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# Scaffolding skeletons using spherical Voronoi diagrams

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## Abstract

Given a skeleton made of line segments we describe how to obtain a coarse mesh (or scaffold) of a surface surrounding it. We emphasize in this abstract the key result that allows us to complete the approach in [3] that could not treat skeletons with loops.

*Keywords:* mesh generation, skeletal models, 3D modeling, Voronoi diagrams, scaffold.

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## 1 Introduction

Skeletons are used in 3D graphics for modeling and animating articulated shapes. The user can design a complex shape by sketching a simple and intuitive geometric object that is the input to a surface generating algorithm. In the context of this paper a skeleton is a set of spatial segments that do not intersect except at the endpoints, then called joints. It is used to mathematically define a surface around the skeleton: by convolution, by offset, or

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other kind of distance surface [13]. Once a surface is properly defined it must be translated into an alternative representation either for visualization (mesh) or for local editing (NURBs, splines, etc). The issue is that usually the new representation is not related to the original skeleton and so the user loses control over the model. To overcome this a skeleton-based mesh generation method is needed. The idea developed here is to construct a “coarse” mesh around the skeleton, as regular as possible, and resembling the structure of the skeleton. This mesh is then used to generate the representation of the mathematical surface. Following the terminology of [11] we call this process *scaffolding* and the corresponding coarse mesh a *scaffold*.

One of the proposed ideas for the mesh reconstruction was the concept of B-meshes [9], it creates a rectangular cage around each piece of skeleton that are stitched together to make a closed mesh. The main drawback of this method is that the stitching process could lead to irregularities at the joints. Recently, Bærentzen et al. [3] proposed a method called SQM (Skeleton to Quad-dominant polygonal Mesh) going in the opposite direction: they first define the mesh points around the joints and then recreate a quad based “tubular” polyhedral surface around each segment. SQM consists of two steps:

- (i) **Partition** of the sphere at the joints into cells, one for each incident segment; each cell is defined by an ordered set of points on the sphere.
- (ii) Create **sleeves**, i.e. link the vertices of the cells at the extremities of each segment.

For the creation of the sleeves the extremal cells must be compatible, that is they must have the same number of vertices. The partition of the sphere in Bærentzen et al. [3] can be recognized to be a Voronoi diagram on the sphere. In order to match the cells they propose an algorithm for inserting additional vertices on the boundary of the cells in such a way that all the sleeves can be simultaneously created. Yet this algorithm doesn’t work in the presence of cycles. Furthermore there is no analysis on the optimality of SQM with respect to the number of links.

The contribution we emphasize in this abstract is that we prove the process outlined above is feasible even in the presence of cycles. We present a method which uses Integer Linear Programming and computes an optimal solution minimizing the number of vertices for the partition of the sphere at the joints. Feasibility is proven thanks to a numerical characterization by Rivin [12] of graphs combinatorially equivalent to inscribed polyhedrons. Our minimization criterion is what gives the “coarsest” mesh, among those based on Voronoi partitions of the spheres at the joints.

It is worth mentioning that another partition of the sphere is introduced in [11], a partition into quadrangles, which therefore makes the matching of cells trivial, the caveat being that the convexity of the cells is not guaranteed.

## 2 Formalization of the geometrical problem

In this paper a *skeleton* is a finite set  $\mathcal{S}$  of spatial line segments satisfying the following property: any two segments intersect at most at one of their endpoints. A skeleton  $\mathcal{S}$  defines naturally a graph  $G_{\mathcal{S}} = (V_{\mathcal{S}}, E_{\mathcal{S}})$  (following the notation on [6]), by identifying the set of vertices  $V_{\mathcal{S}}$  with the set of all endpoints in  $\mathcal{S}$ , and the set of edges  $E_{\mathcal{S}}$  with the set  $\mathcal{S}$  itself. The graph  $G_{\mathcal{S}}$  will be called the *skeleton graph*. An edge  $e \in E_{\mathcal{S}}$  connecting two nodes  $a, b \in V_{\mathcal{S}}$  can be alternatively represented as  $ab$ . If  $e = ab$  we say that  $e$  is *incident* on  $a$  (or  $b$ ) and will be denoted  $e \dashv\!\!\!\dashv a$  (or  $e \dashv\!\!\!\dashv b$ ). The notation  $s_e$  will be used to denote the segment in  $\mathcal{S}$  associated to  $e$ .

The unitary sphere centered at a node  $v \in V_{\mathcal{S}}$  is denoted  $\mathbb{S}_v$ , and  $\mathcal{A}_v = \{s_e \cap \mathbb{S}_v \mid e \in E_{\mathcal{S}}, e \dashv\!\!\!\dashv v\}$  denotes the set of intersection points of the segments incident on  $v$  with the unitary sphere centered at  $v$ .

