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# Space/Time convergence analysis of a class of conservative schemes for linear wave equations

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## Abstract

This paper concerns the space/time convergence analysis of conservative two-steps time discretizations for linear wave equations. Explicit and implicit, second and fourth order schemes are considered, while the space discretization is given and satisfies minimal hypotheses. The convergence analysis is done using energy techniques and holds if the time step is upper-bounded by a quantity depending on space discretization parameters. In addition to showing the convergence for recently introduced fourth order schemes, the novelty of this work consists in the independency of the convergence estimates with respect to the difference between the time step and its greatest admissible value.

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## Résumé

Ce travail concerne l'analyse de convergence espace/temps de schémas en temps à deux pas pour les équations d'onde linéaires. Sont considérés des schémas explicites et implicites, d'ordre deux et quatre, tandis que la discrétisation spatiale est donnée et satisfait des hypothèses minimales. L'analyse de convergence est faite par techniques d'énergie et est valide si le pas de temps est borné par une quantité dépendant des paramètres de discrétisation spatiale. En plus de montrer la convergence pour des schémas d'ordre quatre récemment introduits, la nouveauté de ce travail réside dans le fait que les estimations ne dépendent pas de la différence entre le pas de temps et sa plus grande valeur admissible.

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## Version française abrégée

On considère le problème de propagation d'onde suivant : pour une source donnée  $f \in C^0([0, T], H)$  trouver  $u(t) \in V$  solution, pour tout  $t \in [0, T]$ , de

$$\partial_{tt}u + Au = f \quad \text{dans } V', \quad u(0) = 0 \quad \text{dans } V, \quad \partial_t u(0) = 0 \quad \text{dans } H,$$

où  $V$  et  $H$  sont deux espaces séparables de Hilbert (la norme sur  $H$  est notée  $|\cdot|$ ) et  $V'$  le dual de  $V$ , l'opérateur  $A$  est autoadjoint de  $V$  dans  $V'$ . Considérons donnée une discrétisation en espace s'appuyant sur une famille d'espaces de dimension finie  $\{V_h\}_{h>0}$  avec  $V_h \subset V$ . Le problème semi-discret s'écrit alors : pour un terme source  $f_h \in C^2([0, T], V_h)$  trouver  $u_h(t) \in V_h$ , pour tout  $t \in [0, T]$  satisfaisant

$$\partial_{tt}u_h + A_h u_h = f_h, \quad u_h(0) = 0, \quad \partial_t u_h(0) = 0, \quad \text{dans } V_h,$$

où  $A_h$  est un opérateur de  $V_h$  dans  $V_h$  autoadjoint borné (non uniformément par rapport à  $h$ ) résultant de la discrétisation en espace choisie. Nous supposons qu'il existe une fonction positive  $\delta(h) = o(h)$  telle que pour tout  $h > 0$  on ait

$$\sup_{t \in [0, T]} |u(t) - u_h(t)| \leq \delta(h).$$

Nous étudions les schémas suivants : le  $\theta$ -schéma (noté *TS* et dont le schéma saute-mouton explicite est un cas particulier), le schéma Saute Mouton Stabilisé (noté *SLF*) et le  $(\theta, \varphi)$ -schéma d'ordre quatre (noté *TPS*). Soit  $\Delta t > 0$  le pas de temps et définissons  $t^n := n\Delta t$ . Pour toute suite  $\{v_h^k\}_{k \geq 0} \subset V_h$ , nous notons

$$[v_h^n]_{\Delta t^2} := \frac{v_h^{n+1} - 2v_h^n + v_h^{n-1}}{\Delta t^2}, \quad \{v_h^n\}_\theta := \theta v_h^{n+1} + (1 - 2\theta)v_h^n + \theta v_h^{n-1}, \quad n \geq 1.$$

Nous cherchons une suite  $\{u_h^k\}_{k \geq 0} \subset V_h$  telle que  $u_h^n$  approche  $u_h(t^n)$ . Les deux premiers termes sont supposés donnés et pour tout  $n \geq 1$ ,

- *TS* :  $[u_h^n]_{\Delta t^2} + A_h \{u_h^n\}_\theta = f_h(t^n)$ .
- *SLF* :  $[u_h^n]_{\Delta t^2} + A_h u_h^n + \frac{\Delta t^2}{16} A_h^2 u_h^n = f_h(t^n)$ .
- *TPS* :  $[u_h^n]_{\Delta t^2} + A_h \{u_h^n\}_\theta + \Delta t^2 \left( \theta - \frac{1}{12} \right) A_h^2 \{u_h^n\}_\varphi = f_h(t^n) + \Delta t^2 \left[ \frac{\partial_{tt}}{12} + \left( \theta - \frac{1}{12} \right) A_h \right] f_h(t^n)$ .

