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Strongly Connected Components in graphs, formal proof of Tarjan1972 algorithm

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Plan

• motivation
• algorithm
• pre-/post-conditions
• imperative programs
• conclusion

.. joint work (in progress) with Ran Chen
Motivation

• nice algorithms should have simple formal proofs
• to be fully published in articles or journals
• how to publish formal proofs?
• Coq proofs seem to me unreadable (and by normal human?)
• Why3 allows mix of automatic and interactive proofs
• first-order logic is easy to understand
• algorithms on graphs = a good testbed
A one-pass linear-time algorithm
The algorithm (1/2)

- depth-first search algorithm
- with pushing non visited vertices into a working stack
- and computing oldest vertex reachable by at most a single « back-edge »
- when that oldest vertex is equal to currently visited vertex, a new strongly connected component is in the working stack on top of current vertex.
- then pop working stack until currently visited vertex
The algorithm (2/2)

les valeurs successives de la pile

1 1 1 1 1 1 1 1
2 2 2 2 2 2 2 2
3 3 3 3 3 3 3 3
4 4 4 4 4 4 4 4
5 5 5 8 8 8 8 8
6 6 6 9 9 9 9 9
7 7 7 7 7 7 7 7
8 8 8 8 8 8 8 8
9 9 9 9 9 9 9 9
0 0 0 0 0 0 0 0

sens croissant du rang
The algorithm (2/2)

les valeurs successives de la pile

1 1 1 1 1 1 1 1 1
2 2 2 2 2 2 2 2 2
3 3 3 3 3 3 3 3 3
4 4 4 4 4 4 4 4 4
5 5 5 5 5 5 8 8 8
6 6 6 6 6 6 9 9 9
7 7 7 7 7 7 7 7 7
8 8 8 8 8 8 8 8 8
9 9 9 9 9 9 9 9 9
0 0 0 0 0 0 0 0 0

sens croissant du rang
Proof in algorithms book

• consider the spanning trees (forrest)
• Christmas tree structure of strongly connected components
• 2-3 lemmas about ancestors in spanning trees

Lemma 10. Let $v$ and $w$ be vertices in $G$ which lie in the same strongly connected component. Let $F$ be a spanning forest of $G$ generated by repeated depth-first search. Then $v$ and $w$ have a common ancestor in $F$. Further, if $u$ is the highest numbered common ancestor of $v$ and $w$, then $u$ lies in the same strongly connected component as $v$ and $w$.

• give the program

• that proof program
The program (1/3)

- a functional version with lists and finite sets
- the working stack is a list

```plaintext
function rank (x: vertex) (s: list vertex): int =
    match s with
    | Nil → max_int()
    | Cons y s’ → if x = y && not (lmem x s’) then length s’ else rank x s’
end

function max_int (): int = cardinal vertices

let rec split (x : α) (s: list α) : (list α, list α) =
returns{(s1, s2) → s1 ++ s2 = s}
returns{(s1, _) → lmem x s → is_last_of x s1}
    match s with
    | Nil → (Nil, Nil)
    | Cons y s’ → if x = y then (Cons x Nil, s’) else
      let (s1’, s2) = split x s’ in
      ((Cons y s1’), s2)
end
```
The program (2/3)

- *blacks, grays* are sets of vertices; *sccs* is a set of sets of vertices

- naming conventions:
  - *x, y, z* for vertices; *b* for black sets; *s* for stacks;
  - *cc* for connected components;
  - *sccs* for sets of connected components

```ml
let rec dfs1 x blacks (ghost grays) stack sccs =
  let m = rank x (Cons x stack) in
  let (m1, b1, s1, sccs1) =
    dfs' (successors x) blacks (add x grays) (Cons x stack) sccs in
  if m1 ≥ m then
    let (s2, s3) = split x s1 in
    (max_int(), add x b1, s3, add (elements s2) sccs1)
  else
    (m1, add x b1, s1, sccs1)
```
The program (3/3)

