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CHARACTERIZATIONS OF IDEMPOTENT DISCRETE UNINORMS

MIGUEL COUCEIRO, JIMMY DEVILLET, AND JEAN-LUC MARICHAL

Abstract. In this paper we provide an axiomatic characterization of the idempotent discrete uninorms by means of three conditions only: conservativeness, symmetry, and nondecreasing monotonicity. We also provide an alternative characterization involving the bisymmetry property. Finally, we provide a graphical characterization of these operations in terms of their contour plots, and we mention a few open questions for further research.

1. Introduction

Aggregation functions defined on linguistic scales (i.e., finite chains) have been intensively investigated for about two decades; see, e.g., [3–6, 8–13, 15, 17]. Among these functions, discrete fuzzy connectives (such as discrete uninorms) are binary operations that play an important role in fuzzy logic.

This short paper focuses on characterizations of the class of idempotent discrete uninorms. Recall that a discrete uninorm is a binary operation on a finite chain that is associative, symmetric, nondecreasing (in each variable), and has a neutral element.

A first characterization of the class of idempotent discrete uninorms was given by De Baets et al. [3]. This characterization reveals that any idempotent discrete uninorm is a combination of the minimum and maximum operations. In particular, such an operation is *conservative* in the sense that it always outputs one of the input values.

The outline of this paper is as follows. After presenting some preliminary results on conservative operations in Section 2, we show in Section 3 that the idempotent discrete uninorms are exactly those operations that are conservative, symmetric, and nondecreasing (Theorem 12). This new axiomatic characterization is rather surprising since it requires neither associativity nor the existence of a neutral element. We also present a graphical characterization of these operations in terms of their contour plots (Theorem 15). This graphical characterization shows us a very easy way to generate all the possible idempotent discrete uninorms on a given finite chain. In Section 4 we provide an alternative axiomatic characterization of this class in terms of the bisymmetry property. Specifically, we show that the idempotent discrete uninorms are exactly those operations that are idempotent, bisymmetric, nondecreasing, and have neutral elements. More generally, we also show that the whole class of discrete uninorms can also be axiomatized by simply suppressing

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idempotency in the latter characterization and that this result also holds on arbitrary chains (Theorem 21). Finally, Section 5 is devoted to some concluding remarks and open questions.

2. Preliminaries

In this section we present some basic definitions and preliminary results.

Let X be an arbitrary nonempty set and let $\Delta_X = \{(x,x) \mid x \in X\}$.

Definition 1. An operation $F\colon X^2 \to X$ is said to be

- *idempotent* if $F(x,x) = x$ for all $x \in X$.
- *conservative* (or *selective*) if $F(x,y) \in \{x,y\}$ for all $x,y \in X$.
- *associative* if $F(F(x,y),z) = F(x,F(y,z))$ for all $x,y,z \in X$.

Remark 1. Conservative operations were introduced in [14]. By definition, the output value of such an operation must always be one of the input values. In particular, any conservative operation $F\colon X^2 \to X$ is idempotent. Moreover, such an operation can be "discreticized" in the sense that, for any nonempty discrete subset S of X, its restriction to S^2 ranges in S. More precisely, it can be shown [1] that the following conditions are equivalent:

(i) F is conservative.
(ii) For any $\varnothing \neq S \subseteq X$, we have $F(S^2) \subseteq S$.
(iii) For any $\varnothing \neq S \subseteq X$ and any $x,y \in S$, if $F(x,y) \in S$ then $x \in S$ or $y \in S$.

Definition 2. Let $F\colon X^2 \to X$ be an operation.

- An element $e \in X$ is said to be a *neutral element* of F (or simply a *neutral element*) if $F(x,e) = F(e,x) = x$ for all $x \in X$. In this case we easily show by contradiction that such a neutral element is unique.
- The points (x,y) and (u,v) of X^2 are said to be *connected for* F (or simply *connected*) if $F(x,y) = F(u,v)$. We observe that "being connected" is an equivalence relation. The point (x,y) of X^2 is said to be *isolated for* F (or simply *isolated*) if it is not connected to another point in X^2.