Pour chaque schéma on peut définir une énergie discrète associée, une condition de type CFL assure sa positivité en tant que fonctionnelle :

$$\Delta t^2 \leq \frac{\alpha}{\rho(A_h)}$$

où  $\alpha > 0$  est positif, le plus grand possible, et dépend du schéma. Nous montrons que sous des hypothèses portant sur la discrétisation spatiale du problème ainsi que sous condition CFL, le résultat de convergence espace/temps suivant : il existe  $C > 0$  indépendant de  $\Delta t$  et  $h$  tel que

- $\theta$ -schéma (*TS*) et schéma saute-mouton stabilisé (*SLF*)

$$\sup_{t^n \in [0, T]} |u(t^n) - u_h^n| \leq C [\Delta t^2 T^2 + \delta(h)].$$

- $(\theta, \varphi)$ -schéma (*TPS*)

$$\sup_{t^n \in [0, T]} |u(t^n) - u_h^n| \leq C [\Delta t^4 T^2 + \delta(h)].$$

Notons qu'en particulier ces résultats de convergences sont valables lorsque  $\Delta t^2 = \alpha/\rho(A_h)$ . Pour obtenir ce résultat nous nous ramenons à l'étude de polynômes dont les coefficients sont fonction des paramètres des schémas.

## 1. Abstract and semi-discrete wave propagation problem

In this work, we are interested in the simulation of linear wave propagation problems. The most simple example one could think of is given by the following problem: find  $u(t) \in H_0^1(\Omega)$ , for all  $t \in [0, T]$  such that

$$\partial_{tt}u - \Delta u = f, \quad u(0) = 0, \quad \partial_t u(0) = 0, \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \quad (1)$$

To study wave propagation in a more abstract framework, following [1], chapter XVIII, we assume given separable Hilbert spaces  $H$  and  $V$ . The space  $H$  is equipped with the scalar product  $(\cdot, \cdot)_H$ . The corresponding norms are denoted  $|\cdot|$  and  $\|\cdot\|$  respectively. Moreover we assume that  $V$  is dense in  $H$ ,  $H$  is identified with its dual  $H'$  and  $V$  is continuously embedded in  $H$ . Note that to solve (1) it is standard to choose  $H = L^2(\Omega)$  and  $V = H_0^1(\Omega)$ . We assume given a continuous hermitian bilinear form  $a : V \times V \rightarrow \mathbb{R}$  that satisfies

$$a(v, v) \geq 0, \quad a(v, v) + C_o|v|^2 \geq C_a\|v\|^2, \quad \forall v \in V, \quad (2)$$

where  $(C_a, C_o)$  are real positive scalars. From this bilinear form we construct a self-adjoint positive unbounded operator  $A : V \mapsto V'$ . It is defined by

$$A : u \rightarrow Au \quad \text{such that} \quad \langle Au, v \rangle = a(u, v), \quad \forall v \in V.$$

where  $V'$  denotes the dual of  $V$  and  $\langle \cdot, \cdot \rangle$  denotes the duality product. We consider the following abstract wave propagation problems: for a given  $f \in C^0([0, T], H)$  find  $u(t) \in V$  solution, for all  $t \in [0, T]$ , of

$$\partial_{tt}u + Au = f \quad \text{in } V', \quad u(0) = 0 \quad \text{in } V, \quad \partial_t u(0) = 0 \quad \text{in } H. \quad (3)$$

Existence and uniqueness results for this problem are well known (see again [1], chapter XVIII): there exists a unique function  $u$  solution of (3), it satisfies

$$u \in C^0([0, T], V), \quad \partial_t u \in C^0([0, T], H).$$

We introduce the family of finite dimensional spaces  $\{V_h\}_{h>0}$  with  $V_h \subset V$ . As usual, the subscript  $h$  is devoted to tends to 0 and represents an approximation parameter of  $V_h$  to  $V$ . For each  $h$  we define the operator  $A_h$  as  $A_h : V_h \mapsto V_h$  and

$$A_h : u_h \rightarrow A_h u_h \quad \text{such that} \quad (A_h u_h, v_h)_h = a_h(u_h, v_h), \quad \forall v_h \in V_h,$$

where the scalar product on  $V_h$  denoted by  $(\cdot, \cdot)_h$  as well as the continuous bilinear forms  $a_h(\cdot, \cdot)$  represent some approximation of  $(\cdot, \cdot)_H$  and  $a(\cdot, \cdot)$  respectively that account for instance for the use of quadrature formulae in the computation of integrals. Similarly we denote by  $|\cdot|_h$  the norm induced by the scalar product  $(\cdot, \cdot)_h$ . We assume that there exists a positive constant  $C_H$  such that

$$|v_h| \leq C_H |v_h|_h, \quad \forall v_h \in V_h$$

and we assume that  $a_h$  is an hermitian bilinear form that satisfies the same property as in equation (2)

$$a_h(v_h, v_h) \geq 0, \quad a_h(v_h, v_h) + C_o|v_h|_h^2 \geq C_a\|v_h\|^2, \quad \forall v_h \in V_h.$$

