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Moments estimation of magnetic source terms from partial data

Laurent Baratchart* Sylvain Chevillard* Juliette Leblond*

Eduardo Andrade Lima[†] Dmitry Ponomarev[‡]

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1 Introduction

Estimating the net moment of a magnetization distribution is a fundamental issue in magneto-statics and a basic step of most magnetometric studies, be it in geosciences, medical imaging, material sciences and so on. Classical magnetometers infer the net moment from a set of measurements of the magnetic field taken at some distance from the sample, by comparing the latter to a dipole in which case simple and explicit formulas connect the moment with the field [4]. This is of course an approximation of the ideal situation where the magnetometer lies at infinity, a position from which the sample could indeed be regarded as a pointwise dipole.

While such an approximation is certainly valid if the field is measured with sufficient accuracy far enough from the support of the magnetization, it is not satisfactory when dealing with weakly magnetized objects whose field gets easily blurred by spurious magnetic sources away from the sample. Still, analyzing weak magnetization distributions is of considerable interest, *e.g.*, in paleomagnetism. To this effect, scanning magnetic microscopes were developed in recent years which are capable of measuring very weak fields at submillimetric distance from an object. Such advances generate a need to develop alternative techniques to estimate the net moment.

From a mathematical viewpoint, the problem is to recover the mean of a compactly supported vector field from knowledge of the gradient of the potential of its divergence in some region near the support. Indeed, it follows from Maxwell's equations in the magnetostatic case [5, Ch. 5] that the field and the magnetization distribution are connected through an elliptic partial differential equation of Poisson type. More precisely, the magnetic field is the gradient of a scalar magnetic potential whose Laplacian is the divergence of the magnetization.

The main feature of this inverse problem is the geometry of the measurement set. In the present paper, we consider the case where measurements are taken on a plane which does not intersect the support of the magnetization. For instance this setup is typical of experiments conducted with a superconducting interference device (SQUID) when studying rock samples, see [11, 6]. In practice, two extra-features complicate the situation further. The first is that only a single component of the field can be measured, namely the one which is orthogonal to the measurement plane. This is because the proximity to the sample and the necessary precision of calibration make it difficult to rotate the SQUID. The second is that measurements can only be performed on a finite portion of the measurement plane, close enough to the sample.

In analogy with the expansion of a dipolar field at infinity, which underpins classical magnetometry, the first and most basic issue here is perhaps to get hold of formulas connecting

*Inria, project APICS, 2004 route des Lucioles, BP 93, 06 902 Sophia-Antipolis Cedex, France.

[†]Department of Earth, Atmospheric and Planetary Sciences, Massachusetts Institute of Technology, Cambridge, MA 02139, USA.

[‡]Laboratoire POEMS, ENSTA ParisTech, 828 boulevard des Maréchaux, 91 762 Palaiseau Cedex, France.

the normal component of the magnetic field on a plane to the moment of the magnetization that generates the field, in such a way that knowledge of this component on part of that plane yields approximate formulas for the moment. Surprisingly perhaps, no such formulas seem readily available in the literature and the goal of the present paper is to provide one with some.

We point out that net moment estimation embeds in the larger inverse problem of full magnetization recovery from field measurements. The latter is ill-posed, not even injective. For thin supports (that can be identified with planar sets), non-uniqueness issues are analyzed in [1] and some recovery schemes are considered in [7] for unidirectional magnetizations. In contrast, net moment recovery is well-posed in that magnetizations producing the same normal component for the magnetic field on an open set of a plane must have the same moment (see [2]). Besides, the net moment furnishes valuable information on the magnetization which may be used in full recovery schemes. This motivates investigating recovery schemes for the moment of the magnetization.

Up to a rotation, we may assume for the ease of discussion that the measurement plane is horizontal and that the magnetization is located below this plane. The questions we face are thus: *how can the vertical component of the magnetic field on a portion of horizontal plane buy us an approximation of the moment of the magnetization generating that field? And how does the error decay when that portion of horizontal plane grows large?* In this connection, we mention that such asymptotics for the net moment were taken up in [9, Part III, Sec. 5, 6] for circular measurement areas, using Fourier techniques and tools from harmonic analysis. As we will see, formulas of a similar type can be obtained using elementary properties of homogeneous polynomials and Taylor expansions for rectangular measurement sets. In the present work, we carry out in detail the corresponding computations when the measurement set is a square.

The overview is as follows. The problem is set up in Section 2 and the main approximation results are stated in Section 3. Their proofs are given in Sections 4, 5 and in the Appendix. We discuss in Section 6 how these results can be combined to improve the precision of the moment's estimates. Finally, concluding remarks are provided in Section 7.

2 Notations, problem setting

Given $s, r > 0$, we consider a parallelepiped $\mathcal{A} = [-s, s]^2 \times [0, r] \subset \mathbb{R}^3$ to contain the volume of the sample. Arbitrary points of \mathbb{R}^3 will be denoted as $\mathbf{x} = (x_1, x_2, x_3)$, while $\mathbf{t} = (t_1, t_2, t_3)$ will represent an arbitrary point of \mathcal{A} .

For $i = 1, 2, 3$, we suppose we are given a real-valued function $m_i \in L^1(\mathcal{A})$, the Lebesgue space of summable functions on \mathcal{A} . We denote by \mathbf{m} the magnetization vector field (m_1, m_2, m_3) of components m_i . A volumetric magnetization compactly supported on the slab \mathcal{A} is modeled by the vector field on \mathbb{R}^3 , $\mathbf{x} \mapsto \widetilde{\mathbf{m}}(\mathbf{x})$ where, for $i = 1, 2, 3$, \widetilde{m}_i denotes the function m_i extended by 0 outside \mathcal{A} , i.e., $\widetilde{m}_i(\mathbf{x}) = m_i(\mathbf{x})$ if $\mathbf{x} \in \mathcal{A}$ and $\widetilde{m}_i(\mathbf{x}) = 0$ otherwise. For any $m \in L^1(\mathcal{A})$, we denote by $\langle m \rangle$ the net moment given by the mean value of m :

$$\langle m \rangle = \iiint_{\mathcal{A}} m(\mathbf{t}) \, d\mathbf{t}.$$

More generally, the net moment of m is its 0-th order moment, while the 1-st order moments are the quantities $\langle t_1 m \rangle$, $\langle t_2 m \rangle$, and $\langle t_3 m \rangle$, the 2-nd order moments are the quantities $\langle t_i t_j m \rangle$ (with arbitrary i and j in $\{1, 2, 3\}$), etc.

As recalled in [1], the magnetic field produced by the magnetized slab $(\mathcal{A}, \mathbf{m})$ is $\mathbf{B} = -\mu_0 \nabla \phi$,

where $\mu_0 = 4\pi \times 10^{-7}$ and ϕ is the scalar magnetic potential defined at each point $\mathbf{x} \notin \mathcal{A}$ by

$$\phi(\mathbf{x}) = \frac{1}{4\pi} \iiint_{\mathcal{A}} \frac{\langle \mathbf{m}(\mathbf{t}), \mathbf{x} - \mathbf{t} \rangle}{\|\mathbf{x} - \mathbf{t}\|^3} d\mathbf{t}, \quad (1)$$

with $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ to designate the Euclidean norm and $\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 + x_2y_2 + x_3y_3$ the Euclidean scalar product. Denoting with P_i the function $\mathbf{x} \mapsto \frac{x_i}{\|\mathbf{x}\|^3}$, for $i = 1, 2, 3$, and \star is the convolution product between functions defined on \mathbb{R}^3 , we can express Equation (1) with the following shorter expression:

$$\phi(\mathbf{x}) = \frac{1}{4\pi} (P_1 \star \tilde{m}_1 + P_2 \star \tilde{m}_2 + P_3 \star \tilde{m}_3)(\mathbf{x}). \quad (2)$$

