On Boundary Finite-Time Feedback Control for Heat Equation

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Abstract: The paper deals with boundary finite-time control for heat system. A linear switching control with state dependent switchings is designed based on backstepping procedure. It steers any solution of the heat system to zero in a finite time. The theoretical results are supported by numerical simulations.

1. INTRODUCTION

Finite-time control and estimation problems for plants described by ordinary differential equations are well-known in the literature (see, for example, Haimo (1986), Coron and Praly (1991), Bhat and Bernstein (2000), Levant (2005), Polyakov et al. (2015)). The interest to such control problems is caused by different reasons. On the one hand, finite-time convergence of observer state to the real state of the plant is the simplest way to realize the separation principle for nonlinear control systems. On the other hand, time constraints appear in many control problems and finite-time algorithms is the simplest way to fulfil them. The finite-time control of partial differential equations is a topic of intensive research in the last years (see, e.g. Perrollaz and Rosier (2014); Coron et al. (2015); Coron and Nguyen (2015); Hu and Di Meglio (2015); Alabau-Boussouira et al. (2015)). It is worth stressing that this problem is also related with analysis of exact controllability and exact null controllability of plants described by partial differential equations.

The present paper treats the problem of boundary feedback control design, which steers any solution of the heat system to zero in a finite time. We consider the heat system on the segment $[0, 1]$ with boundary conditions of Dirichlet type. The right boundary condition is controlled while the left one is homogeneous (zero). The similar problem has been studied in Coron and Nguyen (2015), where a linear boundary control with a time varying and piecewise constant "feedback gain" has been designed. The "feedback gain" in this context is a linear continuous functional $K : L^2((0, 1), \mathbb{R})) \to \mathbb{R}$, which maps the current state of the heat system to $\mathbb{R}$ in order to define the control signal (the right boundary condition to the system). This paper develops a novel piecewise linear boundary control with a *state dependent switching* law for the "feedback gain". Analogously to Coron and Nguyen (2015) the backstepping transformation is utilized for feedback design and finite-time convergence analysis of the closed-loop system. We the reader to Boskovic et al. (2001), Krstic and Smyshlyaev (2008) for more details about backstepping approach to boundary control design of PDEs.

The paper is organized as follows. The next section presents problem statement and basic assumptions. In the section 3 the main idea of finite-time stabilization by means of piecewise linear feedback control as well as some related problems are discussed. After that the boundary linear switching feedback control with state dependent switchings is designed, settling time of the closed-loop system is estimated and some modifications of the control allowing global fixed-time stabilization as well as tuning of the settling time are proposed. The numerical simulation results are presented in the end of the section 4. The paper is finished with concluding remarks and a list of bibliographic references.

Notation. The mainly paper uses the standard notations. For instance,

- $\mathbb{R}$ is the field of real numbers and $\mathbb{R}_+ = [0, +\infty)$;
- $\mathbb{Z}$ is the set of integers;
- $\|\cdot\|$ denotes the canonical norm in $L^2([0, 1], \mathbb{R})$.

2. PROBLEM STATEMENT

Let us consider the heat system

$$\frac{\partial v}{\partial t}=\frac{\partial^2 v}{\partial x^2}, \quad v(0)=v_0 \tag{1}$$

with boundary control ζ

$$(v(t))(0)=0 \qquad (v(t))(1)=\zeta(t), t>0 \tag{2}$$

where $v(t)\in L^2((0,1),\mathbb{R})$.

The control aim is to steer the state $v(t)$ of the system (1), (2) *to zero in a finite time* by means of a linear switching feedback control

$$\zeta(t)=K_{\sigma(t)}v(t), \tag{3}$$

where $K_i : L^2((0,1),\mathbb{R})\to\mathbb{R}$, $i\in\mathbb{Z}$ is a family of linear bounded functionals and $\sigma:\mathbb{R}_+\to\mathbb{Z}$ is a state dependent switching function governed by the discrete equation

$$\sigma(t)=G(\sigma(t^-),v(t^-)), \tag{4}$$

with $G:\mathbb{Z}\times L^2((0,1),\mathbb{R})\to\mathbb{Z}$, $t^-=t+0^-$ and $\sigma(0)\in\mathbb{Z}$.