Proposition 3. *Let $F\colon X^2 \to X$ be an idempotent operation. If the point $(x,y) \in X^2$ is isolated, then it lies on Δ_X, that is, $x = y$.*

Proof. Let (x,y) be isolated. From the identity $F(x,y) = F(F(x,y),F(x,y))$ we immediately derive $(x,y) = (F(x,y),F(x,y))$. □

Remark 2. We observe that idempotency is necessary in Proposition 3. Indeed, consider the operation $F\colon X^2 \to X$, where $X = \{a,b\}$, defined as $F(x,y) = a$, if $(x,y) = (a,b)$, and $F(x,y) = b$, otherwise. Then (a,b) is isolated and $a \neq b$. The contour plot of F is represented in Figure 1. Here and throughout, connected points are joined by edges. To keep the figures simple we sometimes omit the edges obtained by transitivity.

Some conservative operations have neutral elements (e.g., $F(x,y) = \max\{x,y\}$ on $X = \{a,b,c\}$ has the neutral element $e = a$) and some have not (e.g., $F(x,y) = x$). The following lemma provides an easy graphical test for checking whether a conservative operation has a neutral element.

Proposition 4. *Let $F\colon X^2 \to X$ be a conservative operation and let $e \in X$. Then e is a neutral element if and only if (e,e) is isolated.*

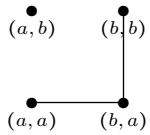

FIGURE 1. A non-idempotent operation

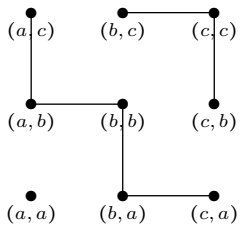

FIGURE 2. An operation with no neutral element

Proof. (Necessity) If (e,e) is not isolated, then there exists $(x,y) \neq (e,e)$ such that $e = F(e,e) = F(x,y) \in \{x,y\}$ and hence $x = e$ or $y = e$. If $x = e$, then $y \neq e$ and $e = F(e,y) = y$, a contradiction. We arrive at a similar contradiction when $y = e$.

(Sufficiency) If e is not a neutral element, then there exists $u \in X \setminus \{e\}$ such that $F(u,e) = e = F(e,e)$ or $F(e,u) = e = F(e,e)$. In both cases, (e,e) is not isolated, a contradiction. □

Corollary 5. *Any isolated point (x,y) of a conservative operation $F\colon X^2 \to X$ is unique and lies on Δ_X. Moreover, $x = y$ is a neutral element.*

Remark 3. Proposition 4 no longer holds if conservativeness is relaxed into idempotency. Indeed, by simply taking $X = \{a,b,c\}$ we can easily construct an idempotent operation with an isolated point on Δ_X and no neutral element (see Figure 2). Also, it is easy to construct an idempotent operation with a neutral element and no isolated point (see Figure 3). It is also noteworthy that there are idempotent operations with more than one isolated point (see Figure 4).

Proposition 6. *An operation $F\colon X^2 \to X$ is conservative if and only if it is idempotent and every point $(x,y) \in X^2 \setminus \Delta_X$ is connected to either (x,x) or (y,y).*

Proof. Clearly, F is conservative if and only if it is idempotent and for every distinct $x,y \in X$ we have either $F(x,y) = x = F(x,x)$ or $F(x,y) = y = F(y,y)$. □

Remark 4. Proposition 6 provides an easy graphical test for checking whether an idempotent operation is conservative. For instance, none of the idempotent operations represented in Figures 2–4 is conservative because in each case the point (a,c) is connected to neither (a,a) nor (c,c).