In the following, we will assume that we have measurements of the vertical component B_3 of \mathbf{B} at a given height $x_3 = z > r$. This defines a function on the plane and we denote by $B_3[\mathbf{m}, z](x_1, x_2) = B_3(x_1, x_2, z)$ its value at a point $(x_1, x_2) \in \mathbb{R}^2$. For $\mathbf{x} \notin \mathcal{A}$, we have:

$$-\frac{4\pi}{\mu_0} B_3[\mathbf{m}, x_3](x_1, x_2) = \partial_{x_1}(P_3 \star \tilde{m}_1 - P_1 \star \tilde{m}_3)(\mathbf{x}) + \partial_{x_2}(P_3 \star \tilde{m}_2 - P_2 \star \tilde{m}_3)(\mathbf{x}). \quad (3)$$

This is easily checked by a direct computation: on the one hand, starting from $B_3[\mathbf{m}, x_3](x_1, x_2) = -\mu_0 \partial_{x_3} \phi(\mathbf{x})$ and computing the derivative with respect to x_3 of ϕ as given in Equation (1); on the other hand, explicitly computing the derivatives with respect to x_1 and x_2 in Equation (3). Another (deeper but more involved) way of seeing it consists in recognizing Poisson and Riesz transforms in Equation (2) and using their properties with respect to differentiation (see [2, 10] for instance, where this is done in the case of a 2D slab).

Finally, for $R > 0$, we introduce the planar measurement areas $Q_R = [-R, R]^2$ (square), $S_R = \{(x_1, x_2) \in \mathbb{R}^2, |x_1| + |x_2| \leq R\}$ (diamond) and $A_R = \{(x_1, x_2) \in \mathbb{R}^2, x_1^2 + x_2^2 \leq R^2\}$ (disk) as illustrated in Figure 1.

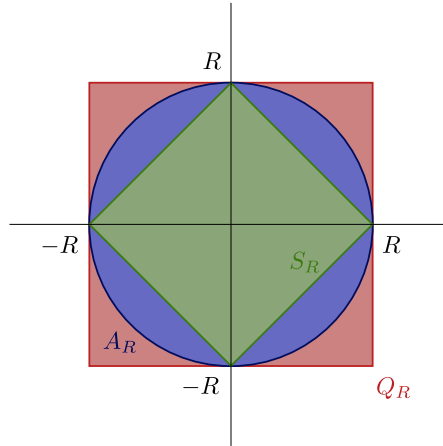


Figure 1: Shapes of Q_R , S_R , A_R .

3 Main results

Our main result is summed up with the following theorem that provides asymptotic expansions (as R goes large) of simple integrals involving $B_3[\mathbf{m}, z]$, in terms of the successive moments of the magnetization \mathbf{m} .

Theorem 1. *Let notation and assumptions be as above. On the square Q_R , it holds that:*

$$\iint_{Q_R} x_1 B_3[\mathbf{m}, z](x_1, x_2) dx_1 dx_2 = \frac{\mu_0}{2} \langle m_1 \rangle + \frac{3\mu_0}{\pi R \sqrt{2}} (\langle t_1 m_3 \rangle + \langle t_3 m_1 \rangle - z \langle m_1 \rangle) + \mathcal{O}\left(\frac{1}{R^3}\right), \quad (4)$$

$$\iint_{Q_R} x_2 B_3[\mathbf{m}, z](x_1, x_2) dx_1 dx_2 = \frac{\mu_0}{2} \langle m_2 \rangle + \frac{3\mu_0}{\pi R \sqrt{2}} (\langle t_2 m_3 \rangle + \langle t_3 m_2 \rangle - z \langle m_2 \rangle) + \mathcal{O}\left(\frac{1}{R^3}\right), \quad (5)$$

$$\begin{aligned} \iint_{Q_R} R B_3[\mathbf{m}, z](x_1, x_2) dx_1 dx_2 &= \frac{2\mu_0}{\pi \sqrt{2}} \langle m_3 \rangle \\ &+ \frac{5\mu_0}{4\pi R^2 \sqrt{2}} \left(-2z^2 \langle m_3 \rangle + 2z (2\langle t_3 m_3 \rangle - \langle t_1 m_1 \rangle - \langle t_2 m_2 \rangle) \right. \\ &\quad \left. + \langle t_1^2 m_3 \rangle + \langle t_2^2 m_3 \rangle - 2\langle t_3^2 m_3 \rangle + 2\langle t_1 t_3 m_1 \rangle + 2\langle t_2 t_3 m_2 \rangle \right) + \mathcal{O}\left(\frac{1}{R^3}\right), \end{aligned} \quad (6)$$

On the diamond S_R , it holds that:

$$\iint_{S_R} x_1 B_3[\mathbf{m}, z](x_1, x_2) dx_1 dx_2 = \frac{\mu_0}{2} \langle m_1 \rangle + \frac{3\mu_0}{\pi R} (\langle t_1 m_3 \rangle + \langle t_3 m_1 \rangle - z \langle m_1 \rangle) + \mathcal{O}\left(\frac{1}{R^3}\right), \quad (7)$$

$$\iint_{S_R} x_2 B_3[\mathbf{m}, z](x_1, x_2) dx_1 dx_2 = \frac{\mu_0}{2} \langle m_2 \rangle + \frac{3\mu_0}{\pi R} (\langle t_2 m_3 \rangle + \langle t_3 m_2 \rangle - z \langle m_2 \rangle) + \mathcal{O}\left(\frac{1}{R^3}\right), \quad (8)$$

$$\begin{aligned} \iint_{S_R} R B_3[\mathbf{m}, z](x_1, x_2) dx_1 dx_2 &= \frac{2\mu_0}{\pi} \langle m_3 \rangle \\ &+ \frac{5\mu_0}{2\pi R^2} \left(-2z^2 \langle m_3 \rangle + 2z (2\langle t_3 m_3 \rangle - \langle t_1 m_1 \rangle - \langle t_2 m_2 \rangle) \right. \\ &\quad \left. + \langle t_1^2 m_3 \rangle + \langle t_2^2 m_3 \rangle - 2\langle t_3^2 m_3 \rangle + 2\langle t_1 t_3 m_1 \rangle + 2\langle t_2 t_3 m_2 \rangle \right) + \mathcal{O}\left(\frac{1}{R^3}\right). \end{aligned} \quad (9)$$

The forthcoming Section 4 will establish Equations (4) and (6). We shall deduce all the other equations from the latter by means of appropriate changes of variable. This is performed in the dedicated Section 5.