The hybrid (switched) linear system (1) - (4) has two components: $v(t)$ - continuous state and $\sigma(t)$ - discrete state. Therefore, its solution is a tuple (v,σ) : $v\in C^0([0,T),L^2((0,1),\mathbb{R}))$ and $\sigma:(0,T)\to\mathbb{Z}$ such that $\sigma(t)$ satisfies the discrete equation (4) for all $t\in(0,T)$ and v is a solution to the heat system (1) - (3) understood in the weak sense:

$$-\int_0^1 v_0(x)\xi(0,x)dx-\int_0^T\int_0^1 v(t,x)\xi_t(t,x)dxdt$$
$$+\int_0^T K_{\sigma(t)}v(t,\cdot)\xi_x(t,1)dt-\int_0^T\int_0^1 v(t,x)\xi_{xx}(t,x)dxdt=0$$

for all $\xi\in C^2([0,T)\times[0,1])$ with compact support in $[0,T)\times[0,1]$ such that ξ vanishes at $[0,T)\times\{0,1\}$.

We refer reader, for example, to Coron and Nguyen (2015) for more details about existence of solutions to (1) - (3) in the case of state independent switchings. All required information about hybrid and switched systems can be found in Goebel and Teel (2012), Liberzon (2003).

3. ON FINITE-TIME STABILIZATION BY MEANS OF SWITCHED LINEAR FEEDBACK

3.1 The basic idea

Let us consider the simplest scalar control system

$$\dot{x}(t)=u(t), \quad t>0, \tag{5}$$

where $x(t)\in\mathbb{R}$ is the system state and $u(t)\in\mathbb{R}$ is the control. To stabilize the origin of this system in a finite time the discontinuous feedback control

$$u(t)=u_d(x(t)):=-\tfrac{x(t)}{|x(t)|}$$

can be utilized. Indeed, the Filippov solution (see Filippov (1988)) to the closed-loop system is given by

$$|x(t)|=\begin{cases}|x(0)|-t & \text{if } t\in[0,x(0)],\\ 0 & \text{if } t>|x(0)|.\end{cases}$$

The problem of finite-time stabilizing control design is well-studied for finite-dimensional plants. For example, in Coron and Praly (1991), Bhat and Bernstein (2005), Andrieu et al. (2008), Levant (2005), Polyakov et al. (2015) non-linear continuous and discontinuous control algorithm are developed for linear systems. The problem of finite-time stabilization of infinite dimensional systems (see Efimov et al. (2015), Perrollaz and Rosier (2014), Coron and Nguyen (2015), Alabau-Boussouira et al. (2015)) is more complicated, since usually an analysis of non-linear infinite dimensional systems is nontrivial. In order to overcome this difficulty a linear time-varying piecewise linear boundary control has been presented Coron and Nguyen (2015) for parabolic PDEs. Linear static feedback usually provides exponential convergence rate to the closed-loop system. In order to provide finite-time (hyper-exponential) stability to the closed-loop system linear system the gains must vary in time (Coron and Nguyen (2015)).

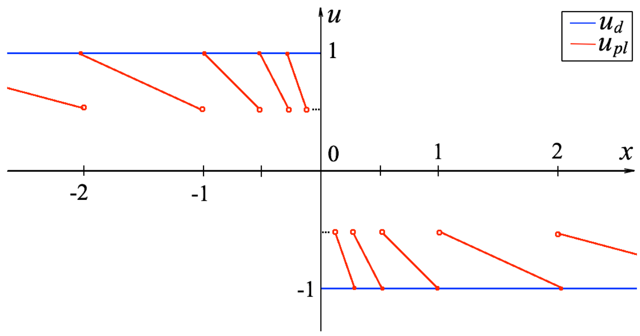

Fig. 1. Relay feedback and its piecewise linear approximation ($r_i=2^{-i}$)

Let us consider a piecewise linear approximation of u_d:

$$u_{pl}(x(t))=-\frac{x(t)}{r_i} \quad \text{if} \quad |x(t)|\in(r_{i+1},r_i], \quad i\in\mathbb{Z}, \tag{6}$$

where $r_i>0$ are some parameters. The presented approximation is a bounded function (see, Fig. 1). The origin of the closed-loop system with control u_{pl} is asymptotically stable. Indeed, since its solution satisfies the identity

$$|x(t_{i+1})|=e^{-\frac{t_{i+1}-t_i}{r_i}}|x(t_i)|,$$

where $t_i:|x(t_i)|=r_i$ and $i\in\mathbb{Z}$, then, obviously, $|x(t_i)|\to 0$. To guarantee finite-time stability the parameters $r_i, i\in\mathbb{Z}$ should be properly selected such that $\sum t_{i+1}-t_i<+\infty$.