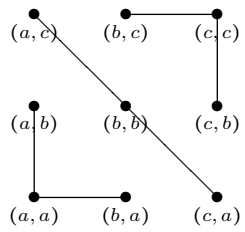

FIGURE 3. An operation with no isolated point

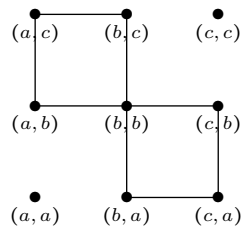

FIGURE 4. An operation with two isolated points

Proposition 7. *An operation $F\colon X^2 \to X$ has a neutral element if and only if there are a vertical section and a horizonal section of X^2 that intersect on Δ_X and such that the restriction of F to each of these sections is the identity function.*

Proof. The result immediately follows from the definition of a neutral element. □

Remark 5. Proposition 7 provides a graphical test for checking the existence of a neutral element. For instance, we can easily see that the operation represented in Figure 3 has b as the neutral element. Note that when the operation is conservative, by Proposition 4 it suffices to search for an isolated point on Δ_X.

3. Main results

We now focus on characterizations of the class of idempotent discrete uninorms. These operations are defined on finite chains. Without loss of generality we will only consider the n-element chains $L_n = \{1, \ldots, n\}$, $n \geq 1$, endowed with the usual ordering relation $\leq$.

Recall that an operation $F\colon L_n^2 \to L_n$ is said to be *nondecreasing in each variable* (or simply *nondecreasing*) if $F(x, y) \leq F(x', y')$ whenever $x \leq x'$ and $y \leq y'$.

Definition 8 (see, e.g., [3])**.** A *discrete uninorm* on L_n is an operation $U\colon L_n^2 \to L_n$ that is associative, symmetric, nondecreasing, and has a neutral element.

A first characterization of the class of idempotent discrete uninorms was established by De Baets et al. [3]. We state this result in the following theorem. Although this characterization is somewhat intricate, it shows, together with Lemma 10 below, that any idempotent discrete uninorm is conservative.

Theorem 9 (see [3, Theorem 3])**.** *An operation $F\colon L_n^2 \to L_n$ with a neutral element $1 < e < n$ is an idempotent discrete uninorm if and only if there exists a nonincreasing map $g\colon [1, e] \to [e, n]$ (nonincreasing means that $g(x) \geq g(y)$ whenever $x \leq y$), with $g(e) = e$, such that*

$$F(x, y) = \begin{cases} \min\{x, y\}, & \text{if } y \leq \overline{g}(x) \text{ and } x \leq \overline{g}(1), \\ \max\{x, y\}, & \text{otherwise,} \end{cases}$$

where $\overline{g}\colon L_n \to L_n$ *is defined by*

$$\overline{g}(x) \;=\; \begin{cases} g(x), & \text{if } x \le e, \\ \max\{z \in [1,e] \mid g(z) \ge x\}, & \text{if } e \le x \le g(1), \\ 1, & \text{if } x > g(1). \end{cases}$$

Remark 6. The fact that any idempotent discrete uninorm is conservative can also be easily proved by following the first few steps of the proof of [2, Theorem 3].

We now show that the idempotent discrete uninorms are exactly those operations that are conservative, symmetric, and nondecreasing (see Theorem 12).

First consider the following lemma, which actually holds on arbitrary, not necessarily finite, chains (i.e., totally ordered sets).

Lemma 10. *If* $F\colon L_n^2 \to L_n$ *is idempotent, nondecreasing, and has a neutral element* $e \in L_n$*, then* $F|_{[1,e]^2} = \min$ *and* $F|_{[e,n]^2} = \max$.

Proof. For any $x, y \in [1,e]$ such that $x \le y$, we have $x = F(x,x) \le F(x,y) \le F(x,e) = x$ and $x = F(x,x) \le F(y,x) \le F(e,x) = x$. This shows that $F|_{[1,e]^2} = \min$. The other identity can be proved similarly. □

Proposition 11. *If* $F\colon L_n^2 \to L_n$ *is conservative, symmetric, and nondecreasing, then it is associative and it has a neutral element.*

