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► To cite this version:

J Tarcisio-Costa, Boniface Nkonga. High order stabilized finite element method for MHD plasma modeling . ICOSAHOM 2016 - International Conference On Spectral and High Order Methods, Jun 2016, Rio de Janeiro, Brazil. hal-01426950

HAL Id: hal-01426950 https://inria.hal.science/hal-01426950

Submitted on 5 Jan 2017

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High order stabilized finite element method for MHD plasma modeling

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ICOSAHOM - Rio de Janeiro June, 28th 2016





- 3 Applications: Internal kink, Bohm B.C.
- 4 Conclusions and perspectives

Motivations

- Go from Reduced to the full MHD model
- Ensure stability of flows dominated by convection
- Boundary conditions

Main goal: use JOREK to simulate Edge-localized modes (ELMS) with the Full-MHD model

Tokamaks

- Tokamak stands for: Toroidal chamber with magnetic coils (From Russian: Toroidal'naya Kamera s Magnitnymi Katushhami).
- Principal tokamaks at present: JET (UK), Asdex-Upgrade (Germany) Tore-Supra (France) and DIIID (USA)
- $\bullet~\mbox{Drifts} \rightarrow \mbox{Nested}$ magnetic field lines $\rightarrow \mbox{Confinement}$ is improved
- The amplification factor of a fusion reactor is defined as

$$Q = rac{P_{fus}}{P_{inj}}$$

- JET: *Q* ≈ 1 ITER: *Q* = 10
- Economically exploitable Q = 50



Tokamak configurations

There are two main configurations for tokamaks





Limiter Material structure into the plasma

Divertor *X-point configuration* Material structure placed outside

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MHD instabilites in Tokamak

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- Nonlinear reduced MHD and full MHD under implementation
- Cubic C1 finite elements (quadrangles)/Fourier (modal)
- Galerkin formulation
- Linearized Crank-Nicholson and Gear: Implicit scheme
- GMRES solver, Hybrid paralelization (MPI/OpenMPI)

Jorek: Numerical method

- Spatial discretization: 4th-order accurate Galerkin method based on Bézier Formalism
- An arbitrary quantity is expanded on a unit square as

$$P(s,t) = \sum_{i=1}^{4} \sum_{j=1}^{4} p_{ij}\sigma_{ij}b_{ij}(s,t)$$

- $(s, t) \in [0, 1], \sigma$ assure continuity of *P*
- Basis functions b_{ij}: Bernstein polynomials

$$\begin{array}{rcl} b_{11} &=& (1-s)^2(1-t)^2(1+2s)(1+2t),\\ b_{12} &=& 3(1-s)^2(1-t)^2(1+2t)s,\\ b_{13} &=& 3(1-s)^2(1-t)^2(1+2s)t,\\ b_{14} &=& 9(1-s)^2(1-t)^2st. \end{array}$$

Meshes

Examples of meshes: poloidal and X-point





Image: A math a math

The single-fluid full MHD system in consideration is

$$\begin{cases} \frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{m} &= \nabla \cdot (\underline{\mathcal{D}} \nabla \rho'), \\ \frac{\partial \boldsymbol{m}}{\partial t} + \nabla \cdot \left(\frac{\boldsymbol{m} \otimes \boldsymbol{m}}{\rho} - \boldsymbol{B} \otimes \boldsymbol{B} \right) + \nabla \left(\boldsymbol{p} + \frac{\boldsymbol{B} \cdot \boldsymbol{B}}{2} \right) &= \nabla \cdot \underline{\boldsymbol{\tau}}, \\ \frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \left(\left(\left(\mathcal{E} + \boldsymbol{p} + \frac{\boldsymbol{B} \cdot \boldsymbol{B}}{2} \right) \frac{\boldsymbol{m}}{\rho} - \frac{\boldsymbol{B} \cdot \boldsymbol{m}}{\rho} \boldsymbol{B} \right) &= \nabla \cdot \boldsymbol{q} + \nabla \cdot \left(\underline{\boldsymbol{\tau}} \frac{\boldsymbol{m}}{\rho} \right), \\ \frac{\partial \boldsymbol{B}}{\partial t} + \nabla \cdot \left(\frac{\boldsymbol{m} \otimes \boldsymbol{B} - \boldsymbol{B} \otimes \boldsymbol{m}}{\rho} \right) &= -\nabla \times (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \end{cases}$$

with $\boldsymbol{m} \equiv \rho \boldsymbol{v}$, $\mathcal{E} = \rho \varepsilon + \frac{1}{2\rho} \boldsymbol{m} \cdot \boldsymbol{m} + \frac{\boldsymbol{B} \cdot \boldsymbol{B}}{2}$, $\boldsymbol{p} = (\gamma - 1)\rho \varepsilon = \rho T$, $\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} = \eta \boldsymbol{J}$, $\boldsymbol{J} \equiv \nabla \times \boldsymbol{B}$.

