Long-time homogenization and asymptotic ballistic transport of classical waves

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Consider an elliptic operator in divergence form with symmetric coefficients. If the diffusion coefficients are periodic, the Bloch theorem allows one to diagonalize the elliptic operator, which is key to the spectral properties of the elliptic operator and the usual starting point for the study of its long-time homogenization. When the coefficients are not periodic (say, quasi-periodic, almost periodic, or random with decaying correlations at infinity), the Bloch theorem does not hold and both the spectral properties and the long-time behavior of the associated operator are unclear. At low frequencies, we may however consider a formal Taylor expansion of Bloch waves (whether they exist or not) based on correctors in elliptic homogenization. The associated Taylor-Bloch waves diagonalize the elliptic operator up to an error term (an "eigendefect"), which we express with the help of a new family of extended correctors. We use the Taylor-Bloch waves with eigendefects to quantify the transport properties and homogenization error over large times for the wave equation in terms of the spatial growth of these extended correctors. On the one hand, this quantifies the validity of homogenization over large times (both for the standard homogenized equation and higher-order versions). On the other hand, this allows us to prove asymptotic ballistic transport of classical waves at low energies for almost periodic and random operators.

CONTENTS

1. INTRODUCTION

Let $\boldsymbol{a}$ be a periodic symmetric coefficient field, and consider the rescaled wave operator $\Box_\varepsilon := \partial_{tt}^2 - \nabla \cdot \boldsymbol{a}(\frac{\cdot}{\varepsilon})\nabla$. It is known since the pioneering works in homogenization that for fixed final time $T < \infty$, the operator $\Box_\varepsilon$ can be replaced by the homogenized wave operator $\Box_{\text{hom}} := \partial_{tt}^2 - \nabla \cdot \boldsymbol{a}_{\text{hom}}\nabla$, where $\boldsymbol{a}_{\text{hom}}$ are the homogenized (and constant) coefficients associated with $\boldsymbol{a}$ through elliptic homogenization (see [13, 22] where the question of the corrector and convergence of the energy is also addressed, and Section 2 for precise definitions). Let $u_0 \in \mathcal{S}(\mathbb{R}^d)$, the Schwartz class, and let $u_\varepsilon \in L^\infty(\mathbb{R}_+, L^2(\mathbb{R}^d))$ be the unique weak solution of

$$\begin{cases} \Box_\varepsilon u_\varepsilon(t,x) &= 0, \\ u_\varepsilon(0,\cdot) &= u_0, \\ \partial_t u_\varepsilon(0,\cdot) &= 0. \end{cases} \tag{1.1}$$

Then for all $T > 0$, $\lim_{\varepsilon\downarrow 0} \sup_{0\le t\le T} \|u_\varepsilon(t,\cdot) - u_{\text{hom}}(t,\cdot)\|_{L^2(\mathbb{R}^d)} = 0$, where u_{hom} solves the homogenized equation

$$\begin{cases} \Box_{\text{hom}} u_{\text{hom}}(t,x) &= 0, \\ u_{\text{hom}}(0,\cdot) &= u_0, \\ \partial_t u_{\text{hom}}(0,\cdot) &= 0. \end{cases} \tag{1.2}$$

Refining this result received much attention in the recent years — and in particular the large-time behavior of u_ε with fixed or oscillating initial conditions. For fixed initial conditions u_0 (independent of ε), one expects dispersive effects — which are not accounted for by (1.2) — to appear at times of order $\varepsilon^{-2}T$ (see [37] for pioneering works in this direction, [33, 17, 18] for the first rigorous results, and [1, 2] for numerical methods). For oscillating initial conditions, the medium interacts with the initial conditions much more, which yields even finer dispersive effects (see [7, 8]). Both refinements crucially rely on spectral properties of the operator $-\nabla \cdot \boldsymbol{a}\nabla$, namely that it is diagonalized by Floquet-Bloch waves (Bloch in short, see [6, 5, 7, 8, 17, 18]): the spectrum of $-\nabla \cdot \boldsymbol{a}\nabla$ is purely absolutely continuous, and extended states are semi-explicit (see below). Hence, there is a clear starting point to study the above questions: project the initial condition on the Bloch wave basis, and treat the wave equation (1.1) as an ODE. From a spectral point of view, the Bloch theory implies that $-\nabla \cdot \boldsymbol{a}\nabla$ has purely absolutely continuous spectrum in form of possibly overlapping bands (the first one including 0).

The Bloch theory crucially relies on the periodicity of $\boldsymbol{a}$, and can be seen as a variant of the Fourier transform (with which it coincides when $\boldsymbol{a}$ is a constant matrix). The main idea is to look for extended states of the operator $-\nabla \cdot \boldsymbol{a}\nabla$ in the form of modulated plane waves $x \mapsto e^{ik\cdot x}\psi_k(x)$, where ψ_k is a periodic function. Such a function ψ_k is then solution of the magnetic eigenvalue problem on the torus

$$-(\nabla + ik) \cdot \boldsymbol{a}(\nabla + ik)\psi_k = \lambda_k \psi_k$$

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for some λ_k. By the Rellich theorem, $-(\nabla + ik) \cdot \boldsymbol{a}(\nabla + ik)$ has compact resolvent, which allows one to define a family of eigenvectors and eigenvalues, on which the Bloch decomposition relies. Replace $\boldsymbol{a}$ by the sum of two periodic functions with incommensurable periods, and the whole picture breaks down: the magnetic operator is now lifted to a higher-dimensional torus, it is hypo-elliptic, and does not have compact resolvent any longer. In particular, we do not know whether the ψ_k exist. For more general coefficients (say almost-periodic, or random), the Bloch theory simply does not hold. Indeed, this theory implies that the elliptic operator $-\nabla \cdot \boldsymbol{a}\nabla$ has purely absolutely continuous spectrum, whereas it is known that this operator has some discrete spectrum in any dimension for some representative examples of $\boldsymbol{a}$, cf. [38, Theorem 3.3.6].

Questions regarding oscillating initial data explore the entire spectrum of the operator $-\nabla \cdot \boldsymbol{a}\nabla$, and we expect a completely different behavior for periodic and non-periodic coefficients, since their spectrum is of different type. This is the realm of challenging questions of spectral analysis [38, 3] and radiative transfer [35, 36]. For non-oscillating initial data however, only the bottom of the spectrum of $-\nabla \cdot \boldsymbol{a}\nabla$ is relevant, and we are in the realm of homogenization. For final times $T < \infty$ independent of ε, qualitative theory for the elliptic operator is enough to prove the convergence of (1.1) to (1.2), and we just need to know that the solution operator $(-\nabla \cdot \boldsymbol{a}(\frac{\cdot}{\varepsilon})\nabla)^{-1}$ converges to the homogenized solution operator $(-\nabla \cdot \boldsymbol{a}_{\text{hom}}\nabla)^{-1}$ as $\varepsilon \downarrow 0$. If we happen to have quantitative information on this convergence in terms of ε (in a broad sense), we might be able to consider larger time frames $[0, \varepsilon^{-\alpha}T]$ (with $\alpha > 0$) and gain information on the large-time behavior of u_ε. The aim of this contribution is to develop such an approach for operators that are beyond the reach of the classical Bloch theory.

As a first and crucial step, we introduce a proxy for the Bloch wave decomposition. Since we are only interested in low frequencies, we only need a proxy for Bloch waves at low frequencies. In the case of periodic coefficients, it is well-known that Bloch waves ψ_k are essentially analytic functions of k, and that their derivatives are related to cell-problems in elliptic homogenization (e.g. [16, 4]). Whereas eigenvectors ψ_k might not exist (even at low frequencies), one may still consider their formal Taylor expansion $\psi_{k,j}$ of order j for all $0 \leq |k| \ll 1$ based on correctors (up to order j) provided the latter exist, which gives rise to what we call Taylor-Bloch waves $x \mapsto e^{ik \cdot x}\psi_{k,j}(x)$. These waves are only "approximate" extended states of the operator $-\nabla \cdot \boldsymbol{a}\nabla$, so that the study of the defect in the eigenvector/eigenvalue relation (which we call the "eigendefect") is equally important as the formula for the Taylor-Bloch wave itself. The study of the Taylor-Bloch expansion is the aim of Section 2, where we introduce a new family of higher-order correctors, that are used to put the eigendefect in a suitable form for the rest of our analysis.

The second step consists in constructing an approximate solution to equation (1.1) using the Taylor-Bloch waves, cf. Section 3. We first replace the initial condition by a well-prepared initial condition in the form of a Taylor-Bloch expansion — which simply amounts to replacing $e^{ik \cdot x}$ by $e^{ik \cdot x}\psi_{\varepsilon k,j}(x)$ in the Fourier inversion formula for u_0. The difference between the solutions of both initial value problems is then proved to be of order ε uniformly in time by energy methods, and it remains to solve the problem with well-prepared initial data. To this aim, we use that Taylor-Bloch waves diagonalize the elliptic operator (up to the eigendefect) to construct an approximate solution by explicit time-integration. Next, we estimate the error due to the eigendefects by energy methods on the wave equation, which yields a control over large times that depend on the growth of

the extended correctors. To conclude, we simplify the approximate solution by throwing away most of the corrections, while keeping sufficient accuracy in $L^2(\mathbb{R}^d)$. This final approximation is accurate up to times $\varepsilon^{-\alpha}T$, where $\alpha > 0$ depends on the growth of the extended correctors.

Equipped with the Taylor-Bloch approximation of the solution to equation (1.1), we turn to the main two results of this article: the long-time homogenization of $\partial_{tt}^2 - \nabla \cdot \boldsymbol{a}(\frac{\cdot}{\varepsilon})\nabla$ (cf. Section 4) and the (asymptotic) ballistic transport properties of $-\nabla \cdot \boldsymbol{a}\nabla$ at low energies (cf. Section 5). The range of application of these results crucially depends on the control we have on the extended correctors. Although this is an important issue, this does not constitute the original part of this article: the analysis uses (rather than develops) methods introduced in recent independent works on quantitative homogenization of linear elliptic operators in divergence form that started with [34, 29, 30, 26, 28], and culminated in [11, 10] and [25, 24, 27], see also [9]. For completeness, and in order to stress the interest of our results, we quickly display in Appendix A the estimates on the growth of the extended correctors that are expected to hold for some almost periodic and random coefficients (the proofs of these results are however not straightforward).

Let us start with the long-time homogenization results. Based on the Taylor-Bloch approximation of the solution to equation (1.1), we prove the validity of the approximation of (1.1) by (1.2) up to times $\varepsilon^{-\alpha}T$ for some $0 < \alpha < 2$ depending on the coefficients and dimension (cf. Appendix A). Provided the second extended correctors are essentially bounded (which is always the case for periodic coefficients, holds under some conditions for almost periodic coefficients, and can only hold in dimensions $d > 4$ for random coefficients with decay of correlations at infinity), we obtain the validity of an approximation involving dispersive effects up to times $\varepsilon^{-\alpha}T$ for some $2 < \alpha < 4$. The interpretation of dispersion in the approximate solution goes through a fourth-order equation, which is a higher-order homogenized equation, and we essentially follow [18] (which deals with periodic coefficients). For non-periodic coefficients, all the results are new. For periodic coefficients, besides we also treat systems, we improve [18] in three respects: we do not require the coefficient field $\boldsymbol{a}$ to be smooth, we obtain error estimates valid over larger times, and we generalize the result to any order (which yields a new family of higher-order homogenized equations parametrized by $k \in \mathbb{N}$ that are valid up to times $\varepsilon^{-2(k+1)^-}T$). Incidentally, our analysis also yields new insight in the homogenization of elliptic problems: it allows to define higher-order homogenized elliptic operators and to control the associated multiscale homogenization error when the RHS is well-prepared. This seems to be new even in the periodic setting and extends (in the symmetric setting) the recent independent work [12] to any order.

We conclude with the asymptotic ballistic transport for $-\nabla \cdot \boldsymbol{a}\nabla$. The characterization of the spectrum of $-\nabla \cdot \boldsymbol{a}\nabla$ (or that of the random Schrödinger operator $-\triangle + V$, with a random potential V), and in particular the understanding of the (expected) transition between discrete and continuous spectrum, is a major open problem of mathematical physics. In the case of the elliptic operator $-\nabla \cdot \boldsymbol{a}\nabla$, the bottom of the spectrum is very peculiar (indeed, spectral localization can be proved at band edges, but not at 0, cf. [21]), and one might expect the spectrum to have a continuous part in the neighborhood of 0. A stronger statement of the existence of continuous spectrum would be the ballistic transport of suitable initial conditions for arbitrarily long times. The approach based on Taylor-Bloch waves yields a first *asymptotic* result in that direction, and allows us

to prove the ballistic transport of initial conditions of "energy" $0 < \varepsilon \ll 1$ on time frames $[0, \varepsilon^{-\alpha}T]$ for some specific $\alpha \geq 0$ depending on the structure of the coefficients and dimension. For periodic coefficients, our estimates can be proved to be uniform, and yield ballistic transport at all times provided $0 < \varepsilon \ll 1$ (thus establishing ballistic transport at small energies without explicitly appealing to the Bloch theorem). For diophantine quasi-periodic coefficients (for which the Bloch theory does not hold), one can choose $\alpha > 0$ arbitrarily large. However, as opposed to the periodic setting, the multiplicative constants in the estimates blow up too fast as $\alpha \uparrow +\infty$ to prove ballistic transport at all times. For random coefficients, α depends on the dimension and the correlations: For Gaussian coefficient fields with integrable correlations, we have some asymptotic ballistic transport in dimensions $d > 2$.

A similar strategy based on approximate spectral analysis (and various notions of approximate Taylor-Bloch waves) can be used to establish asymptotic ballistic transport of quantum waves, and we refer the reader to [20] for such results on the Schrödinger equation with periodic, quasi-periodic, and random potentials.

2. Taylor-Bloch waves

Let $\Lambda \geq 1$. Throughout this contribution we assume that $\boldsymbol{a} : \mathbb{R}^d \to \mathcal{M}_d(\mathbb{R})$ is a measurable uniformly elliptic *symmetric* coefficient field that satifies for all $\xi \in \mathbb{R}^d$ and almost all $x \in \mathbb{R}^d$

$$\xi \cdot \boldsymbol{a}(x)\xi \geq |\xi|^2, \quad |\boldsymbol{a}(x)\xi| \leq \Lambda|\xi|.$$

All the constants in our estimates acquire a dependence on Λ. (In view of the quantitative stochastic homogenization results used in Appendix A, all the results of this contribution hold true in the case of strongly elliptic systems.) This section is inspired by [4], where the authors derive equations satisfied by the derivatives of Bloch waves at 0 for periodic coefficients. The main additional insight compared to [4] is the identification of the structure of these derivatives. In particular, we rewrite them in terms of suitable correctors, which allows us to turn the remainder in the Taylor expansion in divergence form plus a higher-order term, cf. the eigendefect in Definition 2.2. In the periodic setting, this is not fundamental. In other settings however (like almost periodic or random), this allows us to show that the Taylor-Bloch expansion is one order more accurate than expected (cf. Section 3), which is crucial to capture both the correct long-time accuracy in homogenization (cf. Section 4) and the correct dimensions for asymptotic ballistic transport (cf. Section 5).

We assume that $\boldsymbol{a}$ are $\mathbb{Z}^d$-stationary ergodic coefficients (this class includes periodic, quasiperiodic, almost-periodic, and random coefficients with decaying correlation at infinity). We denote by $\mathbb{E}[\cdot]$ the expectation (in the periodic setting, the expectation can be dropped).

2.1. **Extended correctors and higher-order homogenized coefficients.** We start with the definition of a family of extended correctors which will serve as the basis for the definition of the Taylor-Bloch waves, and momentarily fix a unit direction $e \in \mathbb{R}^d$.

Definition 2.1. For all $\ell \geq 0$, we say that $(\varphi_j, \sigma_j, \chi_j)_{0 \leq j \leq \ell}$ are the first ℓ extended correctors in direction e if these functions are locally square-integrable ($\mathbb{Z}^d$-stationary and such that their average on $Q = [0,1)^d$ has vanishing expectation for $0 < j < \ell$) and satisfy the following extended corrector equations on $\mathbb{R}^d$:

- $\varphi_0 \equiv 1$, and for all $j \geq 1$, φ_j is a scalar field that satisfies
$$-\nabla \cdot \boldsymbol{a}\nabla\varphi_j = \nabla \cdot (-\sigma_{j-1}e + \boldsymbol{a}e\varphi_{j-1} + \nabla\chi_{j-1});$$

- for all $j \geq 0$, the symmetric matrix $\tilde{\boldsymbol{a}}_j$, the symmetric $(j+2)$-th order tensor $\bar{\boldsymbol{a}}_j$, and the scalar λ_j are given by
$$\bar{\boldsymbol{a}}_j e^{\otimes(j+1)} = \tilde{\boldsymbol{a}}_j e := \mathbb{E}\left[\int_Q \boldsymbol{a}(\nabla\varphi_{j+1} + e\varphi_j)\right], \quad \lambda_j := e \cdot \tilde{\boldsymbol{a}}_j e, \quad e^{\otimes(j+1)} := \underbrace{e \otimes \cdots \otimes e}_{j+1 \text{ times}};$$

- $\chi_0 \equiv 0$, $\chi_1 \equiv 0$, and for all $j \geq 2$, χ_j is a scalar field that satisfies
$$-\triangle\chi_j = \nabla\chi_{j-1} \cdot e + \sum_{l=1}^{j-1} \lambda_{j-1-l}\varphi_l;$$

- for all $j \geq 1$, q_j is a vector field (a higher-order flux) given by
$$q_j := \boldsymbol{a}(\nabla\varphi_j + e\varphi_{j-1}) - \tilde{\boldsymbol{a}}_{j-1}e + \nabla\chi_{j-1} - \sigma_{j-1}e, \quad \mathbb{E}\left[\int_Q q_j\right] = 0;$$

- $\sigma_0 \equiv 0$, and for all $j \geq 1$, σ_j is a skew-symmetric matrix field (a higher-order flux corrector), i.e. $\sigma_{jkl} = -\sigma_{jlk}$, that satisfies
$$-\triangle\sigma_j = \nabla \times q_j, \quad \nabla \cdot \sigma_j = q_j,$$
with the three-dimensional notation: $[\nabla \times q_j]_{mn} = \nabla_m[q_j]_n - \nabla_n[q_j]_m$, and where the divergence is taken with respect to the second index, i. e. $(\nabla \cdot \sigma_j)_m := \sum_{n=1}^d \partial_n \sigma_{jmn}$.

$\square$

Let us make a few comments on this definition.
- The correctors φ_j are related but do not coincide (for $j > 2$) with the higher-order correctors classically used in the multiscale expansion for periodic coefficients (or random coefficients in [31]), and we refer the reader to [4] for a discussion of these differences in the periodic setting.
- The higher-order flux q_j is chosen to be divergence-free, so that it is an exact $(d-1)$-form and hence admits a "vector potential", that is a $(d-2)$-form, which can be represented by the skew-symmetric tensor σ_j (the equation for σ_j is the natural choice of gauge). These definitions are natural generalizations to any order of the extended correctors (φ, σ) considered in [25] (see below). For the the existence, uniqueness, and properties of these extended correctors (depending on the properties of the field $\boldsymbol{a}$ and the dimension d), we refer the reader to Appendix A. These correctors for $j = 2$ were considered independently by Bella, Fehrmann, Fischer, and Otto in [12].
- The correctors φ_j are variants of those defined in [4]. They are however not normalized the same way (see in particular Remark 2.2 below), which is crucial to consider unbounded higher-order correctors.
- The use of the correctors (σ_j, χ_j) allows one to write the remainder in *some* multiscale expansion in divergence form plus a higher-order term, which turns out to be new (to our knowledge) and directly yields sharp convergence results for the higher-order expansion with well-prepared RHS (which strictly generalizes the quantitative two-scale expansion of [25], as well as the second order expansion of

[12], which both treat non-symmetric coefficients; see also [4, Section 5]), cf. Theorem 3 and Corollary 2.

- Let us quickly show that the first extended correctors (of order $l = 1$) are indeed the standard correctors in elliptic homogenization. The equation satisfied by φ_1 takes the form

$$-\nabla \cdot \boldsymbol{a}(\nabla \varphi_1 + e) = 0$$

so that φ_1 is the classical corrector in stochastic homogenization, i.e. the unique sublinear at infinity solution of $-\nabla \cdot \boldsymbol{a}(\nabla \varphi_1 + e) = 0$ with stationary gradient (for conditions on $\boldsymbol{a}$ under which φ_1 is stationary, we refer to [29, 28, 25, 27]). Thus $\bar{\boldsymbol{a}}_0 = \tilde{\boldsymbol{a}}_0 = \boldsymbol{a}_{\mathrm{hom}}$ (the homogenized coefficients), $\lambda_0 = e \cdot \boldsymbol{a}_{\mathrm{hom}} e$. Hence, $q_1 = \boldsymbol{a}(\nabla \varphi_1 + e) - \boldsymbol{a}_{\mathrm{hom}} e$ (the flux of the corrector minus the homogenized flux), so that σ_1 is nothing but the flux corrector (the existence of which is proved in [25] for stationary ergodic coefficients $\boldsymbol{a}$).

We conclude this paragraph with some important properties of the higher-order homogenized coefficients λ_j. For notational convenience we assume that $\boldsymbol{a}$ enjoys continuum stationarity (for discrete stationarity, it suffices to replace $\mathbb{E}\left[\cdot\right]$ by $\mathbb{E}\left[\int_Q \cdot\right]$).

Proposition 1. *Let $\ell \in \mathbb{N}_0$ and assume that the correctors $(\varphi_j, \sigma_j, \chi_j)_{0 \le j \le \ell}$, their gradients, and $\nabla \varphi_{\ell+1}$ of Definition 2.1 are stationary with bounded second moments, so that the higher-order homogenized coefficients $\{\lambda_j = \mathbb{E}\left[e \cdot \boldsymbol{a}(\nabla \varphi_{j+1} + e\varphi_j)\right]\}_{0 \le j \le \ell}$ are well-defined. Then:*

   (i) *if $0 \le j \le \ell$ is odd, then $\lambda_j = 0$;*
   (ii) *$\lambda_0 > 0$ and $\lambda_2 \ge 0$ (provided $\ell \ge 2$).*

$\square$

For periodic coefficients, this proposition is standard: Statement (i) is due to the symmetry of the coefficient field and follows from spectral theory (see e.g. [6], and the proof of Proposition 4 in Appendix A), and the nice observation (ii) for $j = 2$ was first proved in [15] in the scalar setting. Although the proof of (ii) in [15] essentially extends to the stochastic setting, and the proof of (i) follows from the result in the periodic setting by a suitable approximation argument (the so-called periodization method), we display elementary proofs of both results that do not rely on spectral theory, hold mutatis mutandis for systems, and use in a systematic way the "algebraic" properties of the extended correctors of Definition 2.1. (The proof of $\lambda_{2j+1} = 0$ extends the result $\lambda_1 = 0$ proved in [12].)

Proof of Proposition 1. We start with the proof of (i) by induction, and then turn to the proof of (ii). In this proof we systematically use the symmetry of $\boldsymbol{a}$ to change the order in scalar products without transposing $\boldsymbol{a}$.

Step 1. Proof of (i).
We split the proof of (i) into several substeps.