The right hand sides of the above formulas furnish expansions in powers of $1/R$ of the first order moments of the measurements which are both of theoretical and practical value, see Remarks 1 and 2. They are of particular interest asymptotically, because when the size of the measurement area increases their accuracies at the first order increases as well. More precisely, as R increases, they furnish more and more accurate approximations of the magnetization moments $\langle m_i \rangle$ ($i = 1, 2, 3$) in terms of the first orders moments of $B_3[\mathbf{m}, z]$ (restricted to the considered area), since the higher order terms in $1/R^k$ in the expansions decrease for $k \geq 1$. To turn this into a net moment recovery scheme, it is thus important to reach a trade-off between precision and robustness as these formulas can only be applied experimentally or numerically for fixed R (quantifying the size of the measurement area). Though these issues will be further discussed in Section 6, observe already that these formula depend on the geometry of the measurement area only through multiplicative factors in front of the magnetization moments, as is the case also for other planar domains like the disk A_R centered at 0 and of radius R (see [9, Part III, Sec. 3.5,

3.6]). Indeed, on the disk A_R :

$$\iint_{A_R} x_1 B_3[\mathbf{m}, z](x_1, x_2) dx_1 dx_2 = \frac{\mu_0}{2} \langle m_1 \rangle + \frac{3\mu_0}{4R} (\langle t_1 m_3 \rangle + \langle t_3 m_1 \rangle - z \langle m_1 \rangle) + \mathcal{O}\left(\frac{1}{R^3}\right), \quad (10)$$

$$\iint_{A_R} x_2 B_3[\mathbf{m}, z](x_1, x_2) dx_1 dx_2 = \frac{\mu_0}{2} \langle m_2 \rangle + \frac{3\mu_0}{4R} (\langle t_2 m_3 \rangle + \langle t_3 m_2 \rangle - z \langle m_2 \rangle) + \mathcal{O}\left(\frac{1}{R^3}\right), \quad (11)$$

$$\begin{aligned} \iint_{A_R} R B_3[\mathbf{m}, z](x_1, x_2) dx_1 dx_2 &= \frac{\mu_0}{2} \langle m_3 \rangle \\ &+ \frac{3\mu_0}{8R^2} \left(-2z^2 \langle m_3 \rangle + 2z (2 \langle t_3 m_3 \rangle - \langle t_1 m_1 \rangle - \langle t_2 m_2 \rangle) \right. \\ &\quad \left. + \langle t_1^2 m_3 \rangle + \langle t_2^2 m_3 \rangle - 2 \langle t_3^2 m_3 \rangle + 2 \langle t_1 t_3 m_1 \rangle + 2 \langle t_2 t_3 m_2 \rangle \right) + \mathcal{O}\left(\frac{1}{R^4}\right). \end{aligned} \quad (12)$$

This follows from [9, Part III, Sec. 3.5, (3.75)] and the related formula [9, Part III, Sec. 3.6, p. 146], taking into account the 3-D character of the slab $(\mathcal{A}, \mathbf{m})$.

4 Proofs of Equations (4) and (6)

4.1 Preliminary results

Before going to the actual proof of Equations (4) and (6) we define functions and establish results that will be of constant use in what follows.

Definition 1. We define the \mathbb{R} -valued functions f , g , k and ℓ for $\mathbf{x} \in \mathbb{R}^3$ with $x_3 > 0$ by:

$$\begin{aligned} f(\mathbf{x}) &= \frac{x_2}{x_1^2 + x_3^2} \cdot \frac{1}{\|\mathbf{x}\|}, \\ g(\mathbf{x}) &= \frac{-1}{\|\mathbf{x}\|}, \\ k(\mathbf{x}) &= \frac{1}{x_3} \arctan\left(\frac{x_1 x_2}{x_3 \|\mathbf{x}\|}\right), \\ \ell(\mathbf{x}) &= -\operatorname{arcsinh}\left(\frac{x_2}{(x_1^2 + x_3^2)^{1/2}}\right). \end{aligned}$$

These functions are indefinite integrals of expressions that will naturally come up when rewriting the left hand sides of Equations (4) and (6). This is capsulized in the following proposition whose proof reduces to straightforward computations. Below, the symbol ∂_{x_i} stands for the partial derivative with respect to the coordinate x_i ($i = 1, 2, 3$) while $\partial_{x_1 x_2}^2 = \partial_{x_1} \partial_{x_2}$.

Proposition 1. For any $\mathbf{x} \in \mathbb{R}^3$ with $x_3 > 0$, we have

$$\begin{aligned} \partial_{x_2} f(\mathbf{x}) &= 1/\|\mathbf{x}\|^3, \\ \partial_{x_1} g(\mathbf{x}) &= x_1/\|\mathbf{x}\|^3, \\ \partial_{x_1 x_2}^2 k(\mathbf{x}) &= 1/\|\mathbf{x}\|^3, \\ \partial_{x_1 x_2}^2 \ell(\mathbf{x}) &= x_1/\|\mathbf{x}\|^3. \end{aligned}$$

We will need asymptotic expansions of expressions of the form $f(R - t_1, R - t_2, z - t_3)$, $f(-R - t_1, R - t_2, z - t_3)$, etc., when R goes large. To this effect, it is convenient to introduce companion functions to f , g , k and ℓ as follows.

Definition 2. Let $\mathbf{t} = (t_1, t_2, t_3) \in \mathcal{A}$. We define $F_{\mathbf{t}}$, $G_{\mathbf{t}}$, $K_{\mathbf{t}}$ and $L_{\mathbf{t}}$ from \mathbb{R} to \mathbb{R} by

$$\begin{aligned} F_{\mathbf{t}}(R) &= f(R - t_1, R - t_2, z - t_3), & G_{\mathbf{t}}(R) &= g(R - t_1, R - t_2, z - t_3), \\ K_{\mathbf{t}}(R) &= k(R - t_1, R - t_2, z - t_3), & \text{and } L_{\mathbf{t}}(R) &= \ell(R - t_1, R - t_2, z - t_3). \end{aligned}$$

One easily checks the following proposition.

Proposition 2. For any $(t_1, t_2) \in [-s, s]^2$, any $t_3 > 0$ and any $R \in \mathbb{R}$,

$$\begin{aligned} f(-R - t_1, R - t_2, z - t_3) &= F_{(-t_1, t_2, t_3)}(R), & g(-R - t_1, R - t_2, z - t_3) &= G_{(-t_1, t_2, t_3)}(R), \\ f(-R - t_1, -R - t_2, z - t_3) &= -F_{(-t_1, -t_2, t_3)}(R), & g(-R - t_1, -R - t_2, z - t_3) &= G_{(-t_1, -t_2, t_3)}(R), \\ f(R - t_1, -R - t_2, z - t_3) &= -F_{(t_1, -t_2, t_3)}(R); & g(R - t_1, -R - t_2, z - t_3) &= G_{(t_1, -t_2, t_3)}(R); \\ k(-R - t_1, R - t_2, z - t_3) &= -K_{(-t_1, t_2, t_3)}(R), & \ell(-R - t_1, R - t_2, z - t_3) &= L_{(-t_1, t_2, t_3)}(R), \\ k(-R - t_1, -R - t_2, z - t_3) &= K_{(-t_1, -t_2, t_3)}(R), & \ell(-R - t_1, -R - t_2, z - t_3) &= -L_{(-t_1, -t_2, t_3)}(R), \\ k(R - t_1, -R - t_2, z - t_3) &= -K_{(t_1, -t_2, t_3)}(R); & \ell(R - t_1, -R - t_2, z - t_3) &= -L_{(t_1, -t_2, t_3)}(R). \end{aligned}$$

The essential ingredient for the proof of Equations (4) and (6) is to get asymptotic expansions of the functions $F_{\mathbf{t}}$, $G_{\mathbf{t}}$, $K_{\mathbf{t}}$, $L_{\mathbf{t}}$ (with respect to powers of $1/R$), with explicit error bounds. The important point is that these error bounds are uniform with respect to the variable $\mathbf{t} \in \mathcal{A}$, allowing us to integrate them on \mathcal{A} . Such expansions are given in Lemma 1 below. Before stating it, we need to introduce more notations and to recall some properties of homogeneous polynomials.

We fix once and for all two positive constants ω_s and ω_z such that $\omega_s < 1$, and we pose

$$C = \max \left\{ \frac{s}{\omega_s}, \frac{z}{\omega_z} \right\}, \quad (13)$$

hence $\frac{s}{C} \leq \omega_s < 1$ and $\frac{z}{C} \leq \omega_z$. From now on, we assume that $R \geq C$ with C given by Equation (13).