Proof. Let us first prove that F has a neutral element. We proceed by induction on the size n of the chain. There is nothing to prove if $n = 1$. We can easily see by inspection that there are only two possible operations if $n = 2$ and four possible operations if $n = 3$. The contour plot of these operations are given in Figures 5 and 6, respectively. Now suppose that the result holds for any $(n-1)$-element chain and consider an operation $F\colon L_n^2 \to L_n$ that is conservative, symmetric, and nondecreasing. By conservativeness and symmetry we then have $F(1,n) = F(n,1) \in \{1,n\}$. We may suppose that $F(1,n) = F(n,1) = 1$; the other case can be dealt with dually. By nondecreasing monotonicity, we also have $F(1,x) = F(x,1) = 1$ for all $x \in L_n$. Consider the subchain $L' = L_n \setminus \{1\}$. Clearly, the operation $F' = F|_{L'}$ is conservative, symmetric, and nondecreasing. By the induction hypothesis, F' has a neutral element $e \in L'$. Let us show that e is also a neutral element of F. Suppose that this is not true. Then, by Proposition 4 the point (e,e) is isolated for F' but not for F. This means that there exists $x \in L_n$ such that $1 = F(1,x) = F(x,1) = F(e,e) = e$, which contradicts the fact that $e \in L'$.

Now, let $F\colon L_n^2 \to L_n$ be an operation that is conservative, symmetric, and nondecreasing. We just showed that F must have a neutral element $e \in L_n$. To see that F is associative, let $x, y, z \in L_n$ be arbitrary and let us show that the identity $F(F(x,y),z) = F(x,F(y,z))$ holds. Assume that $x \le y \le z$ (the other five permutations can be treated similarly). We have three cases to examine:

- Suppose $x \le y \le z \le e$ or $e \le x \le y \le z$. Then the result immediately follows from Lemma 10.
- Suppose $x \le y \le e \le z$. By Lemma 10, we have $F(x,y) = \min\{x,y\} = x$.
  - If $F(x,z) = x$, then $F(F(x,y),z) = F(x,z) = x$ and by conservativeness we also have

$$F(x,F(y,z)) \;\in\; \{F(x,y), F(x,z)\} \;=\; \{x\}.$$

FIGURE 5. Possible operations when $n = 2$

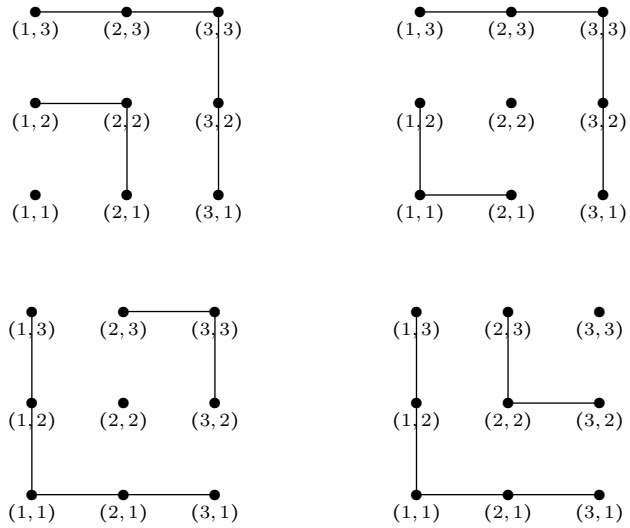

FIGURE 6. Possible operations when $n = 3$

  - If $F(x, z) = z$, then by conservativeness and nondecreasing monotonicity we have $F(y, z) = z$ and hence $F(F(x, y), z) = F(x, z) = F(x, F(y, z))$.
- Suppose that $x \leq e \leq y \leq z$. By Lemma 10, we have $F(y, z) = \max\{y, z\} = z$.
  - If $F(x, y) = x$, then we have $F(F(x, y), z) = F(x, z) = F(x, F(y, z))$.
  - If $F(x, y) = y$, then by conservativeness and nondecreasing monotonicity we have $F(x, z) = z$ and hence $F(F(x, y), z) = F(y, z) = z = F(x, z) = F(x, F(y, z))$.

This completes the proof of the proposition. □

Remark 7. (a) The existence of a neutral element in Proposition 11 is no longer guaranteed if the chain is not finite. For instance, the real operation $F\colon [0, 1]^2 \to [0, 1]$ defined by $F(x, y) = \min\{x, y\}$, if $x, y \in [0, \frac{1}{2})^2$, and $F(x, y) = \max\{x, y\}$, otherwise, is conservative, symmetric, and nondecreasing, but it does not have a neutral element.