Substep 1.1. Proof of the identity: for all $j \ge 1$ and $l \ge j + 1$,

$$\mathbb{E}\left[\nabla \varphi_j \cdot \boldsymbol{a} \nabla \varphi_l - \varphi_{j-1} \varphi_{l-1} e \cdot \boldsymbol{a} e\right] = \mathbb{E}\left[-\nabla \varphi_{j+1} \cdot \boldsymbol{a} \nabla \varphi_{l-1} + \varphi_j \varphi_{l-2} e \cdot \boldsymbol{a} e\right]$$

$$- \mathbb{E}\left[\sum_{m=1}^{j-1} \lambda_{j-1-m} \varphi_{l-1} \varphi_m + \sum_{m=1}^{l-2} \lambda_{l-2-m} \varphi_m \varphi_j\right]. \quad (2.1)$$

Starting point is the equation for φ_l in the form

$$
\begin{aligned}
\mathbb{E}\left[\nabla\varphi_j \cdot \boldsymbol{a}\nabla\varphi_l\right] &= \mathbb{E}\left[-\nabla\varphi_j \cdot (\boldsymbol{a}e\varphi_{l-1} - \sigma_{l-1}e + \nabla\chi_{l-1})\right] \\
&= \mathbb{E}\left[-\nabla\varphi_j \cdot \boldsymbol{a}e\varphi_{l-1}\right] + \mathbb{E}\left[\nabla\varphi_j \cdot \sigma_{l-1}e\right] + \mathbb{E}\left[-\nabla\varphi_j \cdot \nabla\chi_{l-1}\right],
\end{aligned}
$$

and we reformulate the last two RHS terms. By stationarity of $\varphi_j\sigma_{l-1}$ in the form $\mathbb{E}\left[\nabla \cdot (\varphi_j\sigma_{l-1})\right] = 0$, the skew-symmetry of σ_{l-1}, the defining property $\nabla \cdot \sigma_{l-1} = q_{l-1}$, the fact that φ_j has vanishing expectation, and the skew-symmetry of σ_{l-2}, we have

$$
\begin{aligned}
\mathbb{E}\left[\nabla\varphi_j \cdot \sigma_{l-1}e\right] &= \mathbb{E}\left[\varphi_j e \cdot (\nabla \cdot \sigma_{l-1}e)\right] \\
&= \mathbb{E}\left[\varphi_j e \cdot (\boldsymbol{a}(\nabla\varphi_{l-1} + e\varphi_{l-2}) - \lambda_{l-1} + \nabla\chi_{l-2} + \sigma_{l-2}e)\right] \\
&= \mathbb{E}\left[\varphi_j e \cdot (\boldsymbol{a}(\nabla\varphi_{l-1} + e\varphi_{l-2}) + \nabla\chi_{l-2})\right].
\end{aligned}
$$

By the equation for χ_{l-1} tested with φ_j, we also have

$$
\mathbb{E}\left[-\nabla\varphi_j \cdot \nabla\chi_{l-1}\right] = \mathbb{E}\left[-\nabla\chi_{l-2} \cdot e\varphi_j\right] - \mathbb{E}\left[\sum_{m=1}^{l-2} \lambda_{l-2-m}\varphi_m\varphi_j\right].
$$

The combination of these last four identities yields

$$
\mathbb{E}\left[\nabla\varphi_j \cdot \boldsymbol{a}\nabla\varphi_l\right] = \mathbb{E}\left[\varphi_j e \cdot \boldsymbol{a}\nabla\varphi_{l-1} - \nabla\varphi_j \cdot \boldsymbol{a}e\varphi_{l-1}\right] + \mathbb{E}\left[\varphi_j\varphi_{l-2}e \cdot \boldsymbol{a}e - \sum_{m=1}^{l-2} \lambda_{l-2-m}\varphi_m\varphi_j\right]. \tag{2.2}
$$

We then appeal to the equation for φ_{j+1} in the form

$$
\mathbb{E}\left[\nabla\varphi_{l-1} \cdot \boldsymbol{a}\varphi_j e\right] = \mathbb{E}\left[-\nabla\varphi_{l-1} \cdot \boldsymbol{a}\nabla\varphi_{j+1}\right] - \mathbb{E}\left[\nabla\varphi_{l-1} \cdot (\nabla\chi_j - \sigma_j e)\right].
$$

We need to reformulate the second RHS term. By the equation for χ_j, the stationarity of $\varphi_{l-1}\sigma_j$, and the skew-symmetry of σ_j, this term turns into

$$
\begin{aligned}
&-\mathbb{E}\left[\nabla\varphi_{l-1} \cdot (\nabla\chi_j - \sigma_j e)\right] \\
&\qquad = \mathbb{E}\left[-\nabla\chi_{j-1} \cdot e\varphi_{l-1} - \sum_{m=1}^{j-1} \lambda_{j-1-m}\varphi_{l-1}\varphi_m\right] + \mathbb{E}\left[\varphi_{l-1}e \cdot (\nabla \cdot \sigma_j)\right].
\end{aligned}
$$

We then insert the formula $\nabla \cdot \sigma_j = \boldsymbol{a}(\nabla\varphi_j + e\varphi_{j-1}) - \lambda_j + \nabla\chi_{j-1} + \sigma_{j-1}e$ and obtain (since φ_{l-1} has vanishing expectation and σ_{j-1} is skew-symmetric)

$$
-\mathbb{E}\left[\nabla\varphi_{l-1} \cdot (\nabla\chi_j - \sigma_j e)\right] = -\mathbb{E}\left[\sum_{m=1}^{j-1} \lambda_{j-1-m}\varphi_{l-1}\varphi_m\right] + \mathbb{E}\left[\varphi_{l-1}e \cdot \boldsymbol{a}(\nabla\varphi_j + e\varphi_{j-1})\right],
$$

so that (2.2) takes the form

$$
\begin{aligned}
\mathbb{E}\left[\nabla\varphi_j \cdot \boldsymbol{a}\nabla\varphi_l\right] = \mathbb{E}\left[\varphi_{l-1}\varphi_{j-1}e \cdot \boldsymbol{a}e\right] &+ \mathbb{E}\left[-\nabla\varphi_{j+1} \cdot \boldsymbol{a}\nabla\varphi_{l-1} + \varphi_j\varphi_{l-2}e \cdot \boldsymbol{a}e\right] \\
&- \mathbb{E}\left[\sum_{m=1}^{j-1} \lambda_{j-1-m}\varphi_m\varphi_{l-1} + \sum_{m=1}^{l-2} \lambda_{l-2-m}\varphi_m\varphi_j\right].
\end{aligned}
$$

This yields (2.1).

Substep 1.2. Proof of the identity: for all $j \geq 1$,

$$
\mathbb{E}\left[\nabla\varphi_j \cdot \boldsymbol{a}\nabla\varphi_{j+1} - \varphi_{j-1}\varphi_j e \cdot \boldsymbol{a}e\right] = -\mathbb{E}\left[\sum_{m=1}^{j-1} \lambda_{j-1-m}\varphi_m\varphi_j\right]. \tag{2.3}
$$

This identity is a direct consequence of (2.2), which for $l = j + 1$ takes the simpler form

$$\mathbb{E}\left[\nabla\varphi_j \cdot \boldsymbol{a}\nabla\varphi_{j+1}\right] = \underbrace{\mathbb{E}\left[\varphi_j e \cdot \boldsymbol{a}\nabla\varphi_j - \nabla\varphi_j \cdot \boldsymbol{a}e\varphi_j\right]}_{= 0} + \mathbb{E}\left[\varphi_j\varphi_{j-1} e \cdot \boldsymbol{a}e - \sum_{m=1}^{j-1}\lambda_{j-1-m}\varphi_m\varphi_j\right].$$

Substep 1.3. Conclusion.

We are now in the position to prove that for all $j \in \mathbb{N}_0$,

$$\lambda_{2j+1} = \mathbb{E}\left[e \cdot \boldsymbol{a}(\nabla\varphi_{2(j+1)} + e\varphi_{2j+1})\right] = 0.$$

We start by using the equation for φ_1 and the definition $\varphi_0 \equiv 1$ to turn the above into

$$\lambda_{2j+1} = \mathbb{E}\left[-\nabla\varphi_1 \cdot \boldsymbol{a}\nabla\varphi_{2(j+1)} + \varphi_{2j+1}\varphi_0 e \cdot \boldsymbol{a}e\right].$$

We then apply identity (2.1) j times to obtain

$$\begin{aligned}
\lambda_{2j+1} &= \mathbb{E}\left[(-1)^{j+1}\nabla\varphi_{j+1} \cdot \boldsymbol{a}\nabla\varphi_{j+2} + (-1)^j\varphi_j\varphi_{j+1}e \cdot \boldsymbol{a}e\right] \\
&+ \sum_{m_1=1}^{j}(-1)^{m_1+1}\mathbb{E}\left[\sum_{m_2=1}^{m_1-1}\lambda_{m_1-1-m_2}\varphi_{m_2}\varphi_{2(j+1)-m_2} + \sum_{m_2=1}^{2j+1-m_1}\lambda_{2j+1-m_1-m_2}\varphi_{m_2}\varphi_{m_1}\right],
\end{aligned}$$

in which we insert identity (2.3). This yields

$$\begin{aligned}
\lambda_{2j+1} &= (-1)^j\mathbb{E}\left[\sum_{m_2=1}^{j}\lambda_{j-m_2}\varphi_{m_2}\varphi_{j+1}\right] \\
&+ \sum_{m_1=1}^{j}(-1)^{m_1+1}\mathbb{E}\left[\sum_{m_2=1}^{m_1-1}\lambda_{m_1-1-m_2}\varphi_{m_2}\varphi_{2(j+1)-m_1} + \sum_{m_2=1}^{2j+1-m_1}\lambda_{2j+1-m_1-m_2}\varphi_{m_2}\varphi_{m_1}\right],
\end{aligned}$$
(2.4)

and it remains to argue that the RHS vanishes. We proceed by induction and assume that for all $l < j$, $\lambda_{2l+1} = 0$. To initialize the induction, it is enough to note that $\lambda_1 = 0$ since the RHS of (2.4) is obviously zero for $j = 0$. We then rewrite (2.4) in a more suitable form to unravel the cancellations:

$$\lambda_{2j+1} = \sum_{k=1}^{2j+1}(-1)^{k+1}\sum_{l=1}^{2j+1-k}\lambda_{2j+1-k-l}\mathbb{E}\left[\varphi_l\varphi_k\right].$$

If $k + l$ is odd then $(-1)^{l+1} + (-1)^{k+1} = 0$, whereas if $k + l$ is even then $\lambda_{2j+1-k-l} = 0$ by the induction assumption. Hence, for all $k, l \geq 1$ with $k + l \leq 2j + 1$, we have $((-1)^{l+1} + (-1)^{k+1})\lambda_{2j+1-k-l} = 0$. This concludes the proof of (i) since by symmetry one may rewrite the above sum as

$$\lambda_{2j+1} = \frac{1}{2}\sum_{k=1}^{2j+1}\sum_{l=1}^{2j+1-k}((-1)^{l+1} + (-1)^{k+1})\lambda_{2j+1-k-l}\mathbb{E}\left[\varphi_l\varphi_k\right] = 0.$$

Step 2. Proof of (ii).

For $j = 0$, the result reduces to the well-known ellipticity of $\boldsymbol{a}_{\text{hom}}$, which follows from the formula

$$\lambda_0 = \mathbb{E}\left[e \cdot \boldsymbol{a}(\nabla\varphi_1 + e)\right] = \mathbb{E}\left[(\nabla\varphi_1 + e) \cdot \boldsymbol{a}(\nabla\varphi_1 + e)\right] \geq 1,$$

a direct consequence of the corrector equation for φ_1 in the form $\mathbb{E}\left[\nabla\varphi_1 \cdot \boldsymbol{a}(\nabla\varphi_1 + e)\right] = 0$, and of the ellipticity assumption on $\boldsymbol{a}$. We now turn to λ_2, and shall prove that

$$\lambda_2 = \mathbb{E}\left[\nabla(\varphi_2 - \frac{\varphi_1^2}{2}) \cdot \boldsymbol{a}\nabla(\varphi_2 - \frac{\varphi_1^2}{2})\right] \geq 0. \tag{2.5}$$

Starting point is the equation for φ_1 followed by identity (2.1) for $j = 1$ and $l = 3$:

$$\begin{aligned}
\lambda_2 &= \mathbb{E}\left[e \cdot \boldsymbol{a}(\nabla\varphi_3 + e\varphi_2)\right] \\
&= \mathbb{E}\left[-\nabla\varphi_1 \cdot \nabla\varphi_3 + \varphi_0\varphi_2 e \cdot \boldsymbol{a}e\right] \\
&= \mathbb{E}\left[\nabla\varphi_2 \cdot \boldsymbol{a}\nabla\varphi_2 - \varphi_1^2 e \cdot \boldsymbol{a}e + \varphi_1^2\lambda_0\right].
\end{aligned}$$

We reformulate the last two terms using the property $\nabla \cdot \sigma_1 = \boldsymbol{a}(\nabla\varphi_1 + e) - \bar{\boldsymbol{a}}_0 e$ and the skew-symmetry of σ_1:

$$\begin{aligned}
\mathbb{E}\left[-\varphi_1^2 e \cdot \boldsymbol{a}e + \varphi_1^2\lambda_0\right] &= \mathbb{E}\left[-\varphi_1^2 e \cdot (\boldsymbol{a}(\nabla\varphi_1 + e) - \bar{\boldsymbol{a}}_0 e) + \varphi_1^2 e \cdot \boldsymbol{a}\nabla\varphi_1\right] \\
&= \mathbb{E}\left[-\varphi_1^2 e \cdot (\nabla \cdot \sigma_1) + \varphi_1^2 e \cdot \boldsymbol{a}\nabla\varphi_1\right] \\
&= \mathbb{E}\left[-\nabla\varphi_1^2 \cdot \sigma_1 e + \varphi_1^2 e \cdot \boldsymbol{a}\nabla\varphi_1\right].
\end{aligned}$$

We then appeal to the definition formula for σ_2 together with the property $\nabla\cdot(\nabla\cdot\sigma_2) = 0$ in the weak form $\mathbb{E}\left[\nabla\varphi_1^2 \cdot (\nabla \cdot \sigma_2)\right] = 0$, and the equation for φ_1 in the form $\mathbb{E}\left[-\nabla\varphi_1^3 \cdot \boldsymbol{a}e\right] = \mathbb{E}\left[\nabla\varphi_1^3 \cdot \boldsymbol{a}\nabla\varphi_1\right]$, and obtain

$$\begin{aligned}
&\mathbb{E}\left[-\nabla\varphi_1^2 \cdot \sigma_1 e + \varphi_1^2 e \cdot \boldsymbol{a}\nabla\varphi_1\right] \\
&= \mathbb{E}\big[\nabla\varphi_1^2 \cdot \underbrace{(\boldsymbol{a}(\nabla\varphi_2 + \varphi_1 e) - \sigma_1 e)}_{= \nabla \cdot \sigma_2} - \nabla\varphi_1^2 \cdot \boldsymbol{a}\nabla\varphi_2 \underbrace{-\nabla\varphi_1^2 \cdot \boldsymbol{a}e\varphi_1 + \varphi_1^2 e \cdot \boldsymbol{a}\nabla\varphi_1}_{= -\frac{1}{3}\nabla\varphi_1^3 \cdot \boldsymbol{a}e}\big] \\
&= \mathbb{E}\left[-\nabla\varphi_1^2 \cdot \boldsymbol{a}\nabla\varphi_2 + \frac{1}{3}\nabla\varphi_1^3 \cdot \boldsymbol{a}\nabla\varphi_1\right].
\end{aligned}$$

Combining these last three identities, we finally conclude by the Leibniz rule that

$$\lambda_2 = \mathbb{E}\left[\nabla\varphi_2 \cdot \boldsymbol{a}\nabla\varphi_2 - \nabla\varphi_1^2 \cdot \boldsymbol{a}\nabla\varphi_2 + \frac{1}{3}\nabla\varphi_1^3 \cdot \boldsymbol{a}\nabla\varphi_1\right] = \mathbb{E}\left[\nabla(\varphi_2 - \frac{\varphi_1^2}{2}) \cdot \boldsymbol{a}\nabla(\varphi_2 - \frac{\varphi_1^2}{2})\right],$$

as claimed. $\square$

Remark 2.1. Whereas the identity $\lambda_{2j+1} = 0$ is generic, it is unclear to us whether we generically have a sign for λ_{2j}. In particular, arguing as in the proof above we obtain

$$\lambda_4 = \mathbb{E}\left[-\nabla\varphi_3 \cdot \boldsymbol{a}\nabla\varphi_3 + \varphi_2^2 e \cdot \boldsymbol{a}e - \varphi_2^2\lambda_0 + \lambda_2\varphi_1^2\right],$$

which we are presently unable to reformulate as a square (or sum of squares) as in (2.5). $\square$

2.2. **Taylor-Bloch wave, eigenvalue, and eigendefect.** In the following, we assume that extended correctors exist. We define Taylor-Bloch waves as follows.

Definition 2.2. Let $k := \kappa e$ with $\kappa \in \mathbb{R}$ and let $(\varphi_j, \lambda_j, \sigma_j, \chi_j)_{0 \leq j \leq \ell}$ be as in Definition 2.1. The Taylor-Bloch wave $\psi_{k,\ell}$, Taylor-Bloch eigenvalue $\tilde{\lambda}_{k,\ell}$, and Taylor-Bloch eigendefect

$\mathfrak{d}_{k,\ell}$ of order ℓ in direction k are defined by

$$\psi_{k,\ell} := \sum_{j=0}^{\ell}(i\kappa)^j\varphi_j, \quad \tilde{\lambda}_{k,\ell} := \kappa^2\sum_{j=0}^{\ell-1}(i\kappa)^j\lambda_j,$$

$$\mathfrak{d}_{k,\ell} = \nabla\cdot(-\sigma_\ell e + \boldsymbol{a}e\varphi_\ell + \nabla\chi_\ell) + i\kappa\Big(e\cdot\boldsymbol{a}e\varphi_\ell - \sum_{j=1}^{\ell}\sum_{l=\ell-j}^{\ell-1}(i\kappa)^{j+l-\ell}\lambda_l\varphi_j\Big).$$

$\square$

Note that by Proposition 1, $\tilde{\lambda}_{k,\ell}$ is real-valued since $\lambda_{2j+1} = 0$ for all j. The interest of this definition is the following proposition, which establishes that the Taylor-Bloch wave $\psi_{k,\ell}$ is an eigenvector of the magnetic operator $-(\nabla+ik)\cdot\boldsymbol{a}(\nabla+ik)$ on $\mathbb{R}^d$ for the eigenvalue $\tilde{\lambda}_{k,\ell}$ up to the eigendefect $\mathfrak{d}_{k,\ell}$.

Proposition 2. *Let $k = \kappa e$, and $\psi_{k,\ell},\tilde{\lambda}_{k,\ell},\mathfrak{d}_{k,\ell}$ be as in Definition 2.2 for some $\ell \geq 1$. Then we have*

$$-(\nabla+ik)\cdot\boldsymbol{a}(\nabla+ik)\psi_{k,\ell} = \tilde{\lambda}_{k,\ell}\psi_{k,\ell} - (i\kappa)^{\ell+1}\mathfrak{d}_{k,\ell}. \tag{2.6}$$

$\square$

Remark 2.2. In the periodic setting, the Taylor-Bloch wave $\psi_{k,\ell}$ can be compared to the Taylor expansion $\tilde{\psi}_{k,\ell}$ defined in [4, Remark 3.2] of the Bloch wave $\tilde{\psi}_k$ on the unit torus $\mathbb{T}$ (which is known to exist). The latter satisfies $\|\tilde{\psi}_{k,\ell}\|_{L^2(\mathbb{T})} = 1 + O(\kappa^{\ell+1})$ due to the choice of unit normalization of Bloch waves in $L^2(\mathbb{T})$, whereas the former satisfies at best $\|\psi_{k,\ell}\|_{L^2(\mathbb{T})} = 1 + O(\kappa)$. The advantage of proceeding that way is twofold: we do not require the terms of the expansion to have bounded L^2-norm and the algebraic structure is easier to unravel. $\square$

Proof of Proposition 2. We first compute for $\ell \geq 1$

$$-(\nabla+ik)\cdot\boldsymbol{a}(\nabla+ik)\psi_{k,\ell}$$

$$= -(i\kappa)\nabla\cdot\boldsymbol{a}(e+\nabla\varphi_1) - \sum_{j=2}^{\ell}(i\kappa)^j(\nabla\cdot\boldsymbol{a}\nabla\varphi_j + e\cdot\boldsymbol{a}\nabla\varphi_{j-1} + \nabla\cdot(\boldsymbol{a}e\varphi_{j-1}) + e\cdot\boldsymbol{a}e\varphi_{j-2})$$

$$- (i\kappa)^{\ell+1}(e\cdot\boldsymbol{a}\nabla\varphi_\ell + \nabla\cdot(\boldsymbol{a}e\varphi_\ell) + e\cdot\boldsymbol{a}e\varphi_{\ell-1})$$

$$- (i\kappa)^{\ell+2}e\cdot\boldsymbol{a}e\varphi_\ell$$

and, after several resummations:

$$\tilde{\lambda}_{k,\ell}\psi_{k,\ell} = \sum_{j=0}^{\ell}\sum_{l=0}^{\ell-1}\kappa^{2+j+l}i^{j+l}\lambda_l\varphi_j$$

$$= -\sum_{j=0}^{\ell-1}(i\kappa)^{2+j}\sum_{l_1+l_2=j}\lambda_{l_2}\varphi_{l_1} - (i\kappa)^{\ell+2}\sum_{j=1}^{\ell}\sum_{l=\ell-j}^{\ell-1}(i\kappa)^{j+l-\ell}\lambda_l\varphi_j$$

$$= -\sum_{j=2}^{\ell+1}(i\kappa)^j\sum_{l=0}^{j-2}\lambda_l\varphi_{j-2-l} - (i\kappa)^{\ell+2}\sum_{j=1}^{\ell}\sum_{l=\ell-j}^{\ell-1}(i\kappa)^{j+l-\ell}\lambda_l\varphi_j.$$

Hence,

$$
\begin{aligned}
-(\nabla + ik) \cdot \boldsymbol{a}(\nabla + ik)\psi_{k,\ell} &- \tilde{\lambda}_{k,\ell}\psi_{k,\ell} \\
= \; & -(i\kappa)\nabla \cdot \boldsymbol{a}(e + \nabla\varphi_1) \\
& - \sum_{j=2}^{\ell} (i\kappa)^j (\nabla \cdot \boldsymbol{a}\nabla\varphi_j + e \cdot \boldsymbol{a}\nabla\varphi_{j-1} + \nabla \cdot (\boldsymbol{a}e\varphi_{j-1}) + e \cdot \boldsymbol{a}e\varphi_{j-2} - \sum_{l=0}^{j-2}\lambda_l\varphi_{j-2-l}) \\
& - (i\kappa)^{\ell+1}(e \cdot \boldsymbol{a}\nabla\varphi_\ell + \nabla \cdot (\boldsymbol{a}e\varphi_\ell) + e \cdot \boldsymbol{a}e\varphi_{\ell-1} - \sum_{l=0}^{\ell-1}\lambda_l\varphi_{\ell-1-l}) \\
& - (i\kappa)^{\ell+2}(e \cdot \boldsymbol{a}e\varphi_\ell - \sum_{j=1}^{\ell}\sum_{l=\ell-j}^{\ell-1}(i\kappa)^{j+l-\ell}\lambda_l\varphi_j).
\end{aligned}
$$

By definition of φ_1, the term of order κ vanishes. Let us now reformulate the RHS terms of order κ^j for $j = 2, \ldots, \ell+1$. To this aim we note that in view of $\varphi_0 \equiv 1$ and of the definition of χ_{j-2} and χ_{j-1},

$$
\sum_{l=0}^{j-2}\lambda_l\varphi_{j-2-l} = \lambda_{j-2} + \sum_{l=0}^{j-3}\lambda_l\varphi_{j-2-l} = \lambda_{j-2} - \triangle\chi_{j-1} - \nabla\chi_{j-2} \cdot e,
$$

so that, by the skew-symmetry of σ in form of $e \cdot \sigma_{j-2}e = 0$ and by definition of q_{j-1} and σ_{j-1},

$$
\begin{aligned}
e \cdot \boldsymbol{a}\nabla\varphi_{j-1} + e \cdot \boldsymbol{a}e\varphi_{j-2} &- \sum_{l=0}^{j-2}\lambda_l\varphi_{j-2-l} \\
= \; & e \cdot \boldsymbol{a}(\nabla\varphi_{j-1} + e\varphi_{j-2}) - \lambda_{j-2} + \nabla\chi_{j-2} \cdot e + \triangle\chi_{j-1} - e \cdot \sigma_{j-2}e \\
= \; & e \cdot \underbrace{(\boldsymbol{a}(\nabla\varphi_{j-1} + e\varphi_{j-2}) - \tilde{\boldsymbol{a}}_{j-2}e + \nabla\chi_{j-2} \cdot e - \sigma_{j-2}e)}_{= \; q_{j-1}} + \triangle\chi_{j-1} \\
= \; & \nabla \cdot (-\sigma_{j-1}e + \nabla\chi_{j-1}).
\end{aligned}
$$

