HAL open science

Lyapunov criteria for uniform convergence of conditional distributions of absorbed Markov processes

Nicolas Champagnat, Denis Villemonais

To cite this version:

Nicolas Champagnat, Denis Villemonais. Lyapunov criteria for uniform convergence of conditional distributions of absorbed Markov processes. Stochastic Processes and their Applications, 2021, 135, pp.51-74. 10.1016/j.spa.2020.12.005 . hal-01503697v2

HAL Id: hal-01503697

https://inria.hal.science/hal-01503697v2

Submitted on 1 Feb 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Distributed under a Creative Commons Attribution 4.0 International License

Lyapunov criteria for uniform convergence of conditional distributions of absorbed Markov processes

Nicolas Champagnat[1,2,3], Denis Villemonais[1,2,3]

September 30, 2020

Abstract

We study the uniform convergence to quasi-stationarity of multidimensional processes absorbed when one of the coordinates vanishes. Our results cover competitive or weakly cooperative Lotka-Volterra birth and death processes and Feller diffusions with competitive Lotka-Volterra interaction. To this aim, we develop an original non-linear Lyapunov criterion involving two functions, which applies to general Markov processes.

Keywords: stochastic Lotka-Volterra systems; multitype population dynamics; multidimensional birth and death process; multidimensional Feller diffusions; process absorbed on the boundary; quasi-stationary distribution; uniform exponential mixing property; Lyapunov function

2010 Mathematics Subject Classification. Primary: 60J27; 37A25; 60B10. Secondary: 92D25; 92D40.

1 Introduction

We consider a Markov process $(X_t, t \geq 0)$ evolving in a state space $E \cup \partial$, where $\partial \cap E = \emptyset$ and ∂ is absorbing. A quasi-stationary distribution for X is a probability measure ν_{QSD} on E such that

$$\mathbb{P}_{\nu_{QSD}}(X_t \in \cdot \mid t < \tau_\partial) = \nu_{QSD}, \quad \forall t \geq 0,$$

[1]Université de Lorraine, IECL, UMR 7502, Campus Scientifique, B.P. 70239, Vandœuvre-lès-Nancy Cedex, F-54506, France

[2]CNRS, IECL, UMR 7502, Vandœuvre-lès-Nancy, F-54506, France

[3]Inria, TOSCA team, Villers-lès-Nancy, F-54600, France.

E-mail: Nicolas.Champagnat@inria.fr, Denis.Villemonais@univ-lorraine.fr

where τ_∂ is the first hitting time of ∂ and, for any probability measure ν on E, $\mathbb{P}_\nu$ is the law of X with initial distribution ν.

Our goal is to provide a computational method, taking the form of a nonlinear *Lyapunov type condition* (sometimes also referred to as *drift condition*) ensuring the existence and uniqueness of a quasi-stationary distribution and the uniform convergence in total variation of the law of X_t given $X_t \notin \partial$ when $t \to +\infty$ to this quasi-stationary distribution, which means that there exist two constants $\gamma, C > 0$ such that

$$\|\mathbb{P}_\mu(X_t \in \cdot \mid t < \tau_\partial) - \nu_{QSD}\|_{TV} \leq Ce^{-\gamma t}, \quad \forall t \geq 0, \tag{1.1}$$

for all initial distribution μ on E, where $\|\cdot\|_{TV}$ is the usual total variation distance on the set of finite, signed measures on E, defined by $\|\mu\|_{TV} = \sup_{f \in L^\infty(E),\ \|f\|_\infty \leq 1} |\mu(f)|$. We apply this result to two standard models in ecology and evolution, called Lotka-Volterra (or logistic) birth-death or diffusion processes [27, 6, 7], which have attracted a lot of attention in the past and for which the question of uniform convergence toward a quasi-stationary distribution remains largely open in the multi-dimensional case.

Practical (linear) Lyapunov type criteria for convergence to quasi-stationary distributions were also developed in [12]. However, these results usually only entail non-uniform convergence with respect to the initial condition. In the applications we consider here, the results of Sections 4 and 5 in [12] would ensure the existence of two positive functions $\varphi_1 \geq 1$ and $\varphi_2 \leq 1$ on E such that

$$\|\mathbb{P}_\mu(X_t \in \cdot \mid t < \tau_\partial) - \nu_{QSD}\|_{TV} \leq Ce^{-\gamma t}\frac{\mu(\varphi_1)}{\mu(\varphi_2)}, \quad \forall t \geq 0, \tag{1.2}$$

for all initial distribution μ on E. In the cases considered below, the function φ_1 may be taken bounded, but φ_2 is usually not bounded away from zero close to the boundary of E, which leads to a non-uniform convergence result.

As may be expected, the stronger convergence result (1.1) requires a finer control of the behavior of the process near the boundary, uniformly in E. As a consequence, our Lyapunov criteria are more involved than those of [12] and the techniques used here differ: in the present article, we use the control of the derivative of the continuous-time semi-group of the process to localize the conditioned process when it starts close to the boundary (see Proposition 2.3), while [12] makes use of the time-discretisation of the semi-group for this purpose. Additional arguments are also required to control the behavior of the conditioned process close to infinity. Note however that, in the end, our approach relies on the discrete time criterion of [9] and it is

tempting to think that one may use a combination of the present approach and of the discrete time criterion of [12] in order to prove (1.2) with $\varphi_2 = 1$. Although this would lead to more complicated criteria than the one presented below, such an approach could also apply to processes that do not come down from infinity, such as Orstein-Uhlenbeck processes, with uniform convergence among initial distributions in bounded subsets. We leave this question for future research.

The uniform convergence (1.1) is also transferred to other properties of the process. Indeed, it provides uniformity in the time to the so-called *mortality/extinction plateau* (see [30]), uniform convergence of the process conditioned to late survival to the so-called Q-process, uniform exponential ergodicity of the Q-process (see [13]), and uniform convergence of $x \mapsto e^{\lambda_0 t}\mathbb{P}_x(t < \tau_\partial)$ to an eigenfunction $\eta : E \to (0, +\infty)$ for the semigroup, for some positive constant $\lambda_0 > 0$, when $t \to +\infty$ (see [9, Theorem 2.5]).

One of the main tools of our proofs is [9], where we showed that the uniform exponential convergence of conditional distributions in total variation to a unique quasi-stationary distribution is equivalent to the following conditions: there exists a probability measure ν on E such that

(A1) there exist $t_0', c_1' > 0$ such that for all $x \in E$,

$$\mathbb{P}_x(X_{t_0'} \in \cdot \mid t_0' < \tau_\partial) \geq c_1' \nu(\cdot);$$

(A2) there exists $c_2' > 0$ such that for all $x \in E$ and $t \geq 0$,

$$\mathbb{P}_\nu(t < \tau_\partial) \geq c_2' \mathbb{P}_x(t < \tau_\partial).$$

Although it has the merit of generality, this criterion appears to be hard to check in practice [13, 5, 14, 11, 16, 23]. In particular, it lacks computational methods for verification. This is one of the purposes of the new criterion we present. It involves two bounded nonnegative functions V and φ such that $V(x)/\varphi(x) \to +\infty$ when x converges to the boundary of E or to ∞, satisfying

$$-L\varphi \leq C_1 \mathbb{1}_K$$

for some bounded subset K of E and

$$LV + C_2 \frac{V^{1+\varepsilon}}{\varphi^\varepsilon} \leq C_3 \varphi$$

for some $\varepsilon > 0$ and some constants $C_1, C_2, C_3 > 0$, where L denotes (an extension of) the infinitesimal generator of the Markov process X.

We apply this criterion to Lotka-Volterra birth and death processes, and to competitive Lotka-Volterra Feller diffusion processes. The quasi-stationary behavior of (extensions of) these two models have received a lot of attention in the one-dimensional case [34, 33, 25, 29, 3, 14, 15, 22]. We focus here on the multidimensional case, where the processes evolve on the state spaces $E \cup \partial = \mathbb{Z}_+^d$ for birth and death processes (with $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$) and $E \cup \partial = \mathbb{R}_+^d$ for diffusion processes, with $d \geq 2$, and where absorption corresponds to the extinction of a single population. This means that $\partial = \mathbb{Z}_+^d \setminus \mathbb{N}^d$ and $E = \mathbb{N}^d$ (where $\mathbb{N} = \{1, 2, \ldots\}$) for multidimensional birth and death processes and $\partial = \mathbb{R}_+^d \setminus (0, +\infty)^d$ and $E = (0, +\infty)^d$ for multidimensional diffusions. Non-uniform exponential convergence to quasi-stationary distributions for such processes can be obtained using [12], [35] or [20].

Remark 1. The case where absorption corresponds to the extinction of the whole population, i.e. $\partial = \{(0, \ldots, 0)\}$, can be handled combining the results of the present paper and those known in the one-dimensional case [13, 14] following the methods of [4, Thm. 1.1]. This case was also considered in [16].

A Lotka-Volterra birth and death process in dimension $d \geq 2$ is a Markov process $(X_t, t \geq 0)$ on $\mathbb{Z}_+^d$ with transition rates $q_{n,m}$ from $n = (n_1, \ldots, n_d) \in \mathbb{Z}_+^d$ to $m \neq n$ in $\mathbb{Z}_+^d$ given by

$$q_{n,m} = \begin{cases} n_i(\lambda_i + \sum_{j=1}^d \gamma_{ij} n_j) & \text{if } m = n + e_i, \text{ for some } i \in \{1, \ldots, d\} \\ n_i(\mu_i + \sum_{j=1}^d c_{ij} n_j) & \text{if } m = n - e_i, \text{ for some } i \in \{1, \ldots, d\} \\ 0 & \text{otherwise,} \end{cases}$$

where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ where the 1 is at the i-th coordinate. Note that the set $\partial = \mathbb{Z}_+^d \setminus \mathbb{N}^d$ is absorbing for the process. We make the usual convention that

$$q_{n,n} := -q_n := -\sum_{m \neq n} q_{n,m}.$$

From the biological point of view, the constant $\lambda_i > 0$ is the birth rate per individual of type $i \in \{1, \ldots, d\}$, the constant $\mu_i > 0$ is the death rate per individual of type i, $c_{ij} \geq 0$ is the rate of death of an individual of type i from competition with an individual of type j, and $\gamma_{ij} \geq 0$ is the rate of birth of an individual of type i from cooperation with (or predation of) an individual of type j. In general, a Lotka-Volterra process could be explosive if some of the γ_{ij} are positive, but the assumptions of the next theorem ensure that it is not the case and that the process is almost surely absorbed in finite time.

Theorem 1.1. *Consider a competitive Lotka-Volterra birth and death process* $(X_t, t \geq 0)$ *in* $\mathbb{Z}_+^d$ *as above. Assume that the matrix* $(c_{ij} - \gamma_{ij})_{1 \leq i,j \leq d}$ *defines a positive operator on* $\mathbb{R}_+^d$ *in the sense that, for all* $(x_1, \ldots, x_d) \in \mathbb{R}_+^d \setminus \{0\}$, $\sum_{ij} x_i(c_{ij} - \gamma_{ij})x_j > 0$. *Then the process has a unique quasi-stationary distribution* ν_{QSD} *and there exist constants* $C, \gamma > 0$ *such that, for all probability measures* μ *on* $E = \mathbb{N}^d$,

$$\|\mathbb{P}_\mu(X_t \in \cdot \mid t < \tau_\partial) - \nu_{QSD}\|_{TV} \leq Ce^{-\gamma t}, \quad \forall t \geq 0.$$

An important difficulty here is the fact that the absorption rate (i.e. the rate of jump from a state in E to a state in ∂) is not bounded. Birth and death processes with bounded absorption rates are much easier to study, cf. e.g. [9]. The existence of a quasi-stationary distribution for this kind of multi-dimensional birth and death processes can also be obtained using the theory of R-positive matrices, as exposed in [19], but without the uniform exponential convergence (1.1).

