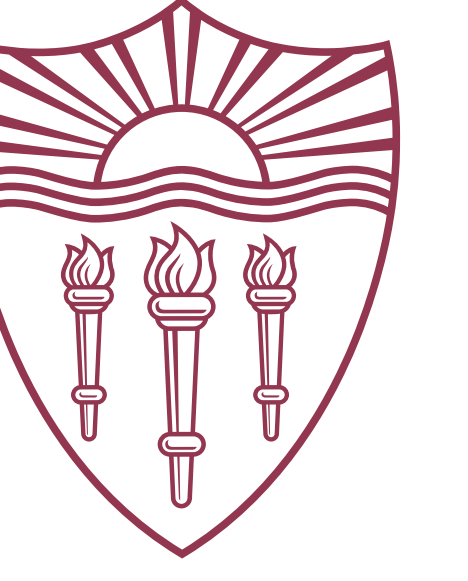


# Towards a Definition of Local Stationarity for Graph Signals

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## Objectives

- Study stochastic signals supported by arbitrary discrete structures rather than Euclidean structures.
- Define a mathematical model for stochastic graph signals with an invariant local spectrum.

## State of the Art: (Global) Stationarity [2]

**Definition 1** (WSS Graph Signal). The first two moments of a WSS graph signal  $\mathbf{x}$  are invariant through the graph translation operator:

$$\mathbb{E}[\mathbf{x}] = \mathbb{E}[T_G \mathbf{x}] \quad \mathbb{E}[\mathbf{x}\mathbf{x}^*] = \mathbb{E}[T_G \mathbf{x} (T_G \mathbf{x})^*]$$

**Theorem 1** (WSS Spectral Characterization). A graph signal  $\mathbf{x}$  is WSS if and only if:

- $\mathbb{E}[\mathbf{x}]$  is colinear to the Fourier mode of lowest frequency (DC Component)
- $\hat{\mathbf{x}}(l)$  and  $\hat{\mathbf{x}}(k)$  are uncorrelated if  $l$  and  $k$  correspond to distinct frequencies

**Definition 2** (Power Spectrum Density (PSD)). The PSD of the graph signal  $\mathbf{x}$  is:

$$\gamma_{\mathbf{x}}(l) = \mathbb{E} [|\hat{\mathbf{x}}(l)|^2].$$

*Remark 1.* Two alternative definitions of (global) stationarity exist, but Definition 1 is the first introduced, and the closest to the classical definition of stationarity. However, Definition 2 remains the same.

## Contribution: Local Stationarity

**Definition 3** (Localized Atoms & Localized Decomposition). Let  $\{g_{i,m}\}_{i,m}$  be a family of atoms with  $g_{i,m}$  localized on vertex  $i$ , and  $m$  an index that is larger when the atom's support is large. The decomposition of the signal  $x$  on this family is given by:

$$w_x(i, m) = \langle x, g_{i,m} \rangle = \sum_j x(j) g_{i,m}^*(j).$$

**Definition 4** (Local Power Spectrum Density (local PSD) and rescaled local PSD).

$$S_{\mathbf{x}}(i, m) = \mathbb{E} [ |w_{\mathbf{x}}(i, m)|^2 ] \quad \overline{S}_{\mathbf{x}}(i, m) = \frac{S_{\mathbf{x}}(i, m)}{\|g_{i,m}\|_2^2}.$$

**Definition 5** (Locally Stationary Signal). A signal  $\mathbf{x}$  is said locally stationary if and only if its (resp. rescaled) local power spectrum  $S_{\mathbf{x}}(i, m)$  (resp.  $\overline{S}_{\mathbf{x}}(i, m)$ ) does not depend on the vertex  $i$ .

Here, we use the *spectral graph wavelets* as the family of localized atoms [3].