For example, if $r_i=2^{-i}$ then $t_{i+1}-t_i=\frac{\ln 2}{2^i}$ and

$$\sum_{i=i_0}^{\infty} t_{i+1}-t_i=\frac{\ln 2}{2^{i_0-1}}<+\infty.$$

In order to stabilize the origin of a control system in a finite time by means of switching linear feedback its gain must tends to infinity as the norm of state tends to zero.

The results of the numerical simulations for the considered switched system with $x(0)=0$ are given at Fig. 2. The simulation has been done using implicit Euler method. The solution is plotted in a logarithmic scale in order to show that it decreases exponentially in-between switching instances.

3.2 Overshooting problem

Note that feasibility of the finite-time control design scheme presented above is not straightforward for multidimensional case. Indeed, let us consider the controlled double integrator

$$\begin{aligned}\dot{x}_1(t)&=x_2(t),\\ \dot{x}_2(t)&=u(t)\end{aligned} \tag{7}$$

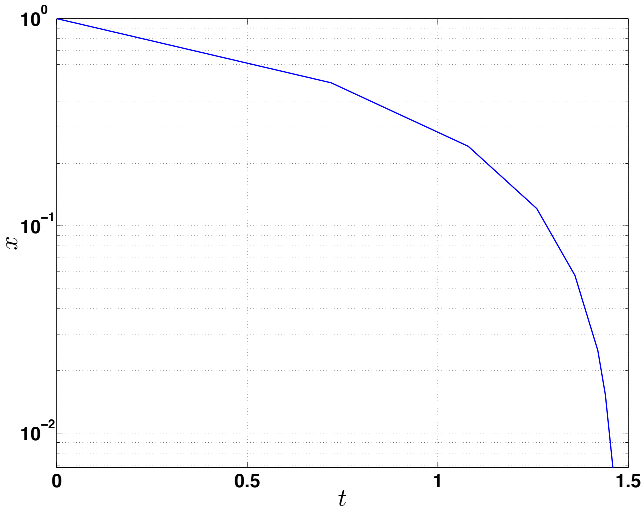

Fig. 2. Solution to scalar system (5) with piecewise linear control (6)

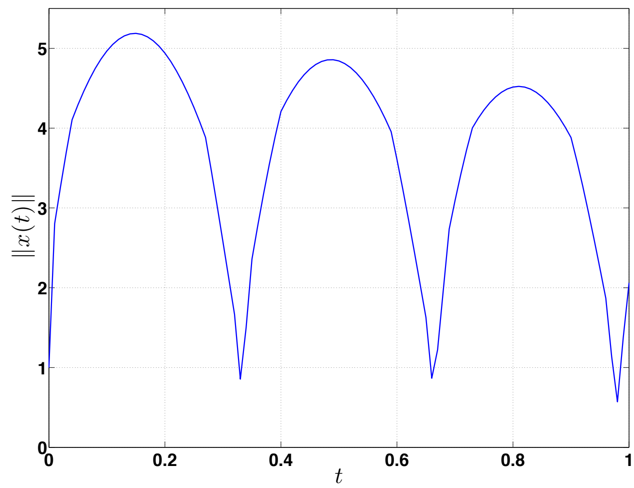

Fig. 3. Solution to scalar system (7) with piecewise linear control (8)

where $x(t) = (x_1(t), x_2(t))^\top \in \mathbb{R}^2$ is the state vector and u is the scalar control input. Following the basic idea given above we apply the piecewise linear switching control

$$u_{sw}(t) = \frac{1}{r_i}(k_1 x_1(t) + k_2 x_2(t)) \quad \text{if} \quad \|x\|_{\mathbb{R}^2} \in (r_{i+1}, r_i]. \quad (8)$$

where the gains $k_1 < 0$ and $k_2 < 0$ and $\{r_i\}$ is a monotone decreasing sequence such that $r_i \to 0$ as $i \to +\infty$.

