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Random Moments for Sketched Mixture Learning

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Abstract—We present a method to solve large-scale mixture learning tasks from a sketch of the data, formed by random generalized empirical moments. We give empirical and theoretical results on $k$-means and Gaussian Mixture Model estimation problems.

I. INTRODUCTION

Consider samples $z_i \in \mathbb{R}^d$, $1 \leq i \leq n$, drawn i.i.d. from a distribution $\pi$. Given a class of hypotheses $\mathcal{H}$ and a loss function $\ell : \mathbb{R}^d \times \mathcal{H} \rightarrow \mathbb{R}$, statistical learning consists in finding the hypothesis $h^* \in \mathcal{H}$ that minimizes the expected risk $\mathcal{R}(h) = \mathbb{E}_\pi(\ell(z, h))$. Since the distribution $\pi$ is not directly available, usual learning procedures minimize the empirical risk instead: $\mathcal{R}_n(h) = \frac{1}{n} \sum z_i \ell(z_i, h)$. This traditional approach is however challenged when samples $z$ are high-dimensional (large $d$) or in large number (large $n$). The first case has been dealt with using random projections [1] or feature selection [2], while the second gave birth to online learning [3] or coresets [4]. We advocate here that when $n$ is large, some learning tasks can be done using only a collection of generalized empirical moments, referred to as sketch, as a (highly) compressed representation of the database. A simple example is Principal Component Analysis (PCA), which can be done with only the empirical covariance. Such sketches can be computed online, and/or in a distributed/parallel manner, and do not require the database to be stored on one single device.

We present here a method to perform $k$-means or Gaussian Mixture Model (GMM) estimation with identity covariance from a sketch formed by a (weighted) random sampling of the characteristic function. Such inverse problems bear similarities with sparse recovery and such inverse problems bear similarities with sparse recovery.

II. MAIN RESULTS

We now present our main results on $k$-means and GMM estimation. In each case, $c_w$ and $\sigma^2$ are appropriately chosen and not detailed in this abstract. Leveraging tools from kernel embeddings of distributions [6] and Random Fourier features [7], our analysis is inspired by Compressive Sensing results [8], adapted to the proposed infinite-dimensional framework.

A. $k$-means

In the $k$-means problem, each hypothesis is a set of centroids $h = \{c_1, \ldots, c_k\}$ and the loss function is $\ell(z, h) = \min_i \|z - c_i\|^2_2$. We restrict to a family of hypotheses where centroids are $2\varepsilon$-separated from each other and contained in a ball of radius $M$, and denote $\mathcal{H}_{k,\varepsilon,M}$ the corresponding class of hypotheses.

Result. Denote $h^* \in \mathcal{H}_{k,\varepsilon,M}$ the true minimizer of the expected risk $\mathcal{R}$ and $\hat{h}$ the hypothesis recovered from the sketch by

$$\hat{h} = \arg\min_{h \in \mathcal{H}_{k,\varepsilon,M}} \min_{\alpha_1,\ldots,\alpha_k} \|\mathbf{y} - A \left( \sum_{l=1}^{k} \alpha_l c_l \right)\|_2$$

where $\alpha_l \geq 0$ and $\sum_{l=1}^{k} \alpha_l = 1$. If $m \geq O \left( k^2 d^2 \log(k,d) \log(1/\rho \cdot M/\varepsilon) \right)$, then with joint probability $1 - \rho$ on the drawing of $z_i$ and $\omega_i$ it holds that

$$\mathcal{R}(\hat{h}) \lesssim \mathcal{R}(h^*) + O \left( \sqrt{kd^2/n} \right).$$

B. Gaussian mixture with identity covariance

In the GMM learning problem, a hypothesis is a set of means and weights $h = \{\mu_1, \ldots, \mu_k, \sigma_1, \ldots, \sigma_k\}$ and the loss function is $\ell(z, h) = -\log \pi_n(z)$, where $\pi_n = \sum_{l=1}^{k} \alpha_l \mathcal{N}(\mu_l, I)$ is a GMM. We restrict to a class of hypotheses where means are separated from each other and contained in a ball of radius $M$, and denote $\mathcal{H}_{k,M}$ the corresponding class of hypotheses. Unlike $k$-means, the separation between means cannot be as small as desired, and there is a trade-off between the required separation and the required number of measurements $m$. A few values are given in Table I.

Result. Denote $h^* \in \mathcal{H}_{k,M}$ the true minimizer of the expected risk $\mathcal{R}$ and $\hat{h}$ the hypothesis recovered from the sketch by solving

$$\hat{h} = \arg\min_{h \in \mathcal{H}_{k,M}} \|\mathbf{y} - A \pi_h\|_2.$$

If the number of measurements $m$ is large enough (see Tab. I), with joint probability $1 - \rho$ on the drawing of $z_i$ and $\omega_i$ it holds that

$$\mathcal{R}(\hat{h}) - \mathcal{R}(h^*) \lesssim \inf_{h \in \mathcal{H}_{k,M}} \|\pi - \pi_h\|_{TV} + O \left( \sqrt{1/n} \right)$$

where the $O$ hides some dependencies in $k, d$ (roughly behaving like $m$ in Tab. I). The bound also involves the best approximation of $\pi$ by a GMM for the TV norm ($L^1$ norm for densities).

III. EXPERIMENTAL RESULTS

The optimization problems [8] and [9] are non-convex and seem hard to solve exactly. Heuristically, a greedy algorithm inspired by sparse recovery referred to as Compressive Learning OMP (CLOMP) [10–12] has been previously shown to perform well. We compare a Matlab implementation of CLOMP available at [13] with Matlab’s kmeans function and VLFeat’s gmm function.

In Fig. I the sketched approach is seen to lead to tremendous savings in time of execution and memory consumption when the number of items $n$ is large, while achieving the same precision as the corresponding traditional approach for a limited number of measurements $m \approx O(kd)$. Fig. further confirms that $m \approx O(kd)$ is empirically sufficient, hence the theoretical guarantees for $m \gtrsim O(k^2d^2)$ are probably pessimistic.

Further work will combine the sketching technique with dimensionality-reduction methods to treat both large $d$ and $n$. 
TABLE I
TRADE-OFF BETWEEN REQUIRED SEPARATION OF MEANS AND NUMBER OF MEASUREMENTS IN THE GMM LEARNING PROBLEM.

<table>
<thead>
<tr>
<th>Separation of means</th>
<th>Number of measurements</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O(\frac{1}{\sqrt{d}})$</td>
<td>$m \geq O\left(k^2 d^2 \exp\left(\frac{k}{2}\right)\right)$</td>
</tr>
<tr>
<td>$O(\sqrt{d + \log k})$</td>
<td>$m \geq O\left((k d)^{2.5} \log k\right)$</td>
</tr>
<tr>
<td>$O(\sqrt{\log k})$</td>
<td>$m \geq O\left(k^2 d^2 \exp\left(\frac{k}{2}\right)\right)$</td>
</tr>
</tbody>
</table>

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REFERENCES

Fig. 1. Relative memory consumption (left), time of estimation (center) and precision (right) for compressive k-means (top) and GMM estimation (bottom) with $k = 10$ components in dimension $d = 10$, compared to MatLab’s kmmeans and VLFeat’s gmm functions (dotted black lines).

Fig. 2. Relative precision for k-means (top) and GMM estimation (bottom) with respect to the relative number of measurements $m/(kd)$. On the left $k = 10$ and on the right $d = 10$. 