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# Circular Dimensional-Permutations and Reliable Broadcasting for Hypercubes and Möbius cubes

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**Abstract.** Reliable broadcasting for interconnection networks can be achieved by constructing multiple independent spanning trees (ISTs) rooted at the same node. In this paper, we prove that there exists  $(n - 1)!$  sets of ISTs rooted at arbitrary node for  $Q_n$  and  $M_n$  based on circular dimensional-permutations of  $0, 1, \dots, n - 1$  and  $n \geq 1$ . At the same time, we give an parallel algorithm, called BCIST, which is the further study of IST problem for  $Q_n$  and  $M_n$  in literature. Furthermore, simulation experiments of ISTs based on JUNG framework and different sets of disjoint paths between node 1 and any node  $v \in V(0-M_4) \setminus \{1\}$  for  $0-M_4$  are also presented.

**Keywords:** dimensional-permutation, reliable broadcasting, hypercube, Möbius cube, independent spanning tree

## 1 Introduction

It is well known that hypercubes are widely used in parallel computing systems, which have many advantageous properties such as lower node degree and diameter, higher connectivity, symmetry, and etc [21], [24]. Furthermore, by changing their links between some nodes, the variants of hypercubes, such as Möbius cubes [9], crossed cubes [20], and twisted cubes [1] were proposed, which have better properties [12], [13], [14], [16], [29].

Independent spanning trees (ISTs for short) have been used in reliable broadcasting, secure message distribution [2], reliable communication protocols [17], one-to-all broadcasting [26], the multi-node broadcasting [3], and diagnosis [6]. Therefore, the problem to construct ISTs for a given network is becoming an important issue.

However, there is a well-known conjecture on the existence of ISTs for any network [17][31]:

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**Conjecture 1.** Let  $G$  be an  $n$ -node-connected network with  $n \geq 1$ . Then, there exist  $n$  node-independent spanning trees rooted at any node for  $G$ .

In what follows, we use independent to represent node-independent. For  $n \leq 4$ , Conjecture 1 was solved [8], [10], [17], [31], but when  $n \geq 5$ , it has remained open. Consequently, researchers are interested in the study of ISTs for various special networks. Conjecture 1 has been solved for some restricted classes of networks, such as planar networks [15], product networks [23], hypercubes [25], [28], [30], Möbius cubes [5], [6], locally twisted cubes [22], crossed cubes [4], [7], twisted cubes [27], even networks [18], odd networks [19], and etc.

We say that a sequence of  $n$  integers is a *permutation* if it contains all integers from 0 to  $n - 1$  exactly once. Considering the results for hypercubes and Möbius cubes, each paper in literature only considers a set of ISTs for the special network and lacks the discussion of the relation between the permutations and ISTs.

**Question 1.** Can all permutations of  $0, 1, \dots, n-1$  be used to construct spanning tree and ISTs for the  $n$ -dimensional hypercube  $Q_n$  and the  $n$ -dimensional Möbius cube  $M_n$ ?

To solve this question, we adopt the definition of circular dimensional-permutation and prove that any circular dimensional-permutation of  $0, 1, \dots, n - 1$  can be used to construct  $n$  ISTs for  $Q_n$  and  $M_n$  in this paper, which is the further discussion of spanning trees and ISTs for  $Q_n$  and  $M_n$  comparing with the results in literature.

The rest of this paper is organized as follows. Section 2 presents some definitions and graph terminologies and notations. Section 3 discusses the IST problem for  $Q_n$  and  $M_n$  rooted at any node. We draw the conclusion of the paper in the last section.

## 2 Preliminaries

### 2.1 Definition of Hypercubes, Möbius Cubes, and ISTs

We use a unique binary string of length  $n$  to denote the address of each node in the  $n$ -dimensional hypercube  $Q_n$  and the  $n$ -dimensional Möbius cube  $M_n$ . In what follows, nodes and their addresses will be used alternatively.  $Q_n$  is a network consists of  $2^n$  nodes. Any two nodes of  $Q_n$  are adjacent whenever their corresponding addresses differ in exactly one place.

$M_n$  is a variant of the  $Q_n$ , which has two types, *0-type*  $n$ -dimensional Möbius cube and *1-type*  $n$ -dimensional Möbius cube. We adopt the following definition of  $M_n$  in [11].

**Definition 1.** [11]  $0-M_1$  and  $1-M_1$  are both the complete graph on two nodes whose addresses are 0 and 1. For any integer  $n$  with  $n \geq 2$ , both  $0-M_n$  and  $1-M_n$  contain one 0-type  $(n - 1)$ -dimensional sub-Möbius cube  $M_{n-1}^0$  and one 1-type  $(n - 1)$ -dimensional sub-Möbius cube  $M_{n-1}^1$ . The nodes in  $M_{n-1}^0$  have a common prefix 0; the nodes in  $M_{n-1}^1$  have a common prefix 1. For two nodes  $x = x_{n-1}x_{n-2} \dots x_0 \in V(M_{n-1}^0)$  and  $y = y_{n-1}y_{n-2} \dots y_0 \in V(M_{n-1}^1)$ , where  $x_{n-1} = \overline{y_{n-1}} = 0$ ,

- (1)  $(x, y) \in E(0-M_n)$  if and only if  $x_i = y_i$ ,  $i = 0, 1, \dots, n-2$ ;  
 (2)  $(x, y) \in E(1-M_n)$  if and only if  $x_i = \bar{y}_i$ ,  $i = 0, 1, \dots, n-2$ .