(b) Associativity of F in Proposition 11 can be established on any chain. This result was proved in the special case where the chain is the real unit interval

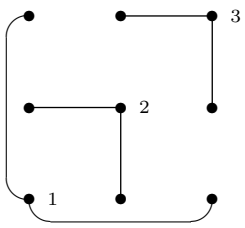

FIGURE 7. An operation that is not nondecreasing

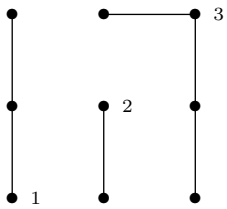

FIGURE 8. An operation that is not symmetric

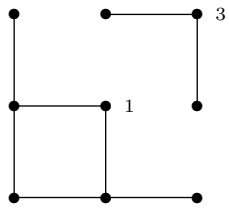

FIGURE 9. An operation that is not conservative

$[0,1]$ in [7, Proposition 2] as a consequence of a sequence of three lemmas. Here we have provided a simpler proof based on the existence of a neutral element.

(c) We observe that conservativeness cannot be relaxed into idempotency in Proposition 11. For instance the operation $F\colon L_3^2 \to L_3$ whose contour plot is depicted in Figure 2 is idempotent, symmetric, and nondecreasing, but one can show that it is not associative and it has no neutral element.

(d) We also observe that each of the conditions of Proposition 11 is necessary. Indeed, we give in Figure 7 an operation that is conservative and symmetric but that is not nondecreasing. We also give in Figure 8 an operation that is conservative and nondecreasing but not symmetric. Finally, we give in Figure 9 an operation that is symmetric and nondecreasing but not conservative. None of these three operations is associative and none has a neutral element.

Theorem 12. *An operation $F\colon L_n^2 \to L_n$ is conservative, symmetric, and nondecreasing if and only if it is an idempotent discrete uninorm.*

Proof. (Necessity) The result immediately follows from Proposition 11.

(Sufficiency) By definition, any idempotent discrete uninorm is symmetric and nondecreasing. It is also conservative. Indeed, this follows from Theorem 9 if $1 < e < n$ and from Lemma 10 if $e = 1$ or $e = n$ (for an alternative proof see Remark 6). □

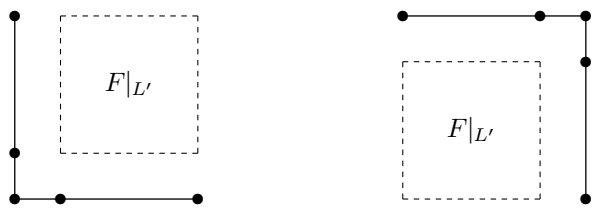

FIGURE 10. Illustration of the proof of Theorem 14

Corollary 13. *Let $F\colon L_n^2 \to L_n$ be an operation that is associative, symmetric, nondecreasing, and has a neutral element. Then F is idempotent if and only if it is conservative.*

Proof. (Necessity) By definition, F is an idempotent discrete uninorm, and hence it is conservative.

(Sufficiency) Trivial. □

The following result gives the exact number of idempotent discrete uninorms on L_n. A proof that essentially relies on Theorem 9 can be found in [3, Theorem 4]. Here we provide an alternative proof based on Theorem 12.

Theorem 14. *There are exactly 2^{n-1} idempotent discrete uninorms on L_n.*

Proof. By Theorem 12 it is enough to count the number c_n of operations on L_n that are conservative, symmetric, and nondecreasing. As already observed in the proof of Proposition 11 we can see by inspection that $c_1 = 1$, $c_2 = 2$, and $c_3 = 4$ (see Figures 5 and 6). Suppose now that $c_{n-1} = 2^{n-2}$ for some integer $n \geq 3$ and let us prove that $c_n = 2^{n-1}$. Let $F\colon L_n^2 \to L_n$ be an arbitrary idempotent discrete uninorm. By conservativeness and symmetry we then have $F(1,n) = F(n,1) \in \{1,n\}$. Suppose first that $F(1,n) = F(n,1) = 1$. By nondecreasing monotonicity, we also have $F(1,x) = F(x,1) = 1$ for all $x \in L_n$. Consider the subchain $L' = L_n \setminus \{1\}$. Clearly, the operation $F' = F|_{L'}$ is conservative, symmetric, and nondecreasing and there are $c_{n-1} = 2^{n-2}$ possible such operations (see Figure 10, on left). We arrive at the same conclusion if $F(1,n) = F(n,1) = n$ (see Figure 10, on right). In total, we then have $c_n = 2^{n-2} + 2^{n-2} = 2^{n-1}$. □