For any fixed $r_i > 0$ the closed-loop system (7)-(8) is asymptotically stable. However, in contrast to the scalar case, we cannot guarantee anymore that the norm of the state will monotonically decrease in time. Moreover, linear control system with high gains may have large deviations from initial conditions.

The results of numerical simulations are depicted at Fig. 3 for the considered planar system with

$$k_1 = 100, \quad k_2 = 1, \quad r_i = 2^{-i}, \quad i \in \mathbb{Z}.$$

They confirm our suspicion showing that the norm $\|x(t)\|$ of the solution to the closed-loop system (7)-(8) oscillates and the switchings of control result sequential decreasing and increasing of the gains. The rigorous stability analysis of such system is complicated, since, for example, it is necessary to prove that the presented switching law does not invokes sliding modes (or Zeno behaviour) or their appearance does not prevent finite-time convergence.

The problem of large deviations of high gain linear control systems is well known in the literature (see e.g. Izmailov (1987), Polyak et al. (2015) and references therein). It has similar nature to the well-known phenomenon called *overshoot*, which may provoke, for example, instability to linear switched systems even if each subsystem is stable (Liberzon, 2003, Chapter 2).

To simplify an analysis and design of finite-time stabilizing control the switching logic should be properly modified for the multidimensional (as well as for infinite dimensional) case. The simplest way is to use time dependent switchings. Such algorithms has been developed, for example, in Coron and Nguyen (2015), Efimov et al. (2016). The main drawback of time dependent approach is insufficient robustness. Since feedback gain increases independently of the state, then small computational errors or measurement noises may induce huge stabilization error at the terminal instant of time.

An alternative solution is presented for planar system in Anan'evskii (2003), where the Lyapunov function has been utilized in (8) instead of Euclidian norm $\|x(t)\|_{\mathbb{R}^2}$. In Polyakov et al. (2015), Polyakov et al. (2016b) the similar scheme has been developed for multidimensional linear systems using a so-called *canonical homogeneous norm* (being implicit Lyapunov function to the system). The switching law is designed in such a way that the homogeneous norm of the closed-loop system decreases monotonically all the time (in disturbance-free case) independently of the switching instances. In Polyakov et al. (2017) this scheme has been adapted to abstract linear evolution equations in Hilbert spaces, where the particular switching law guaranteeing the finite-time convergence of the closed-loop system to zero is given. Unfortunately, the homogeneous norm mentioned above is frequently defined in an implicit way (Polyakov et al. (2015), Polyakov et al. (2016a)). Difficulty of its on-line computation is the main drawback of the control scheme based on the canonical homogeneous norm.

In this paper we design a hybrid finite-time boundary control for heat system. In order to avoid problems mentioned above we introduce a novel state dependent switching law that excludes sliding mode or Zeno behavior (see, e.g. (Liberzon, 2003, Chapter 1)) outside the origin and restricts the overshoot as follows :

$$\|x(t)\| < r_{i-1} \quad \text{for all} \quad t > t_i,$$

where $t_i > 0$ is a state dependent switching instant such that $\|x(t_i)\| = r_i$. It is worth stressing that the switching law uses the conventional L^2-norm of the state simplifying the practical realization of the control scheme.

4. BOUNDARY FINITE-TIME CONTROL FOR HEAT SYSTEM

4.1 Backstepping transformation

Given positive number $\lambda > 0$ the backstepping approach (see, Boskovic et al. (2001), Coron and Nguyen (2015) for the details) introduces the boundary control

$$\zeta(t) = \int_0^1 k(1, y, \lambda) v(t, y) dy \tag{9}$$

where

$$k(x, y, \lambda) = -\lambda y \frac{I_1(\sqrt{\lambda(x^2-y^2)})}{\sqrt{\lambda(x^2-y^2)})}$$

and I_m with $m \in \mathbb{Z}$ is the modified Bessel function of the first kind (see e.g. Watson (1996)). If we denote

$$(F(\lambda)u)(x) = -\int_0^x k(x, y, \lambda) u(y) dy \tag{10}$$

then the state transformation $u = v + F(\lambda)v$ applied to the system (1), (2) yields