Remark 1. Introducing the rescaling quantities ω_s and ω_z is a means to assume that $R \geq C$ whatever the dimensions s and r of \mathcal{A} and the height z of the measurement area (one can simply take $\omega_z = z/R$ and $\omega_s = s/R$, if $R > s$). The assumption $R \geq C$ reflects of course the asymptotic character of the present study. Moreover, the quantities ω_s and ω_z conveniently allow us to specify how large R should be relative to s and z for the error estimates in the expansions below to hold true (see Lemmas to come and Remark 2 at the end of the Appendix).

Let $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}^3$. If $a_n(\boldsymbol{\gamma})$ is a homogeneous polynomial of degree n with real coefficients, i.e.,

$$a_n(\boldsymbol{\gamma}) = \sum_{i=0}^n \sum_{j=0}^{n-i} \alpha_{i,j} \gamma_1^i \gamma_2^j \gamma_3^{n-i-j},$$

where $\alpha_{i,j} \in \mathbb{R}$, we define the associated homogeneous polynomial $A_n(s, z)$ in the variables s and z by:

$$A_n(s, z) = \sum_{i=0}^n \sum_{j=0}^{n-i} |\alpha_{i,j}| s^{i+j} z^{n-(i+j)}. \quad (14)$$

Observe that whenever $\boldsymbol{\gamma} \in [-s, s]^2 \times (0, z]$, we have $|a_n(\boldsymbol{\gamma})| \leq A_n(s, z)$. In particular (recall from Section 2 that $\mathcal{A} = [-s, s]^2 \times [0, r]$ and $z > r$),

$$\forall \mathbf{t} \in \mathcal{A}, \quad |a_n(t_1, t_2, z - t_3)| \leq A_n(s, z). \quad (15)$$

Moreover, if

$$\Delta_n(s, z) = \sum_{k=0}^n b_k s^k z^{n-k}$$

is a homogeneous polynomial of degree n in s and z with $b_k \in \mathbb{R}$, we define another associated homogeneous polynomial $\Delta_{n,1}(s, z)$ of degree $n-1$ by

$$\Delta_{n,1}(s, z) = (|b_0|\omega_z)z^{n-1} + \sum_{k=1}^n (|b_k|\omega_s) s^{k-1} z^{n-k}. \quad (16)$$

Notice that for any $\xi \in [0, 1/C]$ we have $|z\xi| \leq \omega_z$ and $|s\xi| \leq \omega_s$ by Equation (13); therefore $|\Delta_n(s, z) \xi^n| \leq \Delta_{n,1}(s, z) \xi^{n-1}$. We denote by $\Delta_{n,2}(s, z)$ the polynomial obtained by applying the same process to $\Delta_{n,1}(s, z)$, i.e., $\Delta_{n,2}(s, z) = \Delta_{n,1,1}(s, z)$, and more generally we put $\Delta_{n,p+1}(s, z) = \Delta_{n,p,1}(s, z)$. By simple induction, we see that $\Delta_{n,p}(s, z)$ is a homogeneous polynomial of degree $n-p$ and

$$\forall \xi \in [0, 1/C], \quad |\Delta_n(s, z) \xi^n| \leq \Delta_{n,p}(s, z) \xi^{n-p}. \quad (17)$$

As a particular case, observe that $\Delta_{n,n}(s, z)$ is in fact a constant and satisfies $|\Delta_n(s, z) \xi^n| \leq \Delta_{n,n}$.

We can now state our first lemma. We postpone its proof to the appendix at the end of this document.

Lemma 1. *For any $\mathbf{t} \in \mathcal{A}$ and any $R \geq C$, where C is defined by Equation (13), it holds*

$$F_t(R) = \frac{1}{\sqrt{2}} \cdot \frac{1}{R^2} + \frac{5t_1 - t_2}{2\sqrt{2}} \cdot \frac{1}{R^3} + \frac{33t_1^2 - 3t_2^2 - 6t_1t_2 - 10t_3^2 + 20zt_3 - 10z^2}{8\sqrt{2}} \cdot \frac{1}{R^4} + \frac{\delta_1(\mathbf{t}, z, R)}{R^5}$$

$$G_t(R) = \frac{-1}{\sqrt{2}} \cdot \frac{1}{R} - \frac{t_1 + t_2}{2\sqrt{2}} \cdot \frac{1}{R^2} - \frac{t_1^2 + t_2^2 + 6t_1t_2 - 2t_3^2 + 4zt_3 - 2z^2}{8\sqrt{2}} \cdot \frac{1}{R^3} + \frac{\delta_2(\mathbf{t}, z, R)}{R^4}$$

$$K_t(R) = \frac{\pi}{2(z - t_3)} - \sqrt{2} \frac{1}{R} - \frac{\sqrt{2}(t_1 + t_2)}{2} \cdot \frac{1}{R^2} + \frac{\delta_3(\mathbf{t}, z, R)}{R^3},$$

$$L_t(R) = -\operatorname{arcsinh}(1) + \frac{t_2 - t_1}{\sqrt{2}} \cdot \frac{1}{R} - \frac{3t_1^2 - t_2^2 - 2t_1t_2 - 2t_3^2 + 4zt_3 - 2z^2}{4\sqrt{2}} \cdot \frac{1}{R^2} + \frac{\delta_4(\mathbf{t}, z, R)}{R^3},$$

where $|\delta_1(\mathbf{t}, z, R)| \leq \Delta_3^{(1)}(s, z)$, $|\delta_2(\mathbf{t}, z, R)| \leq \Delta_3^{(2)}(s, z)$, $|\delta_3(\mathbf{t}, z, R)| \leq \Delta_2^{(3)}(s, z)$ and $|\delta_4(\mathbf{t}, z, R)| \leq \Delta_3^{(4)}(s, z)$ for some homogeneous polynomials $\Delta_n^{(i)}$ in the variables s and z , of degree n .

Corollary 1. *Under the same hypotheses as in Lemma 1, it holds*

$$F_t(R) = \frac{1}{\sqrt{2}} \cdot \frac{1}{R^2} + \frac{5t_1 - t_2}{2\sqrt{2}} \cdot \frac{1}{R^3} + \frac{\delta_5(\mathbf{t}, z, R)}{R^4},$$

$$G_t(R) = \frac{-1}{\sqrt{2}} \cdot \frac{1}{R} - \frac{t_1 + t_2}{2\sqrt{2}} \cdot \frac{1}{R^2} + \frac{\delta_6(\mathbf{t}, z, R)}{R^3},$$

where $|\delta_5(\mathbf{t}, z, R)| \leq \Delta_2^{(5)}(s, z)$ and $|\delta_6(\mathbf{t}, z, R)| \leq \Delta_2^{(6)}(s, z)$, for some homogeneous polynomials $\Delta_2^{(5)}$ and $\Delta_2^{(6)}$ in the variables s and z , of degree 2.

