'_k} \alpha_k x^{\alpha_k - 1}}{(1 + x^{\alpha_k})^2} = \frac{\alpha'_k - \alpha''_k x^{\alpha_k}}{1 + x^{\alpha_k}}.$$

377 We easily conclude, discussing the sign of the numerator.

378 (ii) The dependence of h w.r.t. ρ is only through the last term in (2.28), namely,
379 $\sum_i |\sigma_i[\rho]/\rho$. By (i) we have that

$$380 \quad (4.16) \quad \left| \frac{\sigma_i^k[\rho]}{\rho} \right| \leq \left\| \frac{\rho_{x_k}}{\rho} \right\|_{\infty} |\sigma_i^k| \leq \alpha_k |\sigma_i^k|, \quad k \in I,$$

381

$$382 \quad (4.17) \quad \left| \frac{\sigma_i^k[\rho]}{\rho} \right| \leq \left\| \frac{x_k \rho_{x_k}}{\rho} \right\|_{\infty} \left| \frac{\sigma_i^k}{x_k} \right| \leq \alpha_k \left| \frac{\sigma_i^k}{x_k} \right|, \quad k \in J.$$

383 In both cases, the choice of α has no influence on the growth order of h . \square

384 **4.3. European option.** In the case of a European option with payoff $u_T(x)$,
385 we need to check that $u_T \in H$, that is, ρ must satisfy

$$386 \quad (4.18) \quad \int_{\Omega} |u_T(x)|^2 h(x)^2 \rho(x) dx < \infty.$$

387 In the framework of the semi-symmetry hypothesis (A.8), we need to check that
388 $u_T \in V$, which gives the additional condition

$$389 \quad (4.19) \quad \sum_{i=1}^{n_{\sigma}} \int_{\Omega} |\sigma_i[u_T](x)|^2 \rho(x) dx < \infty.$$

390 In practice the payoff depends only on s and this allows to simplify the analysis.

391 **5. Applications using the commutator analysis.** The commutator analysis
392 is applied to the general multiple factor model and estimates for the function h char-
393 acterizing the space H (defined in (2.21)) are derived. The estimates are compared
394 to the case when the commutator analysis is not applied. The resulting improvement
395 will be established in the next section.

396 **5.1. Commutator and continuity analysis.** We analyze the general multiple
 397 factor model (1.4), which belongs to the class of models (2.1) with $\Omega \subset \mathbb{R}^{1+N}$, $n_\sigma =$
 398 $2N$, and for $i = 1$ to N :

$$399 \quad (5.1) \quad \sigma_i[v] = f_i(y_i)s^{\beta_i}v_s; \quad \sigma_{N+i}[v] = g_i(y_i)v_i,$$

400 with f_i and g_i of class C^1 over Ω . We need to compute the commutators of the first-
 401 order differential operators associated with the σ_i . The correlations will be denoted
 402 by

$$403 \quad (5.2) \quad \hat{\kappa}_k := \kappa_{k,N+k}, \quad k = 1, \dots, N.$$

404 **REMARK 5.1.** We use many times the following rule. For $\Omega \subset \mathbb{R}^n$, where $n =$
 405 $1 + N$, $u \in H^1(\Omega)$, $a, b \in L^0$, and vector fields $Z[u] := au_{x_1}$ and $Z'[u] := bu_{x_2}$, we
 406 have $Z[Z'[u]] = a(bu_{x_2})_{x_1} = ab_{x_1}u_{x_2} + abu_{x_1x_2}$, so that

$$407 \quad (5.3) \quad [Z, Z'][u] = ab_{x_1}u_{x_2} - ba_{x_2}u_{x_1}.$$

408 We obtain that

$$409 \quad (5.4) \quad [\sigma_i, \sigma_\ell][u] = (\beta_\ell - \beta_i)f_i(y_i)f_\ell(y_\ell)s^{\beta_i+\beta_\ell-1}u_s, \quad 1 \leq i < \ell \leq N,$$

410

$$411 \quad (5.5) \quad [\sigma_i, \sigma_{N+i}][u] = -s^{\beta_i}f'_i(y_i)g_i(y_i)u_s, \quad i = 1, \dots, N,$$

412 and

$$413 \quad (5.6) \quad [\sigma_i, \sigma_{N+\ell}][u] = [\sigma_{N+i}, \sigma_{N+\ell}][u] = 0, \quad i \neq \ell.$$

414 Also,

$$415 \quad (5.7) \quad \begin{aligned} \operatorname{div} \sigma_i + \frac{\sigma_i[\rho]}{\rho} &= f_i(y_i)s^{\beta_i-1}(\beta_i + s\frac{\rho_s}{\rho}), \\ \operatorname{div} \sigma_{N+i} + \frac{\sigma_{N+i}[\rho]}{\rho} &= g'_i(y_i) + g_i(y_i)\frac{\rho_i}{\rho}. \end{aligned}$$

416 **5.1.1. Computation of q .** Remember the definitions of \bar{q} , \hat{q} and q in (2.24) and
 417 (2.25), where δ_{ij} denote the Kronecker operator. We obtain that, for $1 \leq i, j, k \leq N$:

$$418 \quad (5.8) \quad \begin{cases} \bar{q}_{ij0} &= \delta_{ij}\beta_j f_i^2(y_i)s^{2\beta_i-1}; & \bar{q}_{iik} = 0; \\ \bar{q}_{i,N+j} &= 0; \\ \bar{q}_{N+i,j,0} &= \delta_{ij}\hat{\kappa}_i f'_i(y_i)g_i(y_i)s^{\beta_i}; & \bar{q}_{N+i,j,k} = 0; \\ \bar{q}_{N+i,N+j,k} &= \delta_{ijk}g_i(y_i)g'_i(y_i). \end{cases}$$

419 That means, we have for $\hat{q} = \sum_{i,j=1}^{2N} \bar{q}_{ij}$ and $q = \hat{q} - b$ that

$$420 \quad (5.9) \quad \begin{aligned} \hat{q}_0 &= \sum_{i=1}^N (\beta_i f_i^2(y_i)s^{2\beta_i-1} + \hat{\kappa}_i f'_i(y_i)g_i(y_i)s^{\beta_i}); & q_0 &= \hat{q}_0 - rs, \\ \hat{q}_k &= g_k(y_k)g'_k(y_k); & q_k &= \hat{q}_k - \theta_k(\mu_k - y_k), \\ & & k &= 1, \dots, N. \end{aligned}$$