Combined with the defining equation for φ_{j-1}, this shows that the terms of order κ^j for $j = 2, \ldots, \ell$ vanish:

$$
\begin{aligned}
\nabla \cdot \boldsymbol{a}\nabla\varphi_j + e \cdot \boldsymbol{a}\nabla\varphi_{j-1} + \nabla \cdot (\boldsymbol{a}e\varphi_{j-1}) &+ e \cdot \boldsymbol{a}e\varphi_{j-2} - \sum_{l=0}^{j-2}\lambda_l\varphi_{j-2-l} \\
&= \nabla \cdot \boldsymbol{a}\nabla\varphi_j + \nabla \cdot (-\sigma_{j-1}e + \nabla\chi_{j-1} + \boldsymbol{a}e\varphi_{j-1}) = 0.
\end{aligned}
$$

This also implies that the term of order $\kappa^{\ell+1}$ is in divergence form:

$$
e \cdot \boldsymbol{a}\nabla\varphi_\ell + \nabla \cdot (\boldsymbol{a}e\varphi_\ell) + e \cdot \boldsymbol{a}e\varphi_{\ell-1} - \sum_{l=0}^{\ell-1}\lambda_l\varphi_{\ell-1-l} = \nabla \cdot (-\sigma_\ell e + \nabla\chi_\ell + \boldsymbol{a}e\varphi_\ell).
$$

We have thus proved

$$
\begin{aligned}
-(\nabla + ik) \cdot \boldsymbol{a}(\nabla + ik)\psi_\ell - \tilde{\lambda}_{k,\ell}\psi_{k,\ell} = \; & -(i\kappa)^{\ell+1}\nabla \cdot (-\sigma_\ell e + \nabla\chi_\ell + \boldsymbol{a}e\varphi_\ell) \\
& -(i\kappa)^{\ell+2}(e \cdot \boldsymbol{a}e\varphi_\ell - \sum_{j=1}^{\ell}\sum_{l=\ell-j}^{\ell-1}(i\kappa)^{j+l-\ell}\lambda_l\varphi_j),
\end{aligned}
$$

which is the claim by definition of the eigendefect. $\qquad\square$

3. Taylor-Bloch approximate solution of the wave equation

3.1. **Main result and structure of the proof.** Let $u_\varepsilon \in L^\infty(\mathbb{R}_+, L^2(\mathbb{R}^d))$ be the unique weak solution of

$$\begin{cases} \Box_\varepsilon u_\varepsilon &= 0, \\ u_\varepsilon(0,\cdot) &= u_0, \\ \partial_t u_\varepsilon(0,\cdot) &= 0, \end{cases} \tag{3.1}$$

where $u_0 \in \mathcal{S}(\mathbb{R}^d)$ (the Schwartz class). Let $\hat{u}_0 : k \mapsto \mathcal{F}u_0(k) = \int_{\mathbb{R}^d} u_0(x)e^{-ik\cdot x}dx$ be the Fourier transform of u_0 (which is also in $\mathcal{S}(\mathbb{R}^d)$). Let $\ell \geq 1$, and assume that the (higher-order) homogenized tensors $\bar{a}_j$ of Definition 2.1 are well-defined for all $0 \leq j \leq \ell - 1$, and set $\bar{\Gamma}_\ell := \max_{0 \leq j \leq \ell-1} |\bar{a}_j| < \infty$. Since for all $j \geq 0$, $e \cdot \tilde{a}_{2j+1}e = 0$, the quantity $\tilde{\lambda}_{k,\ell}$ (cf. Definition 2.2) is real-valued. On the other hand, since $\bar{a}_0 \geq \mathrm{Id}$ is positive-definite, there exists $K_{\mathrm{max},\ell} > 0$ (which only depends on $\bar{\Gamma}_\ell$) such that $|k|^{-2}\tilde{\lambda}_{k,\ell} \geq \frac{1}{4}$ for all $|k| \leq K_{\mathrm{max},\ell}$. Let $\omega_\ell : \mathbb{R}_+ \to [0,1]$ be a smooth cut-off function which takes value 1 on $[0, \frac{1}{2}K_{\mathrm{max},\ell}]$ and value 0 on $[K_{\mathrm{max},\ell}, \infty)$. We define the approximation $u_{\varepsilon,\ell}$ of u_ε as the following inverse Fourier transform:

$$\begin{aligned} u_{\varepsilon,\ell}(t,x) &:= \mathcal{F}^{-1}\big(\omega_\ell(\varepsilon|\cdot|)\hat{u}_0 \cos(\varepsilon^{-1}\Lambda_\ell(\varepsilon\cdot)t)\big)(x) \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \omega_\ell(\varepsilon|k|)\hat{u}_0(k)e^{ik\cdot x} \cos(\varepsilon^{-1}\Lambda_\ell(\varepsilon k)t)dk, \end{aligned} \tag{3.2}$$

where $\Lambda_\ell(k) := \sqrt{\tilde{\lambda}_{k,\ell}}$, which is well-defined since $\tilde{\lambda}_{\varepsilon k,\ell} \geq 0$ when $\omega_\ell(\varepsilon|k|) \neq 0$. Note that $u_{\varepsilon,\ell}$ is real-valued since $\Lambda_\ell(-k) = \Lambda_\ell(k)$, so that $k \mapsto \omega_\ell(\varepsilon|k|)\hat{u}_0(k)\cos(\varepsilon^{-1}\Lambda_\ell(\varepsilon k)t)$ is Hermitian. The main result of this section establishes the accuracy of the approximation of u_ε by $u_{\varepsilon,\ell}$ over large times depending on the growth of the extended correctors, which we formulate as an assumption (we refer the reader to Appendix A for details).

Hypothesis 1. Let $\ell \geq 1$, and assume that the extended correctors $(\varphi_j, \sigma_j, \chi_j)_{0 \leq j \leq \ell}$ are well-defined for all directions $e \in \mathbb{R}^d$, $|e| = 1$, and satisfy the following growth properties:

- for all $0 \leq j \leq \ell - 1$, $\varphi_j, \sigma_j, \nabla\chi_j$ are $\mathbb{Z}^d$-stationary and satisfy (uniformly over e)

$$\sup_{x \in \mathbb{R}^d} \mathbb{E}\left[\fint_{B(x)} |\varphi_j|^2 + |\sigma_j|^2 + |\nabla\chi_j|^2\right]^{\frac{1}{2}} \lesssim 1 \tag{3.3}$$

  where $B(x)$ denotes the unit ball centered at x;
- for $j = \ell$, there exists $\alpha = (\alpha_1, \alpha_2)$ with $0 \leq \alpha_1 < 1$ and $0 \leq \alpha_2$ such that for all $x \in \mathbb{R}^d$ (uniformly over e)

$$\mathbb{E}\left[\fint_{B(x)} |\varphi_j|^2 + |\sigma_j|^2 + |\nabla\chi_j|^2\right]^{\frac{1}{2}} \lesssim \mu_\alpha(|x|), \tag{3.4}$$

  where $\mu_\alpha(t) = (1+t)^{\alpha_1}\log^{\alpha_2}(2+t)$.

$\qquad\square$

In the rest of this article, $\lesssim$ stands for $\leq C\times$ for a generic constant C that might depend on d, ℓ, $\bar{\Gamma}_\ell$, and the multiplicative constants in (3.3) & (3.4) (but not on T, $\varepsilon > 0$, and u_0).

Theorem 1. *Let $\ell \geq 1$, and assume that $(\varphi_j, \sigma_j, \chi_j)_{0 \leq j \leq \ell}$ satisfy Hypothesis 1 for some $\alpha \in [0, 1) \times \mathbb{R}_+$. Then we have for all $T \geq 0$ and all $1 \geq \varepsilon > 0$,*

$$\sup_{t \leq T} \mathbb{E}\left[\|u_\varepsilon - u_{\varepsilon,\ell}\|^2_{L^2(\mathbb{R}^d)}\right]^{\frac{1}{2}} \lesssim C_\ell(u_0)(\max\{\varepsilon, \varepsilon^\ell \mu_\alpha(\varepsilon^{-1})\} + \varepsilon^\ell T \mu_\alpha(\varepsilon^{-1}T)), \qquad (3.5)$$

where $C_\ell(u_0)$ is a generic norm of u_0 (that may change from line to line in the proofs) which only depends on ℓ and is finite for $u_0 \in \mathcal{S}(\mathbb{R}^d)$. $\qquad\square$

Remark 3.1. Since $\Lambda_{2\ell+1} \equiv \Lambda_{2\ell}$, we have $u_{\varepsilon,2\ell+1} \equiv u_{\varepsilon,2\ell}$, so that if (3.3) & (3.4) hold with ℓ replaced by $2\ell + 1$, estimate (3.5) takes the equivalent form

$$\sup_{t \leq T} \mathbb{E}\left[\|u_\varepsilon - u_{\varepsilon,2\ell}\|^2_{L^2(\mathbb{R}^d)}\right]^{\frac{1}{2}} \lesssim C_{2\ell+1}(u_0)(\varepsilon + \varepsilon^{2\ell+1} T \mu_\alpha(\varepsilon^{-1}T)). \qquad (3.6)$$

$\qquad\square$

Remark 3.2. In Hypothesis 1 the growth of the correctors in measured in terms of their second stochastic moments. There is nothing special about this quantity. If higher stochastic integrability is assumed, the result of Theorem 1 will hold with the corresponding higher stochastic integrability, as well as all the other results of this contribution. We consider this as a separate issue. $\qquad\square$

Remark 3.3. Estimate (3.5) does not necessarily improve as ℓ gets larger for large times. Indeed, there is an interplay between the growth of the corrector and the final time: if the corrector φ_ℓ is unbounded, the $L^2(\mathbb{R}^d)$-norm of the approximation of the solution blows up at large times whereas the $L^2(\mathbb{R}^d)$-norm of the solution remains bounded. $\qquad\square$

The proof of Theorem 1 relies on three arguments:

- the fact that the initial condition can be replaced by a well-prepared initial condition $u_{0,\varepsilon,\ell}$ up to an error uniformly small in time by the assumptions (3.3) & (3.4), cf. Lemma 3.1;
- the fact that the Taylor-Bloch waves almost diagonalize the wave operator, and that the error due to the eigendefects can be controlled by suitable energy estimates on the wave equation with well-prepared initial condition and the assumptions (3.3) & (3.4), cf. Proposition 3;
- the fact that the almost solution $\tilde{v}_{\varepsilon,\ell}$ of the wave equation with well-prepared condition $u_{0,\varepsilon,\ell}$ can be well-approximated by $u_{\varepsilon,\ell}$ by the assumptions (3.3) & (3.4), cf. Lemma 3.2.

In the following, we shall systematically use the notation $k = \kappa e$ with $\kappa \in \mathbb{R}$ and $|e| = 1$, and shall make the dependence of the correctors φ_j upon the direction e explicit, and use the notation φ_j^e. We start with the preparation of the initial condition:

Lemma 3.1. *Let $\ell \geq 1$, and let $u_{0,\varepsilon,\ell} \in L^2(\mathbb{R}^d)$ be defined by*

$$u_{0,\varepsilon,\ell} := \sum_{j=0}^{\ell} \varepsilon^j \varphi_j\left(\frac{x}{\varepsilon}\right) \cdot \nabla^j u_{0,\varepsilon}(x),$$

where $u_{0,\varepsilon} := \mathcal{F}^{-1}(\omega_\ell(\varepsilon|\cdot|)\hat{u}_0)$, and φ_j stands for the (symmetric) j-th order tensor such that $\varphi_j \cdot e^{\otimes j} = \varphi_j^e$, the j-th corrector in direction e. Consider the unique weak solution

$v_{\varepsilon,\ell} \in L^\infty(\mathbb{R}_+, L^2(\mathbb{R}^d))$ *of the initial value problem*

$$\begin{cases} \square_\varepsilon v_{\varepsilon,\ell} &= 0, \\ v_{\varepsilon,\ell}(0,\cdot) &= u_{0,\varepsilon,\ell}, \\ \partial_t v_{\varepsilon,\ell}(0,\cdot) &= 0. \end{cases} \qquad (3.7)$$

Then if Hypothesis 1 holds, we have

$$\sup_{t \geq 0} \mathbb{E}\left[\|u_\varepsilon - v_{\varepsilon,\ell}\|^2_{L^2(\mathbb{R}^d)}\right]^{\frac{1}{2}} \lesssim C_\ell(u_0) \max\{\varepsilon, \varepsilon^\ell \mu_\alpha(\varepsilon^{-1})\}.$$

$\square$

The initial condition $u_{0,\varepsilon,\ell}$ of Lemma 3.1 also takes the following form (cf. Step 1 in the proof of Proposition 3 below)

$$u_{0,\varepsilon,\ell} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \omega_\ell(\varepsilon|k|)\hat{u}_0(k)e^{ik\cdot x}\psi_{\varepsilon k,\ell}\left(\frac{x}{\varepsilon}\right) dk,$$

which shows that it is well-prepared in terms of the Taylor-Bloch expansion. We then turn to the core of the proof: the use of the Taylor-Bloch expansion and the eigendefects.

Proposition 3. *For $\ell \geq 1$, let $\tilde{v}_{\varepsilon,\ell} \in L^\infty(\mathbb{R}_+, L^2(\mathbb{R}^d))$ be the (real-valued) function defined by*

$$\tilde{v}_{\varepsilon,\ell}(t,x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \omega_\ell(\varepsilon|k|)\hat{u}_0(k)e^{ik\cdot x}\cos(\varepsilon^{-1}\Lambda_\ell(\varepsilon k)t)\psi_{\varepsilon k,\ell}\left(\frac{x}{\varepsilon}\right) dk, \qquad (3.8)$$

and let $v_{\varepsilon,\ell}$ be the unique weak solution of (3.7). Then if Hypothesis 1 holds we have for all $T \geq 0$ and $\varepsilon > 0$,

$$\sup_{t \leq T} \mathbb{E}\left[\|v_{\varepsilon,\ell} - \tilde{v}_{\varepsilon,\ell}\|^2_{L^2(\mathbb{R}^d)}\right]^{\frac{1}{2}} \lesssim C_\ell(u_0)\varepsilon^\ell T\mu_\alpha(\varepsilon^{-1}T).$$

$\square$

We conclude with the approximation of $\tilde{v}_{\varepsilon,\ell}$ by neglecting the corrector terms.

Lemma 3.2. *For $\ell \geq 1$, let $u_{\varepsilon,\ell}$ and $\tilde{v}_{\varepsilon,\ell}$ be given by (3.2) and (3.8), respectively. Then if Hypothesis 1 holds we have for all $\varepsilon > 0$,*

$$\sup_{t \geq 0} \mathbb{E}\left[\|u_{\varepsilon,\ell} - \tilde{v}_{\varepsilon,\ell}\|^2_{L^2(\mathbb{R}^d)}\right]^{\frac{1}{2}} \lesssim C_\ell(u_0)\left(\varepsilon + \varepsilon^\ell \mu_\alpha(\varepsilon^{-1}T)\right).$$

$\square$

Theorem 1 is a straightforward consequence of the combination of Lemma 3.1, Proposition 3, and Lemma 3.2, which are proved in the following three subsections.

3.2. **Proof of Lemma 3.1: Preparation of the initial condition.** We split the proof into two steps.

Step 1. Energy estimate for (1.1).
We first derive an energy estimate for the wave equation (1.1). We integrate once (1.1) in time, multiply by u_ε, and integrate over $[0,t] \times \mathbb{R}^d$ to obtain

$$\frac{1}{2}\left(\int_0^t \frac{d}{ds}\|u_\varepsilon(s,\cdot)\|^2_{L^2(\mathbb{R}^d)}ds + \int_0^t \frac{d}{ds}\left\|\sqrt{a_\varepsilon}\nabla\int_0^s u_\varepsilon(r,\cdot)dr\right\|^2_{L^2(\mathbb{R}^d)}ds\right) = 0,$$

from which, by uniform ellipticity of $\boldsymbol{a}$, it immediately follows that

$$\|u_\varepsilon(t,\cdot)\|^2_{L^2(\mathbb{R}^d)} \leq \|u_0\|^2_{L^2(\mathbb{R}^d)}. \tag{3.9}$$

By linearity of (1.1), $u_\varepsilon - v_{\varepsilon,\ell}$ satisfies (1.1) with initial condition $u_0 - u_{0,\varepsilon,\ell}$. In view of the energy estimates (3.9) it is sufficient to estimate $\|u_{0,\varepsilon,\ell} - u_0\|_{L^2(\mathbb{R}^d)}$ to conclude the proof.

Step 2. Estimate of $\|u_{0,\varepsilon,\ell} - u_0\|_{L^2(\mathbb{R}^d)}$.
By definition of $u_{0,\varepsilon,\ell}$,

$$(u_{0,\varepsilon,\ell} - u_0)(x) = (u_{0,\varepsilon} - u_0) + \sum_{j=1}^{\ell} \varepsilon^j \varphi_j\left(\frac{x}{\varepsilon}\right) \cdot \nabla^j u_{0,\varepsilon}(x),$$

so that

$$\|u_{0,\varepsilon,\ell} - u_0\|^2_{L^2(\mathbb{R}^d)} \leq \int_{\mathbb{R}^d} |u_{0,\varepsilon} - u_0|^2 + \sum_{j=1}^{\ell} \varepsilon^{2j} \int_{\mathbb{R}^d} \left(\fint_{B_\varepsilon(x)} \left|\varphi_j^e\left(\frac{\cdot}{\varepsilon}\right)\right|^2\right) \sup_{B_\varepsilon(x)} |\nabla^j u_{0,\varepsilon}|^2 dx.$$

We start with the first RHS term, for which the Plancherel identity yields

$$\begin{aligned}
\int_{\mathbb{R}^d} |u_{0,\varepsilon} - u_0|^2 &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (1 - \omega_\ell(\varepsilon|k|))^2 |\hat{u}_0(k)|^2 dk \\
&\leq \int_{|k| \geq \frac{1}{2\varepsilon} K_{\max,\ell}} |\hat{u}_0(k)|^2 dk \leq \left(\frac{2\varepsilon}{K_{\max,\ell}}\right)^2 \int_{\mathbb{R}^d} |\nabla u_0|^2 \lesssim \varepsilon^2 C_\ell(u_0)^2.
\end{aligned}$$

Taking the expectation of the other RHS terms and using assumption (3.3) for $0 < j \leq \ell-1$ and (3.4) for $j = \ell$, we obtain

$$\begin{aligned}
\mathbb{E}\left[\sum_{j=1}^{\ell} \varepsilon^{2j} \int_{\mathbb{R}^d} \left(\fint_{B_\varepsilon(x)} \left|\varphi_j^e\left(\frac{\cdot}{\varepsilon}\right)\right|^2\right) \sup_{B_\varepsilon(x)} |\nabla^j u_{0,\varepsilon}|^2 dx\right] \\
\lesssim \sum_{j=1}^{\ell-1} \varepsilon^{2j} \int_{\mathbb{R}^d} \sup_{B_\varepsilon(x)} |\nabla^j u_{0,\varepsilon}|^2 dx + \varepsilon^{2\ell} \int_{\mathbb{R}^d} \sup_{B_\varepsilon(x)} |\nabla^j u_{0,\varepsilon}|^2 \mu_\alpha\left(\frac{x}{\varepsilon}\right)^2 dx.
\end{aligned}$$

Using the crude bound $\mu_\alpha(\frac{x}{\varepsilon}) \lesssim \mu_\alpha(\varepsilon^{-1})(1 + |x|)$ together with the bound

$$\begin{aligned}
\int_{\mathbb{R}^d} \sup_{B_\varepsilon(x)} |\nabla^j u_{0,\varepsilon}|^2 (1 + |x|)^2 dx &\leq \int_{\mathbb{R}^d} \left(\sup_{\mathbb{R}^d}\{(1 + |\cdot|)^{\frac{d+3}{2}} |\nabla^j u_{0,\varepsilon}|\}\right)^2 (1 + |x|)^{-d-1} dx \\
&\lesssim \left(\sup_{\mathbb{R}^d}\{(1 + |\cdot|)^{\frac{d+3}{2}} |\nabla^j u_{0,\varepsilon}|\}\right)^2 \leq C_\ell(u_0)^2,
\end{aligned}$$

the claim follows.

3.3. Proof of Proposition 3: Almost diagonalization of the wave operator. We split the proof into three steps.

Step 1. Reformulation of $u_{0,\varepsilon,\ell}$ in term of Taylor-Bloch waves.
We claim that

$$u_{0,\varepsilon,\ell} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{u}_{0,\varepsilon}(k) e^{ik\cdot x} \psi_{\varepsilon k,\ell}\left(\frac{x}{\varepsilon}\right) dk,$$

where $\hat{u}_{0,\ell}(k)$ is a short-hand notation for $\omega_\ell(\varepsilon|k|)\hat{u}_0(k)$ (note that $\psi_{\varepsilon k,\ell}$ is Hermitian by construction). Indeed,

$$
\begin{aligned}
u_{0,\varepsilon,\ell}(x) &= \sum_{j=0}^{\ell} \varepsilon^j \varphi_j\Big(\frac{x}{\varepsilon}\Big) \cdot \nabla^j u_{0,\varepsilon}(x) \\
&= \frac{1}{(2\pi)^d} \sum_{j=0}^{\ell} \varepsilon^j \varphi_j\Big(\frac{x}{\varepsilon}\Big) \cdot \int_{\mathbb{R}^d} \hat{u}_{0,\varepsilon}(k) e^{ik\cdot x}(ik)^{\otimes j}\, dk \\
&= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{u}_{0,\varepsilon}(k) e^{ik\cdot x} \sum_{j=0}^{\ell} (i\varepsilon k)^{\otimes j} \cdot \varphi_j^e\Big(\frac{x}{\varepsilon}\Big) dk,
\end{aligned}
$$

from which the claim follows by definition of $\psi_{\varepsilon k,\ell}$.