Proof. From Lemma 1 we get

$$|\delta_5(\mathbf{t}, z, R)| = \left| \frac{33t_1^2 - 3t_2^2 - 6t_1t_2 - 10(z - t_3)^2}{8\sqrt{2}} + \frac{\delta_1(\mathbf{t}, z, R)}{R} \right| \leq \frac{42s^2 + 10z^2}{8\sqrt{2}} + \frac{\Delta_3^{(1)}(s, z)}{R}.$$

Now, since $1/R \in [0, 1/C]$, Equation (17) implies that $|\Delta_3^{(1)}(s, z) \cdot 1/R| \leq \Delta_{3,1}^{(1)}(s, z)$. Consequently, $|\delta_5(\mathbf{t}, z, R)| \leq \Delta_2^{(5)}(s, z)$ where $\Delta_2^{(5)}(s, z) = \frac{42s^2 + 10z^2}{8\sqrt{2}} + \Delta_{3,1}^{(1)}(s, z)$ is a homogeneous polynomial of degree 2. The result for $G_t(R)$ is obtained similarly. \square

4.2 Proof of Equation (4)

From Equation (3), we get on integrating the term $x_1 \partial_{x_1}$ by parts that :

$$\begin{aligned} -\frac{4\pi}{\mu_0} \iint_{Q_R} x_1 B_3[\mathbf{m}, x_3](x_1, x_2) dx_1 dx_2 &= \int_{-R}^R \left[x_1 (P_3 \star \tilde{m}_1 - P_1 \star \tilde{m}_3)(\mathbf{x}) \right]_{x_1=-R}^R dx_2 \\ &+ \int_{-R}^R \left[x_1 (P_3 \star \tilde{m}_2 - P_2 \star \tilde{m}_3)(\mathbf{x}) \right]_{x_2=-R}^R dx_1 \\ &- \iint_{Q_R} P_3 \star \tilde{m}_1(\mathbf{x}) dx_1 dx_2 \\ &+ \iint_{Q_R} P_1 \star \tilde{m}_3(\mathbf{x}) dx_1 dx_2. \end{aligned}$$

Now, replacing P_1 , P_2 and P_3 by their expressions and using Fubini's theorem to interchange the integration on Q_R and the integration on \mathcal{A} arising from the convolution, we get in view of Proposition 1 that

$$\iint_{Q_R} x_1 B_3[\mathbf{m}, x_3](x_1, x_2) dx_1 dx_2 = -\frac{\mu_0}{4\pi} \iiint_{\mathcal{A}} \left(I_1(\mathbf{t}) + I_2(\mathbf{t}) + I_3(\mathbf{t}) + I_4(\mathbf{t}) \right) d\mathbf{t}, \quad (18)$$

where, for $\mathbf{t} \in \mathcal{A}$:

$$\begin{aligned} I_1(\mathbf{t}) &= \left[\left[(x_1(x_3 - t_3) m_1(\mathbf{t}) - x_1(x_1 - t_1) m_3(\mathbf{t})) f(\mathbf{x} - \mathbf{t}) \right]_{x_1=-R}^R \right]_{x_2=-R}^R, \\ I_2(\mathbf{t}) &= \left[((x_3 - t_3) m_2(\mathbf{t}) - (x_2 - t_2) m_3(\mathbf{t})) \int_{-R}^R \frac{x_1}{\|\mathbf{x} - \mathbf{t}\|^3} dx_1 \right]_{x_2=-R}^R, \\ I_3(\mathbf{t}) &= -(x_3 - t_3) m_1(\mathbf{t}) \left[\left[k(\mathbf{x} - \mathbf{t}) \right]_{x_1=-R}^R \right]_{x_2=-R}^R, \\ I_4(\mathbf{t}) &= m_3(\mathbf{t}) \left[\left[\ell(\mathbf{x} - \mathbf{t}) \right]_{x_1=-R}^R \right]_{x_2=-R}^R. \end{aligned}$$

To simplify $I_2(\mathbf{t})$ further, we can rewrite x_1 as $(x_1 - t_1) + t_1$ and $\|\mathbf{x} - \mathbf{t}\|$ as $\|(x_2 - t_2, x_1 - t_1, x_3 - t_3)\|$, which leads us to

$$\int_{-R}^R \frac{x_1}{\|\mathbf{x} - \mathbf{t}\|^3} dx_1 = \left[g(\mathbf{x} - \mathbf{t}) + t_1 f(x_2 - t_2, x_1 - t_1, x_3 - t_3) \right]_{x_1=-R}^R.$$

Now, grouping terms according to the parity of the powers of x_1 and x_2 , we obtain the following expressions for $I_1(\mathbf{t})$ and $I_2(\mathbf{t})$:

$$\begin{aligned} I_1(\mathbf{t}) &= ((x_3 - t_3) m_1(\mathbf{t}) + t_1 m_3(\mathbf{t})) \left[\left[x_1 f(\mathbf{x} - \mathbf{t}) \right]_{x_1=-R}^R \right]_{x_2=-R}^R - m_3(\mathbf{t}) \left[\left[x_1^2 f(\mathbf{x} - \mathbf{t}) \right]_{x_1=-R}^R \right]_{x_2=-R}^R, \\ I_2(\mathbf{t}) &= ((x_3 - t_3) m_2(\mathbf{t}) + t_2 m_3(\mathbf{t})) \left[\left[g(\mathbf{x} - \mathbf{t}) \right]_{x_1=-R}^R \right]_{x_2=-R}^R \\ &+ t_1 ((x_3 - t_3) m_2(\mathbf{t}) + t_2 m_3(\mathbf{t})) \left[\left[f(x_2 - t_2, x_1 - t_1, x_3 - t_3) \right]_{x_1=-R}^R \right]_{x_2=-R}^R \\ &- m_3(\mathbf{t}) \left[\left[x_2 g(\mathbf{x} - \mathbf{t}) \right]_{x_1=-R}^R \right]_{x_2=-R}^R \\ &- t_1 m_3 \left[\left[x_2 f(x_2 - t_2, x_1 - t_1, x_3 - t_3) \right]_{x_1=-R}^R \right]_{x_2=-R}^R. \end{aligned}$$

From now on, we assume that x_3 is fixed and equal to z and that the hypotheses of Lemma 1 are satisfied.

Asymptotic expansion of I_1 . Using Proposition 2 and Lemma 1, we get

$$\begin{aligned} \left[\left[x_1 f(\mathbf{x} - \mathbf{t})|_{x_3=z} \right]_{x_1=-R}^R \right]_{x_2=-R}^R &= R \left(F_{\mathbf{t}}(R) + F_{(-t_1, t_2, t_3)}(R) + F_{(t_1, -t_2, t_3)}(R) + F_{(-t_1, -t_2, t_3)}(R) \right) \\ &= \frac{4}{\sqrt{2}} \cdot \frac{1}{R} + \frac{\delta_7(\mathbf{t}, z, R)}{R^3}, \\ \left[\left[x_1^2 f(\mathbf{x} - \mathbf{t})|_{x_3=z} \right]_{x_1=-R}^R \right]_{x_2=-R}^R &= R^2 \left[\left[f(\mathbf{x} - \mathbf{t})|_{x_3=z} \right]_{x_1=-R}^R \right]_{x_2=-R}^R \\ &= R^2 \left(F_{\mathbf{t}}(R) - F_{(-t_1, t_2, t_3)}(R) + F_{(t_1, -t_2, t_3)}(R) - F_{(-t_1, -t_2, t_3)}(R) \right) \\ &= \frac{10t_1}{\sqrt{2}} \cdot \frac{1}{R} + \frac{\delta_8(\mathbf{t}, z, R)}{R^3}, \end{aligned}$$

where $|\delta_7(\mathbf{t}, z, R)| \leq 4\Delta_2^{(5)}(s, z)$ and $|\delta_8(\mathbf{t}, z, R)| \leq 4\Delta_3^{(1)}(s, z)$. Therefore,

$$I_1(\mathbf{t}) = \frac{4(z - t_3) m_1(\mathbf{t}) - 6t_1 m_3(\mathbf{t})}{R\sqrt{2}} + \frac{\delta_9(\mathbf{t}, z, R)}{R^3} \quad (19)$$

where $|\delta_9(\mathbf{t}, z, R)| \leq 4z|m_1(\mathbf{t})|\Delta_2^{(5)}(s, z) + 4|m_3(\mathbf{t})|(s\Delta_2^{(5)}(s, z) + \Delta_3^{(1)}(s, z))$.