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \lambda u, \qquad u(t, 0) = 0, \quad u(t, 1) = 0. \tag{11}$$

The *inverse transformation* is defined as

$$v = (I + F(-\lambda))u,$$

where $u, v \in L^2((0, 1), \mathbb{R})$. We refer reader to Boskovic et al. (2001) and Coron and Nguyen (2015) for more details about backstepping transformation.

It is easy to see (using the Lyapunov function defined as $V(u) = \|u\|^2$) that the system (11) is asymptotically stable and

$$\|u(t)\| \leq \|u(0)\| e^{-\lambda t}.$$

Following Coron and Nguyen (2015) we also use the backstepping transformation in order to design a finite-time control of the form (9) *with switching parameter* λ. The system (11) will be utilized as a sort of comparison system in order to prove stability and finite-time convergence of solution to zero as well as to estimate the settling time.

Proposition 1. If $z \in L^2((0, 1), \mathbb{R})$ then

$$\|z + F(\lambda)z\| \leq \Psi_1(\lambda)\|z\| \text{ and } \|z + F(-\lambda)z\| \leq \Psi_{-1}(\lambda)\|z\|,$$

$$\Psi_1(\lambda) = 1 + \frac{\lambda\sqrt{\int_0^1\int_0^x y^2\left(I_0\left(\sqrt{\lambda(x^2-y^2)}\right) - I_2\left(\sqrt{\lambda(x^2-y^2)}\right)\right)^2 dy dx}}{2},$$

$$\Psi_{-1}(\lambda) = 1 + \frac{\lambda\sqrt{\int_0^1\int_0^x y^2\left(J_0\left(\sqrt{\lambda(x^2-y^2)}\right) + J_2\left(\sqrt{\lambda(x^2-y^2)}\right)\right)^2 dy dx}}{2},$$

where J_k is the Bessel function of the first kind.

Proof. The proof of this proposition immediately follows from the identities (Temme, 1966, pages 229, 233 and 234)

$$I_k(p) = \frac{p}{2}(I_{k-1}(p) - I_{k+1}(p),$$

$$J_k(p) = \frac{p}{2}(J_{k-1}(p) + J_{k+1}(p)$$

and

$$I_k(p) = i^{-k} J_k(ip),$$

where $p \in \mathbb{R}$, $i = \sqrt{-1}$ and $k \in \mathbb{Z}$. ■

4.2 The main result

Now let us consider the heat system (1), (2) with the hybrid boundary feedback control

$$\xi(t) = \int_0^1 k\left(1, y, 2^{\sigma(t)}\right) v(t, y) dy, \tag{12}$$

where the switching function σ is governed by the equation (4) with

$$G(\sigma, v) = \begin{cases} i+1 & \text{if } \sigma = i \text{ and } \|v\| \leq r_{i+1}, \\ i & \text{if } \sigma = i \text{ and } r_{i+1} < \|v\| < r_{i-1}, \\ i-1 & \text{if } \sigma = i \text{ and } \|v\| \geq r_{i-1}, \end{cases} \tag{13}$$

where $r_0 = 1, r_i = e^{-q_i} r_{i-1}, i \in \mathbb{Z}$ and the numbers q_i are defined by formula (14).

Proposition 2. If

$$q_i = \ln \Psi_1\left(2^i\right) + \ln \Psi_{-1}\left(2^i\right), \tag{14}$$

then $q_i > 0$ for all $i \in \mathbb{Z}$, $r_i \to 0$ as $i \to +\infty$ and

$$\lim_{i \to +\infty} \frac{q_{i+1}}{q_i} = \sqrt{2}.$$

Proof. Using the representations (see, (Watson, 1996, page 181), (Temme, 1966, page 230))