with dfs’ roots blacks (ghost grays) stack sccs =
  if is_empty roots then
    (max_int(), blacks, stack, sccs)
  else
    let x = choose roots in
    let roots’ = remove x roots in
    let (m1, b1, s1, sccs1) =
      if lmem x stack then
        (rank x stack, blacks, stack, sccs)
      else if mem x blacks then
        (max_int(), blacks, stack, sccs)
      else
        dfs1 x blacks grays stack sccs in
    let (m2, b2, s2, sccs2) =
      dfs’ roots’ b1 grays s1 sccs1 in
    (min m1 m2, b2, s2, sccs2)
Pre-/Post-conditions
Pre/Post-conditions (1/3)

let rec dfs1 x blacks (ghost grays) stack sccs =
requires{mem x vertices} (* R1 *)
requires{access_to grays x} (* R2 *)
requires{not mem x (union blacks grays)} (* R3 *)

(* monotony *)
returns{(_, b, s, _) → ∃s’. s = s’ ++ stack ∧ subset (elements s’) b} (* M1 *)
returns{(_, b, _, _) → subset blacks b} (* M2 *)
returns{(_, _, _, sccs_n) → subset sccs sccs_n} (* M3 *)
Pre/Post-conditions (2/3)

- \( \text{sccs} \subseteq \text{sccs}_n \)
- \( \text{blacks} \subseteq b \)
- \( m \leq \text{rank y stack} \)
- \( m \leq \text{rank x stack} \)
Pre/Post-conditions (3/3)

```plaintext
with dfs' roots blacks (ghost grays) stack sccs =

requires {subset roots vertices} (* R1 *)
requires {∀x. mem x roots → access_to grays x} (* R2 *)

(* post conditions *)
returns {(_, b, _, _) → subset roots (union b grays)} (* E1 *)
returns {(_, m, s, _) → ∀x. mem x roots → m ≤ rank x s} (* E2 *)
returns {(_, m, s, _) → m = max_int() ∨ ∃x. mem x roots ∧ rank_of_reachable m x s} (* E3 *)
returns {(_, m, s, _) → ∀y. crossedge_to s y stack → m ≤ rank y stack} (* E4 *)

(* monotony *)
returns {(_, b, s, _) → ∃s'. s = s' ++ stack ∧ subset (elements s') b} (* M1 *)
returns {(_, b, _, _) → subset blacks b} (* M2 *)
returns {(_, _, sccs_n) → subset sccs sccs_n} (* M3 *)
```
Graphs

type vertex
constant vertices: set vertex
function successors vertex : set vertex
axiom successors_vertices:
   \( \forall x. \text{mem } x \text{ vertices } \rightarrow \text{subset } (\text{successors } x) \text{ vertices } \)
predicate edge (x y: vertex) =
   mem x vertices \land mem y (successors x)
Paths

**inductive** path vertex (list vertex) vertex =
| Path_empty:
  ∀x: vertex. path x Nil x
| Path_cons:
  ∀x y z: vertex, l: list vertex.
    edge x y → path y l z → path x (Cons x l) z

**predicate** reachable (x z: vertex) =
  ∃l. path x l z

**predicate** in_same_scc (x z: vertex) =
  reachable x z ∧ reachable z x

**predicate** is_subscucc (s: set vertex) =
  ∀x z. mem x s → mem z s → in_same_scc x z

**predicate** is_scc (s: set vertex) =
  is_subscucc s ∧ (∀s’. subset s s’ → is_subscucc s’ → s == s’)

Invariants (1/4)

Predicate no_black_to_white (blacks grays: set vertex) =
\( \forall x \ x'. \ \text{edge} \ x \ x' \rightarrow \text{mem} \ x \ \text{blacks} \rightarrow \text{mem} \ x' \ (\text{union} \ \text{blacks} \ \text{grays}) \)