Note that this assumption implies that the operator  $A_h$  is self-adjoint and non-negative. Its spectrum, denoted  $Sp(A_h)$ , is a set of finite number of real non-negative eigenvalues (eigenvalues of multiplicity greater than one are counted accordingly). To each eigenvalue  $\lambda$  we associate a real eigenvector  $w_h^\lambda$  defined by

$$(A_h w_h^\lambda, v_h)_h = \lambda (w_h^\lambda, v_h)_h, \quad \forall v_h \in V_h, \quad |w_h^\lambda|_h = 1,$$

such that the family  $\{w_h^\lambda\}_{\lambda \in Sp(A_h)}$  is an orthonormal basis of  $V_h$ . The semi-discrete equation we consider reads: for any given source term  $f_h \in C^2([0, T], V_h)$  find  $u_h(t) \in V_h$ , for all  $t \in [0, T]$  such that

$$\partial_{tt}u_h + A_h u_h = f_h, \quad u_h(0) = 0, \quad \partial_t u_h(0) = 0, \quad \text{in } V_h. \quad (4)$$

Existence and uniqueness results for this problem are direct consequences of the theory developed in infinite dimensional space (choose  $H = V = V_h$  in the paragraph above): there exists a unique solution  $u_h$  of (4), it satisfies  $u_h \in C^1([0, T], V_h)$ . We introduce the discrepancy error  $e_h(t) = u(t) - u_h(t)$  it depends on the approximation of the scalar product on  $H$ , on the bilinear form  $a_h$ , on the approximation of the space  $V$  itself and on the approximation of the source term  $f$ . It is expected that  $e_h$  vanishes when  $h \rightarrow 0$ . Therefore we make the following assumption

**Hypothesis 1.1** *Convergence of the semi-discrete problem.* *There exists a positive function  $\delta(h) = o(h)$  such that for all  $h > 0$  we have*

$$\sup_{t \in [0, T]} |e_h(t)| \leq \delta(h).$$

Such a result is proved for continuous finite element approximation in [2] in the case  $a(\cdot, \cdot)_h \equiv a(\cdot, \cdot)$ . In [3] the case of spectral elements is given (see [4] for an introduction to spectral elements). For these convergence results to hold, the continuous solution must be regular enough in time and space. We do not detail the assumptions here. In what follows we specify the extra regularity in time and space we require for the semi-discrete solution  $u_h$  in order to state the space/time convergence results, such regularity is a consequence of the space/time regularity of the source term.

**Hypothesis 1.2** *Stability of the semi-discrete problem.* *For a given couple  $(p, q)$  with  $p$  even, we assume that  $\partial_t^q u_h(t) \in V_h$  for all  $t \in [0, T]$  and there exists a constant  $C_{p,q}$  such that for all  $h > 0$  we have*

$$\sup_{t \in [0, T]} |A_h^{p/2} \partial_t^q u_h(t)|_h \leq C_{p,q}.$$

## 2. Analysis of a family of time discretizations

In the sequel we study the following two-steps conservative time discretizations:  $\theta$ -Scheme (*TS*) which is the family of conservative centered Newmark schemes, see [1], such as the classical leap-frog scheme, the Stabilized Leap-Frog scheme (*SLF*) as introduced in [5] and the higher order  $(\theta, \varphi)$ -Scheme (*TPS*) developed in [6]. Let  $\Delta t > 0$  be the time step, a small parameter devoted to tend to zero and let define  $t^n := n\Delta t$ . To shorten the writing we introduce the following notations, for a series  $\{v_h^k\}_{k \geq 0} \subset V_h$ ,

$$[v_h^n]_{\Delta t^2} := \frac{v_h^{n+1} - 2v_h^n + v_h^{n-1}}{\Delta t^2}, \quad \{v_h^n\}_\theta := \theta v_h^{n+1} + (1 - 2\theta)v_h^n + \theta v_h^{n-1}, \quad n \geq 1.$$