Step 2. Equation satisfied by $\widetilde{v}_{\varepsilon,\ell} - v_{\varepsilon,\ell}$.
Let us introduce some short-hand notation:

$$
G_{0,\ell,\varepsilon}(t,x,k) := e^{ik\cdot x}\widetilde{G}_{0,\ell,\varepsilon}(t,k), \ \ \text{where } \widetilde{G}_{0,\ell,\varepsilon}(t,k) := \hat{u}_{0,\varepsilon}(k)\cos(\varepsilon^{-1}\Lambda_\ell(\varepsilon k)t).
$$

In view of Proposition 2, $\widetilde{v}_{\varepsilon,\ell}$ satisfies

$$
\begin{aligned}
\Box_\varepsilon \widetilde{v}_{\varepsilon,\ell} &= -\frac{\varepsilon^{\ell-1}}{(2\pi)^d} \int_{\mathbb{R}^d} (i\kappa)^{\ell+1} G_{0,\ell,\varepsilon}(\nabla\cdot(-\sigma_\ell^e e + \boldsymbol{a}e\varphi_\ell^e + \nabla\chi_\ell^e))\Big(\frac{x}{\varepsilon}\Big)dk \\
&\quad -\frac{\varepsilon^\ell}{(2\pi)^d} \int_{\mathbb{R}^d} (i\kappa)^{\ell+2} G_{0,\ell,\varepsilon}\Big(e\cdot\boldsymbol{a}e\varphi_\ell^e - \sum_{j=1}^{\ell}\sum_{l=\ell-j}^{\ell} (i\varepsilon\kappa)^{j+l-\ell}\lambda_l\varphi_j^e\Big)\Big(\frac{x}{\varepsilon}\Big)dk.
\end{aligned}
$$

Using that $(\nabla\cdot f)\big(\frac{x}{\varepsilon}\big) = \varepsilon\nabla\cdot(f(\frac{x}{\varepsilon}))$, the above turns into

$$
\Box_\varepsilon \widetilde{v}_{\varepsilon,\ell} = -\frac{\varepsilon^\ell}{(2\pi)^d} \int_{\mathbb{R}^d} (i\kappa)^{\ell+1}\widetilde{G}_{0,\ell,\varepsilon}\Big(\nabla\cdot\big(g_{1,\ell}^e\big(\frac{x}{\varepsilon}\big)e^{ik\cdot x}\big) + i\kappa g_{2,\ell,\kappa}^e\big(\frac{x}{\varepsilon}\big)e^{ik\cdot x}\Big)dk \qquad (3.10)
$$

where

$$
g_{1,\ell}^e := -\sigma_\ell^e e + \boldsymbol{a}e\varphi_\ell^e + \nabla\chi_\ell^e, \quad g_{2,\ell,\kappa}^e := e\cdot\boldsymbol{a}e\varphi_\ell^e - \sum_{j=1}^{\ell}\sum_{l=\ell-j}^{\ell} (i\varepsilon\kappa)^{j+l-\ell}\lambda_l\varphi_j^e.
$$

Hence, $\widetilde{v}_{\varepsilon,\ell} - v_{\varepsilon,\ell}$ solves the following equation

$$
\begin{cases}
\Box_\varepsilon(\widetilde{v}_{\varepsilon,\ell} - v_{\varepsilon,\ell})(t,x) &= \varepsilon^\ell(f_{\varepsilon,1,\ell} + f_{\varepsilon,2,\ell}), \\
\partial_t(\widetilde{v}_{\varepsilon,\ell} - v_{\varepsilon,\ell})(0,\cdot) &= 0, \\
(\widetilde{v}_{\varepsilon,\ell} - v_{\varepsilon,\ell})(0,\cdot) &= 0.
\end{cases} \qquad (3.11)
$$

with the source terms given by

$$
\begin{aligned}
f_{\varepsilon,1,\ell}(t,x) &:= -\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (i\kappa)^{\ell+1}\widetilde{G}_{0,\ell,\varepsilon}(t,k)\nabla\cdot\Big(g_{1,\ell}^e\big(\frac{x}{\varepsilon}\big)e^{ik\cdot x}\Big)dk, \\
f_{\varepsilon,2,\ell}(t,x) &:= -\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (i\kappa)^{\ell+2}\widetilde{G}_{0,\ell,\varepsilon}(t,k)g_{2,\ell,\kappa}^e\big(\frac{x}{\varepsilon}\big)e^{ik\cdot x}dk.
\end{aligned}
$$

In the third and last step, we establish energy estimates for equation (3.11).

Step 3. Energy estimates for (3.11).

By linearity of (3.11), it is sufficient to consider the following two auxiliary wave equations:

$$\begin{cases} \Box_\varepsilon v_{\varepsilon,j,\ell}(t,x) & = & \varepsilon^\ell f_{\varepsilon,j,\ell}, \\ \partial_t v_{\varepsilon,j,\ell}(0,\cdot) & = & 0, \\ v_{\varepsilon,j,\ell}(0,\cdot) & = & 0, \end{cases} \quad , j \in \{1,2\}. \tag{3.12}$$

Since for $j = 1$ (3.12) involves a divergence term, the energy estimate is not completely standard and relies very much on the specific form of the RHS. We give a complete proof of the desired estimate in Substeps 3.1 and 3.2. The proof of the estimate for $j = 2$, which takes a similar form as for $j = 1$, is simpler and left to the reader.

Substep 3.1. Estimate of $\mathbb{E}\left[\|\partial_t v_{\varepsilon,1,\ell}(t,\cdot)\|^2_{L^2(\mathbb{R}^d)} + \|\nabla v_{\varepsilon,1,\ell}(t,\cdot)\|^2_{L^2(\mathbb{R}^d)}\right]$.
Multiplying (3.12) by $\partial_t v_{\varepsilon,1,\ell}$ and integrating over $[0,t] \times \mathbb{R}^d$, we obtain by ellipticity of $\boldsymbol{a}$

$$\frac{1}{2}\left(\|\partial_t v_{\varepsilon,1,\ell}(t,\cdot)\|^2_{L^2(\mathbb{R}^d)} + \|\nabla v_{\varepsilon,1,\ell}(t,\cdot)\|^2_{L^2(\mathbb{R}^d)}\right) \leq \frac{\varepsilon^\ell}{(2\pi)^d} I_{1,\ell}, \tag{3.13}$$

where

$$I_{1,\ell} := -\int_{[0,t]\times\mathbb{R}^d} \left(\int_{\mathbb{R}^d} (i\kappa)^{\ell+1} \widetilde{G}_0(s,k,\varepsilon) \nabla \cdot \left(e^{ik\cdot x} g^e_{1,\ell}\left(\frac{x}{\varepsilon}\right)\right) dk\right) \partial_t v_{\varepsilon,1,\ell}(s,x) ds dx.$$

The subtlety is the divergence term which is not bounded uniformly in $L^2(\mathbb{R}^d)$ with respect to ε. To obtain a suitable energy estimate, we first integrate by parts in space. This yields the term $\nabla \partial_t v_{\varepsilon,1,\ell}$, which we do not control a priori. What makes the argument possible is that one may in turn integrate by parts in time, and end up with the quantity $\nabla v_{\varepsilon,1,\ell}$ which we can then absorb in the LHS of (3.13). More precisely, by Fubini's theorem and integration by parts in space and time, $I_{1,\ell}$ takes the form

$$I_{1,\ell}$$
$$= \int_{[0,t]\times\mathbb{R}^d} (i\kappa)^{\ell+1} \widetilde{G}_0(s,k,\varepsilon) \int_{\mathbb{R}^d} \nabla \partial_t v_{\varepsilon,1,\ell}(s,x) \cdot e^{ik\cdot x} g^e_{1,\ell}\left(\frac{x}{\varepsilon}\right) dx ds dk$$
$$= \int_{\mathbb{R}^d} \nabla v_{\varepsilon,1,\ell}(t,x) \cdot \int_{\mathbb{R}^d} g^e_{1,\ell}\left(\frac{x}{\varepsilon}\right) \hat{u}_{0,\varepsilon}(k) e^{ix\cdot k} (i\kappa)^{\ell+1} \cos\left(t\varepsilon^{-1}\Lambda_\ell(\varepsilon k)\right) dk dx$$
$$+ \int_{[0,t]\times\mathbb{R}^d} \nabla v_{\varepsilon,1,\ell}(s,x) \cdot \int_{\mathbb{R}^d} g^e_{1,\ell}\left(\frac{x}{\varepsilon}\right) \hat{u}_{0,\varepsilon}(k) e^{ix\cdot k} (i\kappa)^{\ell+1} \frac{\Lambda_\ell(\varepsilon k)}{\varepsilon} \sin\left(s\varepsilon^{-1}\Lambda_\ell(\varepsilon k)\right) dk dx ds.$$

Let $F_{1,t,\varepsilon,\ell}$ and $F_{2,t,\varepsilon,\ell}$ be the linear operators from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R}^d)$ characterized by their Fourier symbols

$$\hat{F}_{1,t,\varepsilon,\ell}(k) := \omega_\ell(\varepsilon|k|)(i\kappa)^{\ell+1} \cos\left(t\varepsilon^{-1}\Lambda_\ell(\varepsilon k)\right),$$
$$\hat{F}_{2,t,\varepsilon,\ell}(k) := \omega_\ell(\varepsilon|k|)(i\kappa)^{\ell+1} \frac{\Lambda_\ell(\varepsilon k)}{\varepsilon} \sin\left(t\varepsilon^{-1}\Lambda_\ell(\varepsilon k)\right).$$

Note that the decay at infinity of $\hat{F}_{1,t,\varepsilon,\ell}$ and $\hat{F}_{2,t,\varepsilon,\ell}$ are uniform in t and $\varepsilon \leq 1$ and only depends on ℓ, whereas regularity at zero depends on t (this dependence will be crucial in the estimates). Proceeding as in Step 2 of the proof of Lemma 3.1, and using assumptions (3.3)

& (3.4), we then obtain

$$\mathbb{E}\left[\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}g^e_{1,\ell}(\tfrac{x}{\varepsilon})\hat{u}_{0,\varepsilon}(k)e^{ik\cdot x}(i\kappa)^{\ell+1}\cos\left(t\Lambda_\ell(\varepsilon|k|)\right)dk\right|^2 dx\right]$$

$$\lesssim \int_{\mathbb{R}^d}\sup_{B_\varepsilon(x)}|F_{1,t,\varepsilon,\ell}u_0|^2\mu_\alpha(\tfrac{x}{\varepsilon})^2 dx$$

and

$$\mathbb{E}\left[\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}g^e_{1,\ell}(\tfrac{x}{\varepsilon})\hat{u}_{0,\varepsilon}(k)e^{ik\cdot x}(i\kappa)^{\ell+1}\frac{\Lambda_\ell(\varepsilon k)}{\varepsilon}\sin\left(s\varepsilon^{-1}\Lambda_\ell(\varepsilon k)\right)dk\right|^2 dx\right]$$

$$\lesssim \int_{\mathbb{R}^d}\sup_{B_\varepsilon(x)}|F_{2,t,\varepsilon,\ell}u_0|^2\mu_\alpha(\tfrac{x}{\varepsilon})^2 dx,$$

so that

$$\mathbb{E}\left[I_{1,\ell}\right] \lesssim \mathbb{E}\left[\|\nabla v_{\varepsilon,1,\ell}(t,\cdot)\|^2_{L^2(\mathbb{R}^d)}\right]^{\frac{1}{2}}\left(\int_{\mathbb{R}^d}\sup_{B_\varepsilon(x)}|F_{1,t,\varepsilon,\ell}u_0|^2\mu_\alpha(\tfrac{x}{\varepsilon})^2 dx\right)^{\frac{1}{2}}$$

$$+\int_0^t\mathbb{E}\left[\|\nabla v_{\varepsilon,1,\ell}(s,\cdot)\|^2_{L^2(\mathbb{R}^d)}\right]^{\frac{1}{2}}\left(\int_{\mathbb{R}^d}\sup_{B_\varepsilon(x)}|F_{2,s,\varepsilon,\ell}u_0|^2\mu_\alpha(\tfrac{x}{\varepsilon})^2 dx\right)^{\frac{1}{2}}ds.$$

Combined with (3.13) and Young's inequality, this yields the energy estimate for all $T \geq 0$

$$\sup_{0\leq t\leq T}\mathbb{E}\left[\|\partial_t v_{\varepsilon,1,\ell}(t,\cdot)\|^2_{L^2(\mathbb{R}^d)}+\|\nabla v_{\varepsilon,1,\ell}(t,\cdot)\|^2_{L^2(\mathbb{R}^d)}\right]^{\frac{1}{2}}$$

$$\lesssim \varepsilon^\ell\left(\int_{\mathbb{R}^d}\sup_{B(x)}|F_{1,T,\varepsilon,\ell}u_0|^2\mu_\alpha(\tfrac{x}{\varepsilon})^2 dx\right)^{\frac{1}{2}}$$

$$+\varepsilon^\ell\int_0^T\left(\int_{\mathbb{R}^d}\sup_{B(x)}|F_{2,s,\varepsilon,\ell}u_0|^2\mu_\alpha(\tfrac{x}{\varepsilon})^2 dx\right)^{\frac{1}{2}}ds.$$

Noting that multiplication with μ_α essentially acts as an "α-fractional derivative" in Fourier space (which can be made rigorous using Riesz potentials), and in view of the dependence of the Fourier symbols $\hat{F}_{1,t,\varepsilon,\ell}$ and $\hat{F}_{2,t,\varepsilon,\ell}$ on t, this inequality finally turns into

$$\sup_{0\leq t\leq T}\mathbb{E}\left[\|\partial_t v_{\varepsilon,1,\ell}(t,\cdot)\|^2_{L^2(\mathbb{R}^d)}+\|\nabla v_{\varepsilon,1,\ell}(t,\cdot)\|^2_{L^2(\mathbb{R}^d)}\right]^{\frac{1}{2}} \lesssim C_\ell(u_0)\varepsilon^\ell(1+T)\mu_\alpha(\varepsilon^{-1}T), \quad (3.14)$$

where $C_\ell(u_0)$ is a suitable (finite) norm of u_0 that only depends on ℓ (but not on T and ε).

Substep 3.2. Estimate of $\mathbb{E}\left[\|v_{\varepsilon,1,\ell}(t,\cdot)\|^2_{L^2(\mathbb{R}^d)}\right]$.

We integrate (3.12) once in time, test with $v_{\varepsilon,1,\ell}$, and integrate over $[0,t]\times\mathbb{R}^d$, so that to obtain the energy estimate

$$\frac{1}{2}\|v_{\varepsilon,1,\ell}(t,\cdot)\|^2_{L^2(\mathbb{R}^d)} \leq \varepsilon^\ell\int_{[0,t]\times\mathbb{R}^d}\left(\int_0^s f_{\varepsilon,1,\ell}(r,x)dr\right)v_{\varepsilon,1,\ell}(s,x)dsdx. \quad (3.15)$$

The time integration is explicit,

$$\int_0^s f_{\varepsilon,1,\ell}(r,x)dr = -\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}(i\kappa)^{\ell+1}\hat{u}_{0,\varepsilon}(k)\frac{\varepsilon}{\Lambda_\ell(\varepsilon k)}\sin(\frac{\Lambda_\ell(\varepsilon k)}{\varepsilon}s)\nabla\cdot\Big(e^{ik\cdot x}g_{1,\ell}^e\big(\frac{x}{\varepsilon}\big)\Big)\,dk,$$

so that by integration by parts in space, we may rewrite the RHS of the energy estimate in the form

$$\int_{[0,t]\times\mathbb{R}^d}\left(\int_0^s f_{\varepsilon,1,\ell}(r,x)dr\right)v_{\varepsilon,1,\ell}(s,x)dsdx$$

$$= \frac{1}{2\pi}\int_{[0,t]\times\mathbb{R}^d}\nabla v_{\varepsilon,1,\ell}(s,x)\cdot\int_{\mathbb{R}^d}(i\kappa)^{\ell+1}\hat{u}_{0,\varepsilon}(k)\frac{\varepsilon}{\Lambda_\ell(\varepsilon k)}\sin(\frac{\Lambda_\ell(\varepsilon k)}{\varepsilon}s)e^{ik\cdot x}g_{1,\ell}^e\big(\frac{x}{\varepsilon}\big)dk.$$

We then define the linear operator $F_{3,t,\varepsilon,\ell}$ from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R}^d)$ characterized by its Fourier symbol

$$\hat{F}_{3,t,\varepsilon,\ell}(k) \quad := \quad \omega_\ell(\varepsilon|k|)(i\kappa)^{\ell+1}\frac{\varepsilon}{\Lambda_\ell(\varepsilon k)}\sin(\frac{\Lambda_\ell(\varepsilon k)}{\varepsilon}t),$$

and conclude as in Substep 3.1 that

$$\mathbb{E}\left[\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}(i\kappa)^{\ell+1}\hat{u}_{0,\varepsilon}(k)\frac{\varepsilon}{\Lambda_\ell(\varepsilon k)}\sin(\frac{\Lambda_\ell(\varepsilon k)}{\varepsilon}s)e^{ik\cdot x}g_{1,\ell}^e\big(\frac{x}{\varepsilon}\big)dk\right|^2 dx\right]^{\frac{1}{2}}$$

$$\lesssim \left(\int_{\mathbb{R}^d}\sup_{B_\varepsilon(x)}|F_{3,s,\varepsilon,\ell}u_0|^2\mu_\alpha\big(\frac{x}{\varepsilon}\big)^2 dx\right)^{\frac{1}{2}} \lesssim C_\ell(u_0)\mu_\alpha(\varepsilon^{-1}s).$$

Combined with (3.15) and (3.14), this turns into

$$\mathbb{E}\left[\|v_{\varepsilon,1,\ell}(T,\cdot)\|_{L^2(\mathbb{R}^d)}^2\right]$$

$$\lesssim \quad \varepsilon^\ell\sup_{0\le t\le T}\mathbb{E}\left[\|\nabla v_{\varepsilon,1,\ell}(t,x)\|_{L^2(\mathbb{R}^d)}^2\right]^{\frac{1}{2}}\int_0^T C_\ell(u_0)\mu_\alpha(\varepsilon^{-1}s)ds$$

$$\lesssim \quad \Big(C_\ell(u_0)\varepsilon^\ell(1+T)\mu_\alpha(\varepsilon^{-1}T)\Big)^2.$$

Proceeding similarly for the estimate of $\mathbb{E}\left[\|v_{\varepsilon,2,\ell}(T,\cdot)\|_{L^2(\mathbb{R}^d)}^2\right]$, this concludes the proof.

3.4. **Proof of Lemma 3.2: Simplification of the Taylor-Bloch expansion.** By definition,

$$(\tilde{v}_{\varepsilon,\ell}-u_{\varepsilon,\ell})(t,x) \quad = \quad \frac{1}{(2\pi)^d}\sum_{j=1}^\ell \varepsilon^j\varphi_j\big(\frac{x}{\varepsilon}\big)\cdot\int_{\mathbb{R}^d}(ik)^{\otimes j}\omega_\ell(\varepsilon|k|)\hat{u}_0(k)e^{ik\cdot x}\cos(\varepsilon^{-1}\Lambda_\ell(\varepsilon k)t)dk.$$

We then define the linear operators $\{F_{4,t,\varepsilon,\ell,j}\}_{j=1,\ldots,\ell}$ from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R}^d)$ characterized by their Fourier symbols

$$\hat{F}_{4,t,\varepsilon,\ell,j}(k) \quad := \quad \omega_\ell(\varepsilon|k|)(ik)^{\otimes j}\cos(\varepsilon^{-1}\Lambda_\ell(\varepsilon k)t),$$

and conclude as in Substep 3.1 of the proof of Proposition 3 that for all $1 \leq j \leq \ell - 1$,

$$\mathbb{E}\left[\int_{\mathbb{R}^d}\left|\varphi_j(\frac{x}{\varepsilon}) \cdot \int_{\mathbb{R}^d}(ik)^{\otimes j}\omega_\ell(\varepsilon|k|)\hat{u}_0(k)e^{ik\cdot x}\cos(\varepsilon^{-1}\Lambda_\ell(\varepsilon k)t)dk\right|^2 dx\right]^{\frac{1}{2}}$$

$$\lesssim \left(\int_{\mathbb{R}^d}\sup_{B_\varepsilon(x)}|F_{4,t,\varepsilon,\ell,j}u_0|^2 dx\right)^{\frac{1}{2}} \lesssim C_\ell(u_0),$$

whereas for $j = \ell$,

$$\mathbb{E}\left[\int_{\mathbb{R}^d}\left|\varphi_\ell(\frac{x}{\varepsilon}) \cdot \int_{\mathbb{R}^d}(ik)^{\otimes \ell}\omega_\ell(\varepsilon|k|)\hat{u}_0(k)e^{ik\cdot x}\cos(\varepsilon^{-1}\Lambda_\ell(\varepsilon k)t)dk\right|^2 dx\right]^{\frac{1}{2}}$$

$$\lesssim \left(\int_{\mathbb{R}^d}\sup_{B_\varepsilon(x)}|F_{4,t,\varepsilon,\ell,\ell}u_0|^2\mu_\alpha(\frac{x}{\varepsilon})dx\right)^{\frac{1}{2}} \lesssim C_\ell(u_0)\mu_\alpha(\varepsilon^{-1}T).$$

This proves the claim.

4. Long-time homogenization and higher-order homogenized operators

In this section, we draw the consequences of Theorem 1 for the approximation of equation (3.1) by higher-order homogenized equations, extending previous results of [17, 18] for periodic coefficients to higher-order time-scales and to random coefficients. We also give the counterpart of these results for the associated elliptic equation, which extends the recent independent analysis of [12] to any order, and makes quantitative the formal analysis of [4, Section 5].

4.1. **Higher-order homogenized wave equations.** For all $\ell \geq 1$, we define the elliptic operator

$$\tilde{\mathcal{L}}_{\text{hom},\ell,\varepsilon} := -\sum_{j=0}^{\ell-1}\varepsilon^j\bar{a}_j \cdot \nabla^{j+2}, \tag{4.1}$$

where $\bar{a}_j \cdot \nabla^{j+2}v := \prod_{h=1}^{j+2}\sum_{i_h=1}^d[\bar{a}_j]_{i_1,\dots,i_{j+2}}\nabla_{i_1}\dots\nabla_{i_{j+2}}v$ (recall that $\bar{a}_{2j+1} = 0$ for all $j \in \mathbb{N}$), and observe that the function $u_{\varepsilon,\ell}$ defined in (3.2) satisfies

$$\begin{cases} \partial_{tt}^2 u_{\varepsilon,\ell} + \tilde{\mathcal{L}}_{\text{hom},\ell,\varepsilon}u_{\varepsilon,\ell} &= 0, \\ u_{\varepsilon,\ell}(0,\cdot) &= u_0, \\ \partial_t u_{\varepsilon,\ell}(0,\cdot) &= 0. \end{cases} \tag{4.2}$$

For $\ell = 3$, equation (4.2) is not well-posed since the higher-order operator $-\bar{a}_2 \cdot \nabla^4$ in $\tilde{\mathcal{L}}_{\text{hom},3,\varepsilon}$ is non-positive, as first noticed in [15], cf. Proposition 1. In order to circumvent this difficulty, we regularize the operator $\tilde{\mathcal{L}}_{\text{hom},\ell,\varepsilon}$ by a higher-order term, and define for all $\ell \geq 1$

$$\mathcal{L}_{\text{hom},\ell,\varepsilon} := \tilde{\mathcal{L}}_{\text{hom},\ell,\varepsilon} - \gamma_\ell(i\varepsilon)^{2([\frac{\ell-1}{2}]+1)}\text{Id} \cdot \nabla^{2([\frac{\ell-1}{2}]+2)} \tag{4.3}$$

for some $\gamma_\ell \geq 0$ to be chosen below, and consider the higher-order homogenized wave equation

$$\begin{cases} \partial_{tt}^2 w_{\varepsilon,\ell} + \mathcal{L}_{\text{hom},\ell,\varepsilon}w_{\varepsilon,\ell} &= 0, \\ w_{\varepsilon,\ell}(0,\cdot) &= u_0, \\ \partial_t w_{\varepsilon,\ell}(0,\cdot) &= 0. \end{cases} \tag{4.4}$$

The main result of this section is the following long-time homogenization of the wave equation (1.1).

Theorem 2. *Let $\ell \geq 1$, and assume that $(\varphi_j, \sigma_j, \chi_j)_{0 \leq j \leq \ell}$ satisfy Hypothesis 1 for some $\alpha \in [0,1) \times \mathbb{R}_+$. Assume that $\gamma_\ell \geq 0$ is large enough so that $\mathcal{L}_{\mathrm{hom},\ell,\varepsilon}$ is a positive elliptic operator (see Lemma 4.2 below). Let $f \in \mathcal{S}(\mathbb{R}^d)$, and for all $\varepsilon > 0$, let u_ε and $w_{\varepsilon,\ell}$ denote the solutions of (1.1) and (4.4), respectively. Then we have for all $T \geq 0$*

$$\sup_{t \leq T} \mathbb{E}\left[\|u_\varepsilon - w_{\varepsilon,\ell}\|_{L^2(\mathbb{R}^d)}^2\right]^{\frac{1}{2}} \lesssim C_\ell(u_0)\big(\varepsilon + \varepsilon^\ell T \mu_\alpha(\varepsilon^{-1}T)\big),$$

where $C_\ell(u_0)$ is a generic norm of u_0 which only depends on ℓ and is finite for $u_0 \in \mathcal{S}(\mathbb{R}^d)$. $\qquad\square$

As mentioned in the introduction, when $\bar{\boldsymbol{a}}_2 \cdot \nabla^4$ is the higher order operator in (4.2) it is also possible to reformulate this term in a way that yields a well-posed higher-order homogenized wave equation and so that $u_{\varepsilon,\ell}$ remains a "nearly-solution" on sufficiently large times. This approach, due to [17, 18], uses the so-called "Boussinesq trick" and is based on the following algebraic decomposition property:

Lemma 4.1. [18, Lemma 2.5] *Let $\bar{\boldsymbol{a}}_2$ be as in Definition 2.1. There exists a symmetric positive semi-definite second order tensor $\boldsymbol{b}$ and a symmetric positive semi-definite fourth order tensor $\mathbf{c}$ such that*

$$\bar{\boldsymbol{a}}_2 \cdot \nabla^4 = (\boldsymbol{b} \otimes \bar{\boldsymbol{a}}_0) \cdot \nabla^2 - \mathbf{c} \cdot \nabla^4. \tag{4.5}$$

$\qquad\square$

Remark 4.1. The construction of $\boldsymbol{b}$ and $\mathbf{c}$ given in [18] extends mutadis mutandis to systems of equations under the assumption that $\bar{\boldsymbol{a}}_2$ is a symmetric non-negative tensor, and $\bar{\boldsymbol{a}}_0$ is positive-definite (as provided by Proposition 1). $\qquad\square$

In particular, the alternative higher-order homogenized wave equation for $3 \leq \ell \leq 4$ takes the form

$$\begin{cases} \partial_{tt}^2 w_{\varepsilon,\ell} - \bar{\boldsymbol{a}}_0 \cdot \nabla^2 w_{\varepsilon,\ell} - \varepsilon^2 \boldsymbol{b} \cdot \nabla^2 \partial_{tt}^2 w_{\varepsilon,\ell} + \varepsilon^2 \mathbf{c} \cdot \nabla^4 w_{\varepsilon,\ell} &=& 0, \\ w_{\varepsilon,\ell}(0,\cdot) &=& u_0, \\ \partial_t w_{\varepsilon,\ell}(0,\cdot) &=& 0. \end{cases} \tag{4.6}$$

In this case, the following corollary of Theorem 2 holds.

