Strongly Connected Components in graphs, formal proof of Tarjan1972 algorithm

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Strongly Connected Components in graphs, formal proof of Tarjan1972 algorithm
Plan

• motivation
• algorithm
• pre-/post-conditions
• imperative programs
• conclusion

. . . joint work (in progress) with Ran Chen
Motivation

- nice algorithms should have simple formal proofs
- to be fully published in articles or journals
- how to publish formal proofs?
- Coq proofs seem to me unreadable (and by normal human?)
- Why3 allows mix of automatic and interactive proofs
- first-order logic is easy to understand
- algorithms on graphs = a good testbed
A one-pass linear-time algorithm
The algorithm (1/2)

- depth-first search algorithm
- with pushing non visited vertices into a working stack
- and computing oldest vertex reachable by at most a single « back-edge »
- when that oldest vertex is equal to currently visited vertex, a new strongly connected component is in the working stack on top of current vertex.
- then pop working stack until currently visited vertex
The algorithm (2/2)

The algorithm is visualized with a diagram and a table of values. The diagram shows nodes and arrows indicating the flow of values. The table lists the values successively as they are pushed onto the stack in a non-decreasing order of rank.

<table>
<thead>
<tr>
<th>Value</th>
<th>1</th>
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<td>4</td>
<td>5</td>
<td>6</td>
</tr>
</tbody>
</table>
The algorithm (2/2)

les valeurs successives de la pile

1 1 1 1 1 1 1 1 1
2 2 2 2 2 2 2 2
3 3 3 3 3 3 3
4 4 4 4 4 4
5 5 5 8 8
6 6
7
8
9
0
1
2
3
4
5
6

sens croissant du rang
Proof in algorithms book

• consider the spanning trees (forest)

• Christmas tree structure of strongly connected components

• 2-3 lemmas about ancestors in spanning trees

Lemma 10. Let \( v \) and \( w \) be vertices in \( G \) which lie in the same strongly connected component. Let \( F \) be a spanning forest of \( G \) generated by repeated depth-first search. Then \( v \) and \( w \) have a common ancestor in \( F \). Further, if \( u \) is the highest numbered common ancestor of \( v \) and \( w \), then \( u \) lies in the same strongly connected component as \( v \) and \( w \).

• give the program

• that proof program
The program (1/3)

- a functional version with lists and finite sets
- the working stack is a list

```plaintext
function rank (x: vertex) (s: list vertex): int =
    match s with
    | Nil → max_int()
    | Cons y s’ → if x = y && not (lmem x s’) then length s’ else rank x s’
end

function max_int (): int = cardinal vertices

let rec split (x : α) (s: list α) : (list α, list α) =
returns{(s1, s2) → s1 ++ s2 = s}
returns{(s1, _) → lmem x s → is_last_of x s1}
    match s with
    | Nil → (Nil, Nil)
    | Cons y s’ → if x = y then (Cons x Nil, s’) else
        let (s1’, s2) = split x s’ in
        ((Cons y s1’), s2)
end
```
The program (2/3)

- **blacks, grays** are sets of vertices; **sccs** is a set of sets of vertices
- naming conventions:
  - \(x, y, z\) for vertices; \(b\) for black sets; \(s\) for stacks;
  - \(cc\) for connected components;
  - **sccs** for sets of connected components

```ocaml
let rec dfs1 x blacks (ghost grays) stack sccs =
  let m = rank x (Cons x stack) in
  let (m1, b1, s1, sccs1) =
    dfs' (successors x) blacks (add x grays) (Cons x stack) sccs in
  if m1 ≥ m then
    let (s2, s3) = split x s1 in
    (max_int(), add x b1, s3, add (elements s2) sccs1)
  else
    (m1, add x b1, s1, sccs1)
```
The program (3/3)

```plaintext
with dfs’ roots blacks (ghost grays) stack sccs =
  if is_empty roots then
    (max_int(), blacks, stack, sccs)
  else
    let x = choose roots in
    let roots’ = remove x roots in
    let (m1, b1, s1, sccs1) =
      if lmem x stack then
        (rank x stack, blacks, stack, sccs)
      else if mem x blacks then
        (max_int(), blacks, stack, sccs)
      else
        dfs1 x blacks grays stack sccs in
    let (m2, b2, s2, sccs2) =
      dfs’ roots’ b1 grays s1 sccs1 in
    (min m1 m2, b2, s2, sccs2)
```
Pre-/Post-conditions
Pre/Post-conditions (1/3)

```plaintext
let rec dfs1 x blacks (ghost grays) stack sccs =
requires {mem x vertices} (* R1 *)
requires {access_to grays x} (* R2 *)
requires {not mem x (union blacks grays)} (* R3 *)
```