A binary string  $x$  of length  $n$  is denoted by  $x_{n-1}x_{n-2}\dots x_0$ . Suppose that  $u = u_{n-1}u_{n-2}\dots u_0$  and  $v = u_{n-1}u_{n-2}\dots u_l\bar{u}_{l-1}v_{l-2}v_{l-3}\dots v_0$  are two nodes in  $X_n \in \{Q_n, M_n\}$ . We say that  $u$  and  $v$  have a *leftmost differing bit at position*  $l-1$ . We use  $\text{LDF}(u, v)$  to denote the leftmost differing bit of two nodes  $u$  and  $v$ . Given two adjacent nodes  $u$  and  $v$ , if  $\text{LDF}(u, v)=d$ , we say that  $v$  is the *d-neighbor of u* or that the edge  $(u, v)$  is an *edge of dimension d*. For this purpose, let  $N_d(u)$  denote the *d-neighbor of u*. We follow the definitions of *path* and *ancestor* in [7].

Two paths  $P$  and  $P'$  starting from a node  $u$  and ending with another node  $v$  are said to be *internally disjoint* if  $E(P) \cap E(P') = \emptyset$  and  $V(P) \cap V(P') = \{u, v\}$ . Two spanning trees for a network  $G$  are *independent* if they are rooted at the same node, said  $u$ , and for each node  $v \in V(G) \setminus \{u\}$ , the two paths starting at  $u$  and ending with  $v$  are internally disjoint. A set of spanning trees of  $G$  rooted at  $v$  are called *independent spanning trees* if they are pairwise independent.

## 2.2 Definition of Dimensional-Permutation

The sequence of  $n$  integers is called a *dimensional-permutation* if it contains all integers from 0 to  $n-1$  exactly one (Noting that each node in  $Q_n$  or  $M_n$  has  $n$  neighbors, which are 0-neighbor, 1-neighbor,  $\dots$ ,  $(n-1)$ -neighbor). A *circular dimensional-permutation* (CDP for short) is a type of permutation to put all integers from 0 to  $n-1$  along a closed circle in the clockwise order [6]. Suppose that  $\{a_0, a_1, \dots, a_{n-1}\} = \{0, 1, \dots, n-1\}$ . All cyclic permutations of integers are equivalent in the circle,

$$\begin{aligned} a_0 &\rightarrow a_1 \rightarrow \dots \rightarrow a_{n-1}, \\ a_1 &\rightarrow a_2 \rightarrow \dots \rightarrow a_{n-1} \rightarrow a_0, \\ &\dots, \\ a_{n-1} &\rightarrow a_0 \rightarrow a_1 \rightarrow \dots \rightarrow a_{n-2} \end{aligned}$$

belong to the same CDP. The total number of CDPs is  $n!/n = (n-1)!$ . For example, the six CDPs of 0, 1, 2, 3 are

$$\begin{aligned} 3 &\rightarrow 0 \rightarrow 1 \rightarrow 2, \\ 3 &\rightarrow 0 \rightarrow 2 \rightarrow 1, \\ 3 &\rightarrow 1 \rightarrow 0 \rightarrow 2, \\ 3 &\rightarrow 1 \rightarrow 2 \rightarrow 0, \\ 3 &\rightarrow 2 \rightarrow 0 \rightarrow 1, \\ 3 &\rightarrow 2 \rightarrow 1 \rightarrow 0. \end{aligned}$$

## 3 An Reliable Broadcasting Algorithm Based on ISTs for Hypercubes and Möbius Cubes

In this section, we point out that every CDP of  $n$  integers 0, 1, 2,  $\dots$ ,  $n-1$  can be used to construct ISTs for  $X_n \in \{Q_n, M_n\}$ . We now present the following observation.

**Observation 1.** For the  $n$ -dimensional Möbius cube, we proved the correctness of ISTs rooted at any node based on the descending CDP  $n-1, n-2, \dots, 0$  [6]. In essence, the set of optimal ISTs for  $Q_n$  in [25] can be obtained by the ascending CDP  $0, 1, \dots, n-1$ ; the set of ISTs in [30] is similar to that in [28] for  $Q_n$ , which can be constructed by the descending CDP  $n-1, n-2, \dots, 0$ . Thus, the result in this section is the further study of spanning trees and ISTs for  $Q_n$  and  $M_n$ .

### 3.1 ISTs for $Q_n$ and $M_n$ with any Circular Dimensional-Permutation

In what follows, we always let  $u$  denote any node in  $Q_n$  or  $M_n$ . Now we present an algorithm, called BCIST, to construct  $n$  ISTs rooted at an arbitrary node  $u$  for  $X_n \in \{Q_n, M_n\}$ . Fig. 1 demonstrates the construction procedures of  $n$  spanning trees  $T_0, T_1, \dots, T_{n-1}$  rooted at node  $u$  for  $X_n \in \{Q_n, M_n\}$  in radial style.