421 **5.2. Computation of η' and η'' .** The coefficients η' , η'' are solution of (3.16).
 422 We can write $\eta = \hat{\eta} + \tilde{\eta}$, where

$$423 \quad (5.10) \quad \begin{aligned} \hat{q} &= \sum_{i=1}^{n_\sigma} \hat{\eta}_i'' \sigma_i + \sum_{1 \leq i, j \leq n_\sigma} \hat{\eta}'_{ij} [\sigma_i, \sigma_j], \quad \hat{\eta}'_{ij} = 0 \text{ if } i = j. \\ -b &= \sum_{i=1}^{n_\sigma} \tilde{\eta}_i'' \sigma_i + \sum_{1 \leq i, j \leq n_\sigma} \tilde{\eta}'_{ij} [\sigma_i, \sigma_j], \quad \tilde{\eta}'_{ij} = 0 \text{ if } i = j. \end{aligned}$$

424 For $k = 1$ to N , this reduces to

$$425 \quad (5.11) \quad \begin{cases} \hat{\eta}''_{N+k} g_k(y_k) = g'_k(y_k) g_k(y_k); \\ \tilde{\eta}''_{N+k} g_k(y_k) = -\theta_k(\mu_k - y_k). \end{cases}$$

426 So, we have that

$$427 \quad (5.12) \quad \begin{cases} \hat{\eta}''_{N+k} = g'_k(y_k); \\ \tilde{\eta}''_{N+k} = \frac{-\theta_k(\mu_k - y_k)}{g_k(y_k)}. \end{cases}$$

428 For the 0th component, (5.10) can be expressed as

$$429 \quad (5.13) \quad \begin{cases} \sum_{k=1}^N (-\hat{\eta}'_{k, N+k} f'_k(y_k) g_k(y_k) s^{\beta_k} - \hat{\kappa}_k f'_k(y_k) g_k(y_k) s^{\beta_k}) \\ + \sum_{k=1}^N (\hat{\eta}''_k f_k(y_k) s^{\beta_k} - \beta_k f_k^2(y_k) s^{2\beta_k - 1}) \\ + \sum_{k=1}^N (-\tilde{\eta}'_{k, N+k} f'_k(y_k) g_k(y_k) s^{\beta_k} + \tilde{\eta}''_k f_k(y_k) s^{\beta_k}) - rs = 0. \end{cases}$$

430 We choose to set each term in parenthesis in the first two lines above to zero. It
 431 follows that

$$432 \quad (5.14) \quad \hat{\eta}'_{k, N+k} = -\hat{\kappa}_k \in L^\infty(\Omega), \quad \hat{\eta}''_k = \beta_k f_k(y_k) s^{\beta_k - 1}.$$

434 If $N > 1$ we (arbitrarily) choose then to set the last line to zero with

$$435 \quad (5.15) \quad \tilde{\eta}''_k = \tilde{\eta}'_k = 0, \quad k = 2, \dots, N.$$

436 It remains that

$$437 \quad (5.16) \quad \tilde{\eta}''_1 f_1(y_1) s^{\beta_1} - \tilde{\eta}'_{1, N+1} f'_1(y_1) g_1(y_1) s^{\beta_1} = rs.$$

439 Here, we can choose to take either $\tilde{\eta}''_1 = 0$ or $\tilde{\eta}'_{1, N+1} = 0$. We obtain then two
 440 possibilities:

$$441 \quad (5.17) \quad \begin{cases} \text{(i)} & \tilde{\eta}''_1 = 0 \text{ and } \tilde{\eta}'_{1, N+1} = \frac{-rs^{1-\beta_1}}{f'_1(y_1) g_1(y_1)}, \\ \text{(ii)} & \tilde{\eta}''_1 = \frac{rs^{1-\beta_1}}{f_1(y_1)} \text{ and } \tilde{\eta}'_{1, N+1} = 0. \end{cases}$$

442 **5.2.1. Estimate of the h function.** We decide to choose case (i) in (5.17).
 443 The function h needs to satisfy (2.28), (2.29), and (3.17) (instead of (2.30)). Instead
 444 of (2.28), we will rather check the stronger condition (2.34). We compute

$$445 \quad (5.18) \quad h'_\sigma := \sum_{k=1}^N |f_k(y_k)| s^{\beta_k} \left(|(\hat{\kappa}_k)_s| + \left| \frac{\rho_s}{\rho} \right| \right) + |g_k(y_k)| \left(|(\hat{\kappa}_k)_k| + \left| \frac{\rho_k}{\rho} \right| \right) \\
 446 \quad + \sum_{k=1}^N \left(\beta_k |f_k(y_k)| s^{\beta_k-1} + |g'_k(y_k)| \right),$$

$$447 \quad (5.19) \quad h_r := |r|^{\frac{1}{2}},$$

$$448 \quad (5.20) \quad h'_\eta := \hat{h}'_\eta + \tilde{h}'_\eta,$$

450 where we have

$$(5.21)$$

$$451 \quad \hat{h}'_\eta := \sum_{k=1}^N \left(\beta_k |f_k(y_k)| s^{\beta_k-1} + |g'_k(y_k)| + \left| f_k(y_k) |s^{\beta_k} \frac{\partial \hat{\kappa}_k}{\partial s} \right| + \left| g_k(y_k) \frac{\partial \hat{\kappa}_k}{\partial y_k} \right| \right),$$

$$(5.22)$$

$$452 \quad \tilde{h}'_\eta := \sum_{k=1}^N \left| \frac{\theta_k(\mu_k - y_k)}{g_k(y_k)} \right| + \left| r \frac{f_1(y_1)}{f'_1(y_1) g_1(y_1)} \right| + \left| r g_1(y_1) s^{1-\beta_1} \frac{\partial}{\partial y_1} \left[\frac{1}{f'_1(y_1) g_1(y_1)} \right] \right|.$$

454 **REMARK 5.2.** Had we chosen (ii) instead of (i) in (5.17), this would only change
 455 the expression of \tilde{h}'_η that would then be

$$456 \quad (5.23) \quad \tilde{h}'_\eta = \sum_{k=1}^N \left| \frac{\theta_k(\mu_k - y_k)}{g_k(y_k)} \right| + \left| \frac{r s^{1-\beta_1}}{f_1(y_1)} \right|.$$

457 **5.2.2. Estimate of the h function without the commutator analysis.** The
 458 only change in the estimate of h will be the contribution of h'_η and h''_η . We have to
 459 satisfy (2.28)-(2.30). In addition, ignoring the commutator analysis, we would solve
 460 (5.13) with $\hat{\eta}' = 0$, meaning that we choose

$$461 \quad (5.24) \quad \hat{\eta}''_k := \beta_k f_k(y_k) s^{\beta_k-1} + \hat{\kappa}_k \frac{f'_k(y_k) g_k(y_k)}{f_k(y_k)}, \quad k = 1, \dots, N,$$

