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Random Moments for Sketched Mixture Learning

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Abstract—We present a method to solve large-scale mixture learning tasks from a sketch of the data, formed by random generalized empirical moments. We give empirical and theoretical results on k-means and Gaussian Mixture Model estimation problems.

I. INTRODUCTION
Consider samples \( z_i \in \mathbb{R}^d \), \( 1 \leq i \leq n \), drawn i.i.d. from a distribution \( \pi \). Given a class of hypotheses \( \mathcal{H} \) and a loss function \( \ell : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R} \), statistical learning consists in finding the hypothesis \( h^* \in \mathcal{H} \) that minimizes the expected risk \( \mathcal{R}(h) = \mathbb{E}_\pi(\ell(z,h)) \). Since the distribution \( \pi \) is not directly available, usual learning procedures minimize the empirical risk instead: \( \hat{\mathcal{R}}_n(h) = \sum_i \ell(z_i,h) / n \).

This traditional approach is however challenged when samples \( z \) are high-dimensional (large \( d \)) or in large number (large \( n \)). The first case has been dealt with using random projections \([1]\) or feature selection \([2]\), while the second gave birth to online learning \([3]\) or coresets \([4]\). We advocate here that when \( n \) is large, some learning tasks can be done using only a collection of generalised empirical moments, referred to as sketch, as a (highly) compressed representation of the database. A simple example is Principal Component Analysis (PCA), which can be done with only the empirical covariance. Such sketches can be computed online, and/or in a distributed/parallel manner, and do not require the database to be stored on one single device.

We present here a method to perform k-means or Gaussian Mixture Model (GMM) estimation with identity covariance from a sketch formed by a (weighted) random sampling of the characteristic function. Such inverse problems bear similarities with sparse recovery in continuous spaces \([5]\). Define the sketching operator:

\[
\mathcal{A} \pi = \frac{1}{\sqrt{m}} \left[ \sum_{i=1}^{n} \exp(-i \omega_j^T z_i) / c_{\omega_j} \right]_{j=1}^m
\]

(1)

where \( c_{\omega_j} > 0 \) are some weights and frequencies \( \omega_j \in \mathbb{R}^d \) are drawn i.i.d. from a weighted Gaussian distribution \( \Lambda(\omega) \propto c_{\omega}^2 \mathcal{N}(0, \sigma^2 I) \). The empirical sketch used in practice is denoted \( y = \frac{1}{n \sqrt{m}} \left[ \sum_{i=1}^{n} \exp(-i \omega_j^T z_i) / c_{\omega_j} \right]_{j=1}^m \).

II. MAIN RESULTS
We now present our main results on k-means and GMM estimation. In each case, \( c_{\omega} \) and \( \sigma^2 \) are appropriately chosen and not detailed in this abstract. Leveraging tools from kernel embeddings of distributions \([6]\) and Random Fourier features \([7]\), our analysis is inspired by Compressive Sensing results \([8]\), \([9]\) adapted to the proposed infinite-dimensional framework.

A. k-means
In the k-means problem, each hypothesis is a set of centroids \( h = \{ c_1, \ldots, c_k \} \) and the loss function is \( \ell(z,h) = \min_i \| z - c_i \|_2^2 \).

Assumptions. We restrict to a family of hypotheses where centroids are 2\(\varepsilon\)-separated from each other and contained in a ball of radius \( M \), and denote \( \mathcal{H}_{k,M} \) the corresponding class of hypotheses.

Result. Denote \( h^* \in \mathcal{H}_{k,M} \) the true minimizer of the expected risk \( \mathcal{R} \) and \( \hat{h} \) the hypothesis recovered from the sketch by

\[
\hat{h} = \arg\min_{h \in \mathcal{H}_{k,M}} \min_{\alpha_1, \ldots, \alpha_k} \left\| y - A \left( \sum_{i=1}^{k} \alpha_i \delta_{c_i} \right) \right\|_2
\]

(2)

where \( \alpha_i \geq 0 \) and \( \sum_{i=1}^{k} \alpha_i = 1 \).