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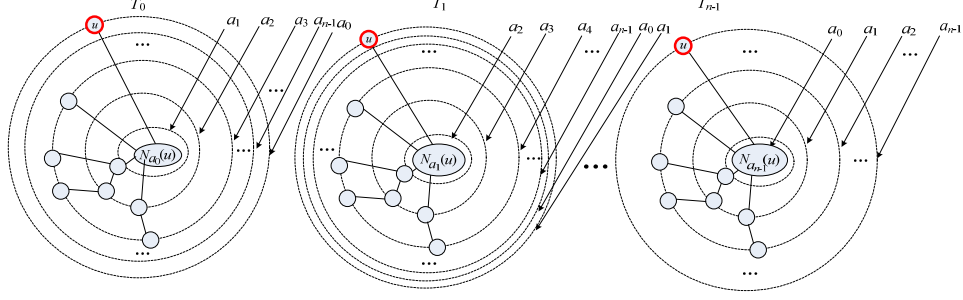
Algorithm BCIST

Input: An array  $S = \{a_0, a_1, \dots, a_{n-1}\}$ , where  
the permutation  $a_0, a_1, \dots, a_{n-1}$  is a CDP of  $0, 1, \dots, n-1$ ;  
an arbitrary node  $u$  in  $X_n \in \{Q_n, M_n\}$ ;  
Output:  $T_0, T_1, \dots, T_{n-1}$  rooted at  $u$  for  $X_n$ ;  
Begin  
1: for  $i = 0$  to  $n-1$  do in parallel  
2:  $V(T_i) = N_{a_i}(u)$ ;  
/\* $N_{a_i}(u)$  denotes the  $a_i$ -neighbor of  $u$ .\*/  
3:  $E(T_i) = \emptyset$ ;  
4: for  $l = 0$  to  $n-1$  do  
5: for each node  $v \in V(T_i)$  do in parallel  
6:  $d = S[(i+l+1) \bmod n]$ ;  
/\*Indexing of  $S$  is counted from 0 to  $n-1$ ,  
where  $n$  is length of the array  $S$ .\*/  
7:  $E(T_i) = E(T_i) \cup \{(v, N_d(v))\}$ ;  
8:  $V(T_i) = V(T_i) \cup \{N_d(v)\}$ ;  
9: end for  
10: end for  
11: end for  
end

---

The construction procedures of  $T_0, T_1, \dots, T_{n-1}$  are similar. Take  $T_0$  for example, the construction procedures are described as follows (See Fig. 1).

At first, there is only one node  $N_{a_0}(u)$  in tree  $T_0$ ; during the 1st iteration ( $l=0$ ), node  $N_{a_0}(u)$  in  $T_0$  is connected to its  $a_1$ -neighbor node  $N_{a_1}(N_{a_0}(u))$ . Therefore,  $V(T_0) = \{N_{a_0}(u), N_{a_1}(N_{a_0}(u))\}$  and  $E(T_0) = \{(N_{a_0}(u), N_{a_1}(N_{a_0}(u)))\}$ ; during the 2nd iteration ( $l=1$ ), each node  $v$  in  $T_0$  is connected to its  $a_2$ -neighbor node  $N_{a_2}(v)$ . Thus, the edges  $(N_{a_1}(N_{a_0}(u)), N_{a_2}(N_{a_1}(N_{a_0}(u))))$  and



**Fig. 1.** Construction procedures based on algorithm BCIST.

$(N_{a_0}(u), N_{a_2}(N_{a_0}(u)))$  are appended to  $T_0$  and  $V(T_0) = \{N_{a_0}(u), N_{a_2}(N_{a_0}(u)), N_{a_1}(N_{a_0}(u)), N_{a_2}(N_{a_1}(N_{a_0}(u)))\}$ ; during the 3rd iteration, each node  $v$  in  $T_0$  is connected to its  $a_3$ -neighbor node  $N_{a_3}(v)$ . As a result, it has doubled the number of nodes in  $T_0$ ; During the  $l$ -th iteration, each node  $v$  in  $T_0$  is connected to its  $S[l+1]$ -neighbor node  $N_{S[l+1]}(v)$  with  $4 \leq l \leq n-2$ ; in the last iteration, each node  $v$  in  $T_0$  is connected to its  $a_0$ -neighbor node  $N_{a_0}(v)$ .

Consequently,  $T_0$  is a spanning tree for  $X_n \in \{Q_n, M_n\}$ .

More examples will be shown in the next subsection. Now we give the following lemma about the relation of adjacent nodes in  $M_n$  and  $Q_n$ .

**Lemma 1.** For any node  $x_{n-1}x_{n-2} \dots x_0$  and its  $k$ -neighbor node  $y_{n-1}y_{n-2} \dots y_0$  with  $0 \leq k \leq n-1$  in  $X_n \in \{M_n, Q_n\}$ , we have  $x_{n-1}x_{n-2} \dots x_{k+1} = y_{n-1}y_{n-2} \dots y_{k+1}$  and  $x_k \neq y_k$ .

Based on Definition 1, the definition of  $Q_n$ , and Lemma 1, we have the following lemma.