In Section 5, we consider general competitive Feller diffusions, which extend the classical example of Lotka-Volterra Feller diffusion process in dimension $d \geq 2$, defined as the solution $X_t = (X_t^1, \ldots, X_t^d)$ to the stochastic differential equation

$$dX_t^i = \sqrt{\gamma_i X_t^i} dB_t^i + X_t^i \left(r_i - \sum_{j=1}^d c_{ij} X_t^j \right) dt, \quad \forall i \in \{1, \ldots, d\}, \tag{1.3}$$

where $(B_t^1, t \geq 0), \ldots, (B_t^d, t \geq 0)$ are independent standard Brownian motions. The Brownian terms and the linear drift terms correspond to classical Feller diffusions, and the quadratic drift terms correspond to Lotka-Volterra interactions between coordinates of the process. The variances per individual γ_i are positive numbers, and the growth rates per individual r_i can be any real number, for all $1 \leq i \leq d$. The parameters c_{ij} are assumed nonnegative for all $1 \leq i, j \leq d$, which corresponds to competitive Lotka-Volterra interaction. It is well known that, in this case, there is global strong existence and pathwise uniqueness for the SDE (1.3), and that it is almost surely absorbed in finite time in $\partial = \mathbb{R}_+^d \setminus (0, +\infty)^d$ if $c_{ii} > 0$ for all $i \in \{1, \ldots, d\}$ (see [4] and Section 5).

Theorem 1.2. *Consider a competitive Lotka-Volterra Feller diffusion* $(X_t, t \geq 0)$ *in* $\mathbb{R}_+^d$ *as above. Assume that* $c_{ii} > 0$ *for all* $i \in \{1, \ldots, d\}$. *Then the process has a unique quasi-stationary distribution* ν_{QSD} *and there exist constants* $C, \gamma > 0$ *such that, for all probability measures* μ *on* $E = (0, \infty)^d$,

$$\|\mathbb{P}_\mu(X_t \in \cdot \mid t < \tau_\partial) - \nu_{QSD}\|_{TV} \leq Ce^{-\gamma t}, \quad \forall t \geq 0. \tag{1.4}$$

This results was previously known [4] when the process (after some transformations) is a Kolmogorov diffusion (i.e. of the form $dY_t = dW_t - \nabla V(Y_t)dt$ for some Brownian motion W and some C^2 function V). In dimension 2, this holds if the constants c_{ij} and γ_{ij} satisfy $c_{12}\gamma_1 = c_{21}\gamma_2$. Our result is valid in any dimension and has no restriction on the coefficients. One can also expect to extend our result to cooperative cases (e.g. with $c_{21} < 0$ and $c_{12} < 0$ as in [4]) by using our abstract Lyapunov criterion with functions combining those used to prove Theorems 1.1 and 1.2. Another motivations of our study comes from [1], where the coming down from infinity of Lotka-Volterra Feller diffusions is studied. It appears that such processes may go extinct far from compact sets for very large initial conditions. Theorem 1.2 proves that this does not prevent the conditioned process to come back to compact sets fast.

Our general non-linear Lyapunov criterion, Theorem 2.4, is stated in Section 2 and proved in Section 3. Sections 4 and 5 are devoted to the study of (extensions of) competitive Lotka-Volterra birth and death processes and competitive Lotka-Volterra Feller diffusions and to the proofs of Theorems 1.1 and 1.2.

Since the publication of the first version of this preprint on Arxiv, the criteria presented here have been applied to other models, including diffusion models of deadlocks in distributed algorithms [8] and stochastic reaction networks [21], with a weakened condition for birth and death processes.

2 A general Lyapunov criterion for uniform exponential convergence of conditional distributions

Our general framework is inspired from [31].

2.1 Definitions and notations

We consider a càdlàg (right continuous with left limits) time-homogeneous strong Markov process $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (X_t, t \geq 0), (\mathbb{P}_x)_{x\in E\cup\{\partial\}})$ with state space $E \cup \partial$, which is assumed to be a metric space with distance function d, equipped with its Borel σ-fields and such that E is measurable and $E \cap \partial = \emptyset$. In what follows, o is a fixed arbitrary point in E and we define the open sets

$$O_n = \{x \in E : d(x, \partial) > 1/(n+1) \text{ and } d(x, o) < n+1\},$$

for all $n \geq 1$.

We assume that $E \cap \partial = \emptyset$, that ∂ is an absorbing set for the process and we introduce

$$\tau_\partial = \inf\{t \geq 0,\ X_t \in \partial\}.$$

We assume that $\tau_\partial < \infty$ a.s. and that the process is regularly absorbed, in the sense that

$$\tau_\partial := \lim_{n \to +\infty} T_n, \text{ where } T_n := \inf\{t \geq 0, X_t \notin O_n\} \tag{2.1}$$

and that, for all $x \in E$ and $t \geq 0$, $\mathbb{P}_x(t < \tau_\partial) > 0$.

We also assume that, for any closed set C, the entry time in C defined by $\tau_C = \inf\{t \geq 0 : X_t \in C\}$, are $(\mathcal{F}_t)_{t \geq 0}$-stopping times.[1] In particular, since $(E \cup \partial) \setminus O_n$ is closed, T_n and thus τ_∂ are $(\mathcal{F}_t)_{t \geq 0}$-stopping times.

We shall make use of the following weakened notion of generator for X, inspired from [31] and which extends the usual weak infinitesimal generator [18].

Definition 2.1. *We say that a measurable function $V : E \cup \partial \to \mathbb{R}$ belongs to the domain $\mathcal{D}(L)$ of the weakened generator L of X if there exists a measurable function $W : E \to \mathbb{R}$ such that, for all $n \in \mathbb{N}$, $t \geq 0$ and $x \in E$,*

$$\mathbb{E}_x \int_0^{t \wedge T_n} |W(X_s)|\, ds < \infty \text{ and } \mathbb{E}_x V(X_{t \wedge T_n}) = V(x) + \mathbb{E}_x \left[\int_0^{t \wedge T_n} W(X_s) ds \right], \tag{2.2}$$

and we define $LV = W$ on E. We also define $LV(x) = 0$ for all $x \in \partial$.

Due to the local form of the above Dynkin formula, it is much easier to check that V belongs to $\mathcal{D}(L)$ than to the usual domain of the weak infinitesimal generator.

We also define the set of admissible functions to which our Lyapunov criterion applies. We will say that a measurable function on E or $E \cup \partial$ is *locally bounded* if it is bounded on O_n for all $n \geq 1$.

Definition 2.2. *We say that a pair (V, φ) of functions V and φ measurable from $E \cup \partial$ to $\mathbb{R}_+$ is an* admissible pair of functions *if*

(i) *V and φ are bounded, identically 0 on ∂, positive on E, and $1/V$ and $1/\varphi$ are locally bounded on E.*

[1] One can easily adapt our proofs to cases where entry times in other sets (e.g. open) are strong Markov times for the process.

(ii) *We have the convergences*

$$\lim_{n\to+\infty} \inf_{x\in E\setminus O_n} \frac{V(x)}{\varphi(x)} = +\infty \tag{2.3}$$

and

$$\lim_{n\to+\infty} V(X_{T_n}) = 0 \quad a.s. \tag{2.4}$$

(iii) *V and φ belong to the domain of the weakened generator L of X, LV is bounded from above and $L\varphi$ is bounded from below.*

The main point of this definition of admissible functions is the following result, whose technical proof is postponed to Section 3.

Proposition 2.3. *Assume (V,φ) is a pair of admissible functions. Then, for all $x \in E$ and $t \geq 0$,*

$$\frac{\mathbb{E}_x[LV(X_s)]}{\mathbb{E}_x[\varphi(X_s)]} - \frac{\mathbb{E}_x[V(X_s)]}{\mathbb{E}_x[\varphi(X_s)]}\frac{\mathbb{E}_x[L\varphi(X_s)]}{\mathbb{E}_x[\varphi(X_s)]} \in L^1([0,t])$$

and

$$\frac{\mathbb{E}_x[V(X_t)]}{\mathbb{E}_x[\varphi(X_t)]} = \frac{V(x)}{\varphi(x)} + \int_0^t \left\{ \frac{\mathbb{E}_x[LV(X_s)]}{\mathbb{E}_x[\varphi(X_s)]} - \frac{\mathbb{E}_x[V(X_s)]}{\mathbb{E}_x[\varphi(X_s)]}\frac{\mathbb{E}_x[L\varphi(X_s)]}{\mathbb{E}_x[\varphi(X_s)]} \right\} ds. \tag{2.5}$$

2.2 A non-linear Lyapunov criterion

In the following assumption, we need a pair of functions, while usual *drift conditions* of Foster-Lyapunov criteria only use one (see for instance [31]). Roughly speaking, the function V is used to control return time in compact sets from neighborhood of the boundary, while the second one, φ, is used to control the absorption rate. One of the main difficulty when checking the following assumption, is that V needs also to be related to the absorption probability, via the inequality (2.7).

Assumption 1. *There exist a pair of admissible functions (V,φ) and constants $k_0 \in \mathbb{N}$, $C_1, C_2, C_3, \varepsilon > 0$ such that*

$$-L\varphi \leq C_1 \mathbb{1}_{O_{k_0}} \text{ and } LV + C_2 \frac{V^{1+\varepsilon}}{\varphi^{\varepsilon}} \leq C_3\varphi. \tag{2.6}$$

and there exist constants $r_0, p_0 > 0$ such that

$$\mathbb{P}_x(r_0 < \tau_\partial) \leq p_0 V(x), \quad \forall x \in E \setminus O_{k_0}. \tag{2.7}$$

The main role of the first part in the above assumption is to bound the derivative computed in Proposition 2.3, which will allow us to show that the quantity $\frac{\mathbb{E}_x[V(X_t)]}{\mathbb{E}_x[\varphi(X_t)]}$ is uniformly bounded in $x \in E$ for t large enough. The second part of this assumption is needed to check that the boundedness of $\frac{\mathbb{E}_x[V(X_t)]}{\mathbb{E}_x[\varphi(X_t)]}$ implies that, conditionally on $t < \tau_\partial$, $X_t \in O_n$ with high probability for n large. This implies that the problem can be localized in this set O_n, so that, in order to check Conditions (A1–A2), it is enough to assume the following local versions of (A1–A2), which are much easier to check.

Assumption 2. *There exists a probability measure ν on E such that, for all $n \geq 1$, there exist $a_n > 0$ and $\theta_n > 0$ satisfying*

$$\mathbb{P}_x(X_{\theta_n} \in \cdot) \geq a_n \nu, \quad \textit{for all } x \in O_n.$$

In addition, for all $n \geq 0$, there exists a constant D_n such that, for all $t \geq 0$,

$$\sup_{x \in O_n} \mathbb{P}_x(t < \tau_\partial) \leq D_n \inf_{x \in O_n} \mathbb{P}_x(t < \tau_\partial). \tag{2.8}$$

The Lyapunov criterion of Assumption 1 will be useful to check that the conditioned process comes back quickly in bounded subsets of E from any neighborhood of the boundary. However, it does not imply this property for initial distribution in a neighborhood of infinity. This is the purpose of the next assumption. Since it only concerns the unconditioned process, it may be proved using usual drift conditions or probabilistic arguments, as we illustrate in the two examples of Sections 4 and 5.

Assumption 3. *For all $\lambda > 0$, there exists $n \geq 1$ such that*

$$\sup_{x \in E} \mathbb{E}_x(e^{\lambda(S_n \wedge \tau_\partial)}) < \infty. \tag{2.9}$$

where $S_n = \inf\{t \geq 0 : X_t \in \overline{O_n}\}$.