The *Voronoi diagram* [2] of  $\mathcal{A}_v$  on the sphere, denoted  $\text{Vor}(\mathcal{A}_v)$ , partitions the sphere into regions  $\{R_e^v\}_{e \dashv\!\!\!\dashv v}$ , where  $(\mathbb{S}_v \cap s_e) \in R_e^v$ , that are delimited by arcs of great-circles [1,10]. A region  $R_e^v$  is thus well defined by an ordered set of points that we call the cell  $C_e^v$ . This set of points consists of the end points of the arcs delimiting  $R_e^v$  and at least one additional point per arc to remove ambiguity. The number of points chosen on a arc of  $C_e^v$  will be called the *number of subdivisions* for that arc. A *scaffold*  $\mathcal{K}_{\mathcal{S}}$  is defined as a pair  $(P_{\mathcal{S}}, \Phi_{\mathcal{S}})$ , satisfying

- (1)  $P_{\mathcal{S}} = \{\mathcal{C}_v \mid v \in V_{\mathcal{S}}\}$  where each  $\mathcal{C}_v = \{C_e^v \mid v \in V_{\mathcal{S}}, e \in E_{\mathcal{S}}, e \dashv\!\!\!\dashv v\}$  is a family of *cells* representing a partition of  $\mathbb{S}_v$  according to  $\text{Vor}(\mathcal{A}_v)$ .
- (2)  $\Phi_{\mathcal{S}} = \{\phi_e \mid e \in E_{\mathcal{S}}\}$  is a family of bijections  $\phi_e$  between  $C_e^a$  and  $C_e^b$  for  $e = ab$ .

For  $e \dashv\!\!\!\dashv v$  and  $C_e^v = (p_1, p_2, \dots, p_n)$  we say that  $p_i$  is *linked* with  $\phi_e(p_i)$ . The term scaffold is “visually” justified by the following: if we try to visualize  $\mathcal{K}_{\mathcal{S}}$  by drawing a segment between any pair of linked elements of  $P_{\mathcal{S}}$  we can see a structure that resembles a set of “scaffolds” around the skeleton  $\mathcal{S}$ . Our goal is to prove that the Voronoi cells can always be subdivided so as to build a scaffold. For that we need the cells  $C_e^a$  and  $C_e^b$  ( $e = ab$ ) to have identical number of points. The point insertion strategy described in [3] does not work in the presence of cycles in the skeleton.

### 3 Feasibility and optimality

We model the problem of matching the cells as a Integer Linear Program (IP) [5]. We shall prove first that the IP is feasible. Thus we can provide an optimal solution using the Branch-and-Bound algorithm [5, page 278].

Dual to the Voronoi diagram  $\text{Vor}(\mathcal{A})$  is the *Delaunay triangulation*  $\text{Del}(\mathcal{A})$  [2]. A Voronoi diagram of a set of points on a 2 dimensional sphere is dual to the spatial convex hull in the surrounding 3 dimensional space [4,8]. This means that the Delaunay triangulation is combinatorically equivalent to the convex hull.

For each  $v \in V_S$  let  $E_v$  be the set of edges of  $\text{Del}(\mathcal{A}_v)$ . Each edge in  $E_v$  represents a common boundary between two regions in  $\text{Vor}(\mathcal{A}_v)$ . For  $f \in E_v$  we define a positive integer  $x_f^v$  called *weight* representing the number of subdivisions to be done to the corresponding arc. The number of elements in a cell  $C_e^v$  ( $v \in V_S$ ,  $e \in E_S$  and  $e \dashv v$ ) is given by

$$|C_e^v| = \sum_{\substack{f \in E_v \\ f \dashv (\mathbb{S}_v \cap s_e)}} x_f^v.$$

For  $e = ab \in E_S$  the bijection  $\phi_e \in \Phi_S$  tells us that the cells  $C_e^a$  and  $C_e^b$  in  $\mathcal{K}_S$  must have the same number of elements. On the other hand the total number of links in  $\mathcal{K}_S$  is given by the sum of all the weights, thus our IP is formalized as follows.

$$\begin{aligned} \text{Minimize: } \sum_{\substack{h \in E_v \\ v \in V_S}} x_h^v \quad \text{Subject to: } & \sum_{\substack{f \in E_a \\ f \dashv (\mathbb{S}_a \cap s_e)}} x_f^a = \sum_{\substack{g \in E_b \\ g \dashv (\mathbb{S}_b \cap s_e)}} x_g^b, \quad \forall \phi_e \in \Phi_S \\ & x_h^v \in \mathbb{Z}, x_h^v \geq 1, \quad \forall h \in E_v, v \in V_S \end{aligned} \quad (1)$$

If this IP is feasible then it will have at least one optimal solution. The feasibility of the system (1) is the main result presented in this abstract, which we state in the following theorem.

**Theorem 3.1** *The set of linear equations (1) has a solution with all the entries  $x_h$  in the set of positive integers.*

The idea of the proof is to show that for  $v \in V_S$ , the *local* linear system

$$\sum_{\substack{f \in E_v \\ f \dashv (\mathbb{S}_a \cap s_e)}} x_f^v = \lambda_v \quad \forall e \dashv v, e = ab \in E_S \quad (2)$$

has a solution in the positive integers with  $\lambda_v$  also a positive integer. The values  $\tilde{x}_f^v$  along with  $\tilde{\lambda}_v$  satisfying (2) will be called *local solution*. Indeed assuming there are local solutions  $(\tilde{x}_f^v, \tilde{\lambda}_v)$  for all  $v \in V_S$  then there is a global solution  $\hat{x}_f$  given by  $\hat{x}_f = \hat{x}_f^v = (\lambda/\tilde{\lambda}_v) \cdot \tilde{x}_f^v$  where  $\lambda = \prod_{u \in V_S} \tilde{\lambda}_u$ .