**Lemma 2.** For any two nodes  $x, y$  in  $V(X_{n-1}^{x_{n-1}})$ , if  $\text{LDF}(x, y) = k$  with  $0 \leq k \leq n-2$  and  $X_{n-1}^{x_{n-1}} \in \{M_{n-1}^{x_{n-1}}, Q_{n-1}^{x_{n-1}}\}$ , then  $\text{LDF}(N_{n-1}(x), N_{n-1}(y)) = k$  and  $N_{n-1}(x), N_{n-1}(y) \in X_{n-1}^{x_{n-1}}$ .

**Lemma 3.** [5] Given a walk  $W: u^{(0)}, u^{(1)} = N_{m_1}(u^{(0)}), u^{(2)} = N_{m_2}(u^{(1)}), \dots, u^{(k)} = N_{m_k}(u^{(k-1)})$  in  $M_n$  for any integer  $k$  with  $1 \leq k \leq n$ , if  $m_1, m_2, \dots, m_k$  differ from one another and  $0 \leq m_i \leq n-1$  for  $i = 1, 2, \dots, k$ , then  $W$  is a path.

For the convenience of proof, we define a vector  $\langle \beta_1, \beta_2, \dots, \beta_n \rangle$  such that the set  $\{\beta_1, \beta_2, \dots, \beta_n\}$  equals to the set  $\{0, 1, \dots, n-1\}$ .

**Lemma 4.** Suppose that  $P: u^{(0)}, u^{(1)} = N_{a_1}(u^{(0)}), u^{(2)} = N_{a_2}(u^{(1)}), \dots, u^{(k)} = N_{a_k}(u^{(k-1)})$  and  $P': u^{(0)}, v^{(1)} = N_{a_1'}(u^{(0)}), v^{(2)} = N_{a_2'}(v^{(1)}), \dots, v^{(t)} = N_{a_t'}(v^{(t-1)})$  are two paths in  $X_n \in \{Q_n, M_n\}$  for any two integers  $k, t$  with  $1 \leq k, t \leq n$  and  $a_1 \neq a_1'$ . If the following conditions hold:

- (1)  $\langle \beta_{i_1}, \beta_{i_2}, \dots, \beta_{i_k} \rangle = \langle a_1, a_2, \dots, a_k \rangle$ , where  $1 \leq i_1 < i_2 < \dots < i_k \leq n$ ;
  - (2)  $\langle \beta_{j_1}, \beta_{j_2}, \dots, \beta_{j_t} \rangle = \langle a_1', a_2', \dots, a_t' \rangle$ , where  $1 \leq j_1 < j_2 < \dots < j_t \leq n$ ,
- then  $V(\text{path}(P, u^{(1)}, u^{(k)})) \cap V(\text{path}(P', v^{(1)}, v^{(t)})) = \emptyset$ .

**Proof.** Suppose that there exists a node  $v$ , such that  $v \in V(P) \cap V(P')$ . We denote  $v$  as  $v = u^{(i)}$  and  $v = v^{(j)}$  with  $1 \leq i \leq k$  and  $1 \leq j \leq t$ . Let  $A =$

$\{a_1, a_2, \dots, a_i\}$  and  $B = \{a_1', a_2', \dots, a_j'\}$ . We have  $M_1 = \max((A \cup B) \setminus (A \cap B))$ . By Lemma 2, we can verify that the  $M_1$ -bit of  $v$  in  $P$  is different from that of  $v$  in  $P'$ , which is a contradiction.  $\square$

Based on Lemma 4, we have the following corollary.

**Corollary 1.** Suppose that  $P: u^{(0)}, u^{(1)} = N_{a_1}(u^{(0)}), u^{(2)} = N_{a_2}(u^{(1)}), \dots, u^{(k)} = N_{a_k}(u^{(k-1)}) = v$  and  $P': v^{(0)}, v^{(1)} = N_{a_1'}(v^{(0)}), v^{(2)} = N_{a_2'}(v^{(1)}), \dots, v^{(t)} = N_{a_t'}(v^{(t-1)}) = v$  are two paths in  $X_n \in \{Q_n, M_n\}$  for any two integers  $k, t$  with  $1 \leq k, t \leq n$  and  $a_k \neq a_t'$ . If the following conditions hold:

- (1)  $\langle \beta_{i_1}, \beta_{i_2}, \dots, \beta_{i_k} \rangle = \langle a_1, a_2, \dots, a_k \rangle$ , where  $1 \leq i_1 < i_2 < \dots < i_k \leq n$ ;
  - (2)  $\langle \beta_{j_1}, \beta_{j_2}, \dots, \beta_{j_t} \rangle = \langle a_1', a_2', \dots, a_t' \rangle$ , where  $1 \leq j_1 < j_2 < \dots < j_t \leq n$ ,
- then  $V(\text{path}(P, u^{(0)}, u^{(k)})) \cap V(\text{path}(P', v^{(0)}, v^{(t)})) = \{v\}$ .

**Lemma 5.** Let the input  $a_0, a_1, \dots, a_{n-1}$  of algorithm BCIST be any dimensional-permutation of integers  $0, 1, \dots, n-1$  and  $X_n \in \{Q_n, M_n\}$ .  $T_i$  obtained by Algorithm BCIST is a spanning tree for  $X_n$  with integer  $i = 0, 1, \dots, n-1$ .