462 and take $\tilde{\eta}''_1$ out of (5.16). Then condition (3.17), with here $\hat{\eta}' = 0$, would give

$$463 \quad (5.25) \quad h \geq c_\eta h_\eta, \quad \text{where } h_\eta := h_{\hat{\eta}} + h_{\tilde{\eta}},$$

464 with

$$465 \quad (5.26) \quad h_{\hat{\eta}} := \sum_{k=1}^N \left(\beta_k |f_k(y_k)| s^{\beta_k-1} + |\hat{\kappa}_k| \left| \frac{f'_k(y_k) g_k(y_k)}{f_k(y_k)} \right| + |g'_k(y_k)| \right),$$

$$466 \quad (5.27) \quad h_{\tilde{\eta}} := \sum_{k=1}^N \left| \frac{\theta_k(\mu_k - y_k)}{g_k(y_k)} \right| + \left| \frac{r s^{1-\beta_1}}{f_1(y_1)} \right|.$$

468 We will see in applications that this is in general worse.

469 **6. Application to stochastic volatility models.** The results of Section 5 are
 470 specified for a subclass of the multiple factor model, in particular for the VAT and
 471 GMH models. We show that the commutator analysis allows to take smaller values
 472 for the function h (and consequently to include a larger class of payoff functions).

473 **6.1. A useful subclass.** Here we assume that

$$474 \quad (6.1) \quad |f_k(y_k)| = |y_k|^{\gamma_k}; \quad |g_k(y_k)| = \nu_k |y_k|^{1-\gamma_k}; \quad \beta_k \in (0, 1]; \quad \nu_k > 0; \quad \gamma_k \in (0, \infty).$$

475 Furthermore, we assume κ to be constant and

$$476 \quad (6.2) \quad |f'_k(y_k)g_k(y_k)| = \text{const} \quad \text{for all } y_k, \quad k = 1, \dots, N.$$

477 Set

$$478 \quad (6.3) \quad \begin{aligned} c_s &:= \|s\rho_s/\rho\|_\infty; \\ c'_k &= \begin{cases} \|\rho_k/\rho\|_\infty & \text{if } \Omega_k = \mathbb{R}, \\ 0 & \text{otherwise.} \end{cases} \\ c''_k &= \begin{cases} 0 & \text{if } \Omega_k = \mathbb{R}, \\ \|y_k\rho_k/\rho\|_\infty & \text{otherwise.} \end{cases} \end{aligned}$$

479 We get, assuming that $\gamma_1 \neq 0$:

$$480 \quad (6.4) \quad h'_\sigma := \sum_{k=1}^N (c_s |y_k|^{\gamma_k} s^{\beta_k-1} + \nu_k c'_k |y_k|^{1-\gamma_k} \\ + \nu_k c''_k |y_k|^{-\gamma_k} + \beta_k |y_k|^{\gamma_k} s^{\beta_k-1} + (1 - \gamma_k) \nu_k |y_k|^{-\gamma_k}),$$

481

$$482 \quad (6.5) \quad \hat{h}'_\eta := \sum_{k=1}^N (\beta_k |y_k|^{\gamma_k} s^{\beta_k-1} + (1 - \gamma_k) \nu_k |y_k|^{-\gamma_k}),$$

$$483 \quad (6.6) \quad \tilde{h}'_\eta := \sum_{k=1}^N \left(\frac{\theta_k |\mu_k - y_k|}{\nu_k |y_k|^{1-\gamma_k}} + \frac{r |y_1|^{\gamma_1}}{\gamma_1 \nu_1} \right).$$

484

485 Therefore when all $y_k \in \mathbb{R}$, we can choose h' as

$$486 \quad (6.7) \quad h' := 1 + \sum_{k=1}^N (|y_k|^{\gamma_k} (1 + s^{\beta_k-1}) + (1 - \gamma_k) |y_k|^{-\gamma_k} + |y_k|^{\gamma_k-1}) \\ + \sum_{k \in I} |y_k|^{1-\gamma_k} + \sum_{k \in J} |y_k|^{-\gamma_k}.$$

487 Without the commutator analysis we would get

$$488 \quad (6.8) \quad \hat{h}_\eta := \sum_{k=1}^N (\beta_k |y_k|^{\gamma_k} s^{\beta_k-1} + \nu_k |\hat{\kappa}_k| |y_k|^{-\gamma_k} + (1 - \gamma_k) \nu_k |y_k|^{-\gamma_k}),$$

$$489 \quad (6.9) \quad \tilde{h}_\eta := \sum_{k=1}^N \left(\theta_k \frac{|\mu_k - y_k|}{\nu_k |y_k|^{1-\gamma_k}} + r s^{1-\beta_1} |y_1|^{-\gamma_1} \right).$$

490

491 Therefore we can choose

$$492 \quad (6.10) \quad h := h''; \quad h'' := h' + rs^{1-\beta_1}/|y_1|^{\gamma_1} + \sum_k \nu_k |\hat{\kappa}_k| |y_k|^{-\gamma_k}.$$

493 So, we always have that $h' \leq h''$, meaning that it is advantageous to use the commu-
494 tator analysis, due to the term $rs^{1-\beta_1}/|y_1|^{\gamma_1}$ above in particular. The last term in
495 the above r.h.s. has as contribution only when $\gamma_k \neq 1$ (since otherwise h' includes a
496 term of the same order).

497 **6.2. Application to the VAT model.** For the variant of the Achdou and
498 Tchou model with multiple factors (VAT), i.e. when $\gamma_k = 1$, for $k = 1$ to N , we can
499 take h equal to

$$500 \quad (6.11) \quad h'_{TA} := 1 + \sum_{k=1}^N |y_k| (1 + s^{\beta_k - 1}),$$

501 when the commutator analysis is used, and when it is not, take h equal to

$$502 \quad (6.12) \quad h_{TA} := h_{TA} + rs^{1-\beta_1}|y_1|^{-1} + \sum_{k=1}^N \nu_k |\hat{\kappa}_k| |y_k|^{-1}.$$

503 Remember that $u_T(s) = (s - K)_+$ for a call option, and $u_T(s) = (K - s)_+$ for a
504 put option, both with strike $K > 0$.