It is well known [31] that this assumption is implied by the existence, for all $\lambda > 0$, of a bounded Lyapunov function $\psi_\lambda : E \cup \{\partial\} \to [1, +\infty)$ such that

$$L\psi_\lambda(x) \leq -\lambda \psi_\lambda(x) + C \mathbb{1}_{\overline{O_n}}$$

for some $n \geq 1$. Indeed in this case,

$$e^{\lambda t}\psi_\lambda(X_t) - \int_0^t e^{\lambda s} \left[\lambda \psi_\lambda(X_s) + L\psi_\lambda(X_s)\right] ds$$

is a local martingale and $e^{\lambda(t\wedge S_n\wedge\tau_\partial)}\psi_\lambda(X_{t\wedge S_n\wedge\tau_\partial})$ is a super-martingale in L^∞. Fatou's lemma then implies that $\mathbb{E}_x(e^{\lambda(S_n\wedge\tau_\partial)}) \leq \psi_\lambda(x)$. Depending on the situations it might be simpler to check Assumption 3 directly using for instance coupling arguments as we do for example for Theorem 5.1 below, or to find a Lyapunov function.

We can now state our main general result. Its proof is given in Section 3.

Theorem 2.4. *Assume that the process $(X_t, t \geq 0)$ is regularly absorbed and that there exists a pair of admissible functions (V, φ) satisfying Assumption 1. Assume also that Assumptions 2 and 3 are satisfied. Then the process X admits a unique quasi-stationary distribution ν_{QSD} and there exist constants $C, \gamma > 0$ such that for all probability measure μ on E,*

$$\|\mathbb{P}_\mu(X_t \in \cdot \mid t < \tau_\partial) - \nu_{QSD}\|_{TV} \leq Ce^{-\gamma t}, \quad \forall t \geq 0. \tag{2.10}$$

3 Proof of the results of Section 2

We first give the proof of Proposition 2.3 in Subsection 3.1 and the proof of Theorem 2.4 in Subsection 3.2. The proofs of two technical lemmas are given in Subsections 3.3 and 3.4.

3.1 Proof of Proposition 2.3

In what follows, we use the classical definition of the integral of a signed function f with respect to a positive measure μ by $\mu(f_+) - \mu(f_-) \in [-\infty, +\infty]$ (where f_+ and f_- denote respectively the positive and negative parts of f), which is well defined as soon as at least one of the two terms is finite. Classical results (as Lebesgue's theorem, Fatou's lemma and Fubini's theorem) still hold in this case.

Using the Definition 2.1 of the weakened infinitesimal generator, we have for all $n \geq 1$

$$\mathbb{E}_x V(X_{t\wedge T_n}) = V(x) + \mathbb{E}_x \int_0^{t\wedge T_n} LV(X_s)ds = V(x) + \mathbb{E}_x \int_0^t LV(X_s)\mathbb{1}_{s<T_n}ds. \tag{3.1}$$

Note that

$$\mathbb{E}_x V(X_{t\wedge T_n}) = \mathbb{E}_x\left(\mathbb{1}_{t<T_n}V(X_t)\right) + \mathbb{E}_x\left(\mathbb{1}_{T_n\leq t}V(X_{T_n})\right).$$

Using (2.1), the Assumption (2.4) and that $V(X_{T_n})$ is uniformly bounded, Lebesgue's theorem implies that

$$\lim_{n\to+\infty} \mathbb{E}_x V(X_{t\wedge T_n}) = \mathbb{E}_x\left(\mathbb{1}_{t<\tau_\partial}V(X_t)\right) = \mathbb{E}_x(V(X_t)). \tag{3.2}$$

Now Fatou's lemma applied to the right-hand side of (3.1) (using that LV is bounded from above) gives

$$\mathbb{E}_x V(X_t) \leq V(x) + \mathbb{E}_x \left[\int_0^{t\wedge\tau_\partial} LV(X_s)\, ds \right] = V(x) + \int_0^t \mathbb{E}_x \left[LV(X_s) \right] ds,$$

where we have used Fubini's theorem for the last inequality. Since $V \geq 0$ and LV is bounded from above, we deduce that $\mathbb{E}_x LV(X_s) \in L^1([0,t])$ and that $V(X_t) \in L^1(\Omega)$. Therefore, we can actually apply Lebesgue's Theorem to the right-hand side of (3.1) and hence

$$\mathbb{E}_x V(X_t) = V(x) + \int_0^t \mathbb{E}_x \left[LV(X_s) \right] ds.$$

The same argument applies to φ (note that (2.3) and (2.4) imply that $\lim \varphi(X_{T_n}) = 0$ a.s.):

$$\mathbb{E}_x \varphi(X_t) = \varphi(x) + \int_0^t \mathbb{E}_x \left[L\varphi(X_s) \right] ds. \tag{3.3}$$

Therefore $\mathbb{E}_x V(X_t)$ and $\mathbb{E}_x \varphi(X_t)$ are continuous with respect to t and (cf. e.g. [2, Lem. VIII.2]), for all $T > 0$, $t \mapsto (\mathbb{E}_x V(X_t), \mathbb{E}_x \varphi(X_t))$ belongs to the Sobolev space $W^{1,1}([0,T], \mathbb{R}^2)$ (the set of functions from $[0,T]$ to $\mathbb{R}^2$ in L^1 admitting a derivative in the sense of distributions in L^1).

In particular, since $\mathbb{P}_x(t < \tau_\partial) > 0$ and hence $\mathbb{E}_x \varphi(X_t) > 0$ for all $t \in [0,T]$, we deduce from the continuity of $t \mapsto \mathbb{E}_x \varphi(X_t)$ that $\inf_{t\in[0,T]} \mathbb{E}_x \varphi(X_t) > 0$. Therefore, we deduce from standard properties of $W^{1,1}$ functions [2, Cor. VIII.9 and Cor. VIII.10] that $t \mapsto \mathbb{E}_x V(X_t)/\mathbb{E}_x \varphi(X_t)$ also belongs to $W^{1,1}([0,T], \mathbb{R})$ and admits as derivative

$$t \mapsto \frac{\mathbb{E}_x LV(X_t)}{\mathbb{E}_x \varphi(X_t)} - \frac{\mathbb{E}_x V(X_t)\, \mathbb{E}_x L\varphi(X_t)}{[\mathbb{E}_x \varphi(X_t)]^2} \in L^1([0,T]).$$

Hence we have proved (2.5).

3.2 Proof of Theorem 2.4

The proof is based on two lemmas. The first one combines Proposition 2.3 and Assumption (2.6) to give uniform (in x) controls on $\frac{\mathbb{E}_x V(X_t)}{\mathbb{E}_x \varphi(X_t)}$ for t large enough. Its proof is given in Subsection 3.3.

Lemma 3.1. *There exists two positive constants A and B such that, for all $x \in E$ and all $s \geq 0$,*

$$\frac{\mathbb{E}_x(LV(X_s))}{\mathbb{E}_x(\varphi(X_s))} - \frac{\mathbb{E}_x(V(X_s))}{\mathbb{E}_x(\varphi(X_s))}\frac{\mathbb{E}_x(L\varphi(X_s))}{\mathbb{E}_x(\varphi(X_s))} \leq A - B\left(\frac{\mathbb{E}_x(V(X_s))}{\mathbb{E}_x(\varphi(X_s))}\right)^{1+\varepsilon}. \tag{3.4}$$

In particular, there exists $t_0 > 0$ such that, for all $x \in E$ and all $t \geq t_0$,

$$\frac{\mathbb{E}_x V(X_t)}{\mathbb{E}_x \varphi(X_t)} \leq \left(\frac{2A}{B}\right)^{1/(1+\varepsilon)} + 2At_0. \tag{3.5}$$

The second lemma makes use of Assumption (2.7) to deduce from Lemma 3.1 the following inequality. Its proof is given in Subsection 3.4.

Lemma 3.2. *Enlarging k_0 in Assumption 1 if necessary, there exists $D > 0$ such that, for all $t \geq t_0$ and all $x \in E$,*

$$\mathbb{E}_x[V(X_t)] \leq D\,\mathbb{E}_x[V(X_t)\mathbb{1}_{X_t \in O_{k_0}}]. \tag{3.6}$$

We prove Theorem 2.4 by checking that the two conditions (A1) and (A2) in the introduction are satisfied (cf. [13]).

Step 1: Proof of (A1).
We first remark that there exists $m_0 \geq 0$ such that $\nu(O_{m_0}) > 0$ (where ν and θ_n below are from Assumption 2) and hence such that, for all $n \geq 1$, all $x \in O_n$ and all $k \geq 1$,

$$\mathbb{P}_x(X_{\theta_n + k\theta_{m_0}} \in \cdot) \geq a_n\,\nu(O_{m_0}) \inf_{y \in O_{m_0}} \mathbb{P}_y(X_{k\theta_{m_0}} \in \cdot) \geq a_n a_{m_0}^k \nu(O_{m_0})^k \nu(\cdot), \tag{3.7}$$

where we used Markov's property and an induction procedure over k.

In particular, we can assume without loss of generality that $\theta_{k_0} \geq r_0$ (where r_0 is from Assumption 1 and k_0 satisfies Assumption 1 and Lemma 3.2). As a consequence, Assumption (2.7), Inequality (3.6) and Markov's property entail that, for all $t \geq t_0$ (where t_0 is from Lemma 3.1),

$$\begin{aligned}\mathbb{P}_x(t + \theta_{k_0} < \tau_\partial) &\leq \mathbb{P}_x(t + r_0 < \tau_\partial) \leq D\,p_0\,\mathbb{E}_x[V(X_t)\mathbb{1}_{X_t \in O_{k_0}}] \\ &\leq D\,p_0 \sup_{O_{k_0}} V\,\mathbb{P}_x(X_t \in O_{k_0}).\end{aligned}$$

On the other hand,

$$\mathbb{P}_x(X_{t+\theta_{k_0}} \in \cdot) \geq \mathbb{P}_x(X_t \in O_{k_0}) \inf_{y \in O_{k_0}} \mathbb{P}_y(X_{\theta_{k_0}} \in \cdot) \geq \mathbb{P}_x(X_t \in O_{k_0})\, a_{k_0}\, \nu(\cdot).$$

Setting $t_0' = t_0 + \theta_{k_0}$, the two above equations imply that, for all $x \in E$,

$$\mathbb{P}_x(X_{t_0'} \in \cdot \mid t_0' < \tau_\partial) \geq \frac{a_{k_0}}{Dp_0 \sup_{O_{k_0}} V}\, \nu(\cdot),$$

and hence that Assumption (A1) holds true.

Step 2: Proof of (A2).
Using (3.7), for all $n \geq m_0$, all $x \in O_n$ and all $k \geq 1$, we obtain that, for all $t \in [\theta_n + k\theta_{m_0}, \theta_n + (k+1)\theta_{m_0})$,

$$\begin{aligned}\mathbb{P}_x(t + s < \tau_\partial) &\geq \mathbb{P}_x(X_{\theta_n + (k+1)\theta_{m_0}} \in O_{m_0}) \inf_{y \in O_{m_0}} \mathbb{P}_y(s < \tau_\partial) \\ &\geq a_n a_{m_0}^{k+1} \nu(O_{m_0})^{k+2} \inf_{y \in O_n} \mathbb{P}_y(s < \tau_\partial).\end{aligned}$$

Setting $\lambda := -\ln(a_{m_0}\nu(O_{m_0}))/\theta_{m_0}$ and using inequality (2.8) of Assumption 2, we deduce that, for all $n \geq 1$ and all $s, t \geq 0$,

$$\inf_{x \in O_n} \mathbb{P}_x(t + s < \tau_\partial) \geq \frac{a_n a_{m_0} \nu(O_{m_0})^2}{D_n} \exp(-\lambda t) \sup_{x \in O_n} \mathbb{P}_x(s < \tau_\partial). \tag{3.8}$$

Now, we apply Assumption 3 for λ as defined above: there exists $n \geq m_0$ such that

$$M := \sup_{x \in E} \mathbb{E}_x[\exp(\lambda(S_{n-1} \wedge \tau_\partial))] < \infty. \tag{3.9}$$

Note that, since X_t is càdlàg, $X_{S_{n-1}} \in \overline{O}_{n-1} \subset O_n$ on the event $\{S_{n-1} < \infty\}$. Hence, using (3.9) and the strong Markov property at time S_{n-1} (which is a stopping time since it is the entry time in a closed set), for all $x \in E$,

$$\begin{aligned}\mathbb{P}_x(t < \tau_\partial) &= \mathbb{P}_x(t < S_{n-1} \wedge \tau_\partial) + \mathbb{P}_x(S_{n-1} \leq t < \tau_\partial) \\ &\leq M e^{-\lambda t} + \int_0^t \sup_{y \in O_n} \mathbb{P}_y(t - s < \tau_\partial) \mathbb{P}_x(S_{n-1} \in ds).\end{aligned}$$

Note that, if $S_{n-1} < \infty$, then $S_{n-1} < \tau_\partial$ and $S_{n-1} = S_n \wedge \tau_\partial)$. Thus, for all $s \leq t$, $\mathbb{P}_y(S_{n-1} \in ds) = \mathbb{P}_y(S_{n-1} \wedge \tau_\partial \in ds,\, S_{n-1} < \infty) \leq \mathbb{P}_y(S_{n-1} \wedge \tau_\partial \in ds)$.