The existence of a real positive solution for the local system (2) comes from the fact that  $\text{Del}(\mathcal{A}_v)$  is combinatorially equivalent to the convex hull of  $\mathcal{A}_v$  which is an inscribable polyhedron, and the following numerical characterization due to Rivin [12] (statement extracted from [7]).

**Proposition 3.2** *If a graph is of inscribable type then weights  $w$  can be assigned to its edges such that:*

- (i) *For each edge  $e$ ,  $0 < w(e) < 1/2$ .*
- (ii) *For each vertex  $v$ , the total weight of all edges incident on  $v$  is equal to 1.*

One then argues that if there is a real positive solution then there must be also a integer positive solution.

## 4 Application and examples

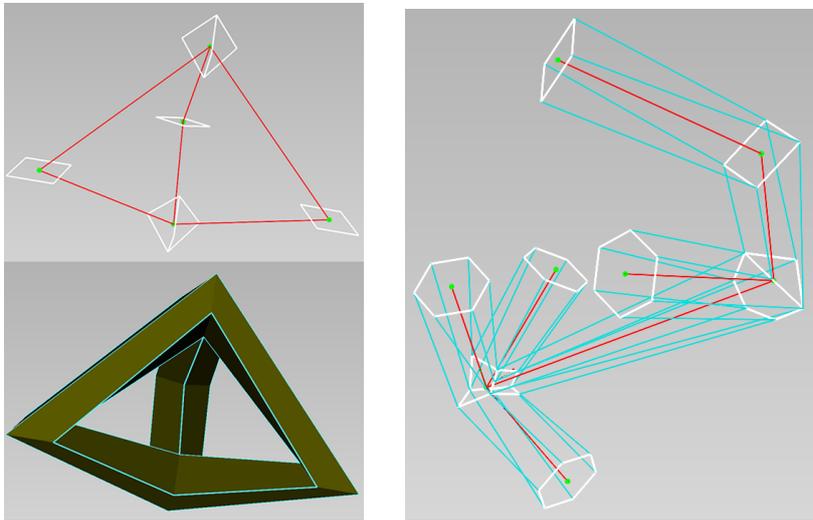


Fig. 1. Example skeletons and scaffolds (skeletons in red, endpoints in green, Voronoi cells in white, links in cyan, scaffold-based mesh in dark green).

The method outlined in this paper has been implemented and used successfully for scaffolding skeletons. It has been tested for a wide variety of models including closed skeletons. It is integrated in a more ambitious project aiming

at the visualization of convolution surfaces, that we wish to report on during the talk. The scaffolds in Figure 1 were obtained using our implementation.

## References

- [1] Augenbaum, J. M. and C. S. Peskin, *On the construction of the Voronoi mesh on a sphere*, Journal of Computational Physics **59** (1985), pp. 177–192.
- [2] Aurenhammer, F., *Voronoi Diagrams A Survey of a Fundamental Data Structure*, ACM Computing Surveys **23** (1991).
- [3] Bærentzen, J. A., M. K. Misztal and K. Welnicka, *Converting skeletal structures to quad dominant meshes*, Computers and Graphics **36** (2012).
- [4] Brown, K. Q., “Geometric Transforms for Fast Geometric Algorithms,” Phd thesis, Carnegie-Mellon University (1979).
- [5] Chen, D.-S., R. G. Batson and Y. Dang, “Applied integer programming: modeling and solution,” John Wiley & Sons, 2011.
- [6] Diestel, R., “Graph Theory,” 1997, 3rd edition.
- [7] Dillencourt, M. B. and W. D. Smith, *Graph-theoretical conditions for inscribability and Delaunay realizability*, Discrete Mathematics **161** (1996), pp. 63–77.
- [8] Grima, C. I. and A. Márquez, “Computational Geometry on Surfaces,” Springer Netherlands, 2001.
- [9] Ji, Z., L. Liu and Y. Wang, *B-Mesh: A modeling system for base meshes of 3D articulated shapes*, Computer Graphics Forum **29** (2010).
- [10] Na, H.-S., C.-N. Lee and O. Cheong, *Voronoi diagrams on the sphere*, Computational Geometry **23** (2002).
- [11] Panotopoulou, A., K. Welker, E. Ross, E. Hubert and G. Morin, *Scaffolding a Skeleton*.
- [12] Rivin, I., *A Characterization of Ideal Polyhedra in Hyperbolic 3-Space*, Annals of Mathematics **143** (1996).
- [13] Zanni, C., “Skeleton-based Implicit Modeling & Applications,” Phd thesis, Université de Grenoble (2013).