Predicate wff_color (blacks grays: set vertex) (s: list vertex)
(sccs: set (set vertex)) =
\( \text{inter} \ \text{blacks} \ \text{grays} = \text{empty} \ \land \)
\( (\text{elements} \ s) = \text{union} \ \text{grays} \ (\text{diff} \ \text{blacks} \ (\text{set_of sccs})) \ \land \)
\( (\text{subset} \ (\text{set_of sccs}) \ \text{blacks}) \ \land \)
\text{no_black_to_white} \ \text{blacks} \ \text{grays}

blacks \ \cap \ \text{grays} = \emptyset

\text{elements} \ s = \ \text{grays} \ \cup \ \text{blacks} - (\text{set_of sccs})

(\text{set_of sccs}) \ \subseteq \ \text{blacks}
Invariants (2/4)

sccs

cc1 cc2 ccn

increasing rank

s

blacks

grays
Invariants (3/4)

predicate wff_stack (blacks gras: set vertex) (s: list vertex) (sccs: set (set vertex)) =

wff_color blacks gras s sccs ∧
simplelist s ∧
subset (elements s) vertices ∧

(∀x y. mem x gras → lmem y s →
  rank x s ≤ rank y s → reachable x y) ∧

(∀y. lmem y s → ∃x. mem x gras ∧
  rank x s ≤ rank y s ∧ reachable y x)
Invariants (4/4)

sccs

cc1  cc2  ccn

s

increasing rank

blacks

grays
let \( m = \text{rank} \ x \ (\text{Cons} \ x \ \text{stack}) \ \text{in} \)
let \( (m_1, b_1, s_1, \text{sccs1}) = \)
dfs' (successors \( x \)) blacks (add \( x \) grays) (Cons \( x \) stack) sccs \text{in} 

\( \text{if} \ m_1 \geq m \ \text{then} \begin{align*}
\text{let} & \quad (s_2, s_3) = \text{split} \ x \ s_1 \ \text{in} \\
& \text{assert}\{s_3 = \text{stack}\}; \\
& \text{assert}\{\text{subset} \ (\text{elements} \ s_2) \ (\text{add} \ x \ b_1)\}; \\
& \text{assert}\{\text{is_subscs} \ (\text{elements} \ s_2) \land \text{mem} \ x \ (\text{elements} \ s_2)\}; \\
& \text{assert}\{\forall y. \ \text{in_same_scc} \ y \ x \rightarrow \text{mem} \ y \ (\text{elements} \ s_2)\}; \\
& \text{assert}\{\text{is_scc} \ (\text{elements} \ s_2)\}; \\
\end{align*} \) 

\( (\text{max_int}(), \text{add} \ x \ b_1, s_3, \text{add} \ (\text{elements} \ s_2) \ \text{sccs1}) \ \text{end} \) 
\( \text{else} \quad \begin{align*}
\end{align*} \) 

\( (m_1, \text{add} \ x \ b_1, s_1, \text{sccs1}) \ \text{end} \)
Coq proof: there exists $x', y'$ with $x' \in s2 \land y' \notin s2 \land \text{edge } x' y'$

$y' \in s3 = \text{stack}$

- $x' = x$  impossible because $m1 \leq \text{rank } y' s1 < \text{rank } x s1$
- $x' \neq x$  impossible because crossedge