505 **LEMMA 6.1.** *For the VAT model, using the commutator analysis, in case of a call*
506 *(resp. put) option with strike $K > 0$, we can take $\rho = \rho_{call}$, (resp. $\rho = \rho_{put}$), with*

$$507 \quad (6.13) \quad \begin{aligned} \rho_{call}(s, y) &:= (1 + s^{3+\varepsilon''})^{-1} \prod_{k=1}^N e^{-\varepsilon N(y_k)}, \\ \rho_{put}(s, y) &:= \frac{s^{\alpha_P}}{1 + s^{\alpha_P}} \prod_{k=1}^N e^{-\varepsilon N(y_k)}, \end{aligned}$$

508 where $\alpha_P := \left(\varepsilon' + 2 \sum_{k=1}^N (1 - \beta_k) - 1 \right)_+$.

509 *Proof.* (i) In the case of a call option, we have that

$$510 \quad (6.14) \quad 1 \geq c_0 s^{\beta_k - 1} \text{ for } c_0 > 0 \text{ small enough over the domain of integration,}$$

511 so that we can as well take

$$512 \quad (6.15) \quad h(s, y) = 1 + \sum_{k=1}^N |y_k| \leq \prod_{k=1}^N (1 + |y_k|).$$

513 So, we need that $\varphi(s, y) \in L^{1,\rho}(\Omega)$, with

$$514 \quad (6.16) \quad \varphi(s, y) = h^2(s, y) u_T^2(s) = (s - K)_+^2 \prod_{k=1}^N (1 + |y_k|)^2.$$

515 By lemma 4.2, where here $J = \{0\}$ and $I = \{1, \dots, N\}$, we may take resp.

$$516 \quad (6.17) \quad \gamma'_0 = 2, \quad \gamma''_0 = 0, \quad \gamma'_k > 0, \quad \gamma''_k > 0, \quad k = 1, \dots, N,$$

517 and so we may choose for $\varepsilon' > 0$ and $\varepsilon'' > 0$:

$$518 \quad (6.18) \quad \alpha'_0 = 0, \quad \alpha''_0 = 3 + \varepsilon'', \quad \alpha'_k = \varepsilon', \quad \alpha''_k = \varepsilon'', \quad k = 1, \dots, N,$$

519 so that setting $\varepsilon := \varepsilon' + \varepsilon''$, we can take $\rho = \rho_{call}$.

520 (ii) For a put option with strike $K > 0$, $1 \leq c_0 s^{\beta_k - 1}$ for big enough $c_0 > 0$, over the
521 domain of integration, so that we can as well take

$$522 \quad (6.19) \quad h(s, y) = 1 + \sum_{k=1}^N |y_k| s^{\beta_k - 1} \leq \prod_{k=1}^N (1 + |y_k| s^{\beta_k - 1})^2 \leq \prod_{k=1}^N s^{2\beta_k - 2} (1 + |y_k|)^2$$

523 and

$$524 \quad (6.20) \quad \varphi(s, y) = h^2(s, y) u_T^2(s) \leq (K - s)_+^2 \prod_{k=1}^N s^{2\beta_k - 2} (1 + |y_k|)^2.$$

525 By lemma 4.2, in the case of a put option and since $(K - s)_+^2$ is bounded, we can take
526 $\gamma'_k, \gamma''_k, \alpha'_k, \alpha''_k$ as before, for $k = 1$ to N , and

$$527 \quad (6.21) \quad \gamma'_0 = 0, \gamma''_0 = 2 \sum_{k=1}^N (1 - \beta_k), \alpha'_0 = \left(\varepsilon' + 2 \sum_{k=1}^N (1 - \beta_k) - 1 \right)_+, \alpha''_0 = 0$$

528 the result follows. \square

529 **REMARK 6.2.** *If we do not use the commutator analysis, then we have a greater*
530 *"h" function; we can check that our previous choice of ρ does not apply any more (so*
531 *we should consider a smaller weight function, but we do not need to make it explicit).*
532 *And indeed, we have then a singularity when say y_1 is close to zero so that the previous*
533 *choice of ρ makes the p integral undefined.*

534 **6.3. Application to the GMH model.** For the generalized multiple factor
535 Heston model (GMH), i.e. when $\gamma_k = 1/2$, $k = 1$ to N , we can take h equal to

$$536 \quad (6.22) \quad h'_H := 1 + \sum_{k=1}^N \left(|y_k|^{\frac{1}{2}} (1 + s^{\beta_k - 1}) + |y_k|^{-\frac{1}{2}} \right),$$

537 when the commutator analysis is used, and when it is not, take h equal to

$$538 \quad (6.23) \quad h_H := h_H + r s^{1 - \beta_1} |y_1|^{-\frac{1}{2}}.$$

539 **LEMMA 6.3.** (i) *For the GMH model, using the commutator analysis, in case of*
540 *a call option with strike K , meaning that $u_T(s) = (s - K)_+$, we can take $\rho = \rho_{call}$,*
541 *with*

$$542 \quad (6.24) \quad \rho_{call}(s, y) := (1 + s^{\varepsilon'' + 3})^{-1} \prod_{k=1}^N y_k^{\varepsilon'} (1 + y_k^{\varepsilon + 2})^{-1}.$$

543 (ii) *For a put option with strike $K > 0$, we can take $\rho = \rho_{put}$, with*

$$544 \quad (6.25) \quad \rho_{put}(s, y) := \prod_{k=1}^N y_k^{\varepsilon'} (1 + y_k^{\varepsilon + 2})^{-1}.$$

545 *Proof.* (i) For the call option, using (6.14) we see that we can as well take

$$546 \quad (6.26) \quad h(s, y) \leq 1 + \sum_{k=1}^N \left(y_k^{1/2} + y_k^{-1/2} \right) \leq (s - K)_+^2 \prod_{k=1}^N (1 + y_k^{1/2} + y_k^{-1/2}).$$

547 So, we need that $\varphi(s, y) \in L^{1, \rho}(\Omega)$, with

$$548 \quad (6.27) \quad \varphi(s, y) = h^2(s, y) u_T^2(s) = (s - K)_+^2 \prod_{k=1}^N (1 + y_k^{1/2} + y_k^{-1/2}).$$

549 By lemma 4.2, where here $J = \{0, \dots, N\}$, we may take resp.

$$550 \quad (6.28) \quad \gamma'_0 = 2, \gamma''_0 = 0, \gamma'_k = 1, \gamma''_k = 1, \quad k = 1, \dots, N,$$

551 and so we may choose for $\varepsilon' > 0$ and $\varepsilon'' > 0$:

$$552 \quad (6.29) \quad \alpha'_0 = 0, \alpha''_0 = 3 + \varepsilon'', \alpha'_k = \varepsilon', \alpha''_k = \varepsilon'' + 2, \quad k = 1, \dots, N,$$

553 so that setting $\varepsilon := \varepsilon' + \varepsilon''$, we can take $\rho = \rho_{call}$.