Hence, using (3.8) twice, we have for all $x \in E$

$$\begin{aligned}
\mathbb{P}_x(t < \tau_\partial) &\leq Me^{-\lambda t} + \int_0^t \sup_{y \in O_n} \mathbb{P}_y(t-s < \tau_\partial)\mathbb{P}_x(S_{n-1} \wedge \tau_\partial \in ds) \\
&\leq \frac{D_n}{a_n a_{m_0} \nu(O_{m_0})^2} \Bigg[M \inf_{y \in O_n} \mathbb{P}_y(t < \tau_\partial) \\
&\qquad\qquad + \inf_{y \in O_n} \mathbb{P}_y(t < \tau_\partial) \int_0^t e^{\lambda s} \mathbb{P}_x(S_{n-1} \wedge \tau_\partial \in ds) \Bigg] \\
&\leq \frac{2MD_n}{a_n a_{m_0} \nu(O_{m_0})^2} \inf_{y \in O_n} \mathbb{P}_y(t < \tau_\partial) \\
&\leq \frac{2MD_n}{a_n a_{m_0} \nu(O_{m_0})^2 \nu(O_n)} \mathbb{P}_\nu(t < \tau_\partial).
\end{aligned}$$

Since $O_n \supset O_{m_0}$, we have $\nu(O_n) \geq \nu(O_{m_0}) > 0$, so the last inequality implies (A2).

3.3 Proof of Lemma 3.1

Step 1: Proof of (3.4).
Fix $x \in E$ and $s \geq 0$. On the one hand, it follows from (2.6) that

$$-\mathbb{E}_x(L\varphi(X_s)) \leq \frac{C_1}{\inf_{O_{k_0}} \varphi} \mathbb{E}_x(\varphi(X_s)).$$

and hence

$$-\frac{\mathbb{E}_x(L\varphi(X_s))}{\mathbb{E}_x(\varphi(X_s))} \mathbb{E}_x(V(X_s)) \leq \frac{C_1}{\inf_{O_{k_0}} \varphi} \mathbb{E}_x(V(X_s)) \tag{3.10}$$

On the other hand, we deduce from (2.6) that

$$\begin{aligned}
\mathbb{E}_x(LV(X_s)) &\leq C_3 \mathbb{E}_x(\varphi(X_s)) - C_2 \mathbb{E}_x\left(\frac{V(X_s)^{1+\varepsilon}}{\varphi(X_s)^\varepsilon}\right) \\
&\leq C_3 \mathbb{E}_x(\varphi(X_s)) - \frac{C_2}{2} \mathbb{E}_x\left(\frac{V(X_s)^{1+\varepsilon}}{\varphi(X_s)^\varepsilon}\right) - \frac{C_2}{2} \frac{\mathbb{E}_x(V(X_s))^{1+\varepsilon}}{\mathbb{E}_x\left(\varphi(X_s)\right)^\varepsilon}.
\end{aligned} \tag{3.11}$$

where we used Hölder's inequality to deduce that

$$\mathbb{E}_x(V(X_s))^{1+\varepsilon} \leq \mathbb{E}_x\left(\frac{V(X_s)^{1+\varepsilon}}{\varphi(X_s)^\varepsilon}\right) \mathbb{E}_x\left(\varphi(X_s)\right)^\varepsilon.$$

Now, because of Assumption (2.3), there exists m large enough such that, for all $y \in E \setminus O_m$,

$$\frac{C_2 V^\varepsilon(y)}{2\varphi^\varepsilon(y)} \geq \frac{C_1}{\inf_{O_{k_0}} \varphi}.$$

Therefore, for such a value of m,

$$\begin{aligned}
&\frac{C_2}{2}\mathbb{E}_x\left(\frac{V(X_s)^{1+\varepsilon}}{\varphi(X_s)^\varepsilon}\right) \\
&\quad = \frac{C_2}{2}\mathbb{E}_x\left(\frac{V(X_s)^{1+\varepsilon}}{\varphi(X_s)^\varepsilon}\mathbb{1}_{X_s\in O_m}\right) + \frac{C_2}{2}\mathbb{E}_x\left(\frac{V(X_s)^{1+\varepsilon}}{\varphi(X_s)^\varepsilon}\mathbb{1}_{X_s\in E\setminus O_m}\right) \\
&\quad \geq \frac{C_1}{\inf_{O_{k_0}} \varphi}\,\mathbb{E}_x\left(V(X_s)\mathbb{1}_{X_s\in E\setminus O_m}\right) \\
&\quad \geq \frac{C_1}{\inf_{O_{k_0}} \varphi}\,\mathbb{E}_x V(X_s) - \frac{C_1 \sup_{O_m} V}{\inf_{O_{k_0}} \varphi \inf_{O_m} \varphi}\,\mathbb{E}_x\varphi(X_s).
\end{aligned}$$

Finally, we obtain from the last inequality, (3.10) and (3.11) that there exists two positive constants $A, B > 0$ such that

$$\frac{\mathbb{E}_x(LV(X_s))}{\mathbb{E}_x(\varphi(X_s))} - \frac{\mathbb{E}_x(V(X_s))}{\mathbb{E}_x(\varphi(X_s))}\frac{\mathbb{E}_x(L\varphi(X_s))}{\mathbb{E}_x(\varphi(X_s))} \leq A - B\left(\frac{\mathbb{E}_x(V(X_s))}{\mathbb{E}_x(\varphi(X_s))}\right)^{1+\varepsilon}$$

Step 2: Proof of (3.5).
We define $a = (2A/B)^{1/(1+\varepsilon)}$ and $t_0 = \frac{4}{\varepsilon B a^\varepsilon}$. Propositions 2.3 and (3.4) imply that, for all $t \geq 0$ and all $x \in E$,

$$\frac{\mathbb{E}_x V(X_t)}{\mathbb{E}_x\varphi(X_t)} \leq \frac{V(x)}{\varphi(x)} + At - B\int_0^t \left(\frac{\mathbb{E}_x V(X_s)}{\mathbb{E}_x\varphi(X_s)}\right)^{1+\varepsilon} ds. \tag{3.12}$$

Since $\varepsilon > 0$, this implies that, for all $x \in E$, there exists $u_x \in [0, t_0]$ such that $\frac{\mathbb{E}_x V(X_{u_x})}{\mathbb{E}_x\varphi(X_{u_x})} < a$ for any $x \in E$. We prove this by contradiction: assume on the contrary that for all $s \in [0, t_0]$, $\frac{\mathbb{E}_x V(X_s)}{\mathbb{E}_x\varphi(X_s)} \geq a$. Then, for all $t \in [0, t_0]$,

$$\frac{\mathbb{E}_x V(X_t)}{\mathbb{E}_x\varphi(X_t)} \leq \frac{V(x)}{\varphi(x)} - \frac{B}{2}\int_0^t \left(\frac{\mathbb{E}_x V(X_s)}{\mathbb{E}_x\varphi(X_s)}\right)^{1+\varepsilon} ds.$$

Integrating this differential inequality up to time t_0 entails

$$\frac{\mathbb{E}_x V(X_{t_0})}{\mathbb{E}_x\varphi(X_{t_0})} \leq \left[\left(\frac{\varphi(x)}{V(x)}\right)^\varepsilon + \frac{\varepsilon B t_0}{2}\right]^{-1/\varepsilon} \leq \left(\frac{2}{\varepsilon B t_0}\right)^{1/\varepsilon} = \frac{a}{2},$$

which gives a contradiction.

Hence, using Proposition 2.3 and (3.4) again, we deduce that, for all $t \in [t_0, 2t_0]$ and all $x \in E$,

$$\begin{aligned}\frac{\mathbb{E}_x(V(X_t))}{\mathbb{E}_x(\varphi(X_t))} &\leq \frac{\mathbb{E}_x V(X_{u_x})}{\mathbb{E}_x \varphi(X_{u_x})} + A(t-u_x) - B\int_{u_x}^t \left(\frac{\mathbb{E}_x V(X_s)}{\mathbb{E}_x \varphi(X_s)}\right)^{1+\varepsilon} ds \\ &\leq a + A(t-u_x) \leq a + A\,2\,t_0.\end{aligned}$$

Using the same argument repetitively between time kt_0 (instead of 0) and $(k+2)t_0$ (instead of $2t_0$) gives the result for all time $t \geq t_0$.

3.4 Proof of Lemma 3.2

Set $a' = \left(\frac{2A}{B}\right)^{1/(1+\varepsilon)} + 3At_0$. Equation (2.3) implies that, enlarging k_0 in Assumption 1 if necessary,

$$\inf_{y \in E \setminus O_{k_0}} \frac{V(y)}{\varphi(y)} \geq 2a'.$$

Lemma 3.1 implies that, for all $t \geq t_0$,

$$\begin{aligned}a' &> \frac{\mathbb{E}_x V(X_t)}{\mathbb{E}_x \varphi(X_t)} \\ &\geq \frac{\mathbb{E}_x[V(X_t)\mathbb{1}_{X_t \in E\setminus O_{k_0}}] + \mathbb{E}_x[V(X_t)\mathbb{1}_{X_t \in O_{k_0}}]}{\sup_{y\in E\setminus O_{k_0}} \frac{\varphi(y)}{V(y)}\mathbb{E}_x[V(X_t)\mathbb{1}_{X_t\in E\setminus O_{k_0}}] + \sup_{y\in E}\frac{\varphi(y)}{V(y)}\mathbb{E}_x[V(X_t)\mathbb{1}_{X_t\in O_{k_0}}]} \\ &\geq \frac{\mathbb{E}_x[V(X_t)\mathbb{1}_{X_t \in E\setminus O_{k_0}}] + \mathbb{E}_x[V(X_t)\mathbb{1}_{X_t \in O_{k_0}}]}{\frac{1}{2a'}\mathbb{E}_x[V(X_t)\mathbb{1}_{X_t\in E\setminus O_{k_0}}] + \sup_{y\in E}\frac{\varphi(y)}{V(y)}\mathbb{E}_x[V(X_t)\mathbb{1}_{X_t\in O_{k_0}}]}.\end{aligned}$$

Therefore,

$$\left(a' \sup_{y\in E}\frac{\varphi(y)}{V(y)} - 1\right)\mathbb{E}_x[V(X_t)\mathbb{1}_{X_t\in O_{k_0}}] \geq \frac{1}{2}\mathbb{E}_x[V(X_t)\mathbb{1}_{X_t\in E\setminus O_{k_0}}].$$

Since $a' > \mathbb{E}_x V(X_t)/\mathbb{E}_x\varphi(X_t) \geq 1/\sup_{y\in E}(\varphi(y)/V(y))$, we deduce that there exists a constant $D > 0$ such that

$$\mathbb{E}_x[V(X_t)] \leq D\,\mathbb{E}_x[V(X_t)\mathbb{1}_{X_t\in O_{k_0}}].$$

4 Application to multidimensional birth and death processes absorbed when one of the coordinates hits 0

We consider general multitype birth and death processes in continuous time, taking values in $\mathbb{Z}_+^d$ for some $d \geq 2$. Let $(X_t, t \geq 0)$ be a Markov process on $\mathbb{Z}_+^d$ with transition rates

$$\text{from } n = (n_1, \ldots, n_d) \text{ to } \begin{cases} n + e_j & \text{with rate } n_j b_j(n), \\ n - e_j & \text{with rate } n_j d_j(n) \end{cases}$$

for all $1 \leq j \leq d$, with $e_j = (0, \ldots, 0, 1, 0, \ldots, 0)$, where the nonzero coordinate is the j-th one, $b(n) = (b_1(n), \ldots, b_d(n))$ and $d(n) = (d_1(n), \ldots, d_d(n))$ are functions from $\mathbb{Z}_+^d$ to $(0, +\infty)^d$. This model represents a density-dependent population dynamics with d types of individuals (say d species), where $b_i(n)$ (resp. $d_i(n)$) represents the reproduction rate (resp. death rate) per individuals of species i when the population is in state n.