## Background: GMRF model

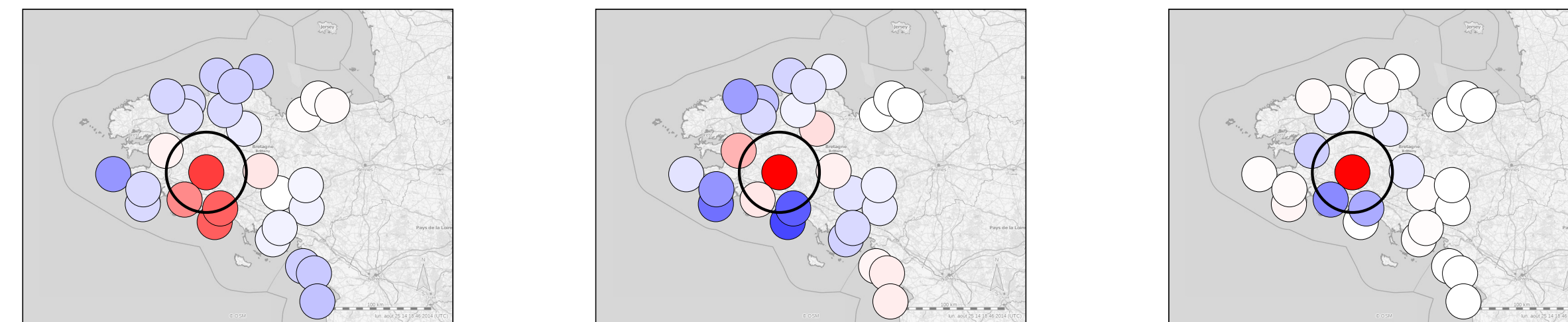
**Definition 6** (Gaussian Markov Random Field Model). A stochastic signal  $\mathbf{x}$  is a GMRF signal if and only if it is WSS with Power Spectrum Density (PSD) [4]:

$$\gamma_{\mathbf{x}}(l) = \frac{b}{a + \lambda_l},$$

where  $\lambda_l$  is the  $l^{\text{th}}$  eigenvalue of the Laplacian matrix of the graph, and  $a, b > 0$ .

## Molène Dataset

- Geographical graph of 28 weather stations in Brittany, FRANCE.
- Weighted by a Gaussian kernel of the geographical distance ( $\sigma^2 = 5.10^9$ )
- Thresholded such that weights are greater than  $10^{-4}$



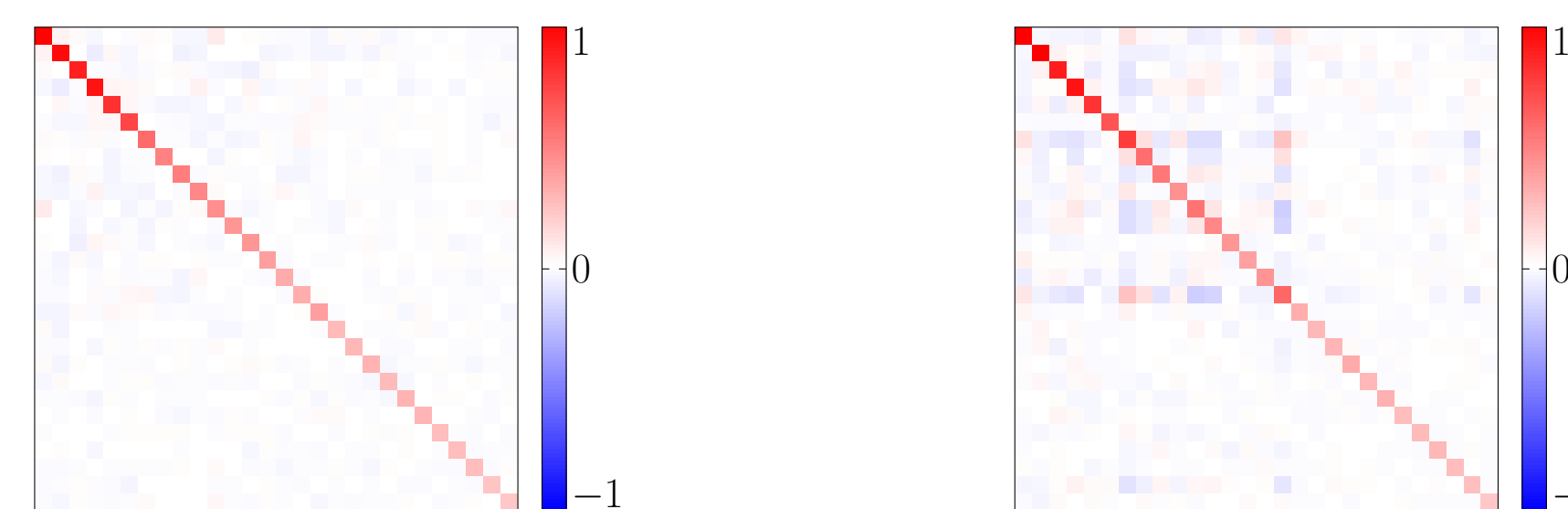
**Figure 1:** Molène Graph, with the anomaly circled, and three atoms ( $m = 5$ ,  $m = 10$  and  $m = 16$ ) centered on the anomaly.