$y' \in \text{SCCS}$  impossible because sccs disjoint from stack

$y'$ is white

- $x' = x$  impossible because successors are black
- $x' \neq x$  impossible because no black to white
let rec dfs1 x blacks (ghost grays) stack sccs =
requires{mem x vertices} (* R1 *)
requires{access_to grays x} (* R2 *)
requires{not mem x (union blacks grays)} (R3 *)
(* invariants *)
requires{wff_stack blacks grays stack sccs} (* I1a *)
requires{∀cc. mem cc sccs ↔ subset cc blacks ∧ is_scc cc} (* I2a *)
returns{(b, s, sccs_n) → wff_stack b grays s stack sccs_n} (* I1b *)
returns{(b, s, sccs_n) → ∀cc. mem cc sccs_n ↔ subset cc b ∧ is_scc cc} (* I2b *)
(* post conditions *)
returns{(b, s, sccs_n) → mem x b} (* E1 *)
returns{(m, s, sccs_n) → m ≤ rank x s} (* E2 *)
returns{(m, s, sccs_n) → m = max_int() ∨ rank_of_reachable m x s} (* E3 *)
returns{(m, s, sccs_n) → ∀y. crossedge_to s y stack → m ≤ rank y stack} (* E4 *)
(* monotony *)
returns{(b, s, sccs_n) → ∃s’. s = s’ ++ stack ∧ subset (elements s’) b} (* M1 *)
returns{(b, s, sccs_n) → subset blacks b} (* M2 *)
returns{(sccs_n) → subset sccs sccs_n} (* M3 *)
Towards imperative program
let rec dfs1 x blacks (ghost grays) stack sccs sn num = 
requires{sn = cardinal (union grays blacks) \&\& subset (union grays blacks) vertices}
(* invariants *)
requires{wff_num sn num stack} (* I3a *)
returns{(_, _, _, s, _, sn_n, num_n) \rightarrow wff_num sn_n num_n s} (* I3b *)
(* post conditions *)
returns{(sn_n, m, _, s, _, _, num_n) \rightarrow \exists y. \text{lmem} y s \&\& sn_n = \text{num}_n[y] \&\& m = \text{rank} y s} (* E5 *)

let m = rank x (Cons x stack) in
let (n1, m1, b1, s1, sccs1, sn1, num1) =
  dfs' (successors x) blacks (add x grays) (Cons x stack) sccs (sn + 1) num[x \leftarrow sn] in
if n1 \geq sn then begin
  let (s2, s3) = split x s1 in
  (max_int(), max_int(), add x b1, s3, add (elements s2) sccs1, sn1, num1) end
else
(n1, m1, add x b1, s1, sccs1, sn1, num1)
predicate wff_num (sn: int) (num: map vertex int) (s: list vertex) =
(∀x. num[x] < sn ≤ max_int()) ∧
(∀x y. lmem x s → lmem y s → num[x] ≤ num[y] ↔ rank x s ≤ rank y s)
let rec dfs1 x blacks (ghost grays) stack sccs sn num =
  let m = rank x (Cons x stack) in
  let n = !sn in
  incr sn; num := !num[x ← n];
  let (n1, m1, b1, s1, sccs1) =
    dfs' (successors x) blacks (add x grays) (Cons x stack) sccs sn num in
  assert{\text{n1} \geq n \leftrightarrow m1 \geq m}; (\ast \ast)
  if n1 \geq n then begin
    let (s2, s3) = split x s1 num in
    assert{s3 = stack};
    assert{\text{subset (elements s2) (add x b1)}};
    assert{\text{is_subsc} \ (\text{elements s2}) \wedge \text{mem x (elements s2)}};
    assert{\forall y. \text{in_same_scc} y x \rightarrow \text{mem y (elements s2)}};
    assert{\text{is_scc (elements s2)}};
    (\text{max_int()}, \text{max_int()}, \text{add x b1, s3, add (elements s2) sccs1}) end
  else begin
    assert{\exists y. \text{mem y grays} \wedge \text{rank y s1 < rank x s1} \wedge \text{reachable x y}};
    (n1, m1, add x b1, s1, sccs1) end
Implementation of graphs
vertices as integers in an array
successors as lists for every vertex

see http://jeanjacqueslevy.net/why3
Conclusion
Conclusion

• readable proofs ?

• simple algorithms should have simple proofs
to be shown with a good formal precision

• compare with other proof systems

• further algorithms (in next talks ?)
  • graphs represented with arrays + lists
  • topological sort, articulation points, sccK, sscT

• Why3 is a beautiful system but not so easy to use !