**Proof.** Without loss of generality, we consider the tree  $T_0$  obtained by algorithm BCIST. After the  $n$  iterations, we have  $1 + 2^0 + 2^1 + \dots + 2^{n-1} = 2^n$  nodes in tree  $T_0$ . Choosing arbitrary two nodes  $v^{(1)}$  and  $v^{(2)}$  from  $T_0$ , the  $\langle N_{a_0}(u), v^{(1)} \rangle$ -path can be denoted by  $N_{a_0}(u), x^{(1)} = N_{a_{i_1}}(N_{a_0}(u)), x^{(2)} = N_{a_{i_2}}(x^{(1)}), \dots, x^{(k)} = N_{a_{i_k}}(x^{(k-1)})$  and the  $\langle N_{a_0}(u), v^{(2)} \rangle$ -path can be denoted by  $N_{a_0}(u), x^{(1)} = N_{a_{j_1}}(N_{a_0}(u)), x^{(2)} = N_{a_{j_2}}(x^{(1)}), \dots, x^{(m)} = N_{a_{j_m}}(x^{(m-1)})$ , where  $0 \leq i_1 < i_2 < \dots < i_k \leq n-1$  and  $0 \leq j_1 < j_2 < \dots < j_m \leq n-1$ . By algorithm BCIST, it is easy to verify  $\langle N_{a_0}(u), v^{(1)} \rangle$ -path and  $\langle N_{a_0}(u), v^{(2)} \rangle$ -path satisfy the conditions in Lemma 4, which implies that  $v^{(1)} \neq v^{(2)}$ .

Thus, we can say that  $T_0$  is a spanning tree rooted at  $N_{a_0}(u)$  for  $X_n \in \{Q_n, M_n\}$ . Noting that  $u$  is the child of node  $N_{a_0}(u)$  and the leaf node in  $T_0$ , that is,  $T_0$  is a spanning tree rooted at  $u$ .  $\square$

**Lemma 6.** [4] Let  $T$  and  $T'$  be two spanning trees rooted at node  $u$  for a network  $G$ .  $T$  and  $T'$  are independent if and only if for every node  $v \in V(G) \setminus \{u\}$ , ancestor  $(v, T) \cap \text{ancestor}(v, T') = \{u\}$  and ancestor  $(v, T) \cup \text{ancestor}(v, T') \supset \{u\}$ .

**Lemma 7.**  $T_0, T_1, \dots, T_{n-1}$  obtained by Algorithm BCIST are  $n$  ISTs for  $X_n \in \{Q_n, M_n\}$ .

**Proof.** We have the following two cases.

**Case 1.**  $X_n$  is  $M_n$ . By Lemma 5,  $T_i$  obtained by Algorithm BCIST is a spanning tree for  $V(M_n)$  for integer  $i = 0, 1, \dots, n-1$ . We only need to prove that  $T_i$  and  $T_j$  are independent for  $0 \leq i < j \leq n-1$ .

The trivial cases is  $n = 1$  and  $n = 2$ . Now we consider  $n$  with  $n \geq 3$ . The longest path  $P_1$  in  $T_i$  and the longest path  $P_2$  in  $T_j$  can be denoted by

$P_1: u, x_0 = N_{a_i}(u), x_1 = N_{a_{i+1}}(x_0), \dots, x_{n-i-1} = N_{a_{n-1}}(x_{n-i-2}), x_{n-i} = N_{a_0}(x_{n-i-1}), x_{n-i+1} = N_{a_1}(x_{n-i}), \dots, x_n = N_{a_i}(x_{n-1})$  and

$P_2: u, y_0 = N_{a_j}(u), y_1 = N_{a_{j+1}}(y_0), \dots, y_{n-j-1} = N_{a_{n-1}}(y_{n-j-2}), y_{n-j} = N_{a_0}(y_{n-j-1}), y_{n-j+1} = N_{a_1}(y_{n-j}), \dots, y_n = N_{a_j}(y_{n-1}),$

respectively, where  $0 \leq i < j \leq n-1$ .

Let  $a_{c_0} = a_i$  and  $a_{d_0} = a_j$ . By lemma 6, we only need to prove that for any  $v \in V(T_i) \cap V(T_j)$ ,  $\text{ancestor}(v, T_i) \cap \text{ancestor}(v, T_j) = \{u\}$  and  $\text{ancestor}(v, T_i) \cup \text{ancestor}(v, T_j) \supset \{u\}$ . Any path in  $T_i$  and any path in  $T_j$  can be denoted by  $P_3$  and  $P_4$ , respectively, as follows.

$P_3$ :  $u, x_0 = N_{a_{c_0}=a_i}(u), x_1' = N_{a_{c_1}(x_i)}, \dots, x_k' = N_{a_{c_k}(x_{k-1}')} , x_{k+1}' = N_{a_{c_0}(x_k')}$  and

$P_4$ :  $u, y_0 = N_{a_{d_0}=a_j}(u), y_1' = N_{a_{d_1}(y_i)}, \dots, y_l' = N_{a_{d_l}(y_{l-1}')} , y_{l+1}' = N_{a_{d_0}(y_l')}$ ,

Without loss of generality, suppose that  $a_j > a_i$  and  $v \in V(P_3) \cap V(P_4)$ . Let  $v = a_{c_w} = a_{d_z}$  where  $1 \leq u \leq k$  and  $1 \leq z \leq m$ . Based on  $P_1, P_2, P_3$ , and  $P_4$ , we define walks  $W_1, W_2, W_3$ , and  $W_4$  as follows.