We seek a series  $\{u_h^k\}_{k \geq 0} \subset V_h$  such that  $u_h^n$  is an approximation of  $u_h(t^n)$  solution of equation (4). The two first terms of this series are considered given <sup>1</sup>. The following terms of the series are computed using one of the schemes below, for  $n \geq 1$ ,

$$\bullet \text{ TS: } [u_h^n]_{\Delta t^2} + A_h \{u_h^n\}_\theta = f_h(t^n). \quad \bullet \text{ SLF: } [u_h^n]_{\Delta t^2} + A_h u_h^n + \frac{\Delta t^2}{16} A_h^2 u_h^n = f_h(t^n).$$

$$\bullet \text{ TPS: } [u_h^n]_{\Delta t^2} + A_h \{u_h^n\}_\theta + \Delta t^2 \left( \theta - \frac{1}{12} \right) A_h^2 \{u_h^n\}_\varphi = f_h(t^n) + \Delta t^2 \left[ \frac{\partial_{tt}}{12} + \left( \theta - \frac{1}{12} \right) A_h \right] f_h(t^n).$$

By construction (SLF) is an explicit scheme, whereas (TS) is explicit when  $\theta = 0$  and (TPS) is explicit when  $(\theta, \varphi) = (0, 0)$ . In this latter case, the scheme corresponds to the modified equation approach

<sup>1</sup>. In practice,  $u_h^0$  and  $u_h^1$  are computed using the initial data and a sufficiently accurate one-step method (such as high-order Runge Kutta methods).

presented in [7]. Notice that for the (TPS) the source term has to be modified in order to get adequate accuracy.

All these schemes can be written

$$P_{\mathcal{K}}(\Delta t^2 A_h) [u_h^n]_{\Delta t^2} + P_{\mathcal{P}}(\Delta t^2 A_h) A_h \{u_h^n\}_{1/4} = f_h^n, \quad (5)$$

with  $f_h^n$  the source term and with the following definitions of the polynomial functions  $P_{\mathcal{K}}$  and  $P_{\mathcal{P}}$

$$\bullet \text{ TS: } P_{\mathcal{K}}(x) = 1 + \left(\theta - \frac{1}{4}\right)x, \quad P_{\mathcal{P}}(x) = 1. \quad \bullet \text{ SLF: } P_{\mathcal{K}}(x) = 1 - \frac{x}{4} + \frac{x^2}{64}, \quad P_{\mathcal{P}}(x) = 1 - \frac{x}{16}.$$

$$\bullet \text{ TPS: } P_{\mathcal{K}}(x) = 1 + \left(\theta - \frac{1}{4}\right)x + \left(\varphi - \frac{1}{4}\right)\left(\theta - \frac{1}{12}\right)x^2, \quad P_{\mathcal{P}}(x) = 1 + \left(\theta - \frac{1}{12}\right)x.$$

In the sequel we aim at establishing energy and error estimates for general schemes of the form (5) under some assumptions on  $P_{\mathcal{K}}$  and  $P_{\mathcal{P}}$ . Let the time discretization error be defined as  $e_h^n := u_h(t^n) - u_h^n$ , it satisfies

$$P_{\mathcal{K}}(\Delta t^2 A_h) [e_h^n]_{\Delta t^2} + P_{\mathcal{P}}(\Delta t^2 A_h) A_h \{e_h^n\}_{1/4} = r_h^n. \quad (6)$$

Assuming sufficient regularity of the semi-discrete solution, there exist  $t^{n-1} \leq t^{n,\diamond}, t^{n,\heartsuit}, t^{n,\clubsuit} \leq t^{n+1}$  such that for each of the considered schemes, the remainder writes

$$\bullet \text{ TS: } r_h^n = \Delta t^2 \left( \frac{1}{12} \partial_t^4 u_h(t^{n,\diamond}) + \theta A_h \partial_{tt} u_h(t^{n,\heartsuit}) \right). \quad \bullet \text{ SLF: } r_h^n = \Delta t^2 \left( \frac{1}{12} \partial_t^4 u_h(t^{n,\diamond}) + \frac{1}{16} A_h^2 u_h(t^n) \right).$$

$$\bullet \text{ TPS: } r_h^n = \Delta t^4 \left( \frac{1}{360} \partial_t^6 u_h(t^{n,\diamond}) + \frac{\theta}{12} A_h \partial_t^4 u_h(t^{n,\heartsuit}) + \varphi \left( \theta - \frac{1}{12} \right) A_h^2 \partial_{tt} u_h(t^{n,\clubsuit}) \right).$$

This suggests that (TS) and (SLF) are second order accurate and (TPS) is fourth order accurate in time.