554 (ii) For a put option with strike $K > 0, 1 \leq c_0 s^{\beta_k - 1}$ for big enough $c_0 > 0$, over the
555 domain of integration, so that we can as well take

$$556 \quad (6.30) \quad h(s, y) = 1 + \sum_{k=1}^N |y_k| s^{\beta_k - 1} \leq \prod_{k=1}^N (1 + |y_k| s^{\beta_k - 1})^2 \leq \prod_{k=1}^N s^{2\beta_k - 2} (1 + |y_k|)^2$$

557 and

$$558 \quad (6.31) \quad \varphi(s, y) = h^2(s, y) u_T^2(s) \leq (K - s)_+^2 \prod_{k=1}^N s^{2\beta_k - 2} (1 + |y_k|)^2.$$

559 By lemma 4.2, in the case of a put option and since $(K - s)_+^2$ is bounded, we can take
560 $\gamma'_k, \gamma''_k, \alpha'_k, \alpha''_k$ as before, for $k = 1$ to N , and

$$561 \quad (6.32) \quad \gamma'_0 = 0, \gamma''_0 = 0, \alpha'_0 = 0, \alpha''_0 = 0.$$

562 the result follows. \square

563 **REMARK 6.4.** *If we do not use the commutator analysis, then, again, we have a*
564 *greater “h” function; we can check that our previous choice of ρ does not apply any*
565 *more (so we should consider a smaller weight function, but we do not need to make it*
566 *explicit). And indeed, by the behaviour of the integral for large s the previous choice*
567 *of ρ makes the p integral undefined.*

568 **Appendix A. Regularity results by Lions and Magenes [15, Ch. 1].**

569 Let H be a Hilbert space identified with its dual and scalar product denoted by
570 (\cdot, \cdot) . Let V be a Hilbert space, densely and continuously embedded in H , with duality
571 product denoted by $\langle \cdot, \cdot \rangle_V$. Set

$$572 \quad (A.1) \quad W(0, T) := \{u \in L^2(0, T; V); \dot{u} \in L^2(0, T; V^*)\}.$$

573 It is known [15, Ch. 1] that

$$574 \quad (A.2) \quad W(0, T) \subset C(0, T; H) \quad \text{with continuous inclusion,}$$

575 and that for any u, v in $W(0, T)$, and $0 \leq t < t' \leq T$, the following integration by
576 parts formula holds:

$$577 \quad (A.3) \quad \int_t^{t'} (\langle \dot{u}(s), v(s) \rangle_V + \langle \dot{v}(s), u(s) \rangle_V) ds = (u(t'), v(t'))_H - (u(t), v(t))_H.$$

578 Equivalently,

$$579 \quad (A.4) \quad 2 \int_t^{t'} \langle \dot{u}(s), u(s) \rangle_V ds = \|u(t')\|_H^2 - \|u(t)\|_H^2, \quad \text{for all } u \in W(0, T).$$

580 Let $A(t) \in L^\infty(0, T; L(V, V^*))$ satisfy the hypotheses of uniform continuity and semi-
581 coercivity, i.e., for some $\alpha > 0$, $\lambda \geq 0$, and $c > 0$:

$$582 \quad (\text{A.5}) \quad \begin{cases} \langle A(t)u, u \rangle_V \geq \alpha \|u\|_V^2 - \lambda \|u\|_H, & \text{for all } u \in V, \text{ and a.a. } t \in [0, T], \\ \|A(t)u\|_{V^*} \leq c \|u\|_V, & \text{for all } u \in V, \text{ and a.a. } t \in [0, T]. \end{cases}$$

583 Given $(f, u_T) \in L^2(0, T; V^*) \times H$, we consider the following (backward) parabolic
584 equation: find u in $W(0, T)$ such

$$585 \quad (\text{A.6}) \quad \begin{cases} -\dot{u}(t) + A(t)u(t) = f & \text{in } L^2(0, T; V^*), \\ u(T) = u_T & \text{in } H, \end{cases}$$

586 and recall classical results from [15, Ch. 1].

587 **PROPOSITION A.1** (first parabolic estimate). *The parabolic equation (A.6) has*
588 *a unique solution $u \in W(0, T)$, and for some $c > 0$ not depending on (f, u_T) :*

$$589 \quad (\text{A.7}) \quad \|u\|_{L^2(0, T; V)} + \|u\|_{L^\infty(0, T; H)} \leq c(\|u_T\|_H + \|f\|_{L^2(0, T; V^*)}).$$

590 We next derive a stronger result with the hypothesis of *semi-symmetry* below:

$$591 \quad (\text{A.8}) \quad \begin{cases} A(t) = A_0(t) + A_1(t), A_0(t) \text{ and } A_1(t) \text{ continuous linear mappings } V \rightarrow V^*, \\ A_0(t) \text{ symmetric and continuously differentiable } V \rightarrow V^* \text{ w.r.t. } t, \\ A_1(t) \text{ is measurable with range in } H, \text{ and for positive numbers } \alpha_0, c_{A,1}: \\ \text{(i) } \langle A_0(t)u, u \rangle_V \geq \alpha_0 \|u\|_V^2, & \text{for all } u \in V, \text{ and a.a. } t \in [0, T], \\ \text{(ii) } \|A_1(t)u\|_H \leq c_{A,1} \|u\|_V, & \text{for all } u \in V, \text{ and a.a. } t \in [0, T], \\ f \in L^2(0, T; H) \text{ and } u_T \in V. \end{cases}$$

592 **PROPOSITION A.2** (second parabolic estimate). *Let (A.8) hold. Then the solu-*
593 *tion $u \in W(0, T)$ of (A.6) belongs to $L^\infty(0, T; V)$, \dot{u} belongs to $L^2(0, T; H)$, and for*
594 *some $c > 0$ not depending on (f, u_T) :*

$$595 \quad (\text{A.9}) \quad \|u\|_{L^\infty(0, T; V)} + \|\dot{u}\|_{L^2(0, T; H)} \leq c(\|u_T\|_V + \|f\|_{L^2(0, T; H)}).$$

596 **Appendix B. Parabolic variational inequalities.**

597 Let $K \subset V$ be a non-empty, closed and convex set, \mathcal{K} be the closure of K in H ,
598 and $u_T \in K$. Let

$$599 \quad (\text{B.1}) \quad \begin{cases} L^2(0, T; K) := \{u \in L^2(0, T; V); u(t) \in K \text{ a.e.}\}, \\ W(0, T; K) := W(0, T) \cap L^2(0, T; K). \end{cases}$$