Corollary 1. *Let $3 \leq \ell \leq 4$, and assume that $(\varphi_j, \sigma_j, \chi_j)_{0 \leq j \leq \ell}$ satisfy Hypothesis 1 for some $\alpha \in [0,1) \times \mathbb{R}_+$. Let $f \in \mathcal{S}(\mathbb{R}^d)$, and for all $\varepsilon > 0$, let u_ε and $w_{\varepsilon,\ell}$ denote the solutions of (1.1) and (4.6), respectively. Then we have for all $T \geq 0$*

$$\sup_{t \leq T} \mathbb{E}\left[\|u_\varepsilon - w_{\varepsilon,\ell}\|_{L^2(\mathbb{R}^d)}^2\right]^{\frac{1}{2}} \lesssim C_\ell(u_0)\big(\varepsilon + \varepsilon^\ell T \mu_\alpha(\varepsilon^{-1}T)\big),$$

where $C_\ell(u_0)$ is a generic norm of u_0 which only depends on ℓ and is finite for $u_0 \in \mathcal{S}(\mathbb{R}^d)$. $\qquad\square$

Let us now turn to the arguments in favor of Theorem 2 and Corollary 1. The following elementary lemma (which can be proved by interpolation in Fourier space, and energy estimates) ensures that $\mathcal{L}_{\mathrm{hom},\ell,\varepsilon}$ is a positive elliptic operator provided γ_ℓ is chosen large enough, and yields the well-posedness of (4.4).

Lemma 4.2. *Let $\ell \geq 1$, and assume that the homogenized tensors $\{\bar{a}_j\}_{0 \leq j \leq \ell-1}$ are well-defined, and set $\bar{\Gamma}_\ell := \max_{0 \leq j \leq \ell-1} |\bar{a}_j| < \infty$. Then there exist $\gamma_\ell \geq 0$ and $c_\ell > 0$ depending only on $\bar{\Gamma}_\ell$ and ℓ such that for all $v \in H^{[\frac{\ell-1}{2}]+2}(\mathbb{R}^d)$ and all $\varepsilon > 0$ we have*

$$(\mathcal{L}_{\mathrm{hom},\ell,\varepsilon} v, v)_{(H^{-([\frac{\ell-1}{2}]+2)}, H^{[\frac{\ell-1}{2}]+2})(\mathbb{R}^d)} \geq c_\ell(\|\nabla v\|^2_{L^2(\mathbb{R}^d)} + \varepsilon^{2[\frac{\ell-1}{2}]+2} \|\nabla^{[\frac{\ell-1}{2}]+2} v\|^2_{L^2(\mathbb{R}^d)}).$$

For $1 \leq \ell \leq 2$, we may choose $\gamma_\ell = 0$.

As a consequence:

- *For all $u_0 \in \mathcal{S}(\mathbb{R}^d)$, equation (4.4) admits a unique solution $w_{\varepsilon,\ell} \in L^\infty(\mathbb{R}_+, L^2(\mathbb{R}^d))$.*
- *For $d \geq 3$ and for all $f \in L^{\frac{2d}{d+2}}(\mathbb{R}^d)$ and all $\varepsilon > 0$, the equation*

$$\mathcal{L}_{\mathrm{hom},\ell,\varepsilon} v_\varepsilon = f \tag{4.7}$$

admits a unique weak solution $v_\varepsilon \in L^{\frac{2d}{d-2}}(\mathbb{R}^d)$ such that $\nabla v_\varepsilon \in H^{2[\frac{\ell-1}{2}]+2}(\mathbb{R}^d)$. In addition, we have

$$\|\nabla v_\varepsilon\|_{L^2(\mathbb{R}^d)} \lesssim \|f\|_{L^{\frac{2d}{d+2}}(\mathbb{R}^d)}, \tag{4.8}$$

where the multiplicative constant is independent of $\varepsilon > 0$.

$\square$

Noting that for all $\alpha \in [0,1) \times \mathbb{R}_+$, $T \geq 0$ and $\varepsilon > 0$, we have $\varepsilon^\ell T \mu_\alpha(\varepsilon^{-1}T) \gtrsim \varepsilon^{2([\frac{\ell-1}{2}]+1)}T$, Theorem 2 and Corollary 1 follow by the triangle inequality from the combination of Theorem 1, the well-posedness result of Lemma 4.2, and the following estimate.

Lemma 4.3. *Let $\ell \geq 1$, and assume that $(\varphi_j, \sigma_j, \chi_j)_{0 \leq j \leq \ell}$ satisfy Hypothesis 1 for some $\alpha \in [0,1) \times \mathbb{R}_+$. Let $f \in \mathcal{S}(\mathbb{R}^d)$, and for all $\varepsilon > 0$, let $u_{\varepsilon,\ell}$ be as in (3.2) and $w_{\varepsilon,\ell}$ be the solution of (4.4) or (4.6) (in which case $3 \leq \ell \leq 4$). Then we have for all $T \geq 0$*

$$\sup_{t \leq T} \|u_{\varepsilon,\ell} - w_{\varepsilon,\ell}\|^2_{L^2(\mathbb{R}^d)} \lesssim C_{2([\frac{\ell-1}{2}]+2)}(u_0)\varepsilon^{2([\frac{\ell-1}{2}]+1)}T,$$

where for all $k \geq 1$,

$$C_k(u_0) := \Big(\int_{\mathbb{R}^d} |\nabla^k u_0(x)|^2 dx\Big)^{\frac{1}{2}}.$$

$\square$

4.2. **Higher-order homogenized elliptic equations.** The approach developed above for the wave equation has a counterpart for elliptic equations and yields the validity of higher-order homogenized equations (which however do *not* coincide with the standard two-scale expansion — except for the terms involving the first two correctors only). Our analysis applies to the equation on the whole space $\mathbb{R}^d$. For convenience, we restrict to $d \geq 3$ in the rest of this section (in which case the Hardy inequality is not critical).

In view of the discussion of the wave operator, the natural candidate $\tilde{\mathcal{L}}_{\mathrm{hom},\ell,\varepsilon}$ defined in (4.1) for the higher-order homogenized elliptic equation is not elliptic for $\ell = 3$, and we rather use the regularized homogenized operator $\mathcal{L}_{\mathrm{hom},\ell,\varepsilon}$ defined in (4.3) for all $\ell \geq 1$ and $\varepsilon > 0$. Lemma 4.2 then ensures that $\mathcal{L}_{\mathrm{hom},\ell,\varepsilon}$ is elliptic for a suitable choice of $\gamma_\ell \geq 0$.

As noticed by Allaire, Briane, and Vanninathan in [4], the formal difference between the higher-order elliptic operator (4.1) (or its regularized version (4.3)) and the one obtained by the standard two-scale expansion is related to the well-preparedness of the RHS of the

original equation. In particular, whereas the usual two-scale expansion is based on the equation

$$-\nabla \cdot \boldsymbol{a}_\varepsilon \nabla u_\varepsilon = f, \qquad (4.9)$$

the Bloch-wave expansion (from which the higher-order elliptic operator (4.3) is derived) is based on the equation will well-prepared RHS

$$-\nabla \cdot \boldsymbol{a}_\varepsilon \nabla u_{\varepsilon,\ell} = \sum_{j=0}^{\ell} \varepsilon^j \varphi_j(\tfrac{\cdot}{\varepsilon}) \cdot \nabla^j f, \qquad (4.10)$$

where φ_j stands for the (symmetric) j-th order tensor such that $\varphi_j \cdot e^{\otimes j} = \varphi_j^e$, given in Definition 2.1 for all $e \in \mathbb{R}^d$. The following theorem quantifies the accuracy of the higher-order homogenized operator for well-prepared data.

Theorem 3. *Let $\ell \geq 1$, and assume that $(\varphi_j, \sigma_j, \chi_j)_{0 \leq j \leq \ell}$ satisfy Hypothesis 1 for some $\alpha \in [0,1) \times \mathbb{R}_+$. Let $f \in \mathcal{S}(\mathbb{R}^d)$, and for all $\varepsilon > 0$, let $u_{\varepsilon,\ell}$ and $u_{\mathrm{hom},\ell,\varepsilon}$ denote the solutions of (4.10) and (4.7), respectively. Then we have*

$$\mathbb{E}\left[\left\| \nabla\big(u_{\ell,\varepsilon} - \sum_{j=0}^{\ell} \varepsilon^j \nabla^j u_{\mathrm{hom},\ell,\varepsilon} \cdot \varphi_j(\tfrac{\cdot}{\varepsilon})\big) \right\|_{L^2(\mathbb{R}^d)}^2 \right]^{\frac{1}{2}} \lesssim \varepsilon^\ell \mu_\alpha(\varepsilon^{-1}) C_{\ell,\varepsilon}(u_{\mathrm{hom},\ell,\varepsilon}), \qquad (4.11)$$

where

$$C_{\ell,\varepsilon}(u_{\mathrm{hom},\ell,\varepsilon}) := \left(\int_{\mathbb{R}^d} \mu_\alpha^2(|x|) \sup_{B_\varepsilon(x)} \{|\nabla^{\ell+1} u_{\mathrm{hom},\ell,\varepsilon}|^2 + (1+|x|)^2 |\nabla^{\ell+2} u_{\mathrm{hom},\ell,\varepsilon}|^2\} dx \right)^{\frac{1}{2}}$$
$$+ \sum_{j=0}^{2[\frac{\ell-1}{2}]} \varepsilon^{j+1} \left(\int_{\mathbb{R}^d} (1+|x|)^2 \sup_{B_\varepsilon(x)} \{|\nabla^{j+\ell+3} u_{\mathrm{hom},\ell,\varepsilon}|^2\} dx \right)^{\frac{1}{2}}.$$

$\square$

To our knowledge, this result is new, even in the periodic setting.

Let us give two corollaries of this result. The first corollary shows that for $\ell \leq 2$, it is not necessary to prepare the RHS of (4.9). On the one hand, this is not surprising since the first two correctors of the Bloch wave expansion coincide with the first two correctors of the usual two-scale expansion (for which the RHS of (4.9) needs not be prepared). On the other hand, the correction (4.10) to the RHS of (4.9) is of order ε whereas the RHS of (4.11) is of order ε^2 (provided the second corrector is essentially bounded), so that there must be subtle cancellations. This result was first proved (in the non-symmetric setting) in [12].

Corollary 2. *Let $1 \leq \ell \leq 2$, and assume that $(\varphi_j, \sigma_j, \chi_j)_{0 \leq j \leq \ell}$ satisfy Hypothesis 1 for some $\alpha \in [0,1) \times \mathbb{R}_+$. Let $f \in \mathcal{S}(\mathbb{R}^d)$, and for all $\varepsilon > 0$, let u_ε and u_{hom} denote the solutions of (4.9) (the RHS of which is not well-prepared) and*

$$-\nabla \cdot \bar{\boldsymbol{a}}_0 \nabla u_{\mathrm{hom}} = f, \qquad (4.12)$$

respectively. Then we have

$$\mathbb{E}\left[\left\|\nabla\big(u_\varepsilon - \sum_{j=0}^{\ell} \varepsilon^j \nabla^j u_{\mathrm{hom}} \cdot \varphi_j(\tfrac{\cdot}{\varepsilon})\big)\right\|^2_{L^2(\mathbb{R}^d)}\right]^{\frac{1}{2}} \lesssim \varepsilon^\ell \mu_\alpha(\varepsilon^{-1}) C_{\ell,\varepsilon}(u_{\mathrm{hom}}), \qquad (4.13)$$

where

$$C_{\ell,\varepsilon}(u_{\mathrm{hom}}) := \left(\int_{\mathbb{R}^d} \mu_\alpha^2(|x|) \sup_{B_\varepsilon(x)} \{|\nabla^{\ell+1} u_{\mathrm{hom}}|^2\} dx\right)^{\frac{1}{2}}.$$

$\square$

The second corollary makes use of the Boussinesq trick to avoid the higher-order regularization of (4.1) for $3 \le \ell \le 4$.

Corollary 3. *Let $3 \le \ell \le 4$, and assume that $(\varphi_j, \sigma_j, \chi_j)_{0 \le j \le \ell}$ satisfy Hypothesis 1 for some $\alpha \in [0,1) \times \mathbb{R}_+$. Let $\boldsymbol{b}$ and $\mathbf{c}$ be the (symmetric positive semi-definite) second and fourth order tensors defined in Lemma 4.1. Let $f \in \mathcal{S}(\mathbb{R}^d)$, and for all $\varepsilon > 0$, let $u_{\varepsilon,\ell}$ denote the solution of (4.10) (the RHS of which is well-prepared), and $u_{\mathrm{hom},\ell,\varepsilon}$ denote the unique solution of*

$$(-\nabla \cdot \bar{\boldsymbol{a}}_0 \nabla + \varepsilon^2 \boldsymbol{c} \cdot \nabla^4) u_{\mathrm{hom},\ell,\varepsilon} = f - \varepsilon^2 \boldsymbol{b} \cdot \nabla^2 f. \qquad (4.14)$$

Then we have

$$\mathbb{E}\left[\left\|\nabla\big(u_{\ell,\varepsilon} - \sum_{j=0}^{\ell} \varepsilon^j \nabla^j u_{\mathrm{hom},\ell,\varepsilon} \cdot \varphi_j(\tfrac{\cdot}{\varepsilon})\big)\right\|^2_{L^2(\mathbb{R}^d)}\right]^{\frac{1}{2}} \lesssim \varepsilon^\ell \mu_\alpha(\varepsilon^{-1}) C_{\ell,\varepsilon}(u_{\mathrm{hom},\ell,\varepsilon}, f), \quad (4.15)$$

where

$$C_{\ell,\varepsilon}(u_{\mathrm{hom},\ell,\varepsilon}, f) := \left(\int_{\mathbb{R}^d} \mu_\alpha^2(|x|) \sup_{B_\varepsilon(x)} \{|\nabla^{\ell+1} u_{\mathrm{hom},\ell,\varepsilon}|^2 + (1+|x|)^2 |\nabla^{\ell+2} u_{\mathrm{hom},\ell,\varepsilon}|^2\} dx\right)^{\frac{1}{2}}$$

$$+ \sum_{j=0}^{2} \varepsilon^j \left(\int_{\mathbb{R}^d} (1+|x|)^2 \sup_{B_\varepsilon(x)} \{|\nabla^{\ell+j} f|^2\} dx\right)^{\frac{1}{2}}$$

$$+ \varepsilon \left(\int_{\mathbb{R}^d} (1+|x|)^2 \sup_{B_\varepsilon(x)} \{|\nabla^{\ell+3} u_{\mathrm{hom},\ell,\varepsilon}|^2\} dx\right)^{\frac{1}{2}}.$$

$\square$

Remark 4.2. The RHS of (4.11) and (4.15) in Theorem 3 and Corollary 3 involve the same number of derivatives of f. For $3 \le \ell \le 4$, (4.11) and (4.15) involve $\ell + 3$ derivatives of f (for (4.11) one needs to differentiate $\ell + 3$ times (4.7), whereas for (4.15) one needs to differentiate $\ell + 1$ times (4.14)). $\square$

4.3. Proof of Lemma 4.3: Higher-order approximation of the wave equation.
We split the proof of this lemma into two steps, and distinguish between (4.6) and (4.4). We start with (4.6).

Step 1. Estimate for (4.6).

Assume that $3 \leq \ell \leq 4$ and let $w_{\varepsilon,\ell}$ solve (4.6). By Lemma 4.1, the error $h_{\varepsilon,\ell} := u_{\varepsilon,\ell} - w_{\varepsilon,\ell}$ satisfies

$$
\begin{cases}
\partial_{tt}^2 h_{\varepsilon,\ell} - \bar{\boldsymbol{a}}_0 \cdot \nabla^2 h_{\varepsilon,\ell} - \varepsilon^2 \boldsymbol{b} \cdot \nabla^2 \partial_{tt}^2 h_{\varepsilon,\ell} + \varepsilon^2 \mathbf{c} \cdot \nabla^4 h_{\varepsilon,\ell} &= \varepsilon^4 (\boldsymbol{b} \otimes \bar{\boldsymbol{a}}_2) \cdot \nabla^6 u_{\varepsilon,\ell}, \\
h_{\varepsilon,\ell}(0,\cdot) &= 0, \\
\partial_t h_{\varepsilon,\ell}(0,\cdot) &= 0.
\end{cases}
\tag{4.16}
$$

As in the proof of Lemma 3.1, we integrate (4.16) once in time, multiply by $h_{\varepsilon,\ell}$ and integrate over $[0,t] \times \mathbb{R}^d$. Since $\bar{\boldsymbol{a}}_0$, $\boldsymbol{b}$, and $\mathbf{c}$ are non-negative, their contributions to the energy estimate on the LSH are non-negative, and we obtain

$$
\|h_{\varepsilon,\ell}(t,\cdot)\|_{L^2(\mathbb{R}^d)}^2 \lesssim \varepsilon^4 \left| \int_0^t \int_{\mathbb{R}^d} h_{\varepsilon,\ell}(s,x) R_{\varepsilon,\ell}(s,x) ds dx \right|,
$$

where, by definiton of $u_{\varepsilon,\ell}$ and integration in time,

$$
R_{\varepsilon,\ell} := \nabla^6 \int_{\mathbb{R}^d} \omega(\varepsilon|k|) \hat{u}_0(k) e^{ik\cdot x} \frac{\varepsilon}{\Lambda_\ell(\varepsilon|k|)} \sin(\varepsilon^{-1} \Lambda_\ell(\varepsilon|k|) s) dk.
$$

We then proceed as in Substep 3.1 of the proof of Proposition 3, and define $F_{s,\varepsilon,\ell}$ as the linear operator from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R}^d)$ characterized by its Fourier symbol

$$
\hat{F}_{s,\varepsilon,\ell}(k) := \omega(\varepsilon|k|)(ik)^{\otimes 6} \frac{\varepsilon}{\Lambda_\ell(\varepsilon|k|)} \sin\left(\varepsilon^{-1} \Lambda_\ell(\varepsilon|k|) s\right),
$$

which is bounded by $|k|^6$ uniformly over $\varepsilon > 0$ and $s \geq 0$ by definition of ω and Λ_ℓ. Hence, by Cauchy-Schwarz inequality,

$$
\begin{aligned}
\|h_{\varepsilon,\ell}(t,\cdot)\|_{L^2(\mathbb{R}^d)}^2 &\lesssim \varepsilon^4 \int_0^t \|h_{\varepsilon,\ell}(s,\cdot)\|_{L^2(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} |F_{s,\varepsilon,\ell} u_0|^2 dx \right)^{\frac{1}{2}} ds, \\
&\lesssim \varepsilon^4 t \sup_{s \leq t} \|h_{\varepsilon,\ell}(s,\cdot)\|_{L^2(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} |\nabla^6 u_0|^2 dx \right)^{\frac{1}{2}}.
\end{aligned}
$$

Since the RHS is non-decreasing in t, we may replace the LHS by $\sup_{s \leq t} \|h_{\varepsilon,\ell}(t,\cdot)\|_{L^2(\mathbb{R}^d)}^2$, and obtain

$$
\sup_{t \leq T} \|h_{\varepsilon,\ell}(t,\cdot)\|_{L^2(\mathbb{R}^d)} \lesssim C_\ell(u_0) \varepsilon^4 T,
$$

which concludes the proof.

Step 2. Estimate for (4.4).
Assume that $\ell \geq 1$ and let $w_{\varepsilon,\ell}$ solve (4.4). By definition, the error $h_{\varepsilon,\ell} := u_{\varepsilon,\ell} - w_{\varepsilon,\ell}$ satisfies

$$
\begin{cases}
\partial_{tt}^2 h_{\varepsilon,\ell} + \mathcal{L}_{\mathrm{hom},\ell,\varepsilon} h_{\varepsilon,\ell} &= \varepsilon^{2([\frac{\ell-1}{2}]+1)} R_{\varepsilon,\ell}, \\
h_{\varepsilon,\ell}(0,\cdot) &= 0, \\
\partial_t h_{\varepsilon,\ell}(0,\cdot) &= 0,
\end{cases}
\tag{4.17}
$$

where

$$
R_{\varepsilon,\ell} := -\gamma_\ell i^{2([\frac{\ell-1}{2}]+1)} \mathrm{Id} \cdot \nabla^{2([\frac{\ell-1}{2}]+2)} u_{\varepsilon,\ell}.
$$

We now establish the energy estimate for (4.17). As in Step 1, we integrate (4.17) once in time, multiply by $h_{\varepsilon,\ell}$, and integrate over $[0,t] \times \mathbb{R}^d$. We then obtain, using that

$$h_{\varepsilon,\ell}(0,\cdot) \equiv 0,$$

$$\|h_{\varepsilon,\ell}(t,\cdot)\|^2_{L^2(\mathbb{R}^d)} + \frac{1}{2}\left(\mathcal{L}_{\mathrm{hom},\ell,\varepsilon}\int_0^t h_{\varepsilon,\ell}(s,\cdot)ds, \int_0^t h_{\varepsilon,\ell}(s,\cdot)ds\right)_{(H^{-([\frac{\ell-1}{2}]+1)}, H^{[\frac{\ell-1}{2}]+1})(\mathbb{R}^d)}$$
$$= \varepsilon^{2([\frac{\ell-1}{2}]+1)}\int_0^t\int_{\mathbb{R}^d} h_{\varepsilon,\ell}(s,x)R_{\varepsilon,\ell}(s,x)dxds.$$

By Lemma 4.2, the second LHS term is non-negative, so that we obtain as in Step 1

$$\sup_{t\leq T}\|h_{\varepsilon,\ell}(t,\cdot)\|^2_{L^2(\mathbb{R}^d)} \lesssim \varepsilon^{4([\frac{\ell-1}{2}]+1)}T^2\sup_{t\leq T}\int_{\mathbb{R}^d} R^2_{\varepsilon,\ell}(t,x)dx.$$

It remains to control the last RHS factor. Proceeding in Step 1, we have in Fourier space,

$$\int_{\mathbb{R}^d} R^2_{\varepsilon,\ell}(t,x)dx \lesssim \int_{\mathbb{R}^d}|k|^{4([\frac{\ell-1}{2}]+2)}|\hat{u}_0(k)|^2 dk \lesssim \int_{\mathbb{R}^d}|\nabla^{2([\frac{\ell-1}{2}]+2)}u_0(x)|^2 dx,$$

and the proof is complete.

4.4. **Proof of Theorem 3: Higher-order approximation of the elliptic equation.**
By scaling, it is enough to consider $\varepsilon = 1$.