For all these schemes, a discrete energy is preserved (in the absence of sources). This energy is computed at the intermediate time step  $t^{n+1/2}$  and is denoted  $\mathcal{E}_h^{n+1/2}$ . It is the sum of a kinetic and potential energies, more precisely  $\mathcal{E}_h^{n+1/2} = \mathcal{E}_{\mathcal{K}}^{n+1/2} + \mathcal{E}_{\mathcal{P}}^{n+1/2}$ . When considering equation (6) satisfied by the error, we have

$$\mathcal{E}_{\mathcal{K}}^{n+1/2} = \frac{1}{2} \left( P_{\mathcal{K}}(\Delta t^2 A_h) \frac{e_h^{n+1} - e_h^n}{\Delta t}, \frac{e_h^{n+1} - e_h^n}{\Delta t} \right)_h \quad (7)$$

$$\text{and} \quad \mathcal{E}_{\mathcal{P}}^{n+1/2} = \frac{1}{2} \left( P_{\mathcal{P}}(\Delta t^2 A_h) A_h \frac{e_h^{n+1} + e_h^n}{2}, \frac{e_h^{n+1} + e_h^n}{2} \right)_h. \quad (8)$$

It can be shown that the total energy satisfies

$$\frac{\mathcal{E}_h^{n+1/2} - \mathcal{E}_h^{n-1/2}}{\Delta t} = \left( r_h^n, \frac{e_h^{n+1} - e_h^{n-1}}{2\Delta t} \right)_h. \quad (9)$$

A necessary condition for the schemes' stability is the non-negativity of the kinetic energy (7) and the potential energy (8). Such a condition is achieved if the following hypothesis holds

**Hypothesis 2.1** *There exists  $\alpha > 0$  such that  $P_{\mathcal{K}}(x)$  and  $P_{\mathcal{P}}(x)$  are non-negative for all  $x \in [0, \alpha]$ .*

For the schemes we consider we have

- TS ( $0 \leq \theta < 1/4$ ):  $\alpha = 4/(1 - 4\theta)$ .    • TS ( $\theta \geq 1/4$ ):  $\alpha = +\infty$ .    • SLF:  $\alpha = 16$ .
- TPS ( $\theta = 0, \varphi = 0$ ):  $\alpha = 12$ .    • TPS ( $\theta \geq 1/4, \varphi \geq 1/4$ ):  $\alpha = +\infty$ .
- TPS  $\forall (\theta, \varphi)$ : see theorem 5.1 of [6].

**Lemma 2.1** Assume Hypothesis 2.1 holds. The energies  $\mathcal{E}_K^{n+1/2}$  and  $\mathcal{E}_P^{n+1/2}$  are non-negative for all  $n \geq 0$  if  $\Delta t$  satisfies

$$\Delta t^2 \leq \frac{\alpha}{\rho(A_h)} \quad (\text{CFL condition}) \quad (10)$$

where  $\rho(A_h)$  is the largest eigenvalue of  $A_h$ .

Note that when  $\alpha$  is finite (resp. infinite) the scheme is conditionally (resp. unconditionally) stable. The non-negativity of the energy  $\mathcal{E}_h^{n+1/2}$  is however not sufficient to ensure the stability of the schemes in  $H$  since it is only a semi-norm. In the sequel, the polynomial functions  $P_P$  and  $P_K$  will be assumed to satisfy the following hypothesis

**Hypothesis 2.2** We assume that there exists a partitioning of  $[0, \alpha]$  into two disjoint subsets  $\mathcal{J}_K$  and  $\mathcal{J}_P$  such that there exist two strictly positive constants  $C_K, C_P$ , such that

$$C_K \leq P_K(x), \quad \forall x \in \mathcal{J}_K, \quad C_P \leq x P_P(x), \quad \forall x \in \mathcal{J}_P.$$

**Définition 2.3** We define for any  $v_h \in V_h$ , the projection operators

$$\Pi_K(v_h) = \sum_{\substack{\Delta t^2 \lambda \in \mathcal{J}_K \\ \lambda \in Sp(A_h)}} (v_h, w_h^\lambda)_h w_h^\lambda \quad \text{and} \quad \Pi_P(v_h) = \sum_{\substack{\Delta t^2 \lambda \in \mathcal{J}_P \\ \lambda \in Sp(A_h)}} (v_h, w_h^\lambda)_h w_h^\lambda$$

The operator  $\Pi_K$  (resp.  $\Pi_P$ ) corresponds to modal projection operators on subsets of  $V_h$  for which the kinetic (resp. potential) part of the energy is a uniform upper bound of the  $H$ -norm with respect to  $h$  and  $\Delta t$ . These ideas are specified in the following Lemma.