If \( m \geq O \left( k^2 d^3 \log \left( k, d \log \left( 1 / \rho \cdot M / \varepsilon \right) \right) \right) \), then with joint probability \( 1 - \rho \) on the drawing of \( z_i \) and \( \omega_j \), it holds that

\[
\mathcal{R}(\hat{h}) \lesssim \mathcal{R}(h^*) + O \left( \sqrt{k d^3 / n} \right).
\]

(3)

B. Gaussian mixture with identity covariance

In the GMM learning problem, a hypothesis is a set of means and weights \( h = \{ \mu_1, \ldots, \mu_k, \alpha_1, \ldots, \alpha_k \} \) and the loss function is \( \ell(z,h) = -\log \pi_k(\zeta) \), where \( \pi_k = \sum_{i=1}^{k} \alpha_i \mathcal{N}(\mu_i, 1) \).

Assumptions. We restrict to a class of hypotheses where means are separated from each other and contained in a ball of radius \( M \), and denote \( \mathcal{H}_{k,M} \) the corresponding class of hypotheses. Unlike k-means, the separation between means cannot be as small as desired, and there is a trade-off between the required separation and the required number of measurements \( m \). A few values are given in Table I.

Result. Denote \( h^* \in \mathcal{H}_{k,M} \) the true minimizer of the expected risk \( \mathcal{R} \) and \( \hat{h} \) the hypothesis recovered from the sketch by solving

\[
\hat{h} = \arg\min_{h \in \mathcal{H}_{k,M}} \| y - A \pi h \|_2.
\]

(4)

If the number of measurements \( m \) is large enough (see Tab. I), with joint probability \( 1 - \rho \) on the drawing of \( z_i \) and \( \omega_j \), it holds that

\[
\mathcal{R}(\hat{h}) - \mathcal{R}(h^*) \lesssim \inf_{h \in \mathcal{H}_{k,M}} \| \pi - \pi h \|_{TV} + O \left( \sqrt{1 / n} \right)
\]

(5)

where the \( \mathcal{O} \) hides some dependencies in \( k, d \) (roughly behaving like \( m \) in Tab. I). The bound also involves the best approximation of \( \pi \) by a GMM for the TV norm (\( L^1 \) norm for densities).

III. EXPERIMENTAL RESULTS
The optimization problems \([3]\) and \([4]\) are non-convex and seem hard to solve exactly. Heuristically, a greedy algorithm inspired by sparse recovery referred to as Compressive Learning OMP (CLOMP) \([10]\)–\([12]\) has been previously shown to perform well. We compare a Matlab implementation of CLOMP available at \([13]\) with Matlab’s \texttt{kmeans} function and VLFeat’s \texttt{gmm} function.

In Fig. I the sketched approach is seen to lead to tremendous savings in time of execution and memory consumption when the number of items \( n \) is large, while achieving the same precision as the corresponding traditional approach for a limited number of measurements \( m \approx O(kd) \). Fig. II further confirms that \( m \approx O(kd) \) is empirically sufficient, hence the theoretical guarantees for \( m \gtrsim O(k^2d^3) \) are probably pessimistic.

Further work will combine the sketching technique with dimensionality-reduction methods to treat both large \( d \) and \( n \).
<table>
<thead>
<tr>
<th>Separation of means</th>
<th>Number of measurements</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{O}(\frac{1}{\sqrt{\log k}})$</td>
<td>$m \geq \mathcal{O}(k^2 d^2 \log(k/d) \log(M/\rho))$</td>
</tr>
<tr>
<td>$\mathcal{O}(\sqrt{\log k} + \log k)$</td>
<td>$m \geq \mathcal{O}(k^3 d^2 \log(k,d) \log(M/\rho))$</td>
</tr>
<tr>
<td>$\mathcal{O}(\sqrt{\log k})$</td>
<td>$m \geq \mathcal{O}(k^2 d^2 e^c \log(k,d) \log(M/\rho))$</td>
</tr>
</tbody>
</table>

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**REFERENCES**


Fig. 1. Relative memory consumption (left), time of estimation (center) and precision (right) for compressive k-means (top) and GMM estimation (bottom) with $k = 10$ components in dimension $d = 10$, compared to Matlab’s `kmeans` and VLFeat’s `gmm` functions (dotted black lines).

Fig. 2. Relative precision for k-means (top) and GMM estimation (bottom) with respect to the relative number of measurements $m/(kd)$. On the left $k = 10$ and on the right $d = 10$. 

![Table I](image-url)