Note that the forms of the birth and death rates imply that, once a coordinate X_t^j of the process hits 0, it remains equal to 0. This corresponds to the extinction of the population of type j. Hence, the set $\partial := \mathbb{Z}_+^d \setminus \mathbb{N}^d$ is absorbing for the process X.

We define for all $k \geq 1$

$$\bar{d}(k) = \sup_{n \in \mathbb{N}^d,\ |n| = k} |n| \sum_{i=1}^{d} \mathbb{1}_{n_i = 1} d_i(n),$$

$$\text{and} \quad \underline{d}(k) = \inf_{n \in \mathbb{N}^d,\ |n| = k} \sum_{i=1}^{d} n_i \left[\mathbb{1}_{n_i \neq 1} d_i(n) - b_i(n) \right],$$

where $|n| := n_1 + \ldots + n_d$. We shall assume

Assumption 4. *There exists $\eta > 0$ small enough such that, for all $k \in \mathbb{N}$ large enough,*

$$\underline{d}(k) \geq \eta \bar{d}(k), \tag{4.1}$$

and

$$\frac{\underline{d}(k)}{k^{1+\eta}} \xrightarrow[k \to +\infty]{} +\infty. \tag{4.2}$$

Note that, since the set O_n is finite for all n, it is standard to check that any function $f : \mathbb{Z}_+^d \to \mathbb{R}$ is in the domain of the weakened infinitesimal

generator of X and, for all $n \in \mathbb{N}^d$,

$$Lf(n) = \sum_{j=1}^{d} [f(n+e_j) - f(n)] n_j b_j(n) + \sum_{j=1}^{d} [f(n-e_j) - f(n)] n_j d_j(n).$$

Under Assumption (4.2), setting $W(n) = |n|$, we have

$$\frac{LW(n)}{W(n)^{1+\eta}} = -\frac{\sum_{j=1}^{d} n_j (d_j(n) - b_j(n))}{W(n)^{1+\eta}} \leq -\frac{\underline{d}(|n|)}{|n|^{1+\eta}} \to -\infty.$$

This classically entails that

$$\sup_{n \in \mathbb{N}^d} \mathbb{E}_n(|X_1|) < +\infty. \tag{4.3}$$

(The argument is very similar to the one used for Lemma 3.1.) In particular, the process is non-explosive and τ_∂ is finite almost surely. Therefore, the process X is regularly absorbed, as defined in Section 2.1. We can now state the main result of the section.

Theorem 4.1. *Under Assumption 4, the multi-dimensional competitive birth and death process $(X_t, t \geq 0)$ absorbed when one of its coordinates hits 0 admits a unique quasi-stationary distribution ν_{QSD} and there exist constants $C, \gamma > 0$ such that, for all probability measure μ on $\mathbb{N}^d$,*

$$\|\mathbb{P}_\mu(X_t \in \cdot \mid t < \tau_\partial) - \nu_{QSD}\|_{TV} \leq Ce^{-\gamma t}, \quad \forall t \geq 0.$$

As will appear in the proof below, to check our conditions, it is sufficient to take functions V and φ of the form $f(|n|)\mathbb{1}_{\mathbb{N}^d}(n)$ for some $f : \mathbb{N} \to \mathbb{R}_+$. More precisely, the first part of Condition (2.6) can be checked for $\varphi(n) = f(|n|)\mathbb{1}_{\mathbb{N}^d}(n)$ with decreasing f and the second part for $V(n) = g(|n|)\mathbb{1}_{\mathbb{N}^d}(n)$ with increasing g, to take advantage of the drift of the birth and death process towards 0 for large $|n|$. However, let us emphasize that, although the choice $\varphi(n) = \mathbb{1}_{\mathbb{N}^d}$ (i.e. $f \equiv 1$) is natural, it cannot satisfy the first part of Condition (2.6), since in this case $-L\varphi(n) = \sum_{i=1}^{d} \mathbb{1}_{n_i=1} d_i(n)$ (the absorption rate) is unbounded. The direct study of $\mathbb{E}(V(X_t) \mid t < \tau_\partial) = \mathbb{E}[V(X_t)]/\mathbb{E}[\mathbb{1}_{n \in \mathbb{N}^d}(X_t)]$ is possible, but only allows to recover particular cases of Theorem 4.1 (cf. [10, Section 4]). Our criterion is more flexible and better adapted to multidimensional birth and death models of interacting populations.

It is easy to check that Assumption 4 is satisfied in the general Lotka-Volterra birth and death process of the introduction. Indeed, we clearly have $\bar{d}(k) \leq Ck^2$ and

$$\begin{aligned}\underline{d}(k) &\geq \inf_{n\in\mathbb{N}^d,\ |n|=k} \sum_{i=1}^{d}(d_i(n) - b_i(n)) - \sup_{n\in\mathbb{N}^d,\ |n|=k} \sum_{i=1}^{n} d_i(n)\mathbb{1}_{n_i=1} \\ &\geq -Ck + \inf_{n\in\mathbb{N}^d,\ |n|=k} \sum_{i,j=1}^{d} n_i(c_{ij} - \gamma_{ij})n_j\end{aligned}$$

for $C = \max_i \mu_i + \max_i \lambda_i + \max_{i,j} c_{ij}$. Under the assumptions of Theorem 1.1, there exists $C' > 0$ such that, for all $n \in \mathbb{N}^d$,

$$\sum_{i,j=1}^{d} n_i(c_{ij} - \gamma_{ij})n_j \geq C'|n|^2.$$

This entails Assumption 4.

Proof of Theorem 4.1. Using (4.3) and copying the arguments of [13, Sec. 4.1.1 and Thm. 4.1], one deduces that Assumption 2 is satisfied with $\nu = \delta_{(1,\dots,1)}$ and that Assumption 3 is also satisfied.

Hence we only have to find a pair of admissible functions (V, φ) satisfying Assumption 1. This pair of functions is given for all $n \in \mathbb{Z}_+^d$ by

$$V(n) = \begin{cases} \sum_{k=1}^{|n|} \frac{1}{k^\alpha} & \text{if } n \in \mathbb{N}^d, \\ 0 & \text{if } n \in \partial \end{cases}$$

and

$$\varphi(n) = \begin{cases} \sum_{k=|n|+1}^{+\infty} \frac{1}{k^\beta} & \text{if } n \in \mathbb{N}^d, \\ 0 & \text{if } n \in \partial, \end{cases}$$

for appropriate choices of $\alpha, \beta > 1$. Note that the two functions are bounded, nonnegative and positive on $\mathbb{N}^d$. Note also that, since $O_n = \{n \in \mathbb{N}^d, |n| \leq n + d\}$ (taking $o = (1, \dots, 1)$), so Conditions (i-ii) of Definition 2.2 are clearly satisfied (in this discrete state space case, the condition (2.4) is trivial since, almost surely, $T_n = \tau_\partial$ for all n large enough). Note also that, since $\inf_{n\in\mathbb{N}^d} V(n) > 0$, Condition (2.7) is also obviously satisfied.

Hence, we only have to check (2.6) since this of course implies that V and φ satisfy Point (iii) of Definition 2.2. So we compute

$$\begin{aligned} L\varphi(n) &= -\sum_{i=1}^{d} \frac{n_i b_i(n)}{(1+|n|)^\beta} + \sum_{i=1}^{d} \frac{n_i \mathbb{1}_{n_i \neq 1} d_i(n)}{|n|^\beta} - \sum_{i=1}^{d} \mathbb{1}_{n_i=1} d_i(n) \sum_{k=|n|+1}^{+\infty} \frac{1}{k^\beta} \\ &\geq \frac{1}{|n|^\beta} \left[\underline{d}(|n|) - \frac{\bar{d}(|n|)}{\beta - 1} \right], \end{aligned}$$

where we used the fact that

$$\sum_{k=x+1}^{+\infty} \frac{1}{k^\beta} \leq \int_x^{+\infty} \frac{dy}{y^\beta} = \frac{1}{(\beta-1)x^{\beta-1}}.$$

Hence it follows from Assumption (4.1) that there exists $\beta > 1$ large enough such that $L\varphi(n) \geq 0$ for all $|n|$ large enough. This entails the first inequality in (2.6).

We fix such a value of β. Using that

$$\sup_{n \in \mathbb{N}^d} V(n) = \sum_{k=1}^{+\infty} \frac{1}{k^\alpha} \leq 1 + \int_1^\infty \frac{dx}{x^\alpha} = \frac{\alpha}{\alpha - 1}$$

and

$$\varphi(n) \geq \int_{|n|+1}^{\infty} \frac{dx}{x^\beta} = \frac{(1+|n|)^{1-\beta}}{\beta - 1} \geq \frac{|n|^{1-\beta}}{2(\beta-1)}$$

for $|n|$ large enough, we compute for such n

$$\begin{aligned} LV(n) + \frac{V^{1+\varepsilon}(n)}{\varphi^\varepsilon(n)} &\leq \sum_{i=1}^{d} \frac{n_i b_i(n)}{(|n|+1)^\alpha} - \sum_{i=1}^{d} \frac{n_i d_i(n) \mathbb{1}_{n_i \neq 1}}{|n|^\alpha} + C|n|^{\varepsilon(\beta-1)} \\ &\leq -\frac{\underline{d}(|n|)}{|n|^\alpha} + C|n|^{\varepsilon(\beta-1)}, \end{aligned}$$

where $C = [\alpha/(\alpha-1)]^{1+\varepsilon}[2(\beta-1)]^\varepsilon$. Choosing $\alpha = 1 + \eta/2$ and $\varepsilon = \eta/[2(\beta-1)]$, Assumption (4.2) implies that $LV(n) + \frac{V^{1+\varepsilon}(n)}{\varphi^\varepsilon(n)} \leq 0$ for $n \notin O_m$ with m large enough. Since $\inf_{n \in O_m} \varphi(n) > 0$, we have the second inequality in (2.6). $\square$

5 Application to multidimensional Feller diffusions absorbed when one of the coordinates hits 0

We consider a general multitype Feller diffusion $(X_t, t \geq 0)$ in $\mathbb{R}_+^d$, solution to the stochastic differential equation

$$dX_t^i = \sqrt{\gamma_i X_t^i} dB_t^i + X_t^i \, r_i(X_t) dt, \quad 1 \leq i \leq d, \tag{5.1}$$

where $(B_t^i, t \geq 0)$ are independent standard Brownian motions, γ_i are positive constants and r_i are measurable maps from $\mathbb{R}_+^d$ to $\mathbb{R}$. From the biological point of view, $r_i(x)$ represents the growth rate per individual of species i in a population of size vector $x \in \mathbb{R}_+^d$. We shall make the following assumption.