**Lemma 2.2** Assume Hypothesis 2.2 holds. Then, for any  $h > 0$  and  $\Delta t > 0$  satisfying the CFL condition (10), and for any  $v_h \in V_h$ ,  $v_h = \Pi_K(v_h) + \Pi_P(v_h)$  and

$$|\Pi_K(v_h)|_h^2 \leq C_K^{-1} (P_K(\Delta t^2 A_h) v_h, v_h)_h, \quad |\Pi_P(v_h)|_h^2 \leq C_P^{-1} (\Delta t^2 A_h P_P(\Delta t^2 A_h) v_h, v_h)_h. \quad (11)$$

Then the following estimates can be written

**Lemma 2.3** Let the series  $\{e_h^n\}$  satisfy (6) and Hypothesis 2.2 holds. Then for all  $h > 0$  and  $\Delta t > 0$  satisfying the CFL condition (10), and for all  $n \geq 1$ ,

$$\sqrt{\mathcal{E}_h^{n+1/2}} \leq \sqrt{\mathcal{E}_h^{1/2}} + \sqrt{2} \gamma \Delta t \sum_{\ell=1}^n |r_h^\ell|_h \quad (12)$$

$$|e_h^{n+1}|_h \leq \sqrt{2} |e_h^1|_h + 2 \gamma t^n \sqrt{2 \mathcal{E}_h^{1/2}} + 4 \gamma^2 \Delta t^2 \sum_{\ell=1}^n \sum_{k=1}^{\ell} |r_h^k|_h, \quad (13)$$

where  $\gamma = C_P^{-1/2} + C_K^{-1/2}/2$ .

It is important to notify here that  $\gamma$  is independent of  $h$  and  $\Delta t$ , this is the key point to obtain uniform estimates in  $\Delta t$ . Indeed as pointed out in Remark 10 of [8] the classical stability proof for the (TS) blows up when the time step approaches its greatest admissible value in inequality (10). The constant  $C_K$  and  $C_P$  (and therefore  $\gamma$ ) are computed for the considered schemes in appendix, we find  $\gamma \leq \sqrt{2}$  for all schemes.

**Proof 2.4 (of Lemma 2.3)** From Cauchy-Schwartz inequality applied to (9) we get

$$\frac{\mathcal{E}_h^{n+1/2} - \mathcal{E}_h^{n-1/2}}{\Delta t} \leq |r_h^n|_h \left| \frac{e_h^{n+1} - e_h^{n-1}}{2\Delta t} \right|_h \leq |r_h^n|_h \underbrace{\left| \Pi_K \left( \frac{e_h^{n+1} - e_h^{n-1}}{2\Delta t} \right) \right|_h}_{\Xi} + |r_h^n|_h \underbrace{\left| \Pi_P \left( \frac{e_h^{n+1} - e_h^{n-1}}{2\Delta t} \right) \right|_h}_{\Phi},$$

where  $\Pi_K$  and  $\Pi_P$  are the projection operators given by definition 2.3. Now we use the two inequalities of (11) on respectively  $\Xi$  and  $\Phi$  and then recognize the square root of the kinetic and potential parts

of the energy. To shorten the notation we introduce the time finite difference of the error:  $d_h^{n+1/2} := (e_h^{n+1} - e_h^n)/\Delta t$ . We have

$$\begin{aligned} 2\Xi &\leq \left| \Pi_{\mathcal{K}}(d_h^{n+1/2}) \right|_h + \left| \Pi_{\mathcal{K}}(d_h^{n-1/2}) \right|_h \\ &\leq C_{\mathcal{K}}^{-1/2} \sqrt{\left( P_{\mathcal{K}}(\Delta t^2 A_h) d_h^{n+1/2}, d_h^{n+1/2} \right)_h} + C_{\mathcal{K}}^{-1/2} \sqrt{\left( P_{\mathcal{K}}(\Delta t^2 A_h) d_h^{n-1/2}, d_h^{n-1/2} \right)_h} \\ &\leq C_{\mathcal{K}}^{-1/2} \left[ \sqrt{2 \mathcal{E}_h^{n+1/2}} + \sqrt{2 \mathcal{E}_h^{n-1/2}} \right], \end{aligned}$$

note that the second inequality is obtained by taking  $v_h = d_h^{n+1/2}$  and  $v_h = d_h^{n-1/2}$  in the first inequality of (11). A similar procedure leads to

$$\Phi \leq C_{\mathcal{P}}^{-1/2} \left[ \sqrt{2 \mathcal{E}_h^{n+1/2}} + \sqrt{2 \mathcal{E}_h^{n-1/2}} \right].$$

By denoting  $\gamma = C_{\mathcal{P}}^{-1/2} + C_{\mathcal{K}}^{-1/2}/2$  we get from

$$\frac{\mathcal{E}_h^{n+1/2} - \mathcal{E}_h^{n-1/2}}{\Delta t} \leq \gamma |r_h^n|_h \left[ \sqrt{2 \mathcal{E}_h^{n+1/2}} + \sqrt{2 \mathcal{E}_h^{n-1/2}} \right],$$

which classically leads to inequality (12). The other part of the proof concerns the upper bound on  $|e_h^{n+1}|_h$ . We write

$$|e_h^{n+1}|_h \leq \underbrace{|\Pi_{\mathcal{K}}(e_h^{n+1})|_h}_{\Omega} + \underbrace{|\Pi_{\mathcal{P}}(e_h^{n+1})|_h}_{\Upsilon}$$