$$I_k(p) = \frac{1}{\pi}\int_0^\pi e^{p\cos(\theta)} \cos(k\theta) d\theta,$$

$$J_k(p) = \frac{1}{\pi}\int_0^\pi \cos(k\theta - p\sin(\theta)) d\theta,$$

it can be shown $1 + C\lambda e^{\sqrt{C\lambda}} \leq \Psi_1(\lambda) \leq 1 + \lambda e^{\sqrt{\lambda}}$ for some $C > 0$ and $1 \leq \Psi_{-1}(\lambda) \leq 1 + \lambda$ (see, also Coron and Nguyen (2015)). Hence, $q_i > 0$ for all $i \in \mathbb{Z}$, $q_i \to +\infty$ as $i \to +\infty$, $r_i \to 0$ as $i \to +\infty$ and

$$\frac{q_{i+1}}{q_i} = \frac{\ln \Psi_1(2^{i+1}) + \ln \Psi_{-1}(2^{i+1})}{\ln \Psi_1(2^i) + 2\ln \Psi_{-1}(2^i)} =$$

$$\frac{\sqrt{2^{i+1}} + \ln\left(\frac{\Psi_1(2^{i+1})}{\exp\left(\sqrt{2^{i+1}}\right)}\right) + \ln \Psi_{-1}\left(2^{i+1}\right)}{\sqrt{2^i} + \ln\left(\frac{\Psi_1(2^i)}{\exp\left(\sqrt{2^i}\right)}\right) + \ln \Psi_{-1}(2^i)} \to \sqrt{2} \text{ as } i \to +\infty.$$

■

Theorem 1. (On finite-time stability). *For any initial condition*

$$v(0) = v_0 \in L^2((0, 1), \mathbb{R}) \tag{15}$$

and

$$\sigma(0) = i_0 \in \mathbb{Z} \quad \textit{with} \quad \|v(0)\| \in (r_{i_0+1}, r_{i_0}], \tag{16}$$

the system (1), (2), (12), (4), (13) *has a unique solution* (v, σ) : $v \in C^0([0, T), L^2((0, 1), \mathbb{R}))$ *and* $\sigma : (0, T) \to \mathbb{Z}$ *such that*

$$\|v(t)\| \to 0 \textit{ as } t \to T^-,$$

where

$$T \leq \sum_{i=i_0}^{+\infty} \frac{q_i + q_{i+1}}{2^i} < +\infty. \tag{17}$$

Proof. I. First of all, let us note that if $\sigma(t) = i, \forall t > 0$ then for any initial condition $v_0 \in L^2((0, 1), \mathbb{R})$ the system (1), (2), (12) has a unique solution $v_i \in C^0([0, +\infty), L^2((0, 1), \mathbb{R}))$ which vanishes at zero as time tends to infinity (see, e.g. Coron and Nguyen (2015)). Hence, if $\|v_i(t_i)\| \in (r_{i+1}, r_i]$ and $\sigma(t) = i$ then there always exists an instance of time $t^* > 0$ separated from t_i such that $\|v_i(t^*)\| = r_{i+1}$ or $\|v_i(t^*)\| = r_{i-1}$. This means that the discrete state σ will be switched to $i + 1$ or to $i - 1$ according to law (4), (13) and switching instant t^* will always be isolated. Therefore, similarly to (Coron and Nguyen, 2015, Lemma 6) it can be shown that for any initial conditions (15), (16) the hybrid system (1), (2), (12), (4), (13) has a unique solution (v, σ) such that $v \in C^0([0, T), L^2((0, 1), \mathbb{R}))$, $T = \sup t_j$ where $t_j > 0$:

$\|v(t_j)\| = r_{i_j}$, $i_j \in \mathbb{Z}, j = 1, 2, \ldots$ are isolated instances of time and $\sigma(t) = i_j$ for $t \in [t_{i_j}, t_{i_{j+1}})$ satisfies the discrete equation (4) with the switching law (13).