Assumption 5. *Assume that, for all $i \in \{1, \ldots, d\}$, r_i is locally Hölder on $\mathbb{R}_+^d$ and that there exist $a > 0$ and $0 < \eta < 1$ such that*

$$r_i(x) \leq a^\eta - x_i^\eta, \tag{5.2}$$

and there exist constants $B_a > a$, $C_a > 0$ and $D_a > 0$ such that

$$\sum_{i=1}^d \mathbb{1}_{x_i \geq B_a} r_i(x) \leq C_a \left(\sum_{i=1}^d \mathbb{1}_{x_i \leq a} r_i(x) + D_a \right), \quad \forall x \in \mathbb{R}_+^d. \tag{5.3}$$

This assumption implies in particular the non-explosion, strong existence and pathwise uniqueness for (5.1). Indeed, since r_i is locally Hölder, standard arguments entail the strong existence and pathwise uniqueness for (5.1) up to the explosion time. Now, Assumption (5.2) and standard comparison results for one-dimensional diffusion processes (see e.g. Theorem 1.1 in [24, Chapter VI]) entail that each coordinate of the process can be upper bounded by the solution of the one-dimensional Feller diffusion

$$d\bar{X}_t^i = \sqrt{\gamma_i \bar{X}_t^i} dB_t^i + \bar{X}_t^i \left[a^\eta - \left(\bar{X}_t^i \right)^\eta \right] dt, \quad 1 \leq i \leq d, \tag{5.4}$$

with initial value $\bar{X}_0^i = X_0^i$. Since $\bar{X}^i$ is a diffusion on $\mathbb{R}_+$ for which $+\infty$ is an entrance boundary and 0 an exit boundary, we deduce that each coordinate is non-explosive and hence that the unique solution to (5.1) is non-explosive. Moreover, the subset $\partial = \mathbb{R}_+^d \setminus (0, +\infty)^d$ is an absorbing boundary for the diffusion process (5.1), so the process X_t is regularly absorbed in the sense of Section 2.2.

Strong existence and pathwise uniqueness imply well-posedness of the martingale problem, hence the strong Markov property hold on the canonical space with respect to the natural filtration (see e.g. [32]). Since the paths

of X are continuous, the hitting times of closed subsets of $\mathbb{R}_+^d$ are stopping times for this filtration. In addition, it follows from Itô's formula and the local boundedness of the coefficients of the SDE that any measurable function $f : \mathbb{R}_+^d \to \mathbb{R}$ twice continuously differentiable on $(0, +\infty)$ belongs to the domain of the weakened generator of X and

$$Lf(x) = \sum_{i=1}^{d} \frac{\gamma_i x_i}{2} \frac{\partial^2 f}{\partial x_i^2}(x) + \sum_{i=1}^{d} x_i r_i(x) \frac{\partial f}{\partial x_i}(x), \quad \forall x \in (0, +\infty)^d.$$

We can now state the main result of the section.

Theorem 5.1. *Under Assumption 5, the multi-dimensional Feller diffusion process $(X_t, t \geq 0)$ absorbed when one of its coordinates hits 0 admits a unique quasi-stationary distribution ν_{QSD} and there exist constants $C, \gamma > 0$ such that, for all probability measure μ on $\mathbb{N}^d$,*

$$\|\mathbb{P}_\mu(X_t \in \cdot \mid t < \tau_\partial) - \nu_{QSD}\|_{TV} \leq Ce^{-\gamma t}, \quad \forall t \geq 0.$$

It is straightforward to check that Assumption 5 is satisfied in the competitive Lotka-Volterra model (1.3), where $r_i(x) = r_i - \sum_{i=1}^d c_{ij} x_j$ with $c_{ij} \geq 0$ and $c_{ii} > 0$ for all $1 \leq i, j \leq d$. In particular, (5.3) holds because

$$\begin{aligned}
\sum_{i=1}^{d} \mathbb{1}_{x_i \geq B_a} r_i(x) &\leq d\bar{r} - \sum_{i=1}^{d} \mathbb{1}_{x_i \geq B_a} c_{ii} x_i \\
&\leq d\bar{r} + \underline{c} \left(dB_a - \sum_{i=1}^{d} x_i \right) \\
&\leq d\bar{r} + \underline{c} dB_a - \frac{\underline{c}}{d\bar{c}} \sum_{i=1}^{d} \mathbb{1}_{x_i \leq a} \sum_{j=1}^{d} c_{ij} x_j \\
&\leq d\bar{r} + \underline{c} dB_a + \frac{\underline{c}}{d\bar{c}} \sum_{i=1}^{d} \mathbb{1}_{x_i \leq a} r_i(x),
\end{aligned}$$

where $\bar{r} = \max_i r_i$, $\bar{c} = \max_{i,j} c_{ij}$ and $\underline{c} = \min_i c_{ii}$. Hence Theorem 1.2 is an immediate corollary of Theorem 5.1.

Assumption 5 allows for other biologically relevant models. For instance, one can consider ecosystems where the competition among individuals only acts when the population size reaches a level $K > 0$, which leads for instance to the SDE

$$dX_t^i = \sqrt{\gamma_i X_t^i}\, dB_t^i + X_t^i \left(r - \sum_{j=1}^{d} c_{ij} \left(X_t^j - K \right)_+ \right) dt, \; X_0^i > 0,$$

where c_{ij} are non-negative constants and $c_{ii} > 0$ for all $i, j \in \{1, \dots, d\}$. Note also that a similar approach (*i.e.* using Theorem 2.4 with Lyapunov functions of the form $\prod_{i=1}^d h(x_i)$) can also be used to handle diffusion processes evolving in bounded boxes.

Before giving the proof of Theorem 5.1, let us give some intuition about the choice of functions V and φ. The behavior of the process when one of the coordinates is close to 0 is similar to the behavior of a one-dimensional diffusion absorbed at 0 started close to 0. This suggests to look for Lyapunov functions of the form $V(x) = \prod_{i=1}^d f(x_i)$ and $\varphi(x) = \prod_{i=1}^d g(x_i)$ where the functions f and g satisfy our criterion for one-dimensional diffusions like (5.4), say

$$dY_t = \sqrt{2Y_t} dB_t + Y_t(r - Y_t^\eta)dt,$$

with generator $Af(x) = xf''(x) + x(r - x^\eta)f'(x)$. In the one dimensional case, Conditions (2.6) are restrictive only close to 0 and close to $+\infty$. When $x \to 0$, assuming $f(x) = x^\alpha$ and $g(x) = x^\beta$ when x is close to 0, we obtain

$$Af(x) \sim \alpha(\alpha - 1)x^{\alpha-1} \quad \text{and} \quad Ag(x) \sim \beta(\beta - 1)x^{\beta-1}.$$

In particular, to satisfy the first part of (2.6) close to 0, we need $\beta > 1$, and to satisfy the second part of (2.6), we need that

$$\alpha(\alpha - 1)x^{\alpha-1} + x^{\alpha-\varepsilon(\beta-\alpha)} \leq x^\alpha$$

in the neighborhood of 0. This holds true if $0 < \alpha < 1$ and $\varepsilon(\beta - \alpha) < 1$. Similarly, assuming $f(x) = a - x^{-\gamma}$ and $g(x) = x^{-\delta}$ close to $+\infty$, with $a, \gamma, \delta > 0$, we obtain

$$Af(x) \sim -\gamma x^{\gamma+\eta} \quad \text{and} \quad Ag(x) \sim \delta x^{\eta-\delta}.$$

For such functions f and g close to $+\infty$, the first part of (2.6) is always satisfied and the second part of (2.6) requires

$$-\gamma x^{\eta-\gamma} + a^{1+\varepsilon} x^{\delta\varepsilon} \leq x^{-\delta},$$

when $x \to +\infty$. This holds true if $\eta - \gamma > \delta\varepsilon$.

This gives the intuition to check our criterion in dimension 1[2]. The multi-dimensional case of Theorem 5.1 requires a more careful study, since we need to consider cases where some of the coordinates of the process are close to 0 or $+\infty$, and the others belong to some compact set.

[2]Note that more efficient criteria exist for one-dimensional diffusions (see [14]). The interest of our criterion comes from the fact that it applies to multi-dimensional processes.

Proof of Theorem 5.1. Up to a linear scaling of the coordinates, we can assume without loss of generality that $\gamma_i = 2$ for all $1 \le i \le d$, so we will only consider this case from now on. Note that Assumption 5 is not modified by the rescaling (up to appropriate changes of the constants a, B_a, C_a and D_a).

We divide the proof into five steps, respectively devoted to the construction of a function φ satisfying the first inequality in (2.6), of a function V satisfying the second inequality in (2.6), to the proof of (2.7), to the proof of a local Harnack inequality (needed to check Assumption 2), and the proofs of Assumption 2 and of Assumption 3.

Step 1: construction of a function φ satisfying the first inequality in (2.6).

Recall the definition of the constants $a > 0$ and $B_a > a$ from Assumption 5. We use the following lemma, whose proof is left to the reader (see Figure 1 for a typical graph of h_β).

Lemma 5.2. *There exists $M > 0$ such that, for all $\beta \ge M$, there exists a function $h_\beta : \mathbb{R}_+ \to \mathbb{R}_+$ twice continuously differentiable on $(0, +\infty)$ such that*

$$h_\beta(x) = \begin{cases} 4x^2/a^2 & \text{if } x \in [0, a/2], \\ B_a^\beta (2x)^{-\beta} & \text{if } x \ge B_a, \end{cases}$$

$h_\beta(x) \ge 1$ for all $x \in [a/2, a]$, h_β is nonincreasing and convex on $[a, +\infty)$,

$$M' := \sup_{\beta \ge M} \sup_{x \in [a/2, a]} |h'_\beta(x)| < +\infty \quad \text{and} \quad M'' := \sup_{\beta \ge M} \sup_{x \in [a/2, a]} |h''_\beta(x)| < +\infty.$$

We set $\beta = M + (2 \vee aM')/C_a + 1$ and

$$\varphi(x) = \prod_{i=1}^d h_\beta(x_i), \quad \forall x \in \mathbb{R}_+^d.$$

We have

$$\frac{L\varphi(x)}{\varphi(x)} = \sum_{i=1}^d \frac{x_i h'_\beta(x_i) r_i(x) + x_i h''_\beta(x)}{h_\beta(x_i)}.$$

Now, it follows from the properties of h_β and Assumptions 5 that, for all $x \in \mathbb{R}_+$ and all $1 \le i \le d$,

$$\frac{x_i h'_\beta(x_i) r_i(x) + x_i h''_\beta(x)}{h_\beta(x_i)} \ge \begin{cases} 0 & \text{if } x_i \ge a, \\ -\beta r_i(x) & \text{if } x_i \ge B_a, \\ 2r_i(x) + \frac{2}{x_i} & \text{if } x_i \le a/2, \\ aM'(r_i(x) - a^\eta) - a^{1+\eta} M' - aM'' & \text{if } a/2 \le x_i \le a, \end{cases}$$

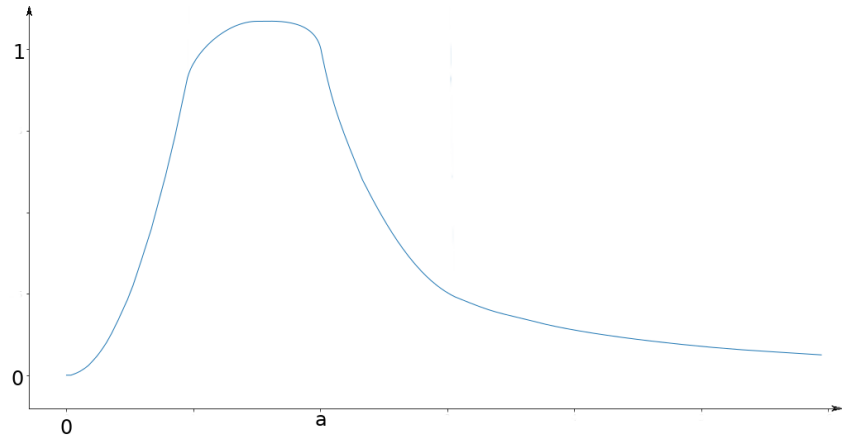

Figure 1: A typical graph of h_β.