Step 1. Representation formula for the residuum.
For all smooth functions v and $k \geq 1$, set $w_k(v) := \sum_{j=0}^k \varphi_j \cdot \nabla^j v$, and for all $m \geq 2$, set

$$S_m(v) := \sum_{p=0}^{m-2}\sum_{j=1}^{m-1-p}(\varphi_j \otimes \bar{\boldsymbol{a}}_p) \cdot \nabla^{p+j+2}v, \tag{4.18}$$

with the understanding that $S_m \equiv 0$ for $m < 2$. We shall prove that that for $1 \leq \ell \leq 2$,

$$-\nabla \cdot \boldsymbol{a}\nabla w_\ell(v) = -\nabla \cdot \bar{\boldsymbol{a}}_0 \nabla v - \nabla \cdot \left[(\boldsymbol{a} \otimes \varphi_\ell - \sigma_\ell) \cdot \nabla^{\ell+1}v\right]. \tag{4.19}$$

whereas for all $\ell \geq 1$,

$$-\nabla \cdot \boldsymbol{a}\nabla w_\ell(v) = -\sum_{j=0}^{\ell-1}\bar{\boldsymbol{a}}_j \cdot \nabla^{j+2} - S_\ell(v)$$
$$+ \nabla\chi_\ell \cdot \nabla^{\ell+2}v - \nabla \cdot \left[(\boldsymbol{a} \otimes \varphi_\ell - \sigma_\ell + \nabla\chi_\ell) \cdot \nabla^{\ell+1}v\right], \tag{4.20}$$

Theorem 3 will then follow from (4.20), whereas Corollary 2 will follow from (4.19). We split the proof of (4.19) & (4.20) into two substeps.

Substep 1.1. First representation formula for the residuum, and proof that for all $k \geq 1$,

$$\nabla \cdot \boldsymbol{a}\nabla w_k(v) = \sum_{j=0}^{k-1}\bar{\boldsymbol{a}}_j \cdot \nabla^{j+2}v + S_{k-1}(v)$$
$$- \nabla\chi_{k-1} \cdot \nabla^{k+1}v + \nabla \cdot \left[(\boldsymbol{a} \otimes \varphi_k - \sigma_k) \cdot \nabla^{k+1}v\right]. \tag{4.21}$$

We proceed by induction. The result for $k = 1$ is by now standard, and the very reason for introducing σ_1 in [25]. More precisely, a direct calculation combined with the defining

equation $-\nabla \cdot \boldsymbol{a}(\mathrm{Id} + \nabla \varphi_1) = 0$ for φ_1, and the definition $\nabla \cdot \sigma_1 = \boldsymbol{a}(\mathrm{Id} + \nabla \varphi_1) - \bar{\boldsymbol{a}}_0$ and skew-symmetry of σ_1 in form of $\nabla^2 v \cdot \nabla \cdot \sigma_1 = -\nabla \cdot (\sigma_1 \cdot \nabla^2 v)$, yields

$$
\begin{aligned}
\nabla \cdot \boldsymbol{a} \nabla w_1(v) &= \nabla \cdot \boldsymbol{a}((\mathrm{Id} + \nabla \varphi_1) \nabla v + \varphi_1 \cdot \nabla^2 v) \\
&= \nabla \cdot \bar{\boldsymbol{a}}_0 \nabla v + \nabla \cdot \big(\boldsymbol{a}(\mathrm{Id} + \nabla \varphi_1) - \bar{\boldsymbol{a}}_0\big) \nabla v + \nabla \cdot \boldsymbol{a}(\varphi_1 \cdot \nabla^2 v) \\
&= \nabla \cdot \bar{\boldsymbol{a}}_0 \nabla v + \nabla^2 v \cdot \big(\boldsymbol{a}(\mathrm{Id} + \nabla \varphi_1) - \bar{\boldsymbol{a}}_0\big) + \nabla \cdot \boldsymbol{a}(\varphi_1 \cdot \nabla^2 v) \\
&= \nabla \cdot \bar{\boldsymbol{a}}_0 \nabla v + \nabla \cdot \big[(\boldsymbol{a} \otimes \varphi_1 - \sigma_1) \cdot \nabla^2 v\big],
\end{aligned}
$$

as claimed. Assume now that (4.21) holds at step $k \geq 1$. Writing $w_{k+1}(v) = w_k(v) + \varphi_{k+1} \cdot \nabla^{k+1} v$, and using (4.21) at step k, we obtain

$$
\begin{aligned}
\nabla \cdot \boldsymbol{a} \nabla w_{k+1}(v) &= \nabla \cdot \boldsymbol{a} \nabla w_k(v) + \nabla \cdot [\boldsymbol{a}(\nabla \varphi_{k+1} \cdot \nabla^{k+1} v)] + \nabla \cdot [(\boldsymbol{a} \otimes \varphi_{k+1}) \cdot \nabla^{k+2} v] \\
&= \sum_{j=0}^{k-1} \bar{\boldsymbol{a}}_j \cdot \nabla^{j+2} v + S_{k-1}(v) - \nabla \chi_{k-1} \cdot \nabla^{k+1} v \\
&\quad + \nabla \cdot \Big[(\boldsymbol{a} \nabla \varphi_{k+1} + \boldsymbol{a} \otimes \varphi_k - \sigma_k) \cdot \nabla^{k+1} v\Big] + \nabla \cdot [(\boldsymbol{a} \otimes \varphi_{k+1}) \cdot \nabla^{k+2} v].
\end{aligned}
$$

$$(4.22)$$

Let us reformulate the third RHS term. We add and substract $\nabla \chi_k$ in order to use the defining equation for φ_{k+1} in form of $\nabla \cdot (\boldsymbol{a} \nabla \varphi_{k+1} + \boldsymbol{a} \otimes \varphi_k - \sigma_k + \nabla \chi_k) = 0$, and rewrite the third RHS as

$$
\begin{aligned}
\nabla \cdot \Big[(\boldsymbol{a} \nabla \varphi_{k+1} + \boldsymbol{a} \otimes \varphi_k - \sigma_k) \cdot \nabla^{k+1} v\Big] &= - \triangle \chi_k \cdot \nabla^{k+1} v \\
&\quad + (\boldsymbol{a} \nabla \varphi_{k+1} + \boldsymbol{a} \otimes \varphi_k - \sigma_k) \cdot \nabla^{k+2} v.
\end{aligned}
$$

We then appeal to the defining formulas $- \triangle \chi_k := \nabla \chi_{k-1} + \sum_{j=1}^{k-1} \varphi_j \otimes \bar{\boldsymbol{a}}_{k-1-j}$ and $q_{k+1} := \boldsymbol{a} \nabla \varphi_{k+1} + \boldsymbol{a} \otimes \varphi_k - \bar{\boldsymbol{a}}_k + \nabla \chi_k - \sigma_k$, and obtain

$$
\begin{aligned}
\nabla \cdot \Big[(\boldsymbol{a} \nabla \varphi_{k+1} + \boldsymbol{a} \otimes \varphi_k - \sigma_k) \cdot \nabla^{k+1} v\Big] &= \sum_{j=1}^{k-1} (\varphi_j \otimes \bar{\boldsymbol{a}}_{k-1-j}) \cdot \nabla^{k+1} v \\
&\quad + \nabla \chi_{k-1} \cdot \nabla^{k+1} v + (q_{k+1} + \bar{\boldsymbol{a}}_k - \nabla \chi_k) \cdot \nabla^{k+2} v.
\end{aligned}
$$

Combining the above with (4.22) then yields

$$
\begin{aligned}
\nabla \cdot \boldsymbol{a} \nabla w_{k+1}(v) = \sum_{j=0}^{k} \bar{\boldsymbol{a}}_j \cdot \nabla^{j+2} v + S_{k-1}(v) + \sum_{j=1}^{k-1} (\varphi_j \otimes \bar{\boldsymbol{a}}_{k-1-j}) \cdot \nabla^{k+1} v - \nabla \chi_k \cdot \nabla^{k+2} v \\
+ q_{k+1} \cdot \nabla^{k+2} v + \nabla \cdot [(\boldsymbol{a} \otimes \varphi_{k+1}) \cdot \nabla^{k+2} v].
\end{aligned}
$$

Using the property

$$
S_{k+1}(v) = S_k(v) + \sum_{j=1}^{k} (\varphi_j \otimes \bar{\boldsymbol{a}}_{k-j}) \cdot \nabla^{k+1} v, \tag{4.23}
$$

we conclude by the defining equation $\nabla \cdot \sigma_{k+1} = q_{k+1}$ and the skew-symmetry of σ_{k+1} in form of $\nabla^{k+2}v \cdot \nabla \cdot \sigma_{k+1} = -\nabla \cdot (\sigma_{k+1} \cdot \nabla^{k+2}v)$ that

$$\nabla \cdot \boldsymbol{a} \nabla w_{k+1}(v) = \sum_{j=0}^{k} \bar{\boldsymbol{a}}_j \cdot \nabla^{j+2}v + S_{k+1}(v) - \nabla \chi_k \cdot \nabla^{k+2}v + \nabla \cdot [(\boldsymbol{a} \otimes \varphi_{k+1} - \sigma_{k+1}) \cdot \nabla^{k+2}v],$$

that is, (4.21) at step $k + 1$.

Substep 1.2. Reformulation of (4.21), and proof of (4.19) & (4.20).
We start with (4.19), which is a direct consequence of (4.21) and the identities $\chi_k \equiv 0$ for all $k \leq 1$ and $\bar{\boldsymbol{a}}_1 = 0$ (cf. Proposition 1). We then turn to (4.20). By the defining formula $-\nabla \chi_{k-1} = \triangle \chi_k + \sum_{j=1}^{k-1} \varphi_j \otimes \bar{\boldsymbol{a}}_{k-1-j}$ for χ_k, (4.21) turns into

$$\nabla \cdot \boldsymbol{a} \nabla w_k(v) = \sum_{j=0}^{k-1} \bar{\boldsymbol{a}}_j \cdot \nabla^{j+2}v + S_{k-1}(v) + \sum_{j=1}^{k-1} (\varphi_j \otimes \bar{\boldsymbol{a}}_{k-1-j}) \cdot \nabla^{k+1}v$$

$$+ \triangle \chi_k \cdot \nabla^{k+1}v + \nabla \cdot \left[(\boldsymbol{a} \otimes \varphi_k - \sigma_k) \cdot \nabla^{k+1}v \right].$$

Combined with (4.23) and the identity $\triangle \chi_k \cdot \nabla^{k+1}v = \nabla \cdot (\nabla \chi_k \cdot \nabla^{k+1}v) - \nabla \chi_k \cdot \nabla^{k+2}v$, this yields (4.20).

Step 2. Proof of Corollary 2: (4.13).
Let u and u_{hom} denote the solutions of (4.9) for $\varepsilon = 1$ and (4.12). With the choice $v = u_{\text{hom}}$, substracting (4.19) from (4.9) for $\varepsilon = 1$ yields

$$-\nabla \cdot \boldsymbol{a} \nabla(u - w_\ell(u_{\text{hom}})) = \nabla \cdot \left[(\boldsymbol{a} \otimes \varphi_\ell - \sigma_\ell + \nabla \chi_\ell) \cdot \nabla^{\ell+1}u_{\text{hom}} \right],$$

so that (4.13) follows from testing this equation with $u - w_\ell(u_{\text{hom}})$, Hypothesis 1, and a scaling argument.

Step 3. Proof of Theorem 3.
Let u_ℓ and $u_{\text{hom},\ell}$ denote the solutions of (4.10) and (4.7) for $\varepsilon = 1$. We split the proof into two substeps: we first derive the equation for the error $h_\ell := u_\ell - u_{\text{hom},\ell}$, and then proceed to the actual estimates.

Substep 3.1. Proof of the identity

$$-\nabla \cdot \boldsymbol{a} \nabla h_\ell = \nabla \cdot \left[(\boldsymbol{a} \otimes \varphi_\ell - \sigma_\ell + \nabla \chi_\ell) \cdot \nabla^{\ell+1}u_{\text{hom},\ell} \right] + \sum_{p=1}^{\ell-1} \sum_{j=\ell-p}^{\ell-1} (\varphi_j \otimes \bar{\boldsymbol{a}}_p) \cdot \nabla^{j+p+2}u_{\text{hom},\ell}$$

$$-\nabla \chi_\ell \cdot \nabla^{\ell+2}u_{\text{hom},\ell} - \gamma_\ell i^{2([\frac{\ell-1}{2}]+1)} \sum_{j=0}^{\ell-1} (\varphi_j \otimes \text{Id}) \cdot \nabla^{j+2([\frac{\ell-1}{2}]+2)}u_{\text{hom},\ell}. \quad (4.24)$$

The starting point is (4.20) with $v = u_{\text{hom},\ell}$:

$$-\nabla \cdot \boldsymbol{a} \nabla w_\ell(u_{\text{hom},\ell}) = -\sum_{j=0}^{\ell-1} \bar{\boldsymbol{a}}_j \cdot \nabla^{j+2}u_{\text{hom},\ell} - S_\ell(u_{\text{hom},\ell})$$

$$+ \nabla \chi_\ell \cdot \nabla^{\ell+2}u_{\text{hom},\ell} - \nabla \cdot \left[(\boldsymbol{a} \otimes \varphi_\ell - \sigma_\ell + \nabla \chi_\ell) \cdot \nabla^{\ell+1}u_{\text{hom},\ell} \right]. \quad (4.25)$$

Using (4.7) for $\varepsilon = 1$, we reformulate the first RHS term as

$$-\sum_{j=0}^{\ell-1} \bar{\boldsymbol{a}}_j \cdot \nabla^{j+2} u_{\mathrm{hom},\ell} = f + \gamma_\ell i^{2([\frac{\ell-1}{2}]+1)} \mathrm{Id} \cdot \nabla^{2([\frac{\ell-1}{2}]+2)} u_{\mathrm{hom},\ell}. \qquad (4.26)$$

In order to reformulate the second RHS term, we take the j-th derivative of (4.7) for $\varepsilon = 1$ in form of

$$- (\varphi_j \otimes \bar{\boldsymbol{a}}_0) \cdot \nabla^{j+2} u_{\mathrm{hom},\ell} = \varphi_j \cdot \nabla^j f + \sum_{p=1}^{\ell-1} (\varphi_j \otimes \bar{\boldsymbol{a}}_p) \cdot \nabla^{j+p+2} u_{\mathrm{hom},\ell}$$
$$+ \gamma_\ell i^{2([\frac{\ell-1}{2}]+1)} (\varphi_j \otimes \mathrm{Id}) \cdot \nabla^{j+2([\frac{\ell-1}{2}]+2)} u_{\mathrm{hom},\ell}, \quad (4.27)$$

which allows us to rewrite $S_\ell(u_{\mathrm{hom},\ell})$ as

$$-S_\ell(u_{\mathrm{hom},\ell}) \stackrel{(4.18)}{=} \sum_{j=1}^{\ell-1} (\varphi_j \otimes \bar{\boldsymbol{a}}_0) \cdot \nabla^{j+2} u_{\mathrm{hom},\ell} + \sum_{p=1}^{\ell-2} \sum_{j=1}^{\ell-1-p} (\varphi_j \otimes \bar{\boldsymbol{a}}_p) \cdot \nabla^{p+j+2} u_{\mathrm{hom},\ell}$$

$$\stackrel{(4.27)}{=} \sum_{j=1}^{\ell-1} \varphi_j \cdot \nabla^j f - \sum_{p=1}^{\ell-1} \sum_{j=\ell-p}^{\ell-1} (\varphi_j \otimes \bar{\boldsymbol{a}}_p) \cdot \nabla^{j+p+2} u_{\mathrm{hom},\ell}$$

$$+ \gamma_\ell i^{2([\frac{\ell-1}{2}]+1)} \sum_{j=1}^{\ell-1} (\varphi_j \otimes \mathrm{Id}) \cdot \nabla^{j+2([\frac{\ell-1}{2}]+2)} u_{\mathrm{hom},\ell}.$$

Combined with (4.25) and (4.26), this yields

$$- \nabla \cdot \boldsymbol{a} \nabla w_\ell(u_{\mathrm{hom},\ell}) = \sum_{j=0}^{\ell-1} \varphi_j \cdot \nabla^j f - \nabla \cdot \left[(\boldsymbol{a} \otimes \varphi_\ell - \sigma_\ell + \nabla \chi_\ell) \cdot \nabla^{\ell+1} u_{\mathrm{hom},\ell} \right]$$

$$- \sum_{p=1}^{\ell-1} \sum_{j=\ell-p}^{\ell-1} (\varphi_j \otimes \bar{\boldsymbol{a}}_p) \cdot \nabla^{j+p+2} u_{\mathrm{hom},\ell} + \nabla \chi_\ell \cdot \nabla^{\ell+2} u_{\mathrm{hom},\ell}$$

$$+ \gamma_\ell i^{2([\frac{\ell-1}{2}]+1)} \sum_{j=0}^{\ell-1} (\varphi_j \otimes \mathrm{Id}) \cdot \nabla^{j+2([\frac{\ell-1}{2}]+2)} u_{\mathrm{hom},\ell}.$$

The claim (4.24) then follows from substracting this identity from (4.10) for $\varepsilon = 1$.

Substep 3.2. Proof of (4.11).
The RHS of (4.24) is the sum of one term in divergence form and three terms in non-divergence form. Testing (4.24) with h_ℓ, making an integration by parts for the terms in divergence form, and using Hardy's inequality on the last three RHS terms in form of

$$\int_{\mathbb{R}^d} |h_\ell(x) g(x)| dx \lesssim \left(\int_{\mathbb{R}^d} |\nabla h_\ell(x)|^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} (1 + |x|)^2 g(x)^2 dx \right)^{\frac{1}{2}},$$

the energy estimate for (4.24) reads

$$\|\nabla h_\ell\|_{L^2(\mathbb{R}^d)}^2 \lesssim I_1^2 + I_2^2 + I_3^2 + I_4^2,$$

where

$$I_1^2 := \int_{\mathbb{R}^d} \Big(\fint_{B(x)} |\varphi_\ell|^2 + |\sigma_\ell|^2 + |\nabla \chi_\ell|^2 \Big) \sup_{B(x)} \{|\nabla^{\ell+1} u_{\mathrm{hom},\ell}|^2\} dx,$$

$$I_2^2 := \sum_{p=1}^{\ell-1} \sum_{j=\ell-p}^{\ell-1} \int_{\mathbb{R}^d} \Big(\fint_{B(x)} |\varphi_j|^2 \Big) (1+|x|)^2 \sup_{B(x)} \{|\nabla^{j+p+2} u_{\mathrm{hom},\ell}|^2\} dx,$$

$$I_3^2 := \int_{\mathbb{R}^d} \Big(\fint_{B(x)} |\nabla \chi_\ell|^2 \Big) (1+|x|)^2 \sup_{B(x)} \{|\nabla^{\ell+2} u_{\mathrm{hom},\ell}|^2\} dx,$$

$$I_4^2 := \sum_{j=0}^{\ell-1} \int_{\mathbb{R}^d} \Big(\fint_{B(x)} |\varphi_j|^2 \Big) (1+|x|)^2 \sup_{B(x)} \{|\nabla^{j+2([\frac{\ell-1}{2}]+2)} u_{\mathrm{hom},\ell}|^2\} dx.$$

Taking the expectation of the energy estimate, and using Hypothesis 1, we obtain the claim for $\varepsilon = 1$. The general result follows by a scaling argument.

Step 4. Proof of Corollary 3: (4.15).
As above, by a scaling argument, it is enough to prove (4.15) for $\varepsilon = 1$. Recall that $u_{\mathrm{hom},\ell}$ is the solution of (4.14) for $\varepsilon = 1$ and $3 \le \ell \le 4$. Starting point is formula (4.20), which we specifically rewrite for $3 \le \ell \le 4$ as

$$-\nabla \cdot \boldsymbol{a} \nabla w_\ell(u_{\mathrm{hom},\ell}) = -\bar{\boldsymbol{a}}_0 \cdot \nabla^2 u_{\mathrm{hom},\ell} - \bar{\boldsymbol{a}}_2 \cdot \nabla^4 u_{\mathrm{hom},\ell} - \sum_{j=1}^{\ell-1} (\varphi_j \otimes \bar{\boldsymbol{a}}_0) \cdot \nabla^{j+2} u_{\mathrm{hom},\ell}$$

$$- \delta_{\ell=4} (\varphi_1 \otimes \bar{\boldsymbol{a}}_2) \cdot \nabla^5 u_{\mathrm{hom},\ell} + \nabla \chi_\ell \cdot \nabla^{\ell+2} u_{\mathrm{hom},\ell}$$

$$- \nabla \cdot \Big[(\boldsymbol{a} \otimes \varphi_\ell - \sigma_\ell + \nabla \chi_\ell) \cdot \nabla^{\ell+1} u_{\mathrm{hom},\ell} \Big],$$

where δ denotes the Kronecker symbol. We then appeal to the decomposition of $\bar{\boldsymbol{a}}_2$ provided by Lemma 4.1, and obtain

$$-\nabla \cdot \boldsymbol{a} \nabla w_\ell(u_{\mathrm{hom},\ell})$$
$$= -\bar{\boldsymbol{a}}_0 \cdot \nabla^2 u_{\mathrm{hom},\ell} + \mathbf{c} \cdot \nabla^4 u_{\mathrm{hom},\ell} - (\boldsymbol{b} \otimes \bar{\boldsymbol{a}}_0) \cdot \nabla^4 u_{\mathrm{hom},\ell}$$
$$- \sum_{j=1}^{\ell-1} (\varphi_j \otimes \bar{\boldsymbol{a}}_0) \cdot \nabla^{j+2} u_{\mathrm{hom},\ell} - \delta_{\ell=4} ((-\varphi_1 \otimes \mathbf{c} + \varphi_1 \otimes \boldsymbol{b} \otimes \bar{\boldsymbol{a}}_0) \cdot \nabla^5 u_{\mathrm{hom},\ell})$$
$$+ \nabla \chi_\ell \cdot \nabla^{\ell+2} u_{\mathrm{hom},\ell} - \nabla \cdot \Big[(\boldsymbol{a} \otimes \varphi_\ell - \sigma_\ell + \nabla \chi_\ell) \cdot \nabla^{\ell+1} u_{\mathrm{hom},\ell} \Big].$$

We now use the defining equation (4.14) for $u_{\mathrm{hom},\ell}$ in form of $\bar{\boldsymbol{a}}_0 \cdot \nabla^2 u_{\mathrm{hom},\ell} = -f + \boldsymbol{b} \cdot \nabla^2 f + \mathbf{c} \cdot \nabla^4 u_{\mathrm{hom},\ell}$ to rewrite the following terms as

$$(\boldsymbol{b} \otimes \bar{\boldsymbol{a}}_0) \cdot \nabla^4 u_{\mathrm{hom},\ell} = -\boldsymbol{b} \cdot \nabla^2 f + \boldsymbol{b}^{\otimes 2} \cdot \nabla^4 f + (\boldsymbol{b} \otimes \mathbf{c}) \cdot \nabla^6 u_{\mathrm{hom},\ell},$$
$$(\varphi_1 \otimes \boldsymbol{b} \otimes \bar{\boldsymbol{a}}_0) \cdot \nabla^5 u_{\mathrm{hom},\ell} = -(\varphi_1 \otimes \boldsymbol{b}) \cdot \nabla^3 f + (\varphi_1 \otimes \boldsymbol{b}^{\otimes 2}) \cdot \nabla^5 f + (\varphi_1 \otimes \boldsymbol{b} \otimes \mathbf{c}) \cdot \nabla^7 u_{\mathrm{hom},\ell}$$
$$(\varphi_j \otimes \bar{\boldsymbol{a}}_0) \cdot \nabla^{j+2} u_{\mathrm{hom},\ell} = -\varphi_j \cdot \nabla^j f + (\varphi_j \otimes \boldsymbol{b}) \cdot \nabla^{j+2} f + (\varphi_j \otimes \mathbf{c}) \cdot \nabla^{j+4} u_{\mathrm{hom},\ell}.$$