Theorems 12 and 14 together enable us to provide the following graphical characterization of the idempotent discrete uninorms in terms of their contour plots.

Theorem 15 (Graphical characterization)**.** *All the idempotent discrete uninorms $F\colon L_n^2 \to L_n$ can be constructed recursively in terms of their contour plots by the following algorithm:*

Step 1. *Choose the neutral element $e \in L_n$ and set $C_1 = \{e\}$. The point (e,e) is necessarily isolated with value e*

Step 2. *For $k = 1, \ldots, n-1$*
   1. *Pick a closest element a_k to C_k in $L_n \setminus C_k$*
   2. *Set $C_{k+1} = \{a_k\} \cup C_k$*
   3. *Connect all the points in $C_{k+1}^2 \setminus C_k^2$ with common value a_k*

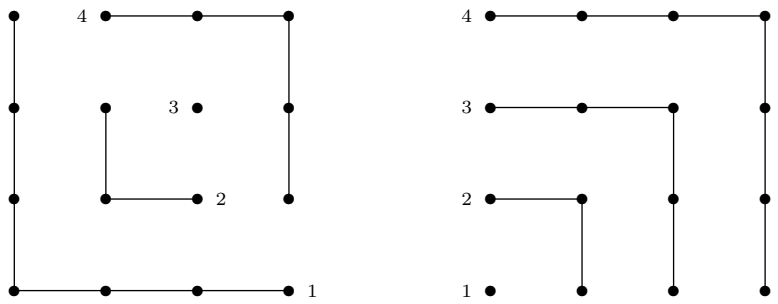

FIGURE 11. Two possible idempotent discrete uninorms

Proof. For every $e \in L_n$ chosen in Step 1, denote by $c_n(e)$ the number of possible operations constructed in Step 2. We show by induction on n that $c_n(e) = \binom{n-1}{e-1}$. We clearly have $c_1(1) = 1$. It is also easy to see that $c_2(1) = c_2(2) = 1$ (see Figure 5). Suppose now that $c_{n-1}(e) = \binom{n-2}{e-1}$ for some integer $n \geq 3$ and let us compute $c_n(e)$ for any $e \in L_n$. We clearly have $c_n(1) = c_n(n) = 1$ so we can assume that $1 < e < n$. It is then easy to see that if $a_1 = e-1$ (resp. $a_1 = e+1$), then the number of possible operations constructed in Step 2 is $c_{n-1}(e-1) = \binom{n-2}{e-2}$ (resp. $c_{n-1}(e) = \binom{n-2}{e-1}$). In total we obtain $c_n(e) = \binom{n-2}{e-2} + \binom{n-2}{e-1} = \binom{n-1}{e-1}$.

Let us now show that the algorithm enables us to generate all the idempotent discrete uninorms on L_n. On the one hand, we clearly see that the algorithm enables us to construct $\sum_{e=1}^{n} c_n(e) = 2^{n-1}$ possible operations. On the other hand, all these operations are clearly conservative, symmetric, and nondecreasing. We then conclude the proof by Theorems 12 and 14. □

Figure 11 gives two possible idempotent discrete uninorms on L_4. All the possible idempotent discrete uninorms on L_2 and L_3 are given in Figures 5 and 6, respectively.

We end this section by giving a graphical test for checking whether a conservative operation is associative.