where we used in the last inequality the fact that $r_i(x) - a^\eta \leq 0$ for all x. Using once again this property, we deduce that, for some constant B independent of $\beta \geq M$,

$$\frac{x_i h'_\beta(x_i) r_i(x) + x_i h''_\beta(x)}{h_\beta(x_i)} \geq \begin{cases} 0 & \text{if } x_i \geq a, \\ -\beta r_i(x) & \text{if } x_i \geq B_a, \\ (2 \vee aM') r_i(x) + \frac{2}{x_i} - B & \text{if } x_i \leq a. \end{cases} \tag{5.5}$$

Hence, for all $x \in \mathbb{R}^d_+$,

$$\frac{L\varphi(x)}{\varphi(x)} \geq -\beta \sum_{i=1}^d \mathbb{1}_{x_i \geq B_a} r_i(x) + \sum_{i=1}^d \mathbb{1}_{x_i \leq a} \left((2 \vee aM') r_i(x) + \frac{2}{x_i} \right) - dB.$$

This and Assumption (5.3) imply that

$$\begin{aligned} \frac{L\varphi(x)}{\varphi(x)} &\geq -\sum_{i=1}^d \mathbb{1}_{x_i \geq B_a} r_i(x) + 2 \sum_{i=1}^d \mathbb{1}_{x_i \leq a} \frac{1}{x_i} - dB - (2 \vee aM') D_a \\ &\geq \sum_{i=1}^d \left(x_i^\eta + \frac{2}{x_i} \right) - B', \end{aligned}$$

for some constant B', where we used Assumption (5.2) in the last inequality.

Hence, there exist $n \geq 1$ and a constant $C > 0$ such that

$$L\varphi(x) \geq -C \mathbb{1}_{x \in O_n},$$

since $O_n = \{x \in [\frac{1}{n+1}, +\infty)^d, \sum x_i \leq n+1\}$ (we equip $\mathbb{R}_+^d$ with the L^1 distance). This ends the proof that φ satisfies the first inequality in (2.6).

Step 2: construction of a function V satisfying (2.6) *and verification that* (V, φ) *is a pair of admissible functions.*

For V, we define

$$V(x) = \prod_{i=1}^{d} g(x_i), \quad \forall x \in \mathbb{R}_+^d,$$

where the function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is twice continuously differentiable on $(0, +\infty)$, increasing concave and such that

$$g(x) = \begin{cases} x^\gamma & \text{if } x \leq 1 \\ \delta - x^{-\eta/2} & \text{if } x \geq 2, \end{cases}$$

for some constants $\gamma < 1$ and $\delta > 0$ and where η is defined in Assumption (5.2). Since $g'(1) = \gamma$ and $g'(2) = \eta 2^{-2-\eta/2}$, it is possible to find $\delta > 0$ such that such a function g exists as soon as $\eta 2^{-2-\eta/2} < \gamma$. Hence, we shall assume that γ belongs to the non-empty interval $(\eta 2^{-2-\eta/2}, 1)$. We have

$$\frac{LV(x)}{V(x)} = \sum_{i=1}^{d} \frac{x_i g'(x_i) r_i(x) + x_i g''(x_i)}{g(x_i)}$$

and

$$\frac{x_i g'(x_i) r_i(x) + x_i g''(x_i)}{g(x_i)} \leq \begin{cases} \gamma r_i(x) - \dfrac{\gamma(1-\gamma)}{x_i} & \text{if } x_i \leq 1, \\ 2a^\eta \sup\limits_{1 \leq x \leq 2} g'(x) & \text{if } 1 \leq x_i \leq 2, \\ \dfrac{\eta r_i(x) x_i^{-\eta/2}}{2(\delta - x_i^{-\eta/2})} & \text{if } x_i \geq 2. \end{cases}$$

We deduce from Assumptions (5.2) that there exist constants $B', B'' > 0$ such that

$$\frac{x_i g'(x_i) r_i(x) + x_i g''(x)}{g(x_i)} \leq B' - \begin{cases} \frac{\gamma(1-\gamma)}{x_i} & \text{if } x_i \leq 1, \\ 0 & \text{if } 1 \leq x_i \leq 2, \\ B'' x_i^{\eta/2} & \text{if } x_i \geq 2. \end{cases} \tag{5.6}$$

Thus, since $h_\beta(x_i) \geq D_\beta \left(x_i^2 \wedge x_i^{-\beta} \right)$ for some constant $D_\beta > 0$ and since $g(x_i) \leq \delta$,

$$\begin{aligned}
\frac{LV(x)}{V(x)} + \frac{V(x)^\varepsilon}{\varphi(x)^\varepsilon} &\leq B'd - \gamma(1-\gamma) \sum_{i=1}^d \frac{\mathbb{1}_{x_i \leq 1}}{x_i} - B'' \sum_{i=1}^d \mathbb{1}_{x_i \geq 2} x_i^{\eta/2} \\
&\quad + \left(\frac{\delta}{D_\beta} \right)^{d\varepsilon} \prod_{i=1}^d \left(x_i^{\varepsilon\beta} \vee x_i^{-2\varepsilon} \right) \\
&\leq B'd + \gamma(1-\gamma) + B''2^{\eta/2} - \gamma(1-\gamma) \left(\inf_i x_i \right)^{-1} - B'' \left(\sup_i x_i \right)^{\eta/2} \\
&\quad + \left(\frac{\delta}{D_\beta} \right)^{d\varepsilon} \left[\left(\sup_i x_i \right)^{\beta d\varepsilon} + \left(\inf_i x_i \right)^{-2d\varepsilon} \right].
\end{aligned}$$

Therefore, choosing $\varepsilon > 0$ such that $d(1+\alpha)\varepsilon < 1$ and $\beta d\varepsilon < \eta/2$, $LV(x) + V(x)^{1+\varepsilon}/\varphi(x)^\varepsilon \leq 0$ for all $x \in \mathbb{R}_+^d$ such that $\inf_i x_i$ is small enough or $\sup_i x_i$ is big enough. Since LV is bounded from above by (5.6) and since V and φ are positive continuous on any compact subset of $(0, +\infty)^d$, we have proved Condition (2.6).

We can now check that (V, φ) is a pair of admissible functions. First, V and φ are both bounded, positive on $(0, +\infty)$ and vanishing on ∂. They both belong to the domain of the weakened infinitesimal generator of X. Since the function g/h_β is positive continuous on $(0, +\infty)$ and

$$\frac{g(x)}{h_\beta(x)} = \begin{cases} x^{\gamma-2} & \text{if } x \leq 1 \wedge a/2, \\ (\delta - x^{-\eta/2})(2x)^\beta / B_a^\beta & \text{if } x \geq B_a \vee 2, \end{cases}$$

we deduce that (2.3) holds true. Condition (2.4) is also clear since $X_{T_n} \to X_{\tau_\partial}$ almost surely. Finally, since LV is bounded from above and $L\varphi$ is bounded from below, we have proved that (V, φ) is a pair of admissible functions.

Step 3: proof of (2.7).

Using the upper bound $X_t^i \leq \bar{X}_t^i$ for all $t \geq 0$ and $1 \leq i \leq d$, where $\bar{X}^i$ is solution to the SDE (5.4), and noting that the processes $(\bar{X}^i)_{1 \leq i \leq d}$ are independent, we have for all $x \in \mathbb{R}_+^d$ and all $t_2 > 0$,

$$\mathbb{P}_x(t_2 < \tau_\partial) \leq \prod_{i=1}^d \mathbb{P}_{x_i}(\bar{X}_{t_2}^i > 0).$$

Now, there exist constants D and D' such that

$$\mathbb{P}_{x_i}(\bar{X}^i_{t_2} > 0) \leq (Dx_i) \wedge 1 \leq D' g(x_i) \quad \text{for all } x_i > 0. \tag{5.7}$$

To prove this, we can consider a scale function s of the diffusion $\bar{X}^i$ such that $s(0) = 0$ and $s(x) > 0$ for $x > 0$. Using the expression of the scale function and the speed measure (see e.g. [32, V.52]), one easily checks that $s(x_i) \sim \alpha x_i$ when $x_i \to 0$ for some $\alpha \neq 0$ and that Proposition 4.9 of [14] is satisfied, so that $\mathbb{P}_{x_i}(\bar{X}^i_{t_2} > 0) \leq M s(x_i)$ for some $M > 0$. Since $s(x_i) \sim \alpha x_i$ when $x_i \to 0$, (5.7) is proved and hence (2.7) holds true.

Step 4: Harnack inequality for u. Consider a bounded nonnegative measurable function f and define the application $u : (t, x) \in \mathbb{R}_+ \times (0, +\infty)^d \mapsto \mathbb{E}_x(f(X_t)\mathbb{1}_{t<\tau_\partial})$. Our aim is to prove that, for all $m \geq 1$, there exist two constants $N_m > 0$ and $\delta_m > 0$, which do not depend on f, such that

$$u(\delta_m^2, x) \leq N_m u(2\delta_m^2, y), \text{ for all } x, y \in O_m \text{ such that } |x - y| \leq \delta_m/2. \tag{5.8}$$

We fix $m \geq 1$ and we will omit the indices m for the constants δ and N for the rest of Step 4. Let K be a compact set with C^∞ boundary such that $O_m \subset K \subset (0, +\infty)^d$ and such that $d(O_m, \partial K) > 0$. We set $\delta = d(O_m, \partial K)/3$ and $\tau = \inf\{t \geq 0 : X_t \in \partial K\}$. Let x and y be fixed in K such that $|x - y| \leq \delta/2$. We define μ_x and μ_y as the joint law of $(\delta^2 - \tau \wedge \delta^2, X_{\tau\wedge\delta^2})$ starting from $X_0 = x$ and $(2\delta^2 - \tau \wedge (2\delta^2), X_{\tau\wedge(2\delta^2)})$ starting from $X_0 = y$, respectively. It follows from Lusin's theorem (see e.g. [17, Thm. 7.5.2]) that there exists a sequence $(h_n)_{n\geq1}$ of bounded C^∞ functions from $([0, \infty[\times\partial K) \cup (\{0\} \times K)$ to $\mathbb{R}_+$ such that $(h_n)_{n\geq1}$ converges bounded pointwisely toward u, $(\mu_x + \mu_y)$-almost everywhere. For all $n \geq 1$, we let $u_n : [0, +\infty[\times K \to \mathbb{R}$ be the solution to the linear parabolic equation

$$\begin{cases} \partial_t u_n = \displaystyle\sum_{i=1}^d x_i \frac{\partial^2 u_n}{\partial x_i^2} + x_i r_i(x) \frac{\partial u_n}{\partial x_i}, \ \forall (t, x) \in (0, \infty) \times \mathrm{int}(K), \\ u_n(t, x) = h_n(t, x), \ \forall (t, x) \in ([0, \infty[\times\partial K) \cup (\{0\} \times K). \end{cases}$$

By [28, Thm 5.1.15], for all $n \geq 1$, u_n is of regularity $C^{1,2}$. In particular, applying Itô's formula to $s \mapsto u_n(\delta^2 - s, X_s)$ at time $\tau \wedge \delta^2$ and taking the expectation, one deduces that,

$$u_n(\delta^2, x) = \mathbb{E}_x\left[u_n(\delta^2 - \tau \wedge \delta^2, X_{\tau\wedge\delta^2})\right] = \mu_x(h_n)$$

By Lebesgue's theorem, the last quantity converges when $n \to +\infty$ to

$$\begin{aligned}\mu_x(u) &= \mathbb{E}_x\left(\mathbb{1}_{\tau\le\delta^2}\, u(\delta^2-\tau, X_\tau)\right) + \mathbb{E}_x\left(\mathbb{1}_{\tau>\delta^2}\, u(0, X_{\delta^2})\right)\\ &= \mathbb{E}_x\left(\mathbb{1}_{\tau\le\delta^2}\, \mathbb{E}_{X_\tau}(f(X_{\delta^2-\tau})\mathbb{1}_{\delta^2-\tau<\tau_\partial})\right) + \mathbb{E}_x\left(\mathbb{1}_{\tau>\delta^2} f(X_{\delta^2})\right)\\ &= \mathbb{E}_x(f(X_{\delta^2})\mathbb{1}_{\delta^2<\tau_\partial}) = u(\delta^2, x),\end{aligned}$$

where we used the strong Markov property at time τ. Similarly, $u_n(2\delta^2, y)$ converges to $u(2\delta^2, y)$.