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The ideal MHD system driving the previous system can be written in the following compact form

$$\frac{\partial \boldsymbol{w}}{\partial t} + \nabla \cdot \boldsymbol{\underline{f}} = \boldsymbol{0},$$

where

 $P \equiv$

$$\boldsymbol{w} = \begin{pmatrix} \rho \\ \boldsymbol{m} \\ \mathcal{E} \\ \boldsymbol{B} \end{pmatrix}, \quad \boldsymbol{\underline{f}} = \begin{pmatrix} \boldsymbol{m}^{T} \\ \boldsymbol{v} \otimes \boldsymbol{m} + \boldsymbol{P} \mathbb{I} - \boldsymbol{B} \otimes \boldsymbol{B} \\ \mathcal{H} \boldsymbol{m}^{T} - (\boldsymbol{B} \cdot \boldsymbol{v}) \boldsymbol{B}^{T} \\ \boldsymbol{v} \otimes \boldsymbol{B} - \boldsymbol{B} \otimes \boldsymbol{v} \end{pmatrix}$$
$$\boldsymbol{p} + \frac{\boldsymbol{B} \cdot \boldsymbol{B}}{2}, \ \mathcal{H} = \frac{\mathcal{E} + \boldsymbol{P}}{\rho}$$

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Quasi-linear form

$$\frac{\partial \boldsymbol{w}}{\partial t} + \underline{\boldsymbol{L}}(\boldsymbol{w}, \boldsymbol{\partial}) \boldsymbol{w} = \boldsymbol{0}$$

The operator $\underline{L}(w, \partial)$

$\underline{L}(\boldsymbol{w},\partial) =$	0	∂^{T}	0	0 ^{<i>T</i>}
	$-rac{\pmb{m}}{ ho^2} \pmb{m} \cdot \pmb{\partial} + rac{\partial P}{\partial ho} \pmb{\partial}$	$rac{1}{ ho}(oldsymbol{m}\otimesoldsymbol{\partial}+oldsymbol{m}\cdotoldsymbol{\partial})$	$\frac{\partial P}{\partial \mathcal{E}} \partial$	$({oldsymbol B}\otimes \partial)^T - {oldsymbol B}\cdot \partial$
		$+\left(rac{\partial P}{\partial oldsymbol{m}}\otimesoldsymbol{m} ight)^{T}$		$+\left(rac{\partial \boldsymbol{P}}{\partial \boldsymbol{B}}\otimes \boldsymbol{m} ight)^{T}$
	$rac{\partial \mathcal{H}}{\partial ho} m{m} \cdot m{\partial} + rac{m{B} \cdot m{m}}{ ho^2} m{B} \cdot m{\partial}$	$\left(rac{\partial \mathcal{H}}{\partial \boldsymbol{m}} ight)^{T} \boldsymbol{m} \cdot \boldsymbol{\partial} + \mathcal{H} \boldsymbol{\partial}^{T}$	$rac{\partial \mathcal{H}}{\partial \mathcal{E}} m{m} \cdot m{\partial}$	$\left(\frac{\partial \mathcal{H}}{\partial \boldsymbol{\mathcal{B}}}\right)^{T} \boldsymbol{m} \cdot \boldsymbol{\partial} - \frac{\boldsymbol{\mathcal{B}} \cdot \boldsymbol{m}}{\rho} \boldsymbol{\partial}^{T}$
		$-rac{oldsymbol{B}^{ op}}{ ho}oldsymbol{B}\cdotoldsymbol{\partial}$		$-\frac{\boldsymbol{m}^T \boldsymbol{B} \cdot \boldsymbol{\partial}}{\rho}$
	$-\frac{\mathbf{B}\mathbf{m}\cdot\partial-\mathbf{m}\mathbf{B}\cdot\partial}{\rho^2}$	$\frac{\mathbf{B}\otimes \partial - \mathbf{B}\cdot \partial}{ ho}$	0	$\frac{\boldsymbol{m}\cdot\boldsymbol{\partial}-\boldsymbol{m}\boldsymbol{\partial}^{T}}{\rho}$

Seven eigenvalues: 2 slow and 2 fast accoustic waves, 2 Alfven waves and 1 material wave

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Rewriting the system as $\boldsymbol{R}(\boldsymbol{w}) = 0$, in which

$$\boldsymbol{R}(\boldsymbol{w}) := \frac{\partial \boldsymbol{w}}{\partial t} + \nabla \cdot \underline{\boldsymbol{f}} - \nabla \cdot \underline{\boldsymbol{g}}, \qquad \underline{\boldsymbol{g}}: \text{ Transport process}$$