Introducing artificially the term  $e_h^n$  we can write

$$\begin{aligned} \Omega &\leq |\Pi_{\mathcal{K}}(e_h^n)|_h + \Delta t \left| \Pi_{\mathcal{K}}(d_h^{n+1/2}) \right|_h \leq |\Pi_{\mathcal{K}}(e_h^n)|_h + \Delta t C_{\mathcal{K}}^{-1/2} \sqrt{\left( P_{\mathcal{K}}(\Delta t^2 A_h) d_h^{n+1/2}, d_h^{n+1/2} \right)_h} \\ &\leq |\Pi_{\mathcal{K}}(e_h^n)|_h + \Delta t C_{\mathcal{K}}^{-1/2} \sqrt{2 \mathcal{E}_h^{n+1/2}} \end{aligned}$$

The same procedure leads to

$$\Upsilon \leq |\Pi_{\mathcal{P}}(e_h^n)|_h + 2 \Delta t C_{\mathcal{P}}^{-1/2} \sqrt{2 \mathcal{E}_h^{n+1/2}}.$$

We use recursively the former inequalities down to  $n = 1$  to get,

$$|e_h^{n+1}|_h \leq \underbrace{|\Pi_{\mathcal{K}}(e_h^1)|_h + |\Pi_{\mathcal{P}}(e_h^1)|_h}_{\leq \sqrt{2} |e_h^1|_h} + 2\sqrt{2} \Delta t \gamma \sum_{\ell=1}^n \sqrt{\mathcal{E}_h^{\ell+1/2}}.$$

We can reuse identity (12) to get (13).

### 3. Space time convergence result

Let  $\varepsilon_h^n$  be the error between the continuous solution and the fully discrete solution. Note that we have  $u(t^n) - u_h^n = e_h(t^n) + e_h^n$ , thanks to this decomposition, assumption 1.1 and lemma 2.3 we can state our final result.



**Theorem 3.1** Assume that  $e_h^0 = e_h^1 = 0$  and assume that Hypothesis 2.2 holds and that Hypothesis 1.2 holds for every sufficiently large  $(p, q)$ , then, for all  $h > 0$  and all  $\Delta t$  satisfying the CFL condition (2.1) the following uniform convergence results hold

- $\theta$ -scheme (TS)

$$\sup_{t^n \in [0, T]} |u(t^n) - u_h^n| \leq C_H \left[ \Delta t^2 \gamma^2 T^2 \left( \frac{C_{0,4}}{6} + 2\theta C_{2,2} \right) + \delta(h) \right].$$

- Stabilized leap-frog scheme (SLF)

$$\sup_{t^n \in [0, T]} |u(t^n) - u_h^n| \leq C_H \left[ \Delta t^2 \gamma^2 T^2 \left( \frac{C_{0,4}}{6} + \frac{C_{4,0}}{8} \right) + \delta(h) \right].$$

- $(\theta, \varphi)$ -scheme (TPS)

$$\sup_{t^n \in [0, T]} |u(t^n) - u_h^n| \leq C_H \left[ \Delta t^4 \gamma^2 T^2 \left( \frac{C_{0,6}}{180} + \theta \frac{C_{2,4}}{6} + 2\varphi \left( \theta - \frac{1}{12} \right) C_{4,2} \right) + \delta(h) \right].$$

Note that, in particular these convergence results hold if  $\Delta t^2 = \alpha/\rho(A_h)$ .

### Appendix: Expression of $\gamma$ for the considered numerical schemes

- TS ( $0 \leq \theta < 1/4$ ). We are looking for a partitioning  $\mathcal{J}_K \cup \mathcal{J}_P$  of the interval  $[0, \frac{4}{(1-4\theta)}]$  and two positive constant  $C_K$  and  $C_P$  such that

$$C_K \leq P_K(x) = 1 + \left( \theta - \frac{1}{4} \right) x, \quad \forall x \in \mathcal{J}_K, \quad C_P \leq x P_P(x) = x, \quad \forall x \in \mathcal{J}_P$$

We see that the first inequality cannot be fulfilled for  $x = \frac{4}{(1-4\theta)} \in \mathcal{J}_K$  and the second one for  $x = 0 \in \mathcal{J}_P$ . The proposed partitioning is the following

$$\mathcal{J}_K = [0, a^2[, \quad C_K = 1 + \left( \theta - \frac{1}{4} \right) a^2, \quad \mathcal{J}_P = \left[ a^2, \frac{1}{(1/4 - \theta)} \right], \quad C_P = a^2$$

where  $a$  can be chosen according to  $\theta$  in order to minimize  $\gamma = C_P^{-1/2} + C_K^{-1/2}/2$ . The peculiar optimal value is given in Lemma 2.3 of [9], it reads

$$a(\theta) = \sqrt{\frac{4}{(1-4\theta)^{2/3} + (1-4\theta)}}; \quad \gamma(\theta) = \frac{1}{a(\theta)} + \frac{1}{\sqrt{4 - (1-4\theta)a(\theta)^2}}$$

Especially if  $\theta = 0$  we get  $\gamma = \sqrt{2}$ .