$W_1$ :  $a_i, a_{i+1}, \dots, a_{n-1}, a_0, a_1, \dots, a_i$ ,

$W_2$ :  $a_j, a_{j+1}, \dots, a_{n-1}, a_0, a_1, \dots, a_j$ ,

$W_3$ :  $a_{c_0} = a_i, a_{c_1}, \dots, a_{c_w}$ , and

$W_4$ :  $a_{d_0} = a_j, a_{d_1}, \dots, a_{d_z}$ .

Since  $a_{c_0} \neq a_{d_0}$ , we have  $x_0 \neq y_0$ , which implies that  $\text{ancestor}(v, P_3) \cup \text{ancestor}(v, P_4) \supset \{u\}$ . We only need to prove that  $\text{ancestor}(v, P_3) \cap \text{ancestor}(v, P_4) = \{u\}$ . We have the following Cases.

**Case 1.1.**  $\max(V(W_3)) \neq \max(V(W_4))$ . Since  $\max(V(W_4)) \geq a_j$ , then we have the following subcases.

**Case 1.1.1.**  $\max(V(W_3)) < a_j$  and  $\max(V(W_4)) = a_j$ . Then, each node in  $V(P_4) \setminus \{y_{m+1}'\}$  and each node in  $V(P_3)$  have a leftmost different bit at position  $a_j$ . Then we have  $v = y_{m+1}'$  and  $\text{ancestor}(v, P_3) \cap \text{ancestor}(v, P_4) = \{u\}$ .

**Case 1.1.2.**  $\max(V(W_3 \cup W_4)) > a_j$ . Then, the  $\max(V(W_3 \cup W_4))$ -bit of  $v$  in  $P_3$  is different from that of  $v$  in  $P_4$ . It is a contradiction.

**Case 1.2.**  $\max(V(W_3)) = \max(V(W_4))$ . Then, we have the following cases.

**Case 1.2.1.**  $\max(V(W_3)) = \max(V(W_4)) = a_j$ . We can verify that  $v \neq y_{m+1}'$  and  $a_j \notin \{a_{d_1}, a_{d_2}, \dots, a_{d_m}\}$ . We can divide path  $\text{ancestor}(v, P_3)$  into  $P_{31}$  and  $P_{32}$  as follows.

$P_{31}$ :  $u, x_0 = N_{a_{c_0}}(u), x_1' = N_{a_{c_1}}(x_0), \dots, x_{f-1}' = N_{a_{c_{f-1}}}(x_{f-2}')$  and

$P_{32}$ :  $x_f' = N_{a_{c_f}=a_j}(x_{f-1}'), x_{f+1}' = N_{a_{c_{f+1}}}(x_f'), \dots, x_w' = N_{a_{c_w}}(x_{w-1}')$ .

By Lemma 2, each node in  $V(P_{31})$  and each node in  $V(P_4)$  have a leftmost different bit at position  $a_j$ . Then,  $V(P_4) \cap V(P_{31}) = \emptyset$ . By Lemma 2,  $\text{LDF}(x_f', y_0) = \max(\{a_{c_0} = a_i, a_{c_1}, \dots, a_{c_{f-1}}\})$ . Let  $A = \{a_{c_{f+1}}, a_{c_{f+2}}, \dots, a_{c_w}\}$  and  $B = \{a_{d_1}, a_{d_2}, \dots, a_{d_z}\}$ . Furthermore, we have the following cases.

**Case 1.2.1.1.**  $\max((A \cup B) \setminus (A \cap B)) > \text{LDF}(x_f', y_0)$ . Then, by Lemma 2, the  $\max((A \cup B) \setminus (A \cap B))$ -bit of  $v$  in  $P_3$  is different from that of  $v$  in  $P_4$ , which is a contradiction.

**Case 1.2.1.2.**  $\max((A \cup B) \setminus (A \cap B)) = \text{LDF}(x_f', y_0)$ . By Algorithm BCIST,  $T_i$  and  $T_j$  are constructed based on the same CDP. Then, we can verify that  $a_{c_w} \neq a_{d_z}$ . By Corollary 1,  $V(P_4) \cap V(P_{32}) = \{v\}$ .

**Case 1.2.1.3.**  $\max((A \cup B) \setminus (A \cap B)) < \text{LDF}(x_f', y_0)$ . Then, by Lemma 2, the  $\text{LDF}(x_f', y_0)$  bit of  $v$  in  $P_3$  is different from that of  $v$  in  $P_4$ , which is a contradiction.