Proposition 16. *Let $F\colon X^2 \to X$ be a conservative operation. Then the following assertions are equivalent.*

(i) *F is not associative.*
(ii) *There exist $a, b, c \in X$ pairwise distinct such that $F(a,b)$, $F(a,c)$, $F(b,c)$ are pairwise distinct.*
(iii) *There exists a rectangle such that one of its vertices is on Δ_X and the three remaining vertices are in $X^2 \setminus \Delta_X$ and pairwise disconnected.*

Proof. The equivalence between (i) and (ii) was shown in [7, Lemma 1 and Corollary 1]. The equivalence between (ii) and (iii) is immediate. Just consider the rectangle constructed on the vertices (a,c), (b,c), (b,b), (a,b) for pairwise distinct $a, b, c \in X$. □

Example 17. As an application of Proposition 16, let us consider the operation $F\colon L_3^2 \to L_3$ defined in Figure 7. We can see that this operation is not associative

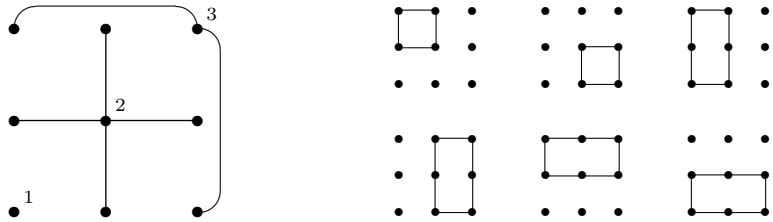

FIGURE 12. An associative operation (left) and the six rectangles to be checked (right)

because the rectangle constructed on the vertices $(2,2)$, $(3,2)$, $(3,1)$, $(2,1)$ has a vertex on Δ_X and the three other vertices are in $X^2 \smallsetminus \Delta_X$ and pairwise disconnected. To give a second example, Figure 12 (left) represents an operation that is conservative, associative, and not nondecreasing. Associativity can be verified by considering the six rectangles shown in Figure 12 (right).

Proposition 18. *If $X = L_n$, then there are exactly $n(n-1)(n-2)$ rectangles satisfying the following property: one of the vertices is on Δ_X and the three remaining vertices are in $X^2 \smallsetminus \Delta_X$.*

Proof. As observed in the proof of Proposition 16, each of these rectangles is constructed on the vertices (a,c), (b,c), (b,b), (a,b) for some pairwise distinct $a,b,c \in L_n$. We then immediately conclude the proof by observing that there are exactly $n(n-1)(n-2)$ triplets $(a,b,c) \in L_n^3$ such that a,b,c are pairwise distinct. □

4. Bisymmetric operations

In this section we provide a characterization of the class of discrete uninorms in terms of the bisymmetry (or mediality) property. From this result we immediately derive a new characterization of the class of idempotent discrete uninorms.

Definition 19. An operation $F\colon X^2 \to X$ is said to be *bisymmetric* if

$$F(F(x,y),F(u,v)) \;=\; F(F(x,u),F(y,v))$$

for all $x,y,u,v \in X$.

Lemma 20. *Let $F\colon X^2 \to X$ be an operation. Then the following assertions hold:*

(a) *If F is bisymmetric and has a neutral element, then it is associative and symmetric.*
(b) *If F is associative and symmetric, then it is bisymmetric.*
(c) *If F is bisymmetric and conservative, then it is associative.*

Proof. (a) The associativity and symmetry were proved for finite chains in [9, Lemma 3.3] and [18, Lemma 3], respectively. These proofs are purely algebraic and work for any nonempty set X.

(b) This result was proved for the real unit interval $[0,1]$ in [16, p. 180]. The same proof works for any nonempty set X.

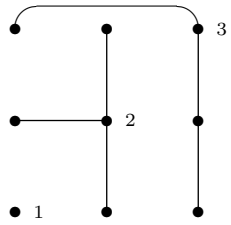

FIGURE 13. An associative operation that is not bisymmetric

(c) Let $x, y, z \in X$. By conservativeness we necessarily have $F(x,z) \in \{x,z\}$. If $F(x,z) = x$, then

$$F(F(x,y),z) \;=\; F(F(x,y),F(z,z)) \;=\; F(F(x,z),F(y,z)) \;=\; F(x,F(y,z)).$$

If $F(x,z) = z$, we have

$$F(F(x,y),z) \;=\; F(F(x,y),F(x,z)) \;=\; F(F(x,x),F(y,z)) \;=\; F(x,F(y,z)).$$

This shows that F is associative. □

Remark 8. (a) Using Lemma 20 we immediately see that if an operation is conservative and symmetric, then it is associative if and only if it is bisymmetric. Combining this observation with Proposition 16 provides a test for bisymmetry under conservativeness and symmetry.