- Hourly readings of January 2014: 744 snapshots of temperature (among other)
- Preprocessing through Empirical Mode Decomposition (see [1])
- **Assumption:** For a given reading, snapshots are realization of a WSS graph signal.
- **Goals:** (i) Check this assumption, (ii) Compute an estimate of the (global) PSD

## Results with a GMRF Model

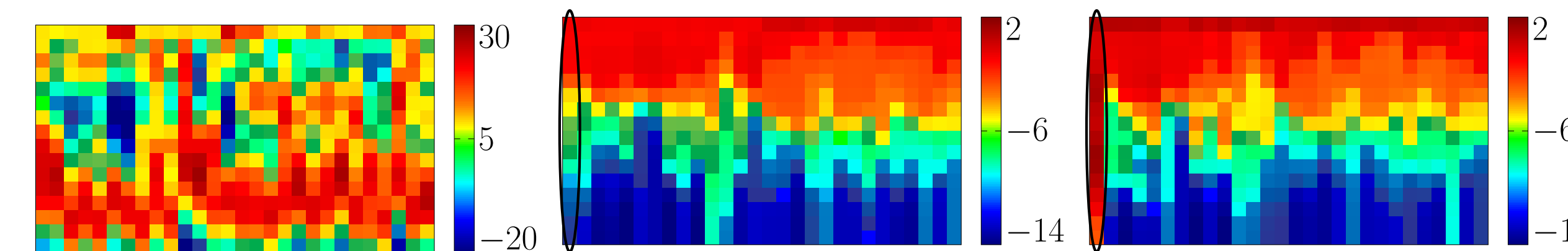
**GMRF:**  $a = b = 1$ . **Non-stationarity:** localized normalized Gaussian noise on vertex 1.