Asymptotic expansion of I_2 . A simple interchange between variables x_1 and x_2 and between variables t_1 and t_2 in the computations of the previous paragraph shows that

$$\begin{aligned} \left[\left[f(x_2 - t_2, x_1 - t_1, z - t_3) \right]_{x_1=-R}^R \right]_{x_2=-R}^R &= \frac{10t_2}{\sqrt{2}} \cdot \frac{1}{R^3} + \frac{\delta_8((t_2, t_1, t_3), z, R)}{R^5}, \\ &= \frac{\delta_{10}(\mathbf{t}, z, R)}{R^3}, \\ \left[\left[x_2 f(x_2 - t_2, x_1 - t_1, z - t_3) \right]_{x_1=-R}^R \right]_{x_2=-R}^R &= \frac{4}{\sqrt{2}} \cdot \frac{1}{R} + \frac{\delta_7((t_2, t_1, t_3), z, R)}{R^3}, \end{aligned}$$

where $|\delta_{10}(\mathbf{t}, z, R)| \leq \frac{10s}{\sqrt{2}} + 4\Delta_3^{(1)}(s, z) \frac{1}{R^2}$. Now, observing that $1/R \in [0, 1/C]$, we obtain $|\delta_{10}(\mathbf{t}, z, R)| \leq \Delta_1^{(10)}(s, z)$ where $\Delta_1^{(10)}(s, z)$ is the homogeneous polynomial of degree 1 defined by $\Delta_1^{(10)}(s, z) = \frac{10s}{\sqrt{2}} + 4\Delta_{3,2}^{(1)}(s, z)$. Here, $\Delta_{3,2}^{(1)}$ is the polynomial constructed from $\Delta_3^{(1)}$ by two successive applications of the process defined by Equation (16).

Moreover, using Proposition 2 and Lemma 1 on $g_{\mathbf{t}}$, we get

$$\begin{aligned} \left[\left[g(\mathbf{x} - \mathbf{t})|_{x_3=z} \right]_{x_1=-R}^R \right]_{x_2=-R}^R &= G_{\mathbf{t}}(R) - G_{(-t_1, t_2, t_3)}(R) - G_{(t_1, -t_2, t_3)}(R) + G_{(-t_1, -t_2, t_3)}(R) \\ &= \frac{\delta_{11}(\mathbf{t}, z, R)}{R^3}, \\ \left[\left[x_2 g(\mathbf{x} - \mathbf{t})|_{x_3=z} \right]_{x_1=-R}^R \right]_{x_2=-R}^R &= R \left(G_{\mathbf{t}}(R) - G_{(-t_1, t_2, t_3)}(R) + G_{(t_1, -t_2, t_3)}(R) - G_{(-t_1, -t_2, t_3)}(R) \right) \\ &= \frac{-2t_1}{\sqrt{2}} \cdot \frac{1}{R} + \frac{\delta_{12}(\mathbf{t}, z, R)}{R^3}, \end{aligned}$$

where $|\delta_{11}(\mathbf{t}, z, R)| \leq 4\Delta_2^{(6)}(s, z)$ and $|\delta_{12}(\mathbf{t}, z, R)| \leq 4\Delta_3^{(2)}(s, z)$. Therefore,

$$I_2(\mathbf{t}) = -\frac{2t_1 m_3(\mathbf{t})}{R\sqrt{2}} + \frac{\delta_{13}(\mathbf{t}, z, R)}{R^3} \quad (20)$$

where

$$\begin{aligned} |\delta_{13}(\mathbf{t}, z, R)| &\leq (4\Delta_2^{(6)}(s, z) + s\Delta_1^{(10)}(s, z)) z|m_2(\mathbf{t})| \\ &\quad + (4s\Delta_2^{(6)}(s, z) + 4\Delta_3^{(2)}(s, z) + 4s\Delta_2^{(5)}(s, z) + s^2\Delta_1^{(10)}(s, z)) m_3(\mathbf{t}) \end{aligned}$$

Asymptotic expansions of I_3 and I_4 . Following the same line of argument as we used for I_1 and I_2 , we get

$$\begin{aligned} \left[\left[k(\mathbf{x} - \mathbf{t})|_{x_3=z} \right]_{x_1=-R}^R \right]_{x_2=-R}^R &= K_{\mathbf{t}}(R) + K_{(-t_1, t_2, t_3)}(R) + K_{(t_1, -t_2, t_3)}(R) + K_{(-t_1, -t_2, t_3)}(R) \\ &= \frac{2\pi}{z - t_3} - 4\sqrt{2} \frac{1}{R} + \frac{\delta_{14}(\mathbf{t}, z, R)}{R^3}, \end{aligned}$$

$$\begin{aligned} \left[\left[\ell(\mathbf{x} - \mathbf{t})|_{x_3=z} \right]_{x_1=-R}^R \right]_{x_2=-R}^R &= L_{\mathbf{t}}(R) - L_{(-t_1, t_2, t_3)}(R) + L_{(t_1, -t_2, t_3)}(R) - L_{(-t_1, -t_2, t_3)}(R) \\ &= \frac{-4t_1}{\sqrt{2}} \cdot \frac{1}{R} + \frac{\delta_{15}(\mathbf{t}, z, R)}{R^3}, \end{aligned}$$

where $|\delta_{14}(\mathbf{t}, z, R)| \leq 4\Delta_2^{(3)}(s, z)$ and $|\delta_{15}(\mathbf{t}, z, R)| \leq 4\Delta_3^{(4)}(s, z)$. Therefore,

$$I_3(\mathbf{t}) = -2\pi m_1(\mathbf{t}) + \frac{4(z - t_3)\sqrt{2} m_1(\mathbf{t})}{R} + \frac{\delta_{16}(\mathbf{t}, z, R)}{R^3}, \quad (21)$$

$$I_4(\mathbf{t}) = \frac{-4t_1 m_3(\mathbf{t})}{R\sqrt{2}} + \frac{\delta_{17}(\mathbf{t}, z, R)}{R^3}, \quad (22)$$

where $|\delta_{16}(\mathbf{t}, z, R)| \leq 4\Delta_2^{(3)}(s, z)z|m_1(\mathbf{t})|$ and $|\delta_{17}(\mathbf{t}, z, R)| \leq 4\Delta_3^{(4)}(s, z)|m_3(\mathbf{t})|$.

Final step. Plugging Equations (19), (20), (21) and (22) into Equation (18), we finally get

$$\iint_{Q_R} x_1 B_3[\mathbf{m}, z](x_1, x_2) dx_1 dx_2 = \frac{\mu_0}{2} \langle m_1 \rangle + \frac{3\mu_0}{\pi R\sqrt{2}} (\langle t_1 m_3 \rangle + \langle t_3 m_1 \rangle - z \langle m_1 \rangle) + \frac{\delta_{18}(s, z, R)}{R^3},$$

where $\delta_{18}(s, z, R) = -\frac{\mu_0}{4\pi} \iiint_{\mathcal{A}} (\delta_9(\mathbf{t}, z, R) + \delta_{13}(\mathbf{t}, z, R) + \delta_{16}(\mathbf{t}, z, R) + \delta_{17}(\mathbf{t}, z, R)) d\mathbf{t}$. From the inequalities obtained in the previous paragraphs, we see that, for any $R \geq C$, where C is the constant given by Equation (13),

$$\begin{aligned} |\delta_{18}(s, z, R)| &\leq \frac{\mu_0}{4\pi} \left(4(\Delta_2^{(5)}(s, z) + \Delta_2^{(3)}(s, z)) z \langle |m_1| \rangle \right. \\ &\quad + (4\Delta_2^{(6)}(s, z) + s\Delta_1^{(10)}(s, z)) z \langle |m_2| \rangle \\ &\quad + (4(\Delta_3^{(1)}(s, z) + \Delta_3^{(2)}(s, z) + \Delta_3^{(4)}(s, z)) \\ &\quad \left. + 4s(2\Delta_2^{(5)}(s, z) + \Delta_2^{(6)}(s, z)) + s^2\Delta_1^{(10)}(s, z)) \langle |m_3| \rangle \right) \end{aligned}$$

This shows that δ_{18} is bounded by a quantity that depends only on s and z but not on R .