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Inserting these identities in the above and rearranging the terms, we obtain the identity valid for $3 \leq \ell \leq 4$:

$$
\begin{aligned}
-\nabla \cdot \boldsymbol{a}\nabla w_\ell(u_{\mathrm{hom},\ell}) \\
= \ f + \sum_{j=1}^{\ell-1} \varphi_j \cdot \nabla^j f + \sum_{j=\ell-2}^{\ell-1} (\varphi_j \otimes \boldsymbol{b}) \cdot \nabla^{j+2} f - \sum_{j=\ell-2}^{\ell-1} (\varphi_j \otimes \mathbf{c}) \cdot \nabla^{j+4} u_{\mathrm{hom},\ell} \\
- \sum_{j=0}^{\ell-2} (\varphi_j \otimes \boldsymbol{b}^{\otimes 2}) \cdot \nabla^{j+4} f - \sum_{j=0}^{\ell-2} (\varphi_j \otimes \boldsymbol{b} \otimes \mathbf{c}) \cdot \nabla^{j+6} u_{\mathrm{hom},\ell} \\
+ \nabla \chi_\ell \cdot \nabla^{\ell+2} u_{\mathrm{hom},\ell} - \nabla \cdot \Big[(\boldsymbol{a} \otimes \varphi_\ell - \sigma_\ell + \nabla\chi_\ell) \cdot \nabla^{\ell+1} u_{\mathrm{hom},\ell} \Big] .
\end{aligned}
$$

Substracting this from (4.9) yields for $h_\ell := u_\ell - u_{\mathrm{hom},\ell}$:

$$
\begin{aligned}
-\nabla \cdot \boldsymbol{a}\nabla h_\ell = \ & \nabla \cdot \Big[(\boldsymbol{a} \otimes \varphi_\ell - \sigma_\ell + \nabla\chi_\ell) \cdot \nabla^{\ell+1} u_{\mathrm{hom},\ell} \Big] \\
& - \nabla \chi_\ell \cdot \nabla^{\ell+2} u_{\mathrm{hom},\ell} + \sum_{j=\ell-2}^{\ell-1} (\varphi_j \otimes \boldsymbol{b}) \cdot \nabla^{j+2} f - \sum_{j=\ell-2}^{\ell-1} (\varphi_j \otimes \mathbf{c}) \cdot \nabla^{j+4} u_{\mathrm{hom},\ell} \\
& + \sum_{j=0}^{\ell-2} (\varphi_j \otimes \boldsymbol{b}^{\otimes 2}) \cdot \nabla^{j+4} f - \sum_{j=0}^{\ell-2} (\varphi_j \otimes \boldsymbol{b} \otimes \mathbf{c}) \cdot \nabla^{j+6} u_{\mathrm{hom},\ell} .
\end{aligned}
$$

Before we proceed with the energy estimate, we reformulate the (higher-order) last RHS term using the homogenized equation to reduce the number of derivatives on $u_{\mathrm{hom},\ell}$ (and ultimately improve the estimate). More precisely, we shall use (4.14) in the form

$$
\nabla^\ell(\mathbf{c} \cdot \nabla^4 u_{\mathrm{hom},\ell}) = \nabla^\ell(\nabla \cdot \bar{\boldsymbol{a}}_0 \nabla u_{\mathrm{hom},\ell} + f - \mathbf{b} \cdot \nabla^2 f),
$$

so that the equation turns into

$$
\begin{aligned}
-\nabla \cdot \boldsymbol{a}\nabla h_\ell = \ & \nabla \cdot \Big[(\boldsymbol{a} \otimes \varphi_\ell - \sigma_\ell + \nabla\chi_\ell) \cdot \nabla^{\ell+1} u_{\mathrm{hom},\ell} \Big] \\
& - \nabla \chi_\ell \cdot \nabla^{\ell+2} u_{\mathrm{hom},\ell} + \sum_{j=\ell-2}^{\ell-1} (\varphi_j \otimes \boldsymbol{b}) \cdot \nabla^{j+2} f - \sum_{j=\ell-2}^{\ell-1} (\varphi_j \otimes \mathbf{c}) \cdot \nabla^{j+4} u_{\mathrm{hom},\ell} \\
& + \sum_{j=0}^{\ell-2} (\varphi_j \otimes \boldsymbol{b}^{\otimes 2}) \cdot \nabla^{j+4} f - \sum_{j=0}^{\ell-3} (\varphi_j \otimes \boldsymbol{b} \otimes \mathbf{c}) \cdot \nabla^{j+6} u_{\mathrm{hom},\ell} \\
& \qquad\qquad - (\varphi_{\ell-2} \otimes \boldsymbol{b}) \cdot \nabla^\ell(\nabla \cdot \bar{\boldsymbol{a}}_0 \nabla u_{\mathrm{hom},\ell} + f - \mathbf{b} \cdot \nabla^2 f) .
\end{aligned}
$$

The desired estimate (4.15) now follows as in Substep 3.2.

5. ASYMPTOTIC BALLISTIC TRANSPORT OF CLASSICAL WAVES

5.1. **Statement of the result.** The following definition introduces the notion of asymptotic ballistic transport for the wave operator $\Box = \partial_{tt}^2 - \nabla \cdot \boldsymbol{a}\nabla$.

Definition 5.1. Denote by $S : \mathbb{R}_+ \times L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d), (t, u_0) \mapsto S_t(u_0)$ the semi-group associated with the initial value problem
$$\begin{cases} \square S_t(u_0) &=& 0, \\ S_0(u_0) &=& u_0, \\ \partial_t S_t(u_0)|_{t=0} &=& 0. \end{cases}$$
For all $\lambda > 0$, let G_λ be the centered Gaussian normalized in $L^2(\mathbb{R}^d)$ and of support of size λ, that is,
$$G_\lambda(x) = \left(\frac{\lambda}{\pi}\right)^{d/2} \exp\left(-\frac{1}{2}\lambda^2|x|^2\right). \tag{5.1}$$
For all $T \geq 0$, we set
$$M(\lambda, T) := \left(\int_{\mathbb{R}^d} (1 + \lambda|x|)^2 S_T(G_\lambda)^2 dx\right)^{\frac{1}{2}}, \quad \mathcal{M}(\lambda, T) := \left(\fint_T^{T+\lambda^{-1}} \mathbb{E}\left[M(\lambda, t)^2\right] dt\right)^{\frac{1}{2}}.$$
We say that $\square$ displays *ballistic transport* at energy $\lambda > 0$ if for all $T \geq 0$,
$$\mathcal{M}(\lambda, \lambda^{-1}T) \gtrsim T. \tag{5.2}$$
We say that $\square$ displays *asymptotic ballistic transport* at 0 of order $\gamma \geq 0$ if there exists $T > 0$ such that for all $0 < \varepsilon \ll 1$ small enough
$$\mathcal{M}(\varepsilon, \varepsilon^{-2-\gamma}T) \gtrsim \varepsilon^{-1-\gamma}T. \tag{5.3}$$
$\square$

Let us comment on this definition. First note that $\mathcal{M}(\lambda, 0) \sim 1$ by a direct calculation. In the definition of $\mathcal{M}(\lambda, T)$ we average in time over $(T, T + \lambda^{-1})$ instead of considering a pointwise-in-time quantity: indeed, the L^2-norm is not a conserved quantity (the invariant quantity involves the kinetic energy as well) and may vanish at some specific times, but not on average (the choice of time λ^{-1} is related to the expected speed λ of the wave). Ballistic transport of an initial wave G_λ takes place if this wave is essentially transported at speed λ (as it is the case for a constant-coefficient wave equation). In particular, if $a \equiv \text{Id}$, a direct calculation yields for all $\lambda > 0$ and $\tau \geq 0$
$$\mathcal{M}(\lambda, \lambda^{-1}T) \sim T,$$
(that is, $\frac{1}{C}T \leq \mathcal{M}(\lambda, \lambda^{-1}T) \leq CT$ for some multiplicative constant C independent of $\lambda > 0$ and $T \geq 1$), which, in view of the weighted norm, illustrates that most of the mass is transported at distance T from the origin. This explains (5.2).

Let us turn to (5.3). Transport is only significant if the support of G_λ has moved, which requires T in (5.2) to be at least of order λ^{-2} (since the support of G_λ has size λ^{-1} and the speed of propagation is λ). This explains the scaling in ε in (5.3), and the wording of *asymptotic ballistic transport*:

- The result is *asymptotic* because the final time $\tau = \varepsilon^{-2-\gamma}T$ one can consider depends on the energy level ε.
- There is effective *transport* because a significant part of the mass has moved by a distance which, measured in the unit ε^{-1} of the typical length-scale at initial time, is bounded by below uniformly in $\varepsilon > 0$. More precisely, by (5.3) and the definition of the weighted norm, this distance is of order $\varepsilon^{-1-\gamma}$, so that the ratio $\frac{\varepsilon^{-1-\gamma}}{\varepsilon^{-1}} = \varepsilon^{-\gamma}$ is isolated from zero as soon as $\gamma \geq 0$. (There would be no asymptotic transport if (5.3) only held for some $\gamma < 0$, as it is the case for the Poisson inclusions in dimensions $d \leq 2$, see below).

- The transport is *ballistic* because it satisfies the ballistic scaling property (5.1).

Note that we could also consider higher-order moments and use $(1 + \lambda|x|)^{2p}$ as a weight instead of $(1 + \lambda|x|)^2$, in which case the RHS of (5.3) would be replaced by $(\varepsilon^{-1-\gamma}T)^p$ in the definition (and in Theorem 4 below, the proof of which adapts straightforwardly).

Our main result is as follows:

Theorem 4. *Let $\ell \geq 1$, and assume that $(\varphi_j, \sigma_j, \chi_j)_{0 \leq j \leq \ell}$ satisfy Hypothesis 1 for some $\alpha = (\alpha_1, \alpha_2) \in [0, 1) \times \mathbb{R}_+$. Then for all $\gamma \geq 0$, we have for all $\varepsilon \ll 1$ and all $T \geq 0$,*

$$\mathcal{M}(\varepsilon, \varepsilon^{-2-\gamma}T) \gtrsim \varepsilon^{-1-\gamma}T(1 - C\varepsilon^{\ell-\gamma}T\mu_\alpha(\varepsilon^{-2-\gamma}T)), \tag{5.4}$$

where the constant $0 < C < \infty$ only depends on $\bar{\Gamma}_\ell := \max_{0 \leq j \leq \ell-1} |\bar{a}_j|$, d, γ, ℓ, and α. In particular, the associated wave operator $\square$ displays asymptotic ballistic transport at 0 provided $\ell = 1$ and $\alpha_1 < \frac{1}{2}$ or $\ell > 1$ (no condition on α), in which case we have for all $0 \leq \gamma < \frac{\ell - 2\alpha_1}{1 + \alpha_1}$, all $T < \infty$, and all $0 < \varepsilon \ll 1$,

$$\mathcal{M}(\varepsilon, \varepsilon^{-2-\gamma}T) \gtrsim \varepsilon^{-1-\gamma}T. \tag{5.5}$$

In the borderline case $\ell = 1$ and $\alpha = (\frac{1}{2}, 0)$, $\square$ displays asymptotic ballistic transport at 0 in the sense that for $\gamma = 0$ and all $0 < T \ll 1$ small enough, we have for all $0 < \varepsilon \ll 1$,

$$\mathcal{M}(\varepsilon, \varepsilon^{-2}T) \gtrsim \varepsilon^{-1}T. \tag{5.6}$$

$\square$

If the extended correctors blow up more rapidly than in the assumptions of Theorem 4, we cannot conclude that the support at final time has moved with respect to the support at initial time in the asymptotic regime $\varepsilon \downarrow 0$. Let us make this result more explicit in three interesting examples:

- For periodic coefficients, one can prove that the multiplicative constant in (5.5) only grows exponentially with γ, so that for $0 < \varepsilon \ll 1$ small enough, one may upgrade (5.5) to $\mathcal{M}(\varepsilon, \varepsilon^{-1}T) \gtrsim T$ for all $T \geq 0$, and obtain ballistic transport at all times (thus recovering this classical result for low frequencies without explicit use of the Bloch theorem).
- For smooth quasi-periodic coefficients satisfying a diophantine condition, there is asymptotic ballistic transport in any dimension at any order $\gamma \geq 0$. This is however not quite enough to prove ballistic transport at all times since the multiplicative constant in (5.5) grows in this case more than exponentially with γ. In particular, (5.5) can be upgraded to $\mathcal{M}(\varepsilon, \varepsilon^{-1}T) \gtrsim T$ for all final times $T \lesssim \exp(|\log \varepsilon|^{1+\mu})$ for some $\mu > 0$ (that can be calculated explicitly). We believe there could be ballistic transport at all times, although our approach currently fails to prove so.
- For Poisson random inclusions (or Gaussian coefficient fields with compactly supported correlations), there is asymptotic ballistic transport in dimensions $d > 2$ (cf. [28, 27] and Appendix A for the desired bounds on the correctors). More precisely,
  - if $d > 2$ is even, for all $T \geq 0$ and all $\gamma < [\frac{d}{2}]$, we have $\mathcal{M}(\varepsilon, \varepsilon^{-2-\gamma}T) \gtrsim \varepsilon^{-1-\gamma}T$;
  - if $d > 2$ is odd, for all $1 \gg T \geq 0$ small enough and for $\gamma = [\frac{d}{2}]$, we have $\mathcal{M}(\varepsilon, \varepsilon^{-2-\gamma}T) \gtrsim \varepsilon^{-1-\gamma}T$.

In particular, the scaling for asymptotic ballistic transport improves with dimension.

Corresponding results for more general statistics of $\boldsymbol{a}$ follow in a straightforward way from Theorem 4 and Appendix A.

Let us emphasize that the choice of a Gaussian initial condition in Theorem 4 is convenient but not essential to the proof: we could indeed consider any properly-rescaled function of the Schwartz class. In terms of spectral interpretation of Theorem 4, (5.5) essentially suggests that if there are localized states at energy $\varepsilon \ll 1$, then their supports are expected to scale like at least as $\varepsilon^{-1-\gamma}$ (in the spirit of the results [14] for the Schrödinger operator).

5.2. **Proof of Theorem 4: Asymptotic ballistic transport.** We split the proof into three steps.

Step 1. Reformulation.
By the hyperbolic rescaling $(t, x) \rightsquigarrow (t', x') = (\varepsilon t, \varepsilon x)$, the moment takes the form

$$\mathcal{M}(\varepsilon, \varepsilon^{-1}T) = \Big(\int_T^{T+1} \mathbb{E} \left[\int_{\mathbb{R}^d} (1 + |x|)^2 u_\varepsilon(t, x)^2 dx \right] dt \Big)^{\frac{1}{2}}$$

where u_ε solves the initial value problem

$$\begin{cases} \Box_\varepsilon u_\varepsilon(t, x) &= 0, \\ u_\varepsilon(0, x) &= G_1(x), \\ \partial_t u_\varepsilon(0, x) &= 0, \end{cases}$$

and G_1 is the Gaussian defined in (5.1). In particular, we are in the realm of large-time homogenization. Recall the approximate solution of Theorem 1 in Section 3, given by

$$u_{\varepsilon, \ell}(t, x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \omega_\ell(\varepsilon|k|) \hat{G}_1(k) e^{ik \cdot x} \cos(\varepsilon^{-1} \Lambda_\ell(\varepsilon k) t) dk,$$

where $\Lambda_\ell(k) := \sqrt{\tilde{\lambda}_{k, \ell}}$, which is well-defined since $\tilde{\lambda}_{\varepsilon k, \ell} \geq 0$ when $\omega_\ell(\varepsilon|k|) \neq 0$. We shall compare the moment $\mathcal{M}(\varepsilon, \varepsilon^{-1}T)$ to some related moment of the approximate solution $u_{\varepsilon, \ell}$. For reasons which will be clear in Step 3 below, we need a localized moment for the approximate solution $u_{\varepsilon, \ell}$. For some $C \gg 1$ that will be fixed in Step 2 (and ultimately only depends on $\bar{\Gamma}_\ell$ and d), we set

$$\mathcal{M}_\ell(\varepsilon, \varepsilon^{-1}T) := \Big(\big(\frac{CT}{2} \big)^d \int_T^{T+1} \int_{\mathbb{R}^d} (1 + |x|)^2 u_{\varepsilon, \ell}(t, x)^2 G_{(CT)^{-1}}^2(x) dx dt \Big)^{\frac{1}{2}},$$

where $G_{(CT)^{-1}}$ is our Gaussian function (5.1). Note that $\sup_{\mathbb{R}^d} (\frac{CT}{2})^d G_{(CT)^{-1}}^2 \lesssim 1$. We shall argue in Step 2 that $\mathcal{M}_\ell(\varepsilon, \varepsilon^{-1}T)$ has the desired ballistic scaling in time provided C is chosen large enough, and then argue in Step 3 that the moment of u_ε is indeed essentially bounded by below by $\mathcal{M}_\ell(\varepsilon, \varepsilon^{-1}T)$ using Theorem 1 and our choice of localizing the moment of the approximate solution $u_{\varepsilon, \ell}$.

Step 2. Moment of the approximate solution.

By Plancherel's formula, we may reformulate the moment of $u_{\varepsilon,\ell}$ as

$$
\begin{aligned}
\mathcal{M}_\ell(\varepsilon, \varepsilon^{-1}T)^2 &= \left(\frac{CT}{2}\right)^d \int_T^{T+1} \int_{\mathbb{R}^d} (1+|x|)^2 u_{\varepsilon,\ell}^2(t,x) G_{(CT)^{-1}}^2(x) dx dt \\
&\geq \left(\frac{CT}{2}\right)^d \int_T^{T+1} \int_{\mathbb{R}^d} |x|^2 u_{\varepsilon,\ell}^2(t,x) G_{(CT)^{-1}}^2(x) dx dt \\
&= \int_T^{T+1} \int_{\mathbb{R}^d} |\nabla_k \hat{u}_{\varepsilon,\ell} * \left(\frac{CT}{4\pi}\right)^{d/2} \hat{G}_{(CT)^{-1}}|^2(t,k) dk dt.
\end{aligned}
$$

Since $\hat{G}_{(CT)^{-1}} = (2\pi)^{d/2} G_{CT}$, we may write the above as

$$
\mathcal{M}_\ell(\varepsilon, \varepsilon^{-1}T)^2 \gtrsim \int_T^{T+1} \int_{\mathbb{R}^d} |\nabla_k \hat{u}_{\varepsilon,\ell} * \bar{G}_{CT}|^2(t,k) dk dt, \tag{5.7}
$$

where $\bar{G}_{CT}(k) := \left(\frac{CT}{\sqrt{2\pi}}\right)^d \exp(-\frac{1}{2}C^2T^2|k|^2)$ has mass unity. Using the following more explicit formula for $u_{\varepsilon,\ell}$

$$
u_{\varepsilon,\ell}(t,x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \omega_\ell(\varepsilon|k|)(2\pi)^{d/2} \exp(-\frac{|k|^2}{2}) e^{ik\cdot x} \cos(\varepsilon^{-1}\Lambda_\ell(\varepsilon k)t) dk,
$$

we have

$$
\begin{aligned}
\nabla_k \hat{u}_{\varepsilon,\ell}(t,k) &= \underbrace{\varepsilon\frac{k}{|k|}\omega_\ell'(\varepsilon|k|)(2\pi)^{d/2} \exp(-\frac{|k|^2}{2}) \cos(\varepsilon^{-1}\Lambda_\ell(\varepsilon k)t) - k\hat{u}_{\varepsilon,\ell}(t,k)}_{=: \mathcal{T}_{1,\varepsilon,k,t}} \\
&\quad \underbrace{- t\nabla\Lambda_\ell(\varepsilon k)\omega_\ell(\varepsilon|k|)(2\pi)^{d/2} \exp(-\frac{|k|^2}{2}) \sin(\varepsilon^{-1}\Lambda_\ell(\varepsilon k)t)}_{=: \mathcal{T}_{2,\varepsilon,k,t}}.
\end{aligned}
$$

We start with the contribution of $\mathcal{T}_{1,\varepsilon,k,t}$, which we shall prove is order 1. On the one hand, by Jensen's inequality,

$$
\int_T^{T+1} \int_{\mathbb{R}^d} |\mathcal{T}_{1,\varepsilon,k,t} * \bar{G}_{CT}|^2(t,k) dk dt \lesssim \int_T^{T+1} \int_{\mathbb{R}^d} |\mathcal{T}_{1,\varepsilon,k,t}|^2(t,k) dk dt.
$$

On the other hand, by the boundedness of ω_ℓ' and $\cos$, this yields for all $\varepsilon \leq 1$

$$
\int_T^{T+1} \int_{\mathbb{R}^d} |\mathcal{T}_{1,\varepsilon,k,t} * \bar{G}_{CT}|^2 dk dt \lesssim \int_T^{T+1} \int_{\mathbb{R}^d} (1+|k|)^2 \exp(-|k|^2) dk dt \lesssim 1. \tag{5.8}
$$

Let us turn to the contribution of $\mathcal{T}_{2,\varepsilon,k,t}$, which is slightly more subtle. We write the convolution as follows:

$$
\mathcal{T}_{2,\varepsilon,k,t} * \bar{G}_{CT} = \mathcal{T}_{2,\varepsilon,k,t} + \int_{\mathbb{R}^d} (\mathcal{T}_{2,\varepsilon,k',t} - \mathcal{T}_{2,\varepsilon,k,t}) \bar{G}_{CT}(k-k') dk'. \tag{5.9}
$$

In order to prove that the second RHS term is higher order, we first estimate the first RHS term by below. By Fubini's theorem,

$$
\begin{aligned}
\int_T^{T+1} \int_{\mathbb{R}^d} &|\mathcal{T}_{2,\varepsilon,k,t}|^2 dk dt \\
&\gtrsim \int_{\mathbb{R}^d} T^2 |\nabla\Lambda_\ell(\varepsilon k)|^2 \omega_\ell^2(\varepsilon|k|) \exp(-|k|^2) \int_T^{T+1} \sin^2(\varepsilon^{-1}\Lambda_\ell(\varepsilon k)t) dt dk.
\end{aligned}
$$

By definition of $K_{\max,\ell}$ and ω_ℓ in Section 3.1, for all $\varepsilon > 0$ and $k \in \mathbb{R}^d$ such that $\omega_\ell(\varepsilon|k|) \neq 0$, we have $(\varepsilon|k|)^{-1}\Lambda_\ell(\varepsilon k) \geq \frac{1}{2}$. In particular, $\varepsilon^{-1}\Lambda_\ell(\varepsilon k) \geq \frac{1}{2}$ for $|k| \geq 1$, so that for all $k \in \mathbb{R}^d$ such that $\omega_\ell(\varepsilon|k|) \neq 0$,

$$\int_T^{T+1} \sin^2(\varepsilon^{-1}\Lambda_\ell(\varepsilon k)t)dt \gtrsim 1_{|k|\geq 1},$$

and therefore

$$\int_T^{T+1} \int_{\mathbb{R}^d} |\mathcal{T}_{2,\varepsilon,k,t}|^2 dk dt \gtrsim T^2 \int_{|k|\geq 1} |\nabla \Lambda_\ell(\varepsilon k)|^2 \omega_\ell^2(\varepsilon|k|) \exp(-|k|^2) dk.$$

Recall that for all $k = \kappa e \in \mathbb{R}^d$ such that $\omega_\ell(|k|) \neq 0$,

$$\Lambda_\ell(k) = \sqrt{\tilde{\lambda}_{k,\ell}} = \kappa \sqrt{\sum_{j\geq 0, 2j < \ell} (-1)^j \kappa^{2j} \lambda_{2j}^e},$$

so that

$$|\nabla \Lambda_\ell(k)| \geq \sqrt{e \cdot \bar{a}_0 e} - c\kappa^{[\ell/2]} \geq 1 - c|k|^{[\ell/2]}$$

for some $0 < c < \infty$ that only depends on the ellipticity constant Λ and the dimension d. Up to slightly reducing $K_{\max,\ell} > 0$, this yields for all $|k| \leq K_{\max,\ell}$, $|\nabla \Lambda_\ell(k)|\omega_\ell(|k|) \gtrsim 1$. Hence, for all $\varepsilon \ll 1$ small enough (where smallness depends only on $\bar{\Gamma}_\ell$ and d), we have

$$\int_T^{T+1} \int_{\mathbb{R}^d} |\mathcal{T}_{2,\varepsilon,k,t}|^2 dk dt \gtrsim T^2 \int_{\varepsilon^{-1}K_{\max,\ell}\geq|k|\geq 1} \exp(-|k|^2)dk \gtrsim T^2. \tag{5.10}$$