(* monotony *)
returns {(_, b, s, _) → ∃s’. s = s’ ++ stack ∧ subset (elements s’) b} (* M1 *)
returns {(_, b, _, _) → subset blacks b} (* M2 *)
returns {(_, _, _, sccs_n) → subset sccs sccs_n} (* M3 *)
Pre/Post-conditions (2/3)

\[ \text{stack} \]

\[
\begin{align*}
\text{sccs} & \subseteq \text{sccs}_n \\
\text{blacks} & \subseteq b
\end{align*}
\]

\[ m \leq \text{rank y stack} \]

\[ m \leq \text{rank x stack} \]
Pre/Post-conditions (3/3)

with dfs' roots blacks (ghost grays) stack sccs =

requires {subset roots vertices} (* R1 *)
requires {\forall x. \text{mem x roots} \rightarrow \text{access_to grays x}} (* R2 *)

(* post conditions *)
returns {(_, b, _, _) \rightarrow \text{subset roots (union b grays)}} (* E1 *)
returns {((m, _, s, _)) \rightarrow \forall x. \text{mem x roots} \rightarrow m \leq \text{rank x s}} (* E2 *)
returns {((m, _, s, _)) \rightarrow m = \text{max_int()} \lor \exists x. \text{mem x roots} \land \text{rank_of_reachable m x s}} (* E3 *)
returns {((m, _, s, _)) \rightarrow \forall y. \text{crossedgeto s y stack} \rightarrow m \leq \text{rank y stack}} (* E4 *)
returns {(_, b, s, _)) \rightarrow \exists s'. s = s' ++ \text{stack} \land \text{subset (elements s') b}} (* M1 *)
returns {(_, b, _, _)) \rightarrow \text{subset blacks b}} (* M2 *)
returns {(_, _, sccs_n) \rightarrow \text{subset sccs sccs_n}} (* M3 *)
Graphs

type vertex
constant vertices: set vertex
function successors vertex : set vertex
axiom successors_vertices:
    \( \forall x. \text{mem } x \text{ vertices } \rightarrow \text{subset } (\text{successors } x) \text{ vertices } \)
predicate edge (x y: vertex) =
    mem x vertices \& mem y (successors x)
Paths

inductive path vertex (list vertex) vertex =
    | Path_empty:
        ∀x: vertex. path x Nil x
    | Path_cons:
        ∀x y z: vertex, l: list vertex.
        edge x y → path y l z → path x (Cons x l) z

predicate reachable (x z: vertex) =
    ∃l. path x l z

predicate in_same_scc (x z: vertex) =
    reachable x z ∧ reachable z x

predicate is_subscscc (s: set vertex) =
    ∀x z. mem x s → mem z s → in_same_scc x z

predicate is_scc (s: set vertex) =
    is_subscscc s ∧ (∀s’. subset s s’ → is_subscscc s’ → s == s’)

Invariants (1/4)

**predicate** no_black_to_white (blacks grays: set vertex) =
\[ \forall x \, x'. \text{ edge } x \, x' \rightarrow \text{mem } x \, \text{blacks} \rightarrow \text{mem } x' \, \text{(union blacks grays)} \]

**predicate** wff_color (blacks grays: set vertex) (s: list vertex) (sccs: set (set vertex)) =
\[
\text{inter } \text{blacks grays } = \text{empty } \land \\
(\text{elements } s) = \text{union } \text{grays} \, \text{diff } \text{blacks} \, \text{set_of_sccs} \land \\
(\text{subset } \text{set_of_sccs} \, \text{blacks}) \land \\
\text{no_black_to_white } \text{blacks grays}
\]

\[
\text{blacks } \cap \text{grays } = \emptyset \\
\text{elements } s = \text{grays } \cup \text{blacks} - \text{set_of_sccs} \\
(\text{set_of_sccs}) \subseteq \text{blacks}
\]
Invariants (2/4)

sccs

cc1 cc2 ccn

increasing rank

blacks

grays
Invariants (3/4)

predicate wff_stack (blacks grays: set vertex) (s: list vertex) (sccs: set (set vertex)) =

  wff_color blacks grays s sccs ∧
  simplelist s ∧
  subset (elements s) vertices ∧

  (∀x y. mem x grays → lmem y s →
   rank x s ≤ rank y s → reachable x y) ∧

  (∀y. lmem y s → ∃x. mem x grays ∧
   rank x s ≤ rank y s ∧ reachable y x)
Invariants (4/4)

sccs

cc1  cc2  ccn

increasing rank

s

blacks

grays

ccn
Assertions

let m = rank x (Cons x stack) in
let (m1, b1, s1, sccs1) =
dfs’ (successors x) blacks (add x grays) (Cons x stack) sccs in

if m1 ≥ m then begin
  let (s2, s3) = split x s1 in
  assert{s3 = stack};
  assert{subset (elements s2) (add x b1)};
  assert{is_subsccc (elements s2) ∧ mem x (elements s2)};
  assert{∀y. in_same_scc y x → mem y (elements s2)};
  assert{is_scc (elements s2)};

  (max_int(), add x b1, s3, add (elements s2) sccs1) end
else begin

  (m1, add x b1, s1, sccs1) end
assert\{∀y. \text{in\_same\_scc}\ y\ x \rightarrow \text{mem}\ y\ (\text{elements}\ s2)\};