4.3 Proof of Equation (6)

We follow the path that led us to Equation (4): starting from Equation (3) we get

$$-\frac{2}{\mu_0} \iint_{Q_R} R B_3[\mathbf{m}, x_3](x_1, x_2) dx_1 dx_2 = \int_{-R}^R R \left[(P_3 \star \tilde{m}_1 - P_1 \star \tilde{m}_3)(\mathbf{x}) \right]_{x_1=-R}^R dx_2 \\ + \int_{-R}^R R \left[(P_3 \star \tilde{m}_2 - P_2 \star \tilde{m}_3)(\mathbf{x}) \right]_{x_2=-R}^R dx_1.$$

Now, replacing P_1 , P_2 and P_3 by their expressions and using Fubini, we obtain

$$\iint_{Q_R} R B_3[\mathbf{m}, x_3](x_1, x_2) dx_1 dx_2 = -\frac{\mu_0}{4\pi} \iiint_{\mathcal{A}} (I_5(\mathbf{t}) + I_6(\mathbf{t})) d\mathbf{t}, \quad (23)$$

where:

$$I_5(\mathbf{t}) = R \left[\left[((x_3 - t_3) m_1(\mathbf{t}) - (x_1 - t_1) m_3(\mathbf{t})) f(\mathbf{x} - \mathbf{t}) \right]_{x_1=-R}^R \right]_{x_2=-R}^R \\ = R((x_3 - t_3) m_1(\mathbf{t}) + t_1 m_3(\mathbf{t})) \left[\left[f(\mathbf{x} - \mathbf{t}) \right]_{x_1=-R}^R \right]_{x_2=-R}^R - R m_3(\mathbf{t}) \left[\left[x_1 f(\mathbf{x} - \mathbf{t}) \right]_{x_1=-R}^R \right]_{x_2=-R}^R, \\ I_6(\mathbf{t}) = R \left[((x_3 - t_3) m_2(\mathbf{t}) - (x_2 - t_2) m_3(\mathbf{t})) \int_{-R}^R \frac{1}{\|\mathbf{x} - \mathbf{t}\|} dx_1 \right]_{x_2=-R}^R \\ = R \left[\left[((x_3 - t_3) m_2(\mathbf{t}) - (x_2 - t_2) m_3(\mathbf{t})) f(x_2 - t_2, x_1 - t_1, x_3 - t_3) \right]_{x_1=-R}^R \right]_{x_2=-R}^R \\ = R((x_3 - t_3) m_2(\mathbf{t}) + t_2 m_3(\mathbf{t})) \left[\left[f(x_2 - t_2, x_1 - t_1, x_3 - t_3) \right]_{x_1=-R}^R \right]_{x_2=-R}^R \\ - R m_3(\mathbf{t}) \left[\left[x_2 f(x_2 - t_2, x_1 - t_1, x_3 - t_3) \right]_{x_1=-R}^R \right]_{x_2=-R}^R.$$

Now, using the same arguments as in Section 4.2, and assuming as before that x_3 is fixed and equal to z and that the hypotheses of Lemma 1 are satisfied, we see that

$$R \left[\left[f(\mathbf{x} - \mathbf{t})|_{x_3=z} \right]_{x_1=-R}^R \right]_{x_2=-R}^R = \frac{10t_1}{\sqrt{2}} \cdot \frac{1}{R^2} + \frac{\delta_{19}(\mathbf{t}, z, R)}{R^3}, \\ R \left[\left[x_1 f(\mathbf{x} - \mathbf{t})|_{x_3=z} \right]_{x_1=-R}^R \right]_{x_2=-R}^R = \frac{4}{\sqrt{2}} + \frac{33t_1^2 - 3t_2^2 - 10(z - t_3)^2}{2\sqrt{2}} \cdot \frac{1}{R^2} + \frac{\delta_{20}(\mathbf{t}, z, R)}{R^3},$$

where $|\delta_{19}(\mathbf{t}, z, R)| \leq 4\Delta_3^{(1)}(s, z)\frac{1}{R}$ and $\delta_{20}(\mathbf{t}, z, R) \leq 4\Delta_3^{(1)}(s, z)$. Therefore,

$$I_5(\mathbf{t}) = \frac{-4m_3(\mathbf{t})}{\sqrt{2}} + \frac{20(z - t_3)t_1 m_1(\mathbf{t}) - 13t_1^2 m_3(\mathbf{t}) + 3t_2^2 m_3(\mathbf{t}) + 10(z - t_3)^2 m_3(\mathbf{t})}{2R^2\sqrt{2}} + \frac{\delta_{21}(\mathbf{t}, z, R)}{R^3},$$

where $|\delta_{21}(\mathbf{t}, z, R)| \leq 4\Delta_3^{(1)}(s, z) \left(|m_3(\mathbf{t})|(1 + \frac{s}{R}) + |m_1(\mathbf{t})|\frac{z}{R} \right)$ and therefore, when $R \geq C$, $|\delta_{21}(\mathbf{t}, z, R)| \leq 4\Delta_3^{(1)}(s, z) \left(|m_3(\mathbf{t})|(1 + \omega_s) + |m_1(\mathbf{t})|\omega_z \right)$ by Equation (13).

Accordingly,

$$I_6(\mathbf{t}) = \frac{-4m_3(\mathbf{t})}{\sqrt{2}} + \frac{20(z - t_3)t_2 m_2(\mathbf{t}) - 13t_2^2 m_3(\mathbf{t}) + 3t_1^2 m_3(\mathbf{t}) + 10(z - t_3)^2 m_3(\mathbf{t})}{2R^2\sqrt{2}} + \frac{\delta_{22}(\mathbf{t}, z, R)}{R^3},$$

where $|\delta_{22}(\mathbf{t}, z, R)| \leq 4\Delta_3^{(1)}(s, z) \left(|m_3(\mathbf{t})|(1 + \omega_s) + |m_2(\mathbf{t})|\omega_z \right)$.

Putting these last two equations together we finally get

$$\begin{aligned} \iint_{Q_R} R B_3[\mathbf{m}, z](x_1, x_2) dx_1 dx_2 &= \frac{2\mu_0}{\pi\sqrt{2}} \langle m_3 \rangle \\ &+ \frac{5\mu_0}{4\pi R^2 \sqrt{2}} \left(-2z^2 \langle m_3 \rangle + z(4\langle t_3 m_3 \rangle - 2\langle t_1 m_1 \rangle - 2\langle t_2 m_2 \rangle) \right. \\ &\quad \left. + \langle t_1^2 m_3 \rangle + \langle t_2^2 m_3 \rangle - 2\langle t_3^2 m_3 \rangle + 2\langle t_1 t_3 m_1 \rangle + 2\langle t_2 t_3 m_2 \rangle \right) + \frac{\delta_{23}(s, z, R)}{R^3}, \end{aligned}$$

where $\delta_{23}(s, z, R) = -\frac{\mu_0}{4\pi} \iiint_{\mathcal{A}} (\delta_{21}(\mathbf{t}, z, R) + \delta_{22}(\mathbf{t}, z, R)) d\mathbf{t}$. Now, δ_{23} is bounded when R goes to infinity, since for any $R \geq C$ (where C is the constant given by Equation (13)) we have

$$|\delta_{23}(s, z, R)| \leq \frac{\mu_0}{\pi} \Delta_3^{(1)}(s, z) \left(2\langle |m_3| \rangle (1 + \omega_s) + \langle |m_1| \rangle \omega_z + \langle |m_2| \rangle \omega_z \right).$$