We now address the second RHS term in (5.9). For all $k \in \mathbb{R}^d$ we write

$$\int_{\mathbb{R}^d} (\mathcal{T}_{2,\varepsilon,k',t} - \mathcal{T}_{2,\varepsilon,k,t})\bar{G}_{CT}(k - k')dk' = \int_{|k'-k|\leq \frac{|k|}{4}\vee 2} (\mathcal{T}_{2,\varepsilon,k',t} - \mathcal{T}_{2,\varepsilon,k,t})\bar{G}_{CT}(k - k')dk'$$

$$+ \int_{|k'-k|> \frac{|k|}{4}\vee 2} (\mathcal{T}_{2,\varepsilon,k',t} - \mathcal{T}_{2,\varepsilon,k,t})\bar{G}_{CT}(k - k')dk'.$$

For the first integral term we use a Lipschitz bound on $k' \mapsto \mathcal{T}_{2,\varepsilon,k',t}$, whereas for the second integral term we exploit the exponential decay of the averaging kernel. Indeed,

$$\nabla \mathcal{T}_{2,\varepsilon,k',t} = t^2 \nabla \Lambda_\ell(\varepsilon k') \otimes \nabla \Lambda_\ell(\varepsilon k')\omega_\ell(\varepsilon|k'|)(2\pi)^{d/2} \exp(-\frac{|k'|^2}{2}) \cos(\varepsilon^{-1}\Lambda_\ell(\varepsilon k')t)$$

$$+ \varepsilon t \nabla \Lambda_\ell(\varepsilon k') \otimes \frac{k'}{|k'|}\omega_\ell'(\varepsilon|k'|)(2\pi)^{d/2} \exp(-\frac{|k'|^2}{2}) \sin(\varepsilon^{-1}\Lambda_\ell(\varepsilon k')t)$$

$$+ \varepsilon t \nabla^2 \Lambda_\ell(\varepsilon k')\omega_\ell(\varepsilon|k'|)(2\pi)^{d/2} \exp(-\frac{|k'|^2}{2}) \sin(\varepsilon^{-1}\Lambda_\ell(\varepsilon k')t)$$

$$+ t \nabla \Lambda_\ell(\varepsilon k') \otimes k'\omega_\ell(\varepsilon|k'|)(2\pi)^{d/2} \exp(-\frac{|k'|^2}{2}) \sin(\varepsilon^{-1}\Lambda_\ell(\varepsilon k')t,$$

so that on the set $\mathcal{K}_k := \{k' : |k' - k| \leq \frac{|k|}{4}\vee 2\}$, we have for all $\varepsilon \leq 1$,

$$\sup_{k'\in\mathcal{K}_k} |\nabla \mathcal{T}_{2,\varepsilon,k',t}| \lesssim t^2 |k|^{\ell+1} \exp(-\frac{9|k|^2}{32}) \lesssim t^2 \exp(-\frac{|k|^2}{4}).$$

We thus obtain

$$\int_{|k'-k|\leq\frac{|k|}{4}\vee 2} |\mathcal{T}_{2,\varepsilon,k',t} - \mathcal{T}_{2,\varepsilon,k,t}|\bar{G}_{CT}(k-k')dk'$$

$$\lesssim \quad t^2 \exp(-\frac{|k|^2}{4})\int_{|k'-k|\leq\frac{|k|}{4}\vee 2} |k-k'|\bar{G}_{CT}(k-k')dk'$$

$$\lesssim \quad \frac{t^2}{CT}\exp(-\frac{|k|^2}{4})\int_{\mathbb{R}^d} CT|k-k'|\bar{G}_{CT}(k-k')dk'$$

$$\lesssim \quad \frac{t^2}{CT}\exp(-\frac{|k|^2}{4}), \tag{5.11}$$

where the multiplicative constant does depend on Λ and d, but not on C. We treat now the second integral term, and simply bound $\mathcal{T}_{2,\varepsilon,k',t}$ by t:

$$|\mathcal{T}_{2,\varepsilon,k',t}| \quad = \quad |t\nabla\Lambda_\ell(\varepsilon k')\omega_\ell(\varepsilon|k'|)(2\pi)^{d/2}\exp(-\frac{|k'|^2}{2})\sin(\varepsilon^{-1}\Lambda_\ell(\varepsilon k')t)|$$

$$\lesssim \quad t|k'|^{[\ell/2]}\exp(-\frac{|k'|^2}{2}) \lesssim t.$$

Hence, we obtain by direct integration of $\bar{G}_{CT}$

$$\int_{|k'-k|>\frac{|k|}{4}\vee 2} |\mathcal{T}_{2,\varepsilon,k',t} - \mathcal{T}_{2,\varepsilon,k,t}|\bar{G}_{CT}(k-k')dk' \quad \lesssim \quad t\int_{|k'|>\frac{|k|}{4}\vee 2} \bar{G}_{CT}(k')dk'$$

$$\lesssim \quad t\exp\left(-\frac{1}{4}C^2T^2(\frac{|k|}{4}\vee 2)^2\right). \tag{5.12}$$

Combining (5.11) and (5.12), we thus obtain

$$\int_T^{T+1}\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}(\mathcal{T}_{2,\varepsilon,k',t} - \mathcal{T}_{2,\varepsilon,k,t})\bar{G}_{CT}(k-k')dk'\right)^2 dkdt$$

$$\lesssim \quad \int_{\mathbb{R}^d}\left(\frac{T^2}{C}\exp(-\frac{|k|^2}{4}) + T^2\exp(-\frac{1}{2}C^2T^2(\frac{|k|}{4}\vee 2)^2)\right)dk, \tag{5.13}$$

where the multiplicative constant only depends on Λ and d (and in particular not on C).

We are in the position to conclude this step. For $\varepsilon \ll 1$ and $T \geq 1$, the four estimates (5.7), (5.8), (5.10), and (5.13) combine to

$$\mathcal{M}_\ell(\varepsilon, \varepsilon^{-1}T)^2 \gtrsim T^2(1 - \frac{1}{C} - \exp(-\frac{1}{2}C^2)),$$

which turns into the desired ballistic estimate

$$\mathcal{M}_\ell(\varepsilon, \varepsilon^{-1}T) \gtrsim T \tag{5.14}$$

provided C is chosen large enough and $T \geq 1$.

Step 3. Control of the error.
Since $\sup_{\mathbb{R}^d}(\frac{CT}{2})^d G^2_{(CT)^{-1}} \lesssim 1$, we have

$$\mathcal{M}(\varepsilon, \varepsilon^{-1}T) \gtrsim \left(\int_T^{T+1}\mathbb{E}\left[\int_{\mathbb{R}^d}|x|^2 u_\varepsilon(t,x)^2(\frac{CT}{2})^d G^2_{(CT)^{-1}}(x)dx\right]dt\right)^{\frac{1}{2}},$$

which, by the triangle inequality and the definition of $G_{(CT)^{-1}}$, yields

$$\mathcal{M}(\varepsilon, \varepsilon^{-1}T) \gtrsim \mathcal{M}_\ell(\varepsilon, \varepsilon^{-1}T)$$
$$-\Big(\int_T^{T+1} \mathbb{E}\Big[\int_{\mathbb{R}^d} |x|^2 |u_\varepsilon(t,x) - u_{\varepsilon,\ell}(t,x)|^2 \exp(-\frac{|x|^2}{C^2T^2})dx\Big] dt\Big)^{\frac{1}{2}}. \quad (5.15)$$

It remains to control the second RHS term. Thanks to the exponential weight, we have

$$\int_T^{T+1} \mathbb{E}\Big[\int_{\mathbb{R}^d} |x|^2 |u_\varepsilon(t,x) - u_{\varepsilon,\ell}(t,x)|^2 \exp(-\frac{|x|^2}{C^2T^2})dx\Big] dt$$
$$\lesssim C^2 T^2 \sup_{T \le t \le T+1} \mathbb{E}\Big[\|u_\varepsilon(t) - u_{\varepsilon,\ell}(t)\|^2_{L^2(\mathbb{R}^d)}\Big].$$

By Theorem 1, this turns into

$$\int_T^{T+1} \mathbb{E}\Big[\int_{\mathbb{R}^d} |x|^2 |u_\varepsilon(t,x) - u_{\varepsilon,\ell}(t,x)|^2 \exp(-\frac{|x|^2}{C^2T^2})dx\Big] dt$$
$$\lesssim C^2 T^2 \big(\max\{\varepsilon, \varepsilon^{\ell+1}\mu_\alpha(\varepsilon^{-1})\} + \varepsilon^{\ell+1}T\mu_\alpha(\varepsilon^{-1}T)\big)^2. \quad (5.16)$$

The desired estimate (5.4) then follows from (5.15), (5.16), and (5.14) with T replaced by $\varepsilon^{-1-\gamma}T$ (for which the condition $\varepsilon^{-1-\gamma}T \ge 1$ for (5.14) in Step 2 is automatically satisfied in the asymptotic regime $\varepsilon \ll 1$).

APPENDIX A. ESTIMATES OF THE EXTENDED HIGHER-ORDER CORRECTORS

A.1. **Existence of higher-order correctors.** Recall that $\varphi_0 \equiv 1, \sigma_0 \equiv 0, \chi_0 \equiv 0, \chi_1 \equiv 0$, and that φ_1 and σ_1 are the classical corrector and flux corrector in homogenization which are well-defined for stationary ergodic coefficients (cf. [25]). As it is standard in stochastic homogenization, one might modify the (higher-order) corrector equations by adding a zero-order term of magnitude T^{-1} for some $T \gg 1$. This would yield existence and uniqueness of stationary approximations of the extended correctors $(\varphi_j, \sigma_j, \nabla\chi_j)$ at any order. In fact, interpreting T as a time-scale, one could even let T depend in a nontrivial way on ε in the various estimates, and work with approximate correctors only. The price to pay to work with these well-defined approximate correctors is that the crucial identity $\nabla \cdot \sigma_j = q_j$ would only hold up to some defect (depending on T). Similarly, there would be an additional defect in the eigenvalue/eigenvector relation for the associated approximate Taylor-Bloch wave. Last, we would have approximations of the tensors $\bar{a}_j$ depending on T. To avoid this additional approximation, we directly work with the higher-order correctors without massive approximation. In this case however, higher-order extended correctors are not necessarily well-defined, and the associated existence/uniqueness theory makes heavy use of quantitative homogenization methods.

Let us start with a soft result: the existence and uniqueness of a (non-stationary) (φ_j, σ_j) provided $\varphi_{j-1}, \sigma_{j-1}, \nabla\chi_{j-1}$ are stationary fields with finite second moment.

Lemma A.1. *Let $\varphi_{j-1}, \sigma_{j-1}, \nabla\chi_{j-1}, q_j$ be as in Definition 2.1. Assume that $\varphi_{j-1}, \sigma_{j-1},$ and $\nabla\chi_{j-1}$ are stationary fields with finite second moments, which implies that q_j is also stationary with finite second moment. Then there exist random fields φ_j, σ_j solving*

$$-\nabla \cdot \boldsymbol{a}\nabla\varphi_j = \nabla \cdot (\sigma_{j-1}e + \boldsymbol{a}e\varphi_{j-1} + \nabla\chi_{j-1}),$$
$$-\triangle\sigma_j = \nabla \times q_j, \quad \nabla \cdot \sigma_j = q_j,$$

such that $\nabla\varphi_j, \nabla\sigma_j$ are uniquely defined stationary fields with finite second moments. $\square$

Let us now distinguish between the assumptions on $\boldsymbol{a}$. We start with the periodic and quasi-periodic setting, then turn to the almost periodic setting, and conclude with the random setting.

A.2. **Periodic and quasi-periodic coefficients.** The following result is a direct consequence of the Poincaré inequality on the torus and of spectral theory.

Proposition 4. *Let $\boldsymbol{a}$ be a measurable periodic coefficient field. Then for all $j \geq 1$, there exist unique periodic extended correctors $\varphi_j, \sigma_j, \chi_j \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ with zero average. If in addition $\boldsymbol{a}$ is symmetric, then for all $j \geq 1$ and unit direction e, $\lambda_{2j+1} = 0$.* $\square$

Proof of Proposition 4. Once the structure of the correctors is clear (only the definition of the flux correctors σ_j is delicate — cf. the discussion on closed forms after Definition 2.1), the only subtle result is that $\lambda_{2j+1} = 0$ for all $j \geq 0$. We give here the classical proof of this fact (which we already proved by a direct approach in Proposition 1). Let $k = \kappa e$, where e is a fixed unit direction and $\kappa \in \mathbb{R}_+$. Since $-(\nabla + i\kappa e) \cdot \boldsymbol{a}(\nabla + i\kappa e)$ has compact resolvent on the torus, we can consider the first eigenvalue $\lambda_1(\kappa)$. As a function of κ, λ_1 is real analytic on a neighborhood of the origin (so that its derivatives are given by the extended correctors), cf. [16]. A direct computation shows that $\lambda_1^{(j)}(0) = i^{j+1}\lambda_{j+1}$ (the j-th derivative of λ_1 at zero is given by $i^{j+1}\lambda_{j+1}$ from Definition 2.1). Since $\lambda_j \in \mathbb{R}$ and $\lambda_1^{(j)}(0) \in \mathbb{R}$, this implies $\lambda_{2j+1} = 0$, as claimed. $\square$

Similar results as in Proposition 4 hold in the case of smooth quasi-periodic coefficient fields first considered by Kozlov [32]. The arguments of [23, Theorem 4], based on an diophantine condition in the form of a weak Poincaré inequality, on Garding's inequality, and elliptic regularity, indeed allow to prove the following.

Proposition 5. *Let $\tilde{\boldsymbol{a}}$ be a smooth coefficient field on a higher-dimensional torus $\mathbb{T}_m$, $m > d$, let M be a winding $m \times d$-matrix, and set $\boldsymbol{a} : \mathbb{R}^d \to \mathcal{M}_d(\mathbb{R}), x \mapsto \boldsymbol{a}(x) := \tilde{\boldsymbol{a}}(Mx)$. If M satisfies a diophantine condition, then for all $j \geq 1$, there exist unique smooth quasi-periodic extended correctors $\varphi_j, \sigma_j, \chi_j \in H^1_{\mathrm{loc}}(\mathbb{R}^d)$ with zero average.* $\square$

We then turn the more general case of almost-periodic coefficient fields $\boldsymbol{a}$.

A.3. **Almost-periodic coefficients.** We first recall the quantitative measure of almost-periodicity introduced in [9]. Given $f : \mathbb{R}^d \to \mathbb{R}^k$ and $y, z \in \mathbb{R}^d$, we define

$$T_z f(x) := f(x + z)$$

and the difference operator

$$\Delta_{yz} f(x) := \frac{1}{2}\left(T_y f(x) - T_z f(x)\right) = \frac{1}{2}\left(f(x + y) - f(x + z)\right). \tag{A.1}$$

Let $\mathcal{T}_k = ((y_1, z_1), \ldots, (y_k, z_k)) \in (\mathbb{R}^d \times \mathbb{R}^d)^k$ be a k-tuple formed by couples $(y_j, z_j) \in \mathbb{R}^d \times \mathbb{R}^d$. For a function $f : \mathbb{R}^d \to \mathbb{R}^{m \times n}$, $m, n \in \mathbb{N}$, we define a difference operator $\Delta_{\mathcal{T}_k}$ acting on f by

$$\Delta_{\mathcal{T}_k} f(x) = \Delta_{y_k z_k} \cdots \Delta_{y_1 z_1} f(x). \tag{A.2}$$

Let $\mathcal{P}_{j,k}$, $j \in \{0, 1, \ldots, k\}$, stand for a set of increasing ordered subsets of $\{1, \ldots, k\}$ with j members. In other words, for $j > 0$ we define

$$\mathcal{P}_{j,k} := \left\{ \zeta \in \{1, \ldots, k\}^j \ : \ \zeta_i < \zeta_{i+1} \ \forall i \in \{1, \ldots, j-1\} \right\}$$

and, for $j = 0$, we set $\mathcal{P}_{0,k} = \emptyset$. By abuse of notation, we also think of $\zeta \in \mathcal{P}_{j,k}$ as being ordered subsets of $\{1, \ldots, k\}$. Then, for $\zeta \in \mathcal{P}_{j,k}$, we denote by ζ^c the unique member of $\mathcal{P}_{k-j,k}$ such that $\{1, \ldots, k\} = \zeta \cup \zeta^c$. By $|\zeta|$ we denote the number of elements in $\zeta \in P_{j,k}$, i.e., $|\zeta| = j$. For $\mathcal{T}_k$ as above and for $\zeta \in \mathcal{P}_{j,k}$ we denote by the j-tuple $\zeta(\mathcal{T}_k)$ the set $\left((y_{\zeta_1}, z_{\zeta_1}), \ldots, (y_{\zeta_j}, z_{\zeta_j}) \right)$ for $j > 0$, and if $\zeta \in \mathcal{P}_{0,k}$, we set $\zeta(\mathcal{T}_k) = \emptyset$ and $\Delta_{\zeta(\mathcal{T}_k)} f = 1$. Furthermore, we let $\mathcal{P}_k$ stand for the family of subsets $(\zeta^1, \ldots, \zeta^k) \in \mathcal{P}_{j_1,k} \times \cdots \times \mathcal{P}_{j_k,k}$ with $\sum_{i=1}^{k} j_i = k$.

We are now in position to recall the quantitative measure of almost periodicity introduced in [9]. For a given $f \in C(\mathbb{R}^d; \mathbb{R}^{m \times n})$, $m, n \in \mathbb{N}$, and $\mathcal{T}_k = \{(y_1, z_1), \ldots, (y_k, z_k)\}$ we define

$$G_k(f, \mathcal{T}_k) := \max_{(\zeta^1, \ldots, \zeta^k) \in \mathcal{P}_k} \left\{ \prod_{j=1}^{k} \left\| \Delta_{\zeta^j(\mathcal{T}_k)} f \right\|_{L^\infty(\mathbb{R}^d; \mathbb{R}^{m \times n})} \right\}, \tag{A.3}$$

that is, the maximum is taken over all (increasing, ordered) partitions of $\mathcal{P}_k$. Then, we define a quantity ρ_k, for each $k \in \mathbb{N}$ and $R \geq 1$, by

$$\rho_k(f, R) := \sup_{y_1 \in \mathbb{R}^d} \inf_{z_1 \in B_R} \cdots \sup_{y_k \in \mathbb{R}^d} \inf_{z_k \in B_R} G_k\left(f, ((y_1, z_1), \ldots, (y_k, z_k)) \right), \tag{A.4}$$

which are the building blocks for the quantitative measure of almost periodicity of [9]:

$$\rho_*(\boldsymbol{a}, R) := \inf_{k \in \mathbb{N} \cap [1, R]} C^k k! \rho_k\left(\boldsymbol{a}, k^{-1} R \right), \tag{A.5}$$

where the constant C in (A.5) only depends on d, Λ. The main quantitative ergodicity assumption that we make on the coefficients is therefore that there exists an exponent $\delta > 0$ and a constant $K \geq 1$ such that, for every $R \geq 1$,

$$\rho_*(\boldsymbol{a}, R) \leq K R^{-\delta}. \tag{A.6}$$

We introduce the following notation: for all $\delta > 0$, integer $j \geq 1$, and $t \geq 0$, we set

$$\nu_{\delta,j}(t) := \begin{cases} 1 & \text{for} \quad j - \delta < 0, \\ \log(2 + t)^{\frac{1}{2}} & \text{for} \quad j - \delta = 0, \\ t^{j-\delta} & \text{for} \quad 0 < j - \delta < 1. \end{cases}$$

Although it is not straightforward, under assumption (A.6), methods similar to [9] (see also [10], with some care for the borderline case $j = \delta$) essentially allow to prove the following control of the extended correctors:

Proposition 6. *Let $\boldsymbol{a}$ be an almost-periodic coefficient field satisfying* (A.6) *for some $K \geq 1$ and $\delta > 0$. Then, for all $1 \leq j < 1 + \delta$, $(\varphi_j, \sigma_j, \nabla \chi_j)$ are well-defined and satisfy for all $x \in \mathbb{R}^d$*

$$|\varphi_j(x)| + |\sigma_j(x)| + |\nabla \chi_j(x)| \lesssim \nu_{\delta,j}(|x|).$$

$\square$

We finally consider random coefficient fields with decaying correlations.

A.4. **Random coefficients.** In this last subsection, we address the representative example of a family of Gaussian coefficient fields. More precisely, we consider Gaussian ensembles of scalar fields $a(x)$. In order to get an example of an ensemble of uniformly elliptic coefficient fields $\boldsymbol{a}$, one applies a pointwise nonlinear Lipschitz transform to possibly several copies of the above. Let $\mathbb{P}'$ (with expectation $\mathbb{E}'$) stand for the distribution of a scalar Gaussian field $a(x)$ that is stationary and centered, and thus characterized by its covariance

$$c(x) := \mathbb{E}'[a(x)a(0)].$$

We assume that the covariance is radial and decays mildly in the sense that there exists $\beta > 0$ such that

$$|c(x)| \lesssim \gamma_\beta(x) := (1 + |x|)^{-\beta}. \tag{A.7}$$

With a slight abuse of notation, we shall say that the law $\mathbb{P}$ (with expectation $\mathbb{E}$) of $\boldsymbol{a}$ is Gaussian with parameter $\beta > 0$. Under this assumption, we have the validity of a weighted logarithmic-Sobolev inequality (cf. [19]), which is key to establish moment bounds on correctors in [25, 24]. We define $\gamma_{\beta,*}(t) := \fint_{|x|<t} \gamma_\beta(|x|)dx$. The latter satisfies

$$\gamma_{\beta,*}(t) \sim \begin{cases} t^{-d} & \text{for} \quad \beta > d, \\ t^{-d}\log(2+t) & \text{for} \quad \beta = d, \\ t^{-\beta} & \text{for} \quad \beta < d. \end{cases}$$

We further introduce the following notation: for all $\beta > 0$, $d \geq 1$, integer $j \geq 1$, and $t \geq 0$, we set

$$\mu_{d,\beta,j}(t) := \left(1 + \int_0^t \gamma_{\beta,*}(\tau)(1+\tau)^{2j-1}d\tau\right)^{\frac{1}{2}}.$$

Although it is not straightforward, proceeding as in [24] (or using a semi-group approach as in [26, 27]), one can prove the following sharp estimates (see also [12] for the case $j = 2$):

Proposition 7. *Let $\boldsymbol{a}$ be a Gaussian coefficient field satisfying* (A.7) *for some $\beta > 0$. Then, for all $1 \leq j < 1 + \frac{\beta \wedge d}{2}$, $(\varphi_j, \sigma_j, \nabla\chi_j)$ are well-defined and satisfy for all $x \in \mathbb{R}^d$*

$$\mathbb{E}\left[|\varphi_j(x)|^2\right]^{\frac{1}{2}} + \mathbb{E}\left[|\sigma_j(x)|^2\right]^{\frac{1}{2}} + \mathbb{E}\left[|\nabla\chi_j(x)|^2\right]^{\frac{1}{2}} \lesssim \mu_{d,\beta,j}(|x|).$$

$\square$

Remark A.1. More precisely, we have for all $1 \leq j < \frac{\beta \wedge d}{2}$

$$\mathbb{E}\left[|\varphi_j(x)|^2\right]^{\frac{1}{2}} + \mathbb{E}\left[|\sigma_j(x)|^2\right]^{\frac{1}{2}} + \mathbb{E}\left[|\nabla\chi_j(x)|^2\right]^{\frac{1}{2}} \lesssim 1,$$

whereas for $\frac{\beta \wedge d}{2} \leq j < 1 + \frac{\beta \wedge d}{2}$,

$$\mathbb{E}\left[|\varphi_j(x)|^2\right]^{\frac{1}{2}} + \mathbb{E}\left[|\sigma_j(x)|^2\right]^{\frac{1}{2}} + \mathbb{E}\left[|\nabla\chi_j(x)|^2\right]^{\frac{1}{2}} \lesssim \begin{cases} \log(2+|x|)^{\frac{1}{2}} & : \quad j \in \{\frac{d}{2}, \frac{\beta}{2}\}, \beta \neq d, \\ \log(2+|x|) & : \quad j = \frac{d}{2} = \frac{\beta}{2}, \\ (1+|x|)^{j-\frac{\beta \wedge d}{2}} & : \quad j > \frac{\beta \wedge d}{2}. \end{cases}$$

$\square$

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