Weak formulation

Divergence free constraint

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Vector potential formulation

$$\mathbf{B} = \nabla \times \mathbf{A}$$
 Coulomb gauge : $\nabla \cdot \mathbf{A} = \mathbf{0}$

 $(\nabla \cdot \mathbf{A})(\nabla \cdot \mathbf{A}^{\star})$ added as a penalization term

It comes to add a contribution

$$\int_{\Omega_{\mathbf{x},\mathbf{h}}} \mathbf{R}(\mathbf{w}) \cdot \mathbf{w}^{\star} + \int_{\Omega_{\mathbf{x},\mathbf{h}}} \mathbf{w}^{'} \cdot (\mathbf{\underline{L}}^{T}(\mathbf{w},\partial)) \mathbf{w}^{\star}) = 0, \quad \forall \mathbf{w}^{\star} \in \overrightarrow{\mathcal{W}}_{\mathbf{h}}(\Omega_{\mathbf{x},\mathbf{h}})$$

where \boldsymbol{w}' is vector of subscales and $\underline{\boldsymbol{L}}^T(\boldsymbol{w},\partial)$ is the adjoint of $\underline{\boldsymbol{L}}(\boldsymbol{w},\partial)$ \boldsymbol{w}' can be approximated by

$$\mathbf{w}^{'} \approx \mathbf{\mathcal{T}}(\mathbf{\underline{L}}(\mathbf{w}, \mathbf{\partial}) \delta \mathbf{w})$$

Several options for \mathcal{T} , choice based on heuristic arguments

$$\underline{\mathcal{T}} = \alpha \frac{h_{e}}{2||\boldsymbol{a}||} \Rightarrow \frac{||\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{\zeta}}||_{2}}{||\boldsymbol{\Lambda}||_{\infty}}\mathbb{I}$$

Stabilized weak formulation

Set of interpolated variables

$$\boldsymbol{w} = \begin{pmatrix} \rho & & \\ \boldsymbol{m} & = & \rho \boldsymbol{v} \\ \mathcal{E} & = & \frac{\rho T}{\gamma - 1} + \frac{1}{2} \boldsymbol{v} \cdot \boldsymbol{v} + \frac{1}{2} \boldsymbol{B} \cdot \boldsymbol{B} \\ \boldsymbol{B} & & \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{Y} = \begin{pmatrix} \rho \\ \boldsymbol{v} \\ \boldsymbol{T} \\ \boldsymbol{A} \end{pmatrix}$$

Stabilized weak formulation

$$\int_{\Omega_{\mathbf{x},\mathbf{h}}} (\underline{\mathbf{M}}(\mathbf{Y})\partial_t \mathbf{Y}) \cdot \widetilde{\mathbf{w}}^{\star} - \int_{\overline{\Omega}_{\mathbf{x},\mathbf{h}}} \underline{\mathbf{F}} : (\partial \odot \mathbf{w}^{\star}) + \int_{\Omega_{\mathbf{x},\mathbf{h}}} \underline{\mathbf{D}} : (\partial \odot \mathbf{w}^{\star}) - \frac{1}{\epsilon} \int_{\Omega_{\mathbf{x},\mathbf{h}}} (\nabla \cdot \mathbf{A}) (\nabla \cdot \mathbf{A}^{\star}) \\ + \int_{\partial \overline{\Omega}_{\mathbf{x},\mathbf{h}}} \underline{\mathbf{F}} : (\mathbf{n} \odot \mathbf{w}^{\star}) - \int_{\partial \overline{\Omega}_{\mathbf{x},\mathbf{h}}} \underline{\mathbf{D}} : (\mathbf{n} \odot \mathbf{w}^{\star}) - \frac{1}{\epsilon} \int_{\partial \Omega_{\mathbf{x},\mathbf{h}}} (\mathbf{n} \times \partial_t \mathbf{A}) \cdot \mathbf{B}^{\star} + \frac{1}{\epsilon} \int_{\partial \Omega_{\mathbf{x},\mathbf{h}}} \mathbf{S}(\mathbf{w}, \mathbf{w}_b, \mathbf{w}^{\star}) \\ = -\int_{\Omega_{\mathbf{x},\mathbf{h}}} (\underline{\mathbf{L}}^T(\mathbf{w}, \partial)) \mathbf{w}^{\star})$$