II. Let us show that $\|v(t)\| < r_{i_j - 1}$ for $t > t_j$ if $\|v(t_j)\| \leq r_{i_j}$ and $\sigma(t) = i_j$. Since due to backstepping transformation the function $u_{i_j} = (I + F(2^{i_j}))v$ is the solution to (11) with $\lambda = 2^{i_j}$ then it is easy to see that

$$\|u_{i_j}(t)\| \leq e^{-2^{i_j}(t-t_j)}\|u_{i_j}(t_j)\|,$$

i.e. $\|u_{i_j}(t)\| < \|u_{i_j}(t_j)\|$ for $t > t_j$. On the other hand, since $\|u_{i_j}(t)\| \leq \Psi_1(2^{i_j})\|v(t)\|$ and $\|v(t)\| \leq \Psi_{-1}(2^{i_j})\|u_{i_j}(t)\|$ then

$$\|v(t)\| < \Psi_1(2^{i_j})\Psi_{\text{-}1}(2^{i_j})\|v(t_j)\| = e^{q_{i_j}}\|v(t_j)\|$$
$$\leq e^{q_{i_j}} r_{i_j} = r_{i_j - 1} \quad \text{for} \quad t > t_j.$$

This immediately implies that $i_{j+1} = i_j + 1$ and $\|v(t)\| \to 0$ as $t \to T^-$.

III. Let us show that $T < +\infty$. It has been noted above that $\|u_{i_j}(t)\| \leq e^{-2^{i_j}(t-t_{i_j})}\|u_{i_j}(t_{i_j})\|$. Hence, we derive

$$t_{j+1} - t_j \leq \tfrac{1}{2^{i_j}} \ln \tfrac{\|u_{i_j}(t_j)\|}{\|u_{i_j}(t_{j+1})\|} \leq \tfrac{1}{2^{i_j}} \ln \tfrac{\Psi_1(2^{i_j})\Psi_{-1}(2^{i_j})\|v(t_j)\|}{\|v(t_{j+1})\|}.$$

Taking into account $\|v(t_j)\| = r_{i_j}$, $\|v(t_{j+1})\| = r_{i_{j+1}} = e^{-q_{i_{j+1}}} r_{i_j}$, $i_{j+1} = i_j + 1$ and the formula (14) we obtain

$$T \leq \sum_{j=0}^{+\infty} t_{j+1} - t_j \leq \sum_{j=0}^{+\infty} \frac{1}{2^{i_j}} \ln \frac{e^{q_{i_j}} r_{i_j}}{r_{i_j+1}} = \sum_{i=i_0}^{+\infty} \frac{q_i + q_{i+1}}{2^i}$$
$$= 3\left(\sum_{i=i_0}^{+\infty} \frac{q_i}{2^i}\right) - 2\frac{q_{i_0}}{2^{i_0}}.$$

Due to Proposition 2 we have

$$\lim_{i+\infty} \frac{2^{-(i+1)} q_{i+1}}{2^{-i} q_i} = \frac{1}{2} \lim_{i+\infty} \frac{q_{i+1}}{q_i} = \frac{1}{\sqrt{2}} < 1.$$

and $2^{-i} q_i \to 0$ as $i \to \infty$ since $q_i = O(\sqrt{2^i})$ for $i \to +\infty$. Hence, the ratio test for convergent series guarantees $T < +\infty$. ■

Small modification of the presented control law allows us to stabilize the heat system in a fixed time independently of the initial conditions.

Corollary 1. (On fixed-time stability). *If the parameters q_i are redefined for $i < 0$ as follows $q_{-k} := q_k$, $k = 1, 2, 3, \ldots$ and*

$$\xi(t) = \int_0^1 k\left(1, y, 2^{|\sigma(t)|}\right) v(t, y) dy, \qquad (18)$$

then Theorem 1 remains true but the settling time T becomes globally bounded

$$T \leq \sum_{i=-\infty}^{+\infty} \frac{q_i + q_{i+1}}{2^i} < +\infty.$$

independently of the initial condition $v_0 \in L^2((0,1), \mathbb{R})$.