## Empirical Spectral Correlation Matrix



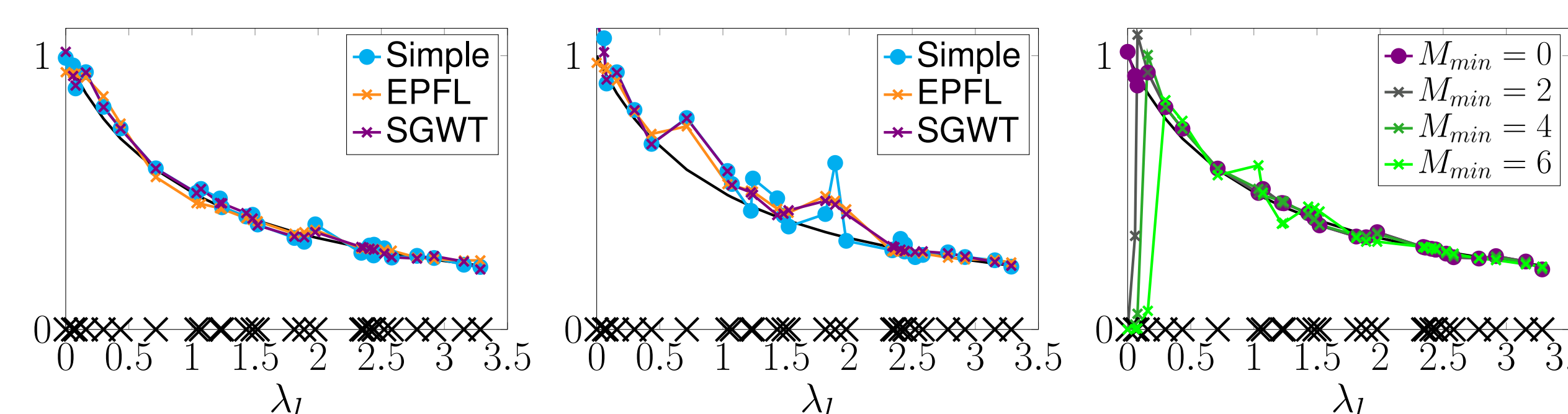
**Figure 2:** GMRF Model (left) and GMRF with non-stationarity (right) spectral correlation matrices: Vertex 1 can not be identified as the anomaly.

## Local PSD: Anomaly Detection



**Figure 3:** Local PSD (coarsest scales on top, first vertex on the left) of a GMRF Model (left) rescaled local PSD (middle) and rescaled PSD of the same model with a non-stationarity on vertex 1 (right).  $M = 16$  scales. Logarithmic color scales. Note that the anomaly has an impact on local PSDs of other vertices, but to a smaller extent.

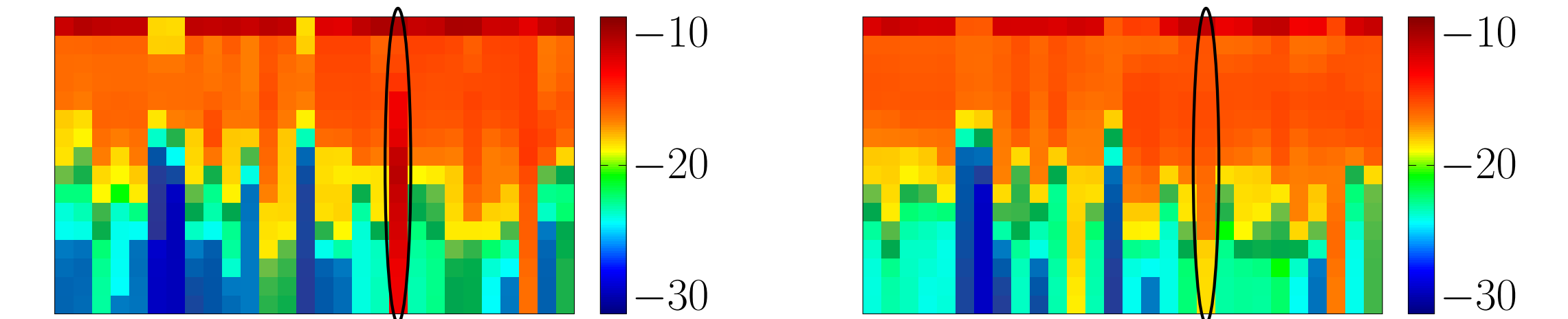
## Global PSD Estimation (Least Squares)



**Figure 4:** Global PSD Estimation from the Local PSD against alternative estimators (left) same results with the non-stationarity (middle) and impact of removing the coarsest scales (right). Black: Ground truth. EPFL refers to the estimator of [5].