(b) We observe that the conjunction of bisymmetry and symmetry implies neither associativity nor the existence of a neutral element (take for instance the arithmetic mean over the reals).

(c) Also, the conjunction of conservativeness and associativity does not imply bisymmetry, even in the presence of a neutral element. We give in Figure 13 an operation that is conservative, associative, and has 1 as the neutral element. However it is not bisymmetric since $F(F(1,2),F(3,2)) \neq F(F(1,3),F(2,2))$.

Using Lemma 20, we immediately derive the following two characterizations. The first one was already established in [18, Corollary 1] in the special setting of aggregation operators (i.e., operations $F\colon L_n^2 \to L_n$ that are nondecreasing and that satisfy $F(1,1) = 1$ and $F(n,n) = n$).

Theorem 21. *An operation $F\colon L_n^2 \to L_n$ is bisymmetric, nondecreasing, and has a neutral element if and only if it is a discrete uninorm.*

Corollary 22. *An operation $F\colon L_n^2 \to L_n$ is idempotent (or conservative), bisymmetric, nondecreasing, and has a neutral element if and only if it is an idempotent discrete uninorm.*

Since Theorem 21 and Corollary 22 are derived from Lemma 20 only, we immediately see that these results still hold on any chain, or even on any ordered set, provided the discrete uninorm is replaced with a uninorm (i.e., a binary operation that is associative, symmetric, nondecreasing, and has a neutral element).

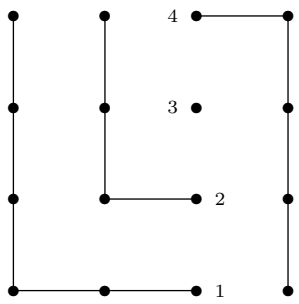

FIGURE 14. An operation that is neither associative nor symmetric

Remark 9. We observe that bisymmetry is necessary in Corollary 22. For instance the operation $F\colon L_4^2 \to L_4$ whose contour plot is depicted in Figure 14 is conservative, nondecreasing, and has a neutral element. However, it is neither associative nor symmetric.

5. Concluding remarks

In this paper we established three main characterizations of the class of idempotent discrete uninorms on finite chains. Two of them are of axiomatic nature (Theorem 12 and Corollary 22) while the third one is of graphical nature (Theorem 15). These axiomatic characterizations are essentially based on conservativeness, which is a rather strong property that can be easily justified in some contexts (see Remark 1). The graphical characterization is a rather surprising result that shows that the idempotent discrete uninorms on a given finite chain can be very easily generated from a graphical viewpoint. This result contrasts with the rather intricate descriptive characterization given in Theorem 9.

In this work we put a particular emphasis on the graphical properties of operations by looking into their contour plots. In particular, we have presented graphical tests to verify whether an operation:

- is conservative (Proposition 6),
- has a neutral element (Proposition 7),
- is an idempotent discrete uninorm (Theorem 15),

and whether a conservative operation:

- has a neutral element (Proposition 4),
- is associative (Proposition 16).

In view of these results, some questions emerge naturally, and we end this paper by listing a few below.

(a) Enumerate and/or generate all the conservative (resp. conservative and associative, etc.) operations on a finite chain.
(b) Provide a graphical test for checking whether a conservative operation is bisymmetric. Using Remark 8(a) we already have such a test for symmetric operations.

(c) Knowing that $F\colon L_n^2 \to L_n$ is bisymmetric and symmetric, does it imply that it is associative and that it has a neutral element? We know from Remark 8(b) that this is not true on an arbitrary chain.

Acknowledgments

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