Using the Harnack inequality provided by [26, Theorem 1.1] (with $\theta = 2$ and $R = \delta$), we deduce that there exists a constant $N > 0$ which does not depend on f, $x, y \in K$ such that $|x-y| \le \delta/2$ nor on n such that

$$u_n(\delta^2, x) \le N u_n(2\delta^2, y).$$

Hence, we deduce that

$$u(\delta^2, x) \le N u(2\delta^2, y), \text{ for all } x, y \in K \text{ such that } |x-y| \le \delta/2 \tag{5.9}$$

and (5.8) is proved.

Step 5 : proof that Assumptions 2 and 3 are satisfied. Fix $x_1 \in O_1$ and let ν denote the conditional law $\mathbb{P}_{x_1}(X_{\delta_1^2} \in \cdot \mid \delta_1^2 < \tau_\partial)$. Then the Harnack inequality (5.8) entails that, for all $x \in O_1$ such that $|x - x_1| \le \delta_1/2$ and all measurable nonnegative bounded f on $(0, +\infty)^d$,

$$\mathbb{E}_x\left[f(X_{2\delta_1^2})\mathbb{1}_{2\delta_1^2<\tau_\partial}\right] \ge \frac{1}{N_1}\mathbb{E}_{x_1}\left[f(X_{\delta_1^2})\mathbb{1}_{\delta_1^2<\tau_\partial}\right].$$

This means that

$$\mathbb{P}_x(X_{2\delta_1^2} \in \cdot) \ge \frac{\mathbb{P}_{x_1}(\delta_1^2 < \tau_\partial)}{N_1}\,\nu.$$

Now, let $m \ge 1$. Since O_m is bounded, connected and at a positive distance of ∂, $\mathbb{P}_x(X_1 \in O_1 \cap B(x_1, \delta_1/2))$ is uniformly bounded from below in O_m by a positive constant M_m. Therefore, Markov's property implies that, for all $x \in O_m$,

$$\mathbb{P}_x(X_{1+2\delta_1^2} \in \cdot) \ge \frac{\mathbb{P}_{x_1}(\delta_1^2 < \tau_\partial)\, M_n}{N_1}\,\nu.$$

This is the first part of Assumption 2.

The second part of Assumption 2 is also a consequence of (5.8). Indeed, for any fixed m and for all $t \geq 2\delta_m^2$, this equation applied to $f(x) = \mathbb{P}_x(t - 2\delta_m^2 < \tau_\partial)$ and the Markov property entail that

$$\mathbb{P}_x(t - \delta_m^2 < \tau_\partial) \leq N_m \mathbb{P}_y(t < \tau_\partial), \text{ for all } x, y \in O_m \text{ such that } |x - y| \leq \delta_m/2.$$

Since $s \mapsto \mathbb{P}_x(s < \tau_\partial)$ is non-increasing, we deduce that

$$\mathbb{P}_x(t < \tau_\partial) \leq N_m \mathbb{P}_y(t < \tau_\partial), \text{ for all } x, y \in O_m \text{ such that } |x - y| \leq \delta_m/2.$$

Since O_m has a finite diameter and is connected, we deduce that there exists N'_m such that, for all $t \geq 2\delta_m^2$,

$$\mathbb{P}_x(t < \tau_\partial) \leq N'_m \mathbb{P}_y(t < \tau_\partial), \text{ for all } x, y \in O_m.$$

Now, for $t \leq 2\delta_m^2$, we simply use the fact that $x \mapsto \mathbb{P}_x(2\delta_m^2 < \tau_\partial)$ is uniformly bounded from below on O_m by a constant $1/N''_m > 0$. In particular,

$$\mathbb{P}_x(t < \tau_\partial) \leq 1 \leq N''_m \mathbb{P}_y(2\delta_m^2 < \tau_\partial) \leq N''_m \mathbb{P}_y(t < \tau_\partial), \text{ for all } x, y \in O_m.$$

As a consequence, the second part of Assumption 2 is satisfied.

Assumption 3 is a direct consequence of the domination by solutions to (5.4), since these solutions come down from infinity and hit 0 in finite time almost surely (cf. e.g. [3]).

Finally, we deduce from Steps 1, 2, 3 and 5 that all the assumptions of Theorem 2.4 are satisfied. This concludes the proof of Theorem 5.1. □

References

[1] V. Bansaye. Approximation of stochastic processes by nonexpansive flows and coming down from infinity. *Ann. Appl. Probab.*, 29(4):2374–2438, 08 2019.

[2] H. Brezis. *Analyse fonctionnelle.* Collection Mathématiques Appliquées pour la Maîtrise. [Collection of Applied Mathematics for the Master's Degree]. Masson, Paris, 1983. Théorie et applications. [Theory and applications].

[3] P. Cattiaux, P. Collet, A. Lambert, S. Martínez, S. Méléard, and J. San Martín. Quasi-stationary distributions and diffusion models in population dynamics. *Ann. Probab.*, 37(5):1926–1969, 2009.

[4] P. Cattiaux and S. Méléard. Competitive or weak cooperative stochastic Lotka-Volterra systems conditioned on non-extinction. *J. Math. Biol.*, 60(6):797–829, 2010.

[5] N. Champagnat, K. A. Coulibaly-Pasquier, and D. Villemonais. Criteria for exponential convergence to quasi-stationary distributions and applications to multi-dimensional diffusions. *Séminaire de Probabilités XLIX*, pages 165–182, 2018.

[6] N. Champagnat, R. Ferrière, and S. Méléard. Unifying evolutionary dynamics: From individual stochastic processes to macroscopic evolution. *Theor. Pop. Biol.*, 69:297–321, 2006.

[7] N. Champagnat, R. Ferrière, and S. Méléard. Individual-based probabilistic models of adaptive evolution and various scaling approximations. In *Seminar on Stochastic Analysis, Random Fields and Applications V*, volume 59 of *Progr. Probab.*, pages 75–113. Birkhäuser, Basel, 2008.

[8] N. Champagnat, R. Schott, and D. Villemonais. Probabilistic Non-asymptotic Analysis of Distributed Algorithms. *ArXiv e-prints*, Feb. 2018.

[9] N. Champagnat and D. Villemonais. Exponential convergence to quasi-stationary distribution and Q-process. *Probab. Theory Related Fields*, 164(1):243–283, 2016.

[10] N. Champagnat and D. Villemonais. Population processes with unbounded extinction rate conditioned to non-extinction. *ArXiv e-prints*, Nov. 2016. Unpublished.

[11] N. Champagnat and D. Villemonais. Exponential convergence to quasi-stationary distribution for absorbed one-dimensional diffusions with killing. *ALEA Lat. Am. J. Probab. Math. Stat.*, 14(1):177–199, 2017.

[12] N. Champagnat and D. Villemonais. General criteria for the study of quasi-stationarity. *arXiv e-prints*, page arXiv:1712.08092, Dec 2017.

[13] N. Champagnat and D. Villemonais. Uniform convergence to the Q-process. *Electron. Commun. Probab.*, 22:Paper No. 33, 7, 2017.

[14] N. Champagnat and D. Villemonais. Uniform convergence of conditional distributions for absorbed one-dimensional diffusions. *Adv. in Appl. Probab.*, 50(1):178–203, 2018.

[15] J.-R. Chazottes, P. Collet, and S. Méléard. Sharp asymptotics for the quasi-stationary distribution of birth-and-death processes. *Probab. Theory Related Fields*, 164(1-2):285–332, 2016.

[16] J.-R. Chazottes, P. Collet, and S. Méléard. On time scales and quasi-stationary distributions for multitype birth-and-death processes. *ArXiv e-prints*, Feb. 2017.

[17] R. M. Dudley. *Real analysis and probability*, volume 74 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2002. Revised reprint of the 1989 original.

[18] E. B. Dynkin. *Markov processes. Vols. I, II*, volume 122 of *Translated with the authorization and assistance of the author by J. Fabius, V. Greenberg, A. Maitra, G. Majone. Die Grundlehren der Mathematischen Wissenschaften, Bände 121*. Academic Press Inc., Publishers, New York, 1965.

[19] P. A. Ferrari, H. Kesten, and S. Martínez. R-positivity, quasi-stationary distributions and ratio limit theorems for a class of probabilistic automata. *Ann. Appl. Probab.*, 6(2):577–616, 1996.

[20] G. Ferré, M. Rousset, and G. Stoltz. More on the long time stability of Feynman-Kac semigroups. *arXiv e-prints*, page arXiv:1807.00390, Jul 2018.

[21] M. C. Hansen and C. Wiuf. Existence of a Unique Quasi-stationary Distribution for Stochastic Reaction Networks. *ArXiv e-prints*, Aug. 2018.

[22] A. Hening and M. Kolb. Quasistationary distributions for one-dimensional diffusions with singular boundary points. *Stochastic Processes and their Applications*, 129(5):1659 – 1696, 2019.

[23] E. Horton, A. E. Kyprianou, and D. Villemonais. Stochastic Methods for the Neutron Transport Equation I: Linear Semigroup asymptotics. *ArXiv e-prints*, Oct. 2018.

[24] N. Ikeda and S. Watanabe. *Stochastic differential equations and diffusion processes*. North-Holland Mathematical Library, 1989.

[25] M. Kolb and D. Steinsaltz. Quasilimiting behavior for one-dimensional diffusions with killing. *Ann. Probab.*, 40(1):162–212, 2012.

[26] N. V. Krylov and M. V. Safonov. A property of the solutions of parabolic equations with measurable coefficients. *Izv. Akad. Nauk SSSR Ser. Mat.*, 44(1):161–175, 239, 1980.

[27] A. Lambert. The branching process with logistic growth. *Ann. Appl. Probab.*, 15(2):1506–1535, 2005.

[28] A. Lunardi. *Analytic semigroups and optimal regularity in parabolic problems*, volume 16 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Verlag, Basel, 1995.

[29] S. Martínez, J. San Martín, and D. Villemonais. Existence and uniqueness of a quasistationary distribution for markov processes with fast return from infinity. *J. Appl. Probab.*, 51(3):756768, 2014.

[30] S. Méléard and D. Villemonais. Quasi-stationary distributions and population processes. *Probab. Surv.*, 9:340–410, 2012.

[31] S. P. Meyn and R. L. Tweedie. Stability of Markovian processes. III. Foster-Lyapunov criteria for continuous-time processes. *Adv. in Appl. Probab.*, 25(3):518–548, 1993.

[32] L. C. G. Rogers and D. Williams. *Diffusions, Markov processes, and martingales. Vol. 2.* Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2000. Itô calculus, Reprint of the second (1994) edition.

[33] D. Steinsaltz and S. N. Evans. Quasistationary distributions for one-dimensional diffusions with killing. *Trans. Amer. Math. Soc.*, 359(3):1285–1324 (electronic), 2007.

[34] E. A. van Doorn. Quasi-stationary distributions and convergence to quasi-stationarity of birth-death processes. *Adv. in Appl. Probab.*, 23(4):683–700, 1991.

[35] A. Velleret. Unique Quasi-Stationary Distribution, with a possibly stabilizing extinction. *arXiv e-prints*, page arXiv:1802.02409, Feb 2018.