600 We consider parabolic variational inequalities as follows: find $u \in W(0, T; K)$ such
601 that

$$602 \quad (\text{B.2}) \quad \begin{cases} \langle -\dot{u}(t) + A(t)u(t) - f(t), v - u(t) \rangle_V \geq 0 & \text{for all } v \in K, \text{ a.a. } t, \\ u(T) = u_T & \text{in } H. \end{cases}$$

603 Take $v \in W(0, T; K)$. Adding to the previous inequality the integration by parts
604 formula

$$605 \quad (\text{B.3}) \quad - \int_0^T \langle \dot{v}(s) - \dot{u}(s), v(s) - u(s) \rangle_V ds = \frac{1}{2} \|u(0) - v(0)\|_H^2 - \frac{1}{2} \|u(T) - v(T)\|_H^2$$

606 and since $u(T) = u_T$ we find that

(B.4)

$$607 \quad \begin{cases} \int_0^T \langle -\dot{v}(t) + A(t)u(t) - f(t), v - u(t) \rangle_V \geq \frac{1}{2} \|u(0) - v(0)\|_H^2 - \frac{1}{2} \|u(T) - v(T)\|_H^2 \\ \text{for all } v \in W(0, T; K), \quad u(T) = u_T. \end{cases}$$

608 It can be proved that the two formulation (B.2) and (B.4) are equivalent (they have the
609 same set of solutions), and that they have at most one solution. The weak formulation
610 is as follows: find $u \in L^2(0, T; K) \cap C(0, T; H)$ such that

(B.5)

$$611 \quad \begin{cases} \int_0^T \langle -\dot{v}(t) + A(t)u(t) - f(t), v - u(t) \rangle_V \geq -\frac{1}{2} \|u(T) - v(T)\|_H^2 \\ \text{for all } v \in L^2(0, T; K), \quad u(T) = u_T. \end{cases}$$

612 Clearly a solution of the strong formulation (B.2) is solution of the weak one.

613 **PROPOSITION B.1** (Brézis [6]). *The following holds:*

614 (i) *Let $u_T \in K$ and $f \in L^2(0, T; V^*)$. Then the weak formulation (B.5) has a*
615 *unique solution u and, for some $c > 0$, given $v_0 \in K$:*

(B.6)

$$616 \quad \|u\|_{L^\infty(0, T; H)} + \|u\|_{L^2(0, T; V)} \leq c(\|u_T\|_H + \|f\|_{L^2(0, T; V^*)} + \|v_0\|_V).$$

617 (ii) *Let in addition the semi-symmetry hypothesis (A.8) hold, and let u_T belong*
618 *to K . Then $u \in L^\infty(0, T; V)$, $\dot{u} \in L^2(0, T; H)$, and u is the unique solution*
619 *of the original formulation (B.2). Furthermore, for some $c > 0$:*

(B.7)

$$620 \quad \|u\|_{L^\infty(0, T; V)} + \|\dot{u}\|_{L^2(0, T; H)} \leq c(\|u_T\|_V + \|f\|_{L^2(0, T; H)}).$$

621 **Appendix C. Monotonicity.** Assume that H is an Hilbert lattice, i.e., is
622 endowed with an order relation \succeq compatible with the vector space structure:

(C.1)

$$623 \quad x_1 \succeq x_2 \text{ implies that } \gamma x_1 + x \succeq \gamma x_2 + x, \text{ for all } \gamma \geq 0 \text{ and } x \in H,$$

624 such that the maxima and minima denoted by $\max(x_1, x_2)$ and $\min(x_1, x_2)$ are well
625 defined, the operator \max, \min be continuous, with $\min(x_1, x_2) = -\max(-x_1, -x_2)$.
626 Setting $x_+ := \max(x, 0)$ and $x_- := -\min(x, 0)$ we have that $x = x_+ - x_-$. Assuming
627 that the maximum of two elements of V belong to V we see that we have an induced
628 lattice structure on V . The induced dual order over V^* is as follows: for v_1^* and v_2^* in
629 V^* , we say that $v_1^* \geq v_2^*$ if $\langle v_1^* - v_2^*, v \rangle_V \geq 0$ whenever $v \geq 0$.

630 Assume that we have the following extension of the integration by parts formula
631 (B.3): for all u, v in $W(0, T)$ and $0 \leq t < t' \leq T$,

(C.2)

$$632 \quad 2 \int_t^{t'} \langle \dot{u}(s), u_+(s) \rangle_V ds = \|u_+(t')\|_H^2 - \|u_+(t)\|_H^2.$$

633 and that

(C.3)

$$634 \quad \langle A(t)u, u_+ \rangle_V = \langle A(t)u_+, u_+ \rangle_V.$$

635 **PROPOSITION C.1.** *Let u_i be solution of the parabolic equation (A.6) for $(f, u_T) =$*
636 *(f^i, u_T^i) , $i = 1, 2$. If $f^1 \geq f^2$ and $u_T^1 \geq u_T^2$, then $u_1 \geq u_2$.*

637 This type of result may be extended to the case of variational inequalities. If K
 638 and K' are two subsets of V , we say that K *dominates* K' if for any $u \in K$ and
 639 $u' \in K'$, $\max(u, u') \in K$ and $\min(u, u') \in K'$.

640 **PROPOSITION C.2.** *Let u_i be solution of the weak formulation (B.5) of the parabolic*
 641 *variational inequality for $(f, u_T, K) = (f^i, u_T^i, K^i)$, $i = 1, 2$. If $f_1 \geq f_2$, $u_T^1 \geq u_T^2$, and*
 642 *K^1 dominates K^2 , then $u_1 \geq u_2$.*

643 The monotonicity w.r.t. the convex K is due to Haugazeau [13] (in an elliptic
 644 setting, but the result is easily extended to the parabolic one). See also Brézis [7].