- TS ( $\theta \geq 1/4$ ). In this case the polynomial functions  $P_K(x)$  and  $P_P(x)$  are positive on the whole interval  $[0, +\infty]$  and moreover  $P_K(x) \geq 1 \forall x \geq 0$ . Therefore one can choose  $\mathcal{J}_K = [0, +\infty]$  and  $\mathcal{J}_P = \{\emptyset\}$ . As a consequence one can set  $C_K = 1$  and since  $C_P$  has no influence and can be chosen formally equal to  $+\infty$ . We get  $\gamma = 1/2$ .

- SLF. We are looking for a partitioning  $\mathcal{J}_K \cup \mathcal{J}_P$  of the interval  $[0, 16]$  and two constant  $C_K$  and  $C_P$  such that

$$C_K \leq \left( 1 - \frac{x}{8} \right)^2 = 1 - \frac{x}{4} + \frac{x^2}{64}, \quad \forall x \in \mathcal{J}_K, \quad C_P \leq x \left( 1 - \frac{x}{16} \right), \quad \forall x \in \mathcal{J}_P$$

Out of symmetry with respect to  $x = 8$ , we look for  $\mathcal{J}_{\mathcal{K}} = [0, 8(1 - a) \cup 8(1 + a), 16]$  and  $\mathcal{J}_{\mathcal{P}} = [8(1 - a), 8(1 + a)]$ . Then  $C_{\mathcal{K}} = a^2$  and  $C_{\mathcal{P}} = 4(1 - a^2)$ . Therefore

$$\gamma = C_{\mathcal{P}}^{-1/2} + \frac{C_{\mathcal{K}}^{-1/2}}{2} = \frac{1}{2\sqrt{1 - a^2}} + \frac{1}{2a}.$$

The choice of  $a$  that minimizes the value of  $\gamma$  is  $a = \frac{1}{\sqrt{2}}$  and leads to  $\gamma = \sqrt{2}$ .

- *TPS* ( $\theta = 0, \varphi = 0$ ). The approach is similar to the one presented above. We are looking for a partitioning  $\mathcal{J}_{\mathcal{K}} \cup \mathcal{J}_{\mathcal{P}}$  of the interval  $[0, 12]$  and two constant  $C_{\mathcal{K}}$  and  $C_{\mathcal{P}}$  such that

$$C_{\mathcal{K}} \leq 1 - \frac{x}{4} + \frac{x^2}{48}, \quad \forall x \in \mathcal{J}_{\mathcal{K}}, \quad C_{\mathcal{P}} \leq x \left(1 - \frac{x}{12}\right), \quad \forall x \in \mathcal{J}_{\mathcal{P}}$$

We look for  $\mathcal{J}_{\mathcal{K}} = [0, 6(1 - a) \cup 6(1 + a), 12]$  and  $\mathcal{J}_{\mathcal{P}} = [6(1 - a), 6(1 + a)]$ , this choice gives some values for  $C_{\mathcal{K}}$  and  $C_{\mathcal{P}}$  that can be optimized to minimize  $\gamma$ . The optimal choice is  $a = \frac{1}{\sqrt{3}}$  and  $\gamma = \sqrt{2}$ .

- *TPS* ( $\theta \geq 1/4, \varphi \geq 1/4$ ). In the case of the  $(\theta, \varphi)$ -scheme with  $\theta \geq 1/4$  and  $\varphi \geq 1/4$ , the polynomial functions  $P_{\mathcal{K}}(x)$  and  $P_{\mathcal{P}}(x)$  read

$$P_{\mathcal{K}}(x) = 1 + \left(\theta - \frac{1}{4}\right)x + \left(\varphi - \frac{1}{4}\right)\left(\theta - \frac{1}{12}\right)x^2, \quad P_{\mathcal{P}}(x) = 1 + \left(\theta - \frac{1}{12}\right)x.$$

The polynomial  $P_{\mathcal{K}}(x)$  and  $P_{\mathcal{P}}(x)$  are positive on the whole interval  $[0, +\infty]$  and  $P_{\mathcal{K}}(x) \geq 1 \quad \forall x \geq 0$ . Therefore one can choose  $\mathcal{J}_{\mathcal{K}} = [0, +\infty]$  and  $\mathcal{J}_{\mathcal{P}} = \{\emptyset\}$ . We get  $\gamma = 1/2$ .

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