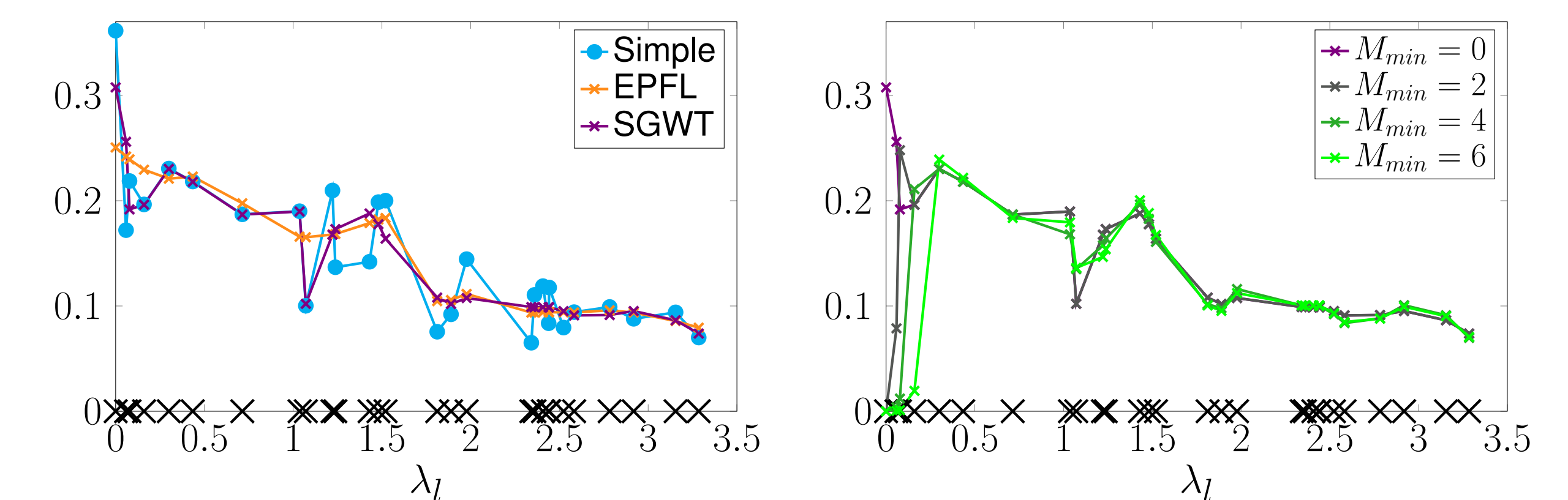
## Results with Temperature Data

### Anomaly Detection



**Figure 5:** Local PSD (coarsest scales on top, first vertex on the left) of the preprocessed temperature data (left) and of a model following the same empirical PSD (right). Studying the data, this station shows the sharpest short-term variations (higher variance).

### PSD Estimation



**Figure 6:** Global PSD Estimation from the Local PSD against alternative estimators (left) and impact of removing the coarsest (right) for the preprocessed temperature data. Note that EPFL estimator is smoother in low frequencies, while our estimator allows for more richness. It can be shown that our estimator is however smoother in high frequencies.

## Conclusion and Perspectives

- New definition of local stationarity for graph signals using the newly introduced concept of local power spectrum.
- Anomaly detection using the local PSD.
- New (global) PSD estimator with more expressivity for low frequencies, and smooth PSD in the high frequencies.
- Perspective: Work on  $\{g_{i,m}\}$  to obtain better localization.
- Perspective: Stationarity test.

## References

- [1] B. Girault. *Signal Processing on Graphs - Contributions to an Emerging Field*. Phd thesis, Ecole normale supérieure de lyon - ENS LYON, Dec. 2015.
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- [3] D. K. Hammond, P. Vandergheynst, and R. Gribonval. Wavelets on graphs via spectral graph theory. *Applied and Computational Harmonic Analysis*, 30(2):129–150, 2011.
- [4] E. Pavez and A. Ortega. Generalized Laplacian precision matrix estimation for graph signal processing. In *2016 IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP)*, pages 6350–6354. IEEE, 2016.
- [5] N. Perraudin and P. Vandergheynst. Stationary signal processing on graphs. *arXiv preprint arXiv:1601.02522*, 2016.