Note also that the settling time T can be tuned by means of minor change of the control law

$$\xi(t) = \int_0^1 k\left(1, y, \rho 2^{\sigma(t)}\right) v(t, y) dy,$$

and the parameters

$$q_i = \ln \Psi_1\left(\rho 2^i\right) + \ln \Psi_{-1}\left(\rho 2^i\right)$$

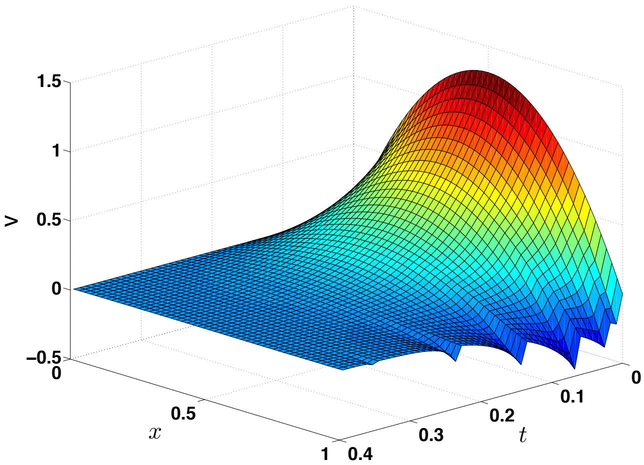

Fig. 4. Evolution of heat equation with boundary finite-time control

where $\rho > 0$ is a tuning parameter. In this case the estimate (17) becomes

$$T \leq \frac{1}{\rho} \sum_{i=i_0}^{+\infty} \frac{q_i + q_{i+1}}{2^i} < +\infty.$$

4.3 Numerical example

The simulation result for boundary finite-time control application to heat system with the initial condition $v(0, x) = 5.35x(1-x)$ is depicted at Fig. 4. For numerical simulations the state system has been discretized by divided differences on a uniform grid with the step $h = 0.02$ for the space variable. The discretization with respect to time has been done using implicit Euler scheme with the step size $\tau = h^2$. All integrals are approximated by means of trapezoidal rule. It can bee seen from the Fig.4 that the switching of the "feedback gain" implies the jump in the right boundary condition, but the solution to the controlled heat system remains continuous in L_2-norm.

The Fig. 5 presents the evolution of the norm $\|v(t)\|$ in time with a logarithmic scale in order to confirm on simulations that the convergence to zero is faster than exponential one. Note that the numerical simulations also demonstrates "overshooting" (fast transition) after each switching instance and confirms the fact rigorously proven in Theorem 1 : *an overshooting after each switching instant $t_j > 0$: $\|v(t_j)\| = r_{i_j}$ is always less than the precedent switching level $r_{i_j - 1}$.*

5. CONCLUSIONS

In the paper the boundary linear switching feedback control with state dependent switchings is designed for heat system on the segment $[0, 1]$. It steers any solution of the closed-loop system to zero in a finite-time. Switching nature of the feedback invokes a sort of overshooting phenomenon: fast relatively large deviations of the solution from the state at switching instant of time during transient process. This phenomenon is well known even for finite-time dimensional control systems. In our case the switching law is designed in such a way that the overshooting is sufficiently small to avoid unnecessary switchings and to

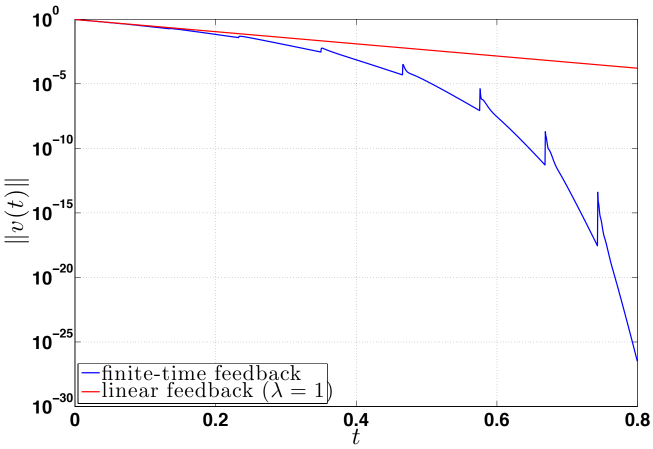

Fig. 5. Evolution of L^2-norms with finite-time and linear control laws

guarantee finite-time convergence of the solution to zero. This theoretical result is supported by numerical simulations where the overshooting has been visually observed when evolution of the norm of the solution is plotted in the logarithmic scale (Fig. 5). Finite-time feedback control with state dependent switchings is expected to be robust (in ISS sense) with respect to different types of uncertainties/disturbances. A rigorous study of this issue can be considered as a possible direction for future research.

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