645 **Appendix D. Link with American options.** An American option is the
 646 right to get a payoff $\Psi(t, x)$ at any time $t < T$ and u_T at time T . We can motivate
 647 as follows the derivation of the associated variational inequalities. If the option can
 648 be exercised only at times $t_k = hk$, with $h = T/M$ and $k = 0$ to M (Bermudean
 649 option), then the same PDE as for the European option holds over (t_k, t_{k+1}) , $k = 0$
 650 to $M - 1$. Denoting by \tilde{u}_k the solution of this PDE, we have that $u(t_k) = \max(\Psi, \tilde{u}_k)$.
 651 Assuming that A does not depend on time and that there is a flux $f(t, x)$ of dividends,
 652 we compute the approximation u_k of $u(t_k)$ as follows. Discretizing the PDE with the
 653 implicit Euler scheme we obtain the continuation value \hat{u}_k solution of

$$654 \quad (D.1) \quad \frac{\hat{u}_k - u_{k+1}}{h} + A\hat{u}_k = f(t_k, \cdot), \quad k = 0, \dots, M - 1; \quad u_M = \max(\Psi, 0),$$

655 so that $u_k = u_{k+1} - hA\hat{u}_k + hf(t_k, \cdot)$, we find that

$$656 \quad (D.2) \quad u_k = \max(\hat{u}_k, \Psi) = \max(u_{k+1} - hA\hat{u}_k + hf(t_k, \cdot), \Psi),$$

657 which is equivalent to

$$658 \quad (D.3) \quad \min(u_k - \Psi, \frac{u_k - u_{k+1}}{h} + A\hat{u}_k - f(t_k, \cdot)) = 0.$$

659 This suggest for the continuous time model and general operators A and r.h.s. f the
 660 following formulation

$$661 \quad (D.4) \quad \min(u(t, x) - \Psi(x), -\dot{u}(t, x) + A(t, x)u(t, x) - f(t, x)) = 0, \quad (t, x) \in (0, T) \times \Omega.$$

662 The above equation has a rigorous mathematical sense in the context of viscosity
 663 solution, see Barles [5]. However we rather need the variational formulation which
 664 can be derived as follows. Let $v(x)$ satisfy $v(x) \geq \Psi(x)$ a.e., be smooth enough. Then

$$665 \quad (D.5) \quad \int_{\Omega} (-\dot{u}(t, x) + A(t, x)u(t, x) - f(t, x)) (v(x) - u(t, x)) dx = \\ \int_{\{u(t, x) = \Psi(x)\}} (-\dot{u}(t, x) + A(t, x)u(t, x) - f(t, x)) (v(x) - u(t, x)) dx \\ + \int_{\{u(t, x) > \Psi(x)\}} (-\dot{u}(t, x) + A(t, x)u(t, x) - f(t, x)) (v(x) - u(t, x)) dx.$$

666 The first integrand is nonnegative, being a product of nonnegative terms, and the
 667 second integrand is equal to 0 since by (D.3), $-\dot{u}(t, x) + A(t, x)u(t, x) - f(t, x) = 0$
 668 a.e. when $u(t, x) > \Psi(x)$. So we have that, for all $v \geq \Psi$ smooth enough:

$$669 \quad (D.6) \quad \int_{\Omega} (-\dot{u}(t, x) + A(t, x)u(t, x) - f(t, x)) (v(x) - u(t, x)) dx \geq 0.$$

670 We see that this is of the same nature as a parabolic variational inequality, where K
671 is the set of functions greater or equal to Ψ (in an appropriate Sobolev space).

672 **Appendix E. Some one dimensional problems.** It is not always easy to
673 characterize the space \mathcal{V} . Let us give a detailed analysis in a simple case.

674 **E.1. The Black-Scholes setting.** For the Black-Scholes model with zero inter-
675 est rate (the extension to a constant nonzero interest rate is easy) and unit volatility
676 coefficient, we have that $Au = -\frac{1}{2}x^2u''(x)$, with $x \in (0, \infty)$. In the case of a put
677 option: $u_T(x) = (K - x)_+$ we may take $H := L^2(\mathbb{R}_+)$. For $v \in \mathcal{D}(0, \infty)$ and u
678 sufficiently smooth we have that $-\frac{1}{2} \int_0^\infty x^2u''(x)dx = a(u, v)$ with

$$679 \quad (E.1) \quad a(u, v) := \frac{1}{2} \int_0^\infty x^2u'(x)v'(x)dx + \int_0^\infty xu'(x)v(x)dx.$$

680 This bilinear form a is continuous and semi coercive over the set

$$681 \quad (E.2) \quad V := \{u \in H; xu'(x) \in H\}.$$

682 It is easily checked that $\bar{u}(x) := x^{-1/3}/(1+x)$ belongs to V . So, some elements of V
683 are unbounded near zero.

684 We now claim that $\mathcal{D}(0, \infty)$ is a dense subset of V . First, it follows from a
685 standard truncation argument and the dominated convergence theorem that $V_\infty :=$
686 $V \cap L^\infty(0, \infty)$ is a dense subset of V . Note that elements of V are continuous over
687 $(0, \infty)$. Given $\varepsilon > 0$ and $u \in V_\infty$, define

$$688 \quad (E.3) \quad u_\varepsilon(x) := \begin{cases} 0 & \text{if } x \in (0, \varepsilon), \\ u(2\varepsilon)(x/\varepsilon - 1) & \text{if } x \in [\varepsilon, 2\varepsilon], \\ u(2\varepsilon) & \text{if } x > 2\varepsilon. \end{cases}$$

689 Obviously $u_\varepsilon \in V_\infty$. By the dominated convergence theorem, $u_\varepsilon \rightarrow u$ in H . Set for
690 $w \in V$

$$691 \quad (E.4) \quad \Phi_\varepsilon(w) := \int_0^{2\varepsilon} x^2w'(x)^2dx.$$

692 Since Φ_ε is quadratic and $v_\varepsilon \rightarrow u$ in H , we have that:

$$693 \quad (E.5) \quad \frac{1}{2} \int_0^\infty x^2(u'_\varepsilon - u')^2dx = \frac{1}{2}\Phi_\varepsilon(u_\varepsilon - u) \leq \Phi_\varepsilon(u_\varepsilon) + \Phi_\varepsilon(u).$$

694 Since $u \in V$, $\Phi_\varepsilon(u) \rightarrow 0$ and

$$695 \quad (E.6) \quad \Phi_\varepsilon(u_\varepsilon) \leq \|u\|_\infty^2 \int_0^{2\varepsilon} \varepsilon^{-2}x^2dx = O(\|u\|_\infty^2\varepsilon).$$

696 So, the l.h.s. of (E.5) has limit 0 when $\varepsilon \downarrow 0$. We have proved that the set V^0 of
697 functions in V^∞ equal to zero near zero, is a dense subset of V . Now define for $N > 0$

$$698 \quad (E.7) \quad \varphi_N(x) = \begin{cases} 1 & \text{if } x \in (0, N), \\ 1 - \log(x/N) & \text{if } x \in [N, eN], \\ 0 & \text{if } x > eN. \end{cases}$$

699 Given $u \in V_0$, set $u_N := u\varphi_N$. Then $u_N \in H$ and, by a dominated convergence
700 argument, $u_N \rightarrow u$ in H . The weak derivative of u_N is $u'_N = u'\varphi_N + u\varphi'_N$. By a

701 dominated convergence argument, $xu'\varphi_N \rightarrow xu'$ in $L^2(\mathbb{R}_+)$. It remains to prove that
 702 $xu'\varphi'_N \rightarrow 0$ in $L^2(\mathbb{R}_+)$. But φ'_N is equal to $1/x$ over its support, so that

$$703 \quad (\text{E.8}) \quad \|xu'\varphi'_N\|_{L^2(\mathbb{R}_+)}^2 = \int_N^{eN} u^2(x)dx \leq \int_N^\infty u^2(x)dx \rightarrow 0$$

704 when $N \uparrow \infty$. The claim is proved.