• Coq proof: there exists $x', y'$ with $x' \in s2 \land y' \not\in s2 \land \text{edge}\ x'\ y'$

  $y' \in s3 = \text{stack}$

  $x' = x$  impossible because $m1 \leq \text{rank}\ y'\ s1 < \text{rank}\ x\ s1$

  $x' \neq x$  impossible because crossedge

  $y' \in \text{SCCS}$  impossible because sccs disjoint from stack

$y'$ is white

  $x' = x$  impossible because successors are black

  $x' \neq x$  impossible because no black to white
Pre/Post-conditions (1/3)

let rec dfs1 x blacks (ghost grays) stack sccs =
requires{mem x vertices} (* R1 *)
requires{access_to grays x} (* R2 *)
requires{not mem x (union blacks grays)} (* R3 *)
(* invariants *)
requires{wff_stack blacks grays stack sccs} (* I1a *)
requires{∀cc. mem cc sccs ⇔ subset cc blacks ∧ is_scc cc} (* I2a *)
returns{(_, b, s, sccs_n) → wff_stack b grays s sccs_n} (* I1b *)
returns{(_, b, _, sccs_n) → ∀cc. mem cc sccs_n ⇔ subset cc b ∧ is_scc cc} (* I2b *)
(* post conditions *)
returns{(_, b, _, _) → mem x b} (* E1 *)
returns{(_, m, s, _) → m ≤ rank x s} (* E2 *)
returns{(_, m, s, _) → m = max_int() ∨ rank_of_reachable m x s} (* E3 *)
returns{(_, m, s, _) → ∀y. cross_edge_to s y stack → m ≤ rank y stack} (* E4 *)
(* monotony *)
returns{(_, b, s, _) → ∃s’. s = s’ ++ stack ∧ subset (elements s’) b} (* M1 *)
returns{(_, b, _, _) → subset blacks b} (* M2 *)
returns{(_, _, _, sccs_n) → subset sccs sccs_n} (* M3 *)
Towards imperative program
let rec dfs1 x blacks (ghost grays) stack sccs sn num =
requires{sn = cardinal (union grays blacks) ∨ subset (union grays blacks) vertices} 
(* invariants *)
requires{wff_num sn num stack} (* I3a *)
returns{(_, _, _, s, _, sn_n, num_n) → wff_num sn_n num_n s} (* I3b *)
(* post conditions *)
returns{(sn_n, m, _, s, _, _, num_n) → sn_n = m = max_int() ∨ 
  ∃y. lmem y s ∧ sn_n = num_n[y] ∧ m = rank y s} (* E5 *)

let m = rank x (Cons x stack) in
let (n1, m1, b1, s1, sccs1, sn1, num1) =
  dfs' (successors x) blacks (add x grays) (Cons x stack) sccs (sn + 1) num[x ← sn] in
if n1 ≥ sn then begin
  let (s2, s3) = split x s1 in
  (max_int(), max_int(), add x b1, s3, add (elements s2) sccs1, sn1, num1) end
else
  (n1, m1, add x b1, s1, sccs1, sn1, num1)
**Assertions**

\[
\text{predicate wff\_num} (\text{sn: int}) (\text{num: map vertex int}) (\text{s: list vertex}) = \\
(\forall x. \text{num}[x] < \text{sn} \leq \text{max\_int}()) \land \\
(\forall x \ y. \text{lmem} x \ s \rightarrow \text{lmem} y \ s \rightarrow \text{num}[x] \leq \text{num}[y] \leftrightarrow \text{rank} x \ s \leq \text{rank} y \ s)
\]
let rec dfs1 x blacks (ghost grays) stack sccs sn num =
  let m = rank x (Cons x stack) in
  let n = !sn in
  incr sn; num := !num[x ← n];
  let (n1, m1, b1, s1, sccs1) =
      dfs' (successors x) blacks (add x grays) (Cons x stack) sccs sn num in
  assert{ n1 ≥ n ↔ m1 ≥ m }; (* *)
  if n1 ≥ n then begin
    let (s2, s3) = split x s1 num in
    assert{s3 = stack};
    assert{subset (elements s2) (add x b1)};
    assert{is_subscs (elements s2) ∧ mem x (elements s2)};
    assert{∀y. in_same_scc y x → mem y (elements s2)};
    assert{is_scc (elements s2)};
    (max_int(), max_int(), add x b1, s3, add (elements s2) sccs1) end
  else begin
    assert{∃y. mem y grays ∧ rank y s1 < rank x s1 ∧ reachable x y};
    (n1, m1, add x b1, s1, sccs1) end
• implementation of graphs
• vertices as integers in an array
• successors as lists for every vertex

• see http://jeanjacqueslevy.net/why3
Conclusion
Conclusion

- readable proofs?

- simple algorithms should have simple proofs to be shown with a good formal precision

- compare with other proof systems

- further algorithms (in next talks?)
  - graphs represented with arrays + lists
  - topological sort, articulation points, sccK, sscT

- Why3 is a beautiful system but not so easy to use!