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In the previous equation

$$\underline{\boldsymbol{F}} = \begin{pmatrix} \rho \boldsymbol{v} \\ \rho \boldsymbol{v} \otimes \boldsymbol{v} + P \boldsymbol{\mathcal{I}} - \boldsymbol{\mathcal{B}} \otimes \boldsymbol{\mathcal{B}} \\ \rho \mathcal{H} \boldsymbol{v} - \boldsymbol{\mathcal{B}} \cdot \boldsymbol{v} \boldsymbol{\mathcal{B}} \\ - \boldsymbol{v} \times \boldsymbol{\mathcal{B}} \end{pmatrix}, \quad \underline{\boldsymbol{D}} = \begin{pmatrix} \underline{\boldsymbol{\mathcal{D}}} \partial \rho' \\ \underline{\boldsymbol{\mathcal{T}}} \\ \underline{\boldsymbol{\tau}} \\ - (\boldsymbol{\boldsymbol{\mathcal{E}}} + \boldsymbol{v} \times \boldsymbol{\mathcal{B}}) \end{pmatrix}$$

and the boundary conditions

$$\boldsymbol{S}(\boldsymbol{w}, \boldsymbol{w}_{b}, \boldsymbol{w}^{\star}) = \begin{pmatrix} (\rho - \rho_{b})\rho^{\star}\chi_{\rho} \\ \rho(\boldsymbol{v} - \boldsymbol{v}_{b}) \cdot \boldsymbol{m}^{\star}\chi_{\boldsymbol{v}} + (\rho\boldsymbol{v} - \boldsymbol{m}_{b}) \cdot \boldsymbol{m}^{\star}\chi_{\boldsymbol{m}} \\ \rho(T - T_{b})\mathcal{E}^{\star}\chi_{T} + (\rho - \rho_{b})\mathcal{E}^{\star}\chi_{\rho} \\ (\boldsymbol{A} - \boldsymbol{A}_{b}) \cdot \boldsymbol{J}^{\star}\chi_{\boldsymbol{A}} + (\boldsymbol{B} - \boldsymbol{B}_{b}) \cdot \boldsymbol{B}^{\star}\chi_{\boldsymbol{B}} \end{pmatrix}$$

For further details, please refer to the Research Report: hal.inria.fr/hal-01294788

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Resistive internal kink

viscosity -0.5 -1.0 1.0 0.0 VMS-Stabilization -0.5



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MHD instabilites in Tokamak

Resistive internal kink dynamics



 $S \propto \eta^{-1}$: Magnetic Reynolds number.

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Resistive internal kink: growth rate



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Resistive internal kink: magnetic reconnection



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Resistive internal kink: magnetic reconnection



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Resistive internal kink: magnetic reconnection



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Boundary Conditions: Bohm

We need to compute the following integrals at the boundary

$$-\int_{\partial\Omega_{x,h}} \pm \rho \frac{\sqrt{\gamma T}}{||\boldsymbol{B}||} \boldsymbol{B} \cdot \boldsymbol{n} \rho^{\star},$$
$$-\int_{\partial\Omega_{x,h}} \left(\pm \rho \boldsymbol{v} \frac{\sqrt{\gamma T}}{||\boldsymbol{B}||} \boldsymbol{B} \cdot \boldsymbol{n} \right) \cdot \boldsymbol{m}^{\star},$$
$$-\int_{\partial\Omega_{x,h}} \pm (\boldsymbol{p} \boldsymbol{n} + \pi \boldsymbol{n} - \boldsymbol{B} (\boldsymbol{B} \cdot \boldsymbol{n})) \cdot \boldsymbol{m}^{\star}$$
$$-\int_{\partial\Omega_{x,h}} \pm \boldsymbol{p} \frac{\sqrt{\gamma T}}{||\boldsymbol{B}||} \boldsymbol{B} \cdot \boldsymbol{n} \boldsymbol{\mathcal{E}}^{\star}$$

Where *n* is the outward normal and

$$\pm \equiv \frac{\boldsymbol{B} \cdot \boldsymbol{n}}{|\boldsymbol{B} \cdot \boldsymbol{n}|}$$

so that $\boldsymbol{v}_b \cdot \boldsymbol{n}$ is always positive

Also, a penalization term
$$\frac{1}{\epsilon} \int_{\partial \Omega_{x,h}} \left(\rho \mathbf{v} \mp \rho \sqrt{\gamma T} \frac{\mathbf{B}}{||\mathbf{B}||} \right) \cdot \mathbf{m}^{\star}$$

X-point results: Initial density and velocity

 $t = \mathbf{1} \tau_{\mathbf{A}}$



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X-point results: Initial density and velocity



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X-point results: Velocity field

 $t = 1\tau_A$ $t = 400 \tau_{A}$ 0.0001000 0.2501 0.5000 0.7500 1.000 0.0001000 0.2501 0.5000 0.7500 1.000

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X-point results: Quasi-steady state

Evolution of the Kinetic Energy: n = 0



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Conclusions

- Full MHD equations successfully implemented for simulating circular plasmas
- Stabilization scheme has improved the quality of the results
- Quasi-steady state for X-point geometry with Bohm B.C.

Perspectives

• Simulation of ELMs using the full model