705 **E.2. The CIR setting.** In the Cox-Ingersoll-Ross model [8] the stochastic pro-
 706 cess satisfies

$$707 \quad (\text{E.9}) \quad ds(t) = \theta(\mu - s(t))dt + \sigma\sqrt{s}dW(t), \quad t \geq 0$$

708 We assume the coefficients θ , μ and σ to be constant and positive. The associated
 709 PDE is given by

$$710 \quad (\text{E.10}) \quad \begin{cases} Au := -\theta(\mu - x)u' - \frac{1}{2}x\hat{\sigma}^2u'' = 0 & (x, t) \in \mathbb{R}^+ \times (0, T), \\ u(x, T) = u_T(x) & x \in \mathbb{R}^+. \end{cases}$$

711 Again for the sake of simplicity we will take $\rho(x) = 1$, which is well-adapted in the
 712 case of a payoff with compact support in $(0, \infty)$. For $v \in \mathcal{D}(0, \infty)$ and u sufficiently
 713 smooth we have that $\int_0^\infty Au(x)v(x)dx = a(u, v)$ with

$$714 \quad (\text{E.11}) \quad a(u, v) := \theta \int_0^\infty (\mu - x)u'(x)v(x)dx + \frac{1}{2}\hat{\sigma}^2 \int_0^\infty xu'(x)v'(x)dx + \frac{1}{2}\hat{\sigma}^2 \int_0^\infty u'(x)v(x)dx.$$

715 So one should take \mathcal{V} of the form

$$716 \quad (\text{E.12}) \quad \mathcal{V} := \{u \in H; \sqrt{x}u'(x) \in L^2(\mathbb{R}_+)\}.$$

717 We next determine H by requiring that the bilinear form is continuous; by the Cauchy-
 718 Schwarz inequality

$$719 \quad (\text{E.13}) \quad \left| \int_0^\infty u'(x)v(x)dx \right| \leq \|x^{1/2}u'\|_2 \|x^{-1/2}v\|_2; \quad \left| \int_0^\infty xu'(x)v(x)dx \right| \leq \|x^{1/2}u'\|_2 \|x^{1/2}v\|_2.$$

720 We easily deduce that the bilinear form a is continuous and semi coercive over \mathcal{V} ,
 721 when choosing

$$722 \quad (\text{E.14}) \quad H := \{v \in L^2(\mathbb{R}_+); (x^{1/2} + x^{-1/2})v \in L^2(\mathbb{R}_+)\},$$

723 Note that then the integrals below are well defined and finite for any $v \in \mathcal{V}$:

$$724 \quad (\text{E.15}) \quad \int_0^\infty (x^{1/2}v')(x^{-1/2}v) = \int_0^\infty vv' = \frac{1}{2} \int_0^\infty (v^2)'.$$

725 So $w := v^2$ is the primitive of an integrable function and therefore has a limit at zero.
 726 Since v is continuous over $(0, \infty)$ it follows that v has a limit at zero.

727 However if this limit is nonzero we get a contradiction with the condition that
 728 $x^{-1/2}v \in L^2(\mathbb{R}_+)$. So, every element of \mathcal{V} has zero value at zero.

729 We now claim that $\mathcal{D}(0, \infty)$ is a dense subset of \mathcal{V} . First, $\mathcal{V}_\infty := \mathcal{V} \cap L^\infty(0, \infty)$ is
 730 a dense subset of \mathcal{V} . Note that elements of \mathcal{V} are continuous over $(0, \infty)$. Given $\varepsilon > 0$

731 and $u \in \mathcal{V}_\infty$, define $u_\varepsilon(x)$ as in (E.3). Then $u_\varepsilon \in \mathcal{V}_\infty$. By the dominated convergence
732 theorem, $u_\varepsilon \rightarrow u$ in H . Set for $w \in \mathcal{V}$

$$733 \quad (\text{E.16}) \quad \Phi_\varepsilon(w) := \int_0^{2\varepsilon} xw'(x)^2 dx.$$

734 Since Φ_ε is quadratic and $u_\varepsilon \rightarrow u$ in H , we have that:

$$735 \quad (\text{E.17}) \quad \frac{1}{2} \int_0^\infty x^2 (u'_\varepsilon - u')^2 dx = \frac{1}{2} \Phi_\varepsilon(u_\varepsilon - u) \leq \Phi_\varepsilon(u_\varepsilon) + \Phi_\varepsilon(u).$$

736 Since $u \in \mathcal{V}$, $\Phi_\varepsilon(u) \rightarrow 0$ and

$$737 \quad (\text{E.18}) \quad \Phi_\varepsilon(u_\varepsilon) \leq \varepsilon^{-2} u(2\varepsilon)^2 \int_0^{2\varepsilon} x dx = 2u(2\varepsilon)^2 \rightarrow 0.$$

738 So, the l.h.s. of (E.17) has limit 0 when $\varepsilon \downarrow 0$. We have proved that the set \mathcal{V}^0 of
739 functions in \mathcal{V}^∞ equal to zero near zero, is a dense subset of \mathcal{V} . Define φ_N as in (E.7)

740 Given $u \in \mathcal{V}_0$, set $u_N := u\varphi_N$. As before, $u_N \rightarrow u$ in H , is $u'_N = u'\varphi_N + u\varphi'_N$,
741 $xu'\varphi_N \rightarrow xu$ in $L^2(\mathbb{R}_+)$, and it remains to prove that $xu\varphi'_N \rightarrow 0$ in $L^2(\mathbb{R}_+)$. But φ'_N
742 is equal to $1/x$ over its support, so that when $N \uparrow \infty$:

$$743 \quad (\text{E.19}) \quad \|x^{1/2}u\varphi'_N\|_{L^2(\mathbb{R}_+)}^2 = \int_N^{eN} x^{-1}u^2(x) dx \leq \int_N^\infty u^2(x) dx \rightarrow 0.$$

744 The claim is proved.

745

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