5 Proofs of the remaining equations

5.1 Generalities

Consider a linear isometry Ψ of \mathbb{R}^2 and define the linear isometry $\bar{\Psi}$ of \mathbb{R}^3 by $\bar{\Psi}(x_1, x_2, x_3) = (\Psi(x_1, x_2), x_3)$. We define $Q'_R = \Psi(Q_R)$. Denoting by Ψ_i the i -th component of Ψ (where $i \in \{1, 2\}$), we have, by the change of variable $(x'_1, x'_2) = \Psi(x_1, x_2)$,

$$\iint_{Q'_R} x'_i B_3[\mathbf{m}, z](x'_1, x'_2) dx'_1 dx'_2 = \iint_{Q_R} \Psi_i(\mathbf{x}) B_3[\mathbf{m}, z](\Psi(x_1, x_2)) dx_1 dx_2. \quad (24)$$

Moreover, according to Equation (1), we have, for any $\mathbf{x} \in \mathbb{R}^3$ such that $x_3 > r$,

$$B_3[\mathbf{m}, x_3](\Psi(x_1, x_2)) = \frac{-\mu_0}{4\pi} \partial_{x_3} \left(\iiint_{\mathbb{R}^3} \frac{\langle \widetilde{\mathbf{m}}(\mathbf{t}'), \bar{\Psi}(\mathbf{x}) - \mathbf{t}' \rangle}{\|\bar{\Psi}(\mathbf{x}) - \mathbf{t}'\|^3} d\mathbf{t}' \right).$$

Then, using the change of variable $\mathbf{t}' = \bar{\Psi}(\mathbf{t})$ and the fact that $\bar{\Psi}$ is a linear isometry (and hence preserves the inner product and the norm), the above expression becomes

$$\frac{-\mu_0}{4\pi} \partial_{x_3} \left(\iiint_{\mathbb{R}^3} \frac{\langle \bar{\Psi}^{-1}(\widetilde{\mathbf{m}}(\bar{\Psi}(\mathbf{t}))), \mathbf{x} - \mathbf{t} \rangle}{\|\mathbf{x} - \mathbf{t}\|^3} d\mathbf{t} \right).$$

Finally, defining M_1, M_2 and M_3 by $\mathbf{M}(\mathbf{t}) = \bar{\Psi}^{-1}(\widetilde{\mathbf{m}}(\bar{\Psi}(\mathbf{t})))$, we observe that

$$B_3[\mathbf{m}, x_3](\Psi(x_1, x_2)) = B_3[\mathbf{M}, x_3](x_1, x_2). \quad (25)$$

Now, putting together Equations (24) and (25) and using the linearity of Ψ_1 and Ψ_2 , we conclude that

$$\begin{pmatrix} \iint_{Q'_R} x'_1 B_3[\mathbf{m}, z](x'_1, x'_2) dx'_1 dx'_2 \\ \iint_{Q'_R} x'_2 B_3[\mathbf{m}, z](x'_1, x'_2) dx'_1 dx'_2 \end{pmatrix} = \Psi \begin{pmatrix} \iint_{Q_R} x_1 B_3[\mathbf{M}, z](x_1, x_2) dx_1 dx_2 \\ \iint_{Q_R} x_2 B_3[\mathbf{M}, z](x_1, x_2) dx_1 dx_2 \end{pmatrix}. \quad (26)$$

Accordingly,

$$\iint_{Q'_R} R B_3[\mathbf{m}, z](x'_1, x'_2) dx'_1 dx'_2 = \iint_{Q_R} R B_3[\mathbf{M}, z](x_1, x_2) dx_1 dx_2. \quad (27)$$

We now express certain moments of M_1 , M_2 and M_3 . Let $i \in \{1, 2, 3\}$ and let us denote by $\bar{\Psi}_i^{-1}$ the i -th component of $\bar{\Psi}^{-1}$. Also, we consider an arbitrary bounded function $\theta : \mathbb{R} \rightarrow \mathbb{R}$. Using successively the definition of M_i , the change of variable $\mathbf{t}' = \bar{\Psi}(\mathbf{t})$ together with the fact that $\bar{\Psi}$ is a linear isometry, the fact that $\bar{\Psi}_3^{-1}(\mathbf{t}') = t'_3$ and the linearity of $\bar{\Psi}_i^{-1}$, we get

$$\langle \theta(t_3) M_i \rangle = \iiint_{\mathbb{R}^3} \theta(t_3) \bar{\Psi}_i^{-1}(\widetilde{\mathbf{m}}(\bar{\Psi}(\mathbf{t}))) d\mathbf{t} = \iiint_{\mathbb{R}^3} \theta(\bar{\Psi}_3^{-1}(\mathbf{t}')) \bar{\Psi}_i^{-1}(\widetilde{\mathbf{m}}(\mathbf{t}')) d\mathbf{t}' = \bar{\Psi}_i^{-1} \begin{pmatrix} \langle \theta(t_3) m_1 \rangle \\ \langle \theta(t_3) m_2 \rangle \\ \langle \theta(t_3) m_3 \rangle \end{pmatrix}.$$

Observing that, for any $\mathbf{x} \in \mathbb{R}^3$, $\bar{\Psi}^{-1}(\mathbf{x}) = (\Psi^{-1}(x_1, x_2), x_3)$, we thus obtain:

$$\begin{pmatrix} \langle \theta(t_3) M_1 \rangle \\ \langle \theta(t_3) M_2 \rangle \end{pmatrix} = \Psi^{-1} \begin{pmatrix} \langle \theta(t_3) m_1 \rangle \\ \langle \theta(t_3) m_2 \rangle \end{pmatrix} \quad \text{and} \quad \langle \theta(t_3) M_3 \rangle = \langle \theta(t_3) m_3 \rangle. \quad (28)$$

Accordingly, we obtain $\langle t_i \theta(t_3) M_3 \rangle = \iiint_{\mathbb{R}^3} \bar{\Psi}_i^{-1}(\mathbf{t}') \theta(t'_3) \tilde{m}_3(\mathbf{t}') d\mathbf{t}' = \bar{\Psi}_i^{-1} \begin{pmatrix} \langle t_1 \theta(t_3) m_3 \rangle \\ \langle t_2 \theta(t_3) m_3 \rangle \\ \langle t_3 \theta(t_3) m_3 \rangle \end{pmatrix}$, whence

$$\begin{pmatrix} \langle t_1 \theta(t_3) M_3 \rangle \\ \langle t_2 \theta(t_3) M_3 \rangle \end{pmatrix} = \Psi^{-1} \begin{pmatrix} \langle t_1 \theta(t_3) m_3 \rangle \\ \langle t_2 \theta(t_3) m_3 \rangle \end{pmatrix} \quad \text{and} \quad \langle t_3 \theta(t_3) M_3 \rangle = \langle t_3 \theta(t_3) m_3 \rangle. \quad (29)$$

5.2 Proof of Equation (5)

To prove Equation (5), we define $\Psi : \mathbf{x} \mapsto (x_2, x_1)$. It is a linear involutive isometry, hence Equation (26) gives in this context

$$\iint_{Q'_R} x'_2 B_3[\mathbf{m}, z](x'_1, x'_2) dx'_1 dx'_2 = \