**Case 1.2.2.**  $\max(V(W_3)) = \max(V(W_4)) = M_1 > a_j$ . We can divide path sub-path  $\langle u, v \rangle$ -path of  $P_3$  into  $P_{31}$  and  $P_{32}$  as follows.

$$P_{31} : u, x_0 = N_{a_{c_0}}(u), x_1' = N_{a_{c_1}}(x_0), \dots, x_{f-1}' = N_{a_{c_{f-1}}}(x_{f-2}')$$

$$P_{32} : x_f' = N_{a_{c_f=M_1}}(x_{f-1}'), x_{f+1}' = N_{a_{c_{f+1}}}(x_f'), \dots, x_w' = N_{a_{c_w}}(x_{u-1}').$$

We can divide path sub-path  $\langle u, v \rangle$ -path of  $P_4$  into  $P_{41}$  and  $P_{42}$  as follows.

$$P_{41} : u, y_j = N_{a_{d_0=a_j}}(u), y_1' = N_{a_{d_1}}(y_j), \dots, y_{h-1}' = N_{a_{d_{h-1}}}(y_{h-2}'),$$

$$P_{42} : y_h' = N_{a_{d_h}}(y_{h-1}'), y_{h+1}' = N_{a_{d_{h+1}}}(y_h'), \dots, y_z' = N_{a_z}(y_{z-1}'),$$

By Lemma 2,  $V(P_{31}) \cap V(P_{41}) = \{u\}$ . Clearly, the  $M_1$ -bit of each node in  $V(P_{42})$  is different from each node in  $V(P_{31})$ , thus  $V(P_{42}) \cap V(P_{31}) = \emptyset$ . Similarly, we have  $V(P_{32}) \cap V(P_{41}) = \emptyset$ . We only need to prove that  $V(P_{32}) \cap V(P_{42}) = \emptyset$ . Similarly to Case 1.2.1, we can verify that  $a_{c_w} \neq b_{d_z}$ . By Corollary 1,  $V(P_{32}) \cap V(P_{42}) = \{v\}$ .

As a consequence, we have ancestor  $(v, P_3) \cap \text{ancestor}(v, P_4) = \{u\}$ .

**Case 2.**  $X_n$  is  $Q_n$ . The proof is similar to that of Case 1.

Based on the above discussion, by Lemma 6, the lemma holds.  $\square$

Since there are  $(n-1)!$  CDPs of  $0, 1, \dots, n-1$ , based on Lemma 7, we have the following theorem.

**Theorem 1.** Based on Algorithm BCIST, there are  $(n-1)!$  sets of ISTs for  $X_n \in \{Q_n, M_n\}$ .

Comparing with the result in [25], **all the  $(n-1)!$  sets of ISTs can provide optimal reliable broadcasting for  $Q_n$** . As far as the symmetry is concerned, **all the  $(n-1)!$  sets of ISTs can also provide optimal reliable broadcasting for  $M_n$** .

### 3.2 Simulation Experiments of ISTs and Disjoint Paths for $0-M_4$

As it is well-known that JUNG framework is a software library which provides a common and extendible language for the modeling, analysis, and visualization of data that can be represented as a network. Using Java multi-thread and JUNG framework, we can easily construct ISTs for  $Q_n$  and  $M_n$ . For example, Fig. 2 illustrates some construction procedures of four ISTs rooted at 1 for  $0-M_4$ . The four trees can be obtained by four steps with parallel fashion.

Based on different CDPs, we can obtain multiple sets of ISTs. Thus, there are different sets of disjoint paths between arbitrary two nodes. Fig. 3(a)–(f) show six sets of ISTs rooted at 1 for  $0-M_4$  based on the six CDPs mentioned in Section 2.2. Clearly, there are six sets of  $n$  disjoint paths between node 1 and any node  $v \in V(0-M_4) \setminus \{1\}$ , among which the total length of four disjoint paths between node 1 and node  $v$  may be different. For example, the four disjoint paths between nodes 1 and 13 in Fig. 3(a) are

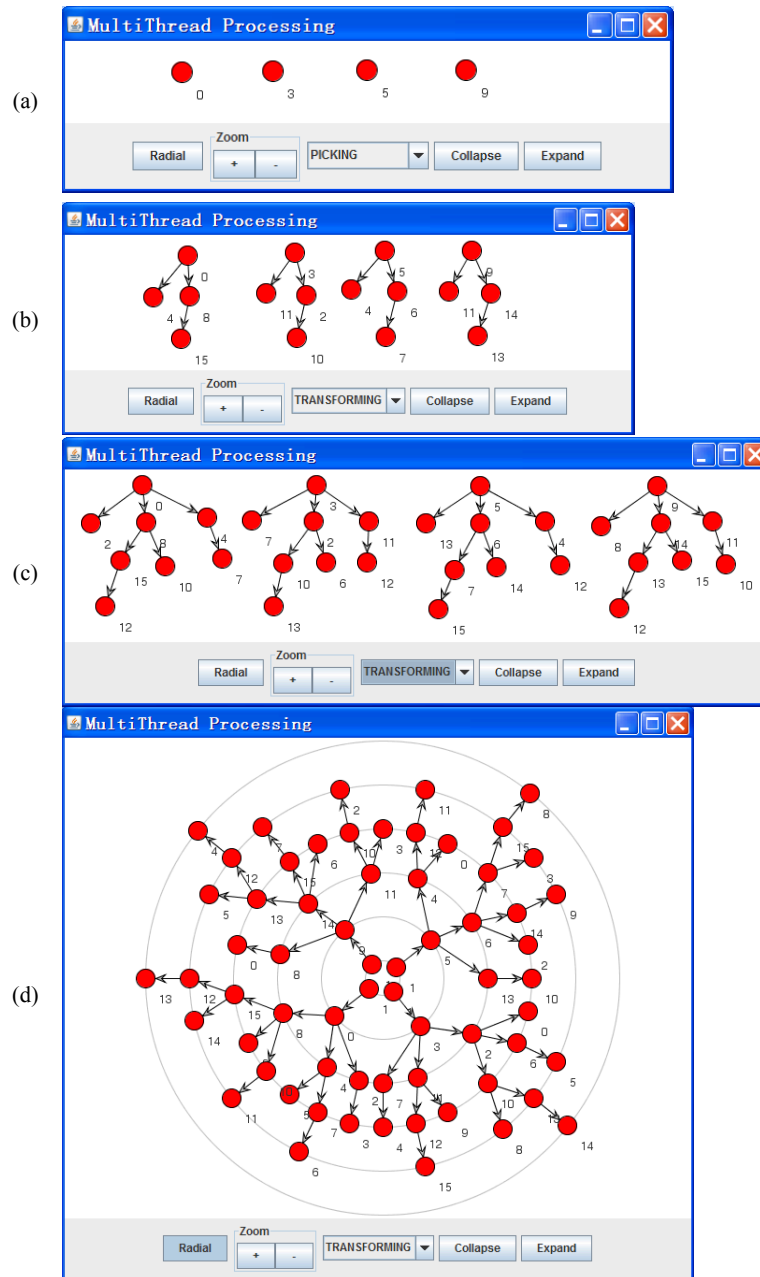
$$1 \rightarrow 0 \rightarrow 8 \rightarrow 15 \rightarrow 12 \rightarrow 13,$$

$$1 \rightarrow 3 \rightarrow 2 \rightarrow 10 \rightarrow 13,$$

$$1 \rightarrow 5 \rightarrow 13, \text{ and}$$

$$1 \rightarrow 9 \rightarrow 14 \rightarrow 13.$$

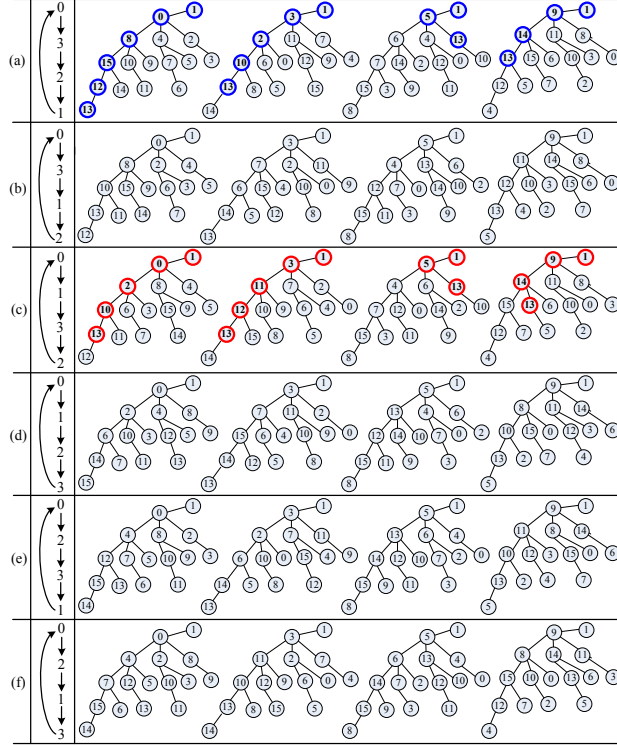
The four disjoint paths between nodes 1 and 13 in Fig. 3(c) are



**Fig. 2.** Simulation experiments of ISTs based on JUNG framework. (a) Initialization; (b) The 2nd iteration; (c) The 3rd iteration; (d) The 4th iteration.

- 1 → 0 → 2 → 10 → 13,
- 1 → 3 → 11 → 12 → 13,
- 1 → 5 → 13, and
- 1 → 9 → 14 → 13.

The total length of four disjoint paths between nodes 1 and 13 in Fig. 3(c) is shorter than that in Fig. 3(a). Thus, basing on CDPs, we may easily obtain a set of  $n$  disjoint paths with better performance between some pairs of nodes.



**Fig. 3.** Example of ISTs rooted at node 1 and disjoint paths between nodes 1 and 13 in  $0-M_4$ .

## 4 Conclusions

In this paper, we study the circular dimensional-permutations and reliable broadcasting based on ISTs for the  $n$ -dimensional hypercube  $Q_n$  and the  $n$ -dimensional Möbius cube  $M_n$ . We prove that any circular dimensional-permutation of  $0, 1, \dots, n - 1$  can be used to construct  $n$  ISTs, which is the further discussion of IST problem for  $Q_n$  and  $M_n$  comparing with the results in literature. Moreover,

we also conduct simulation experiments of ISTs based on JUNG framework and list some disjoint paths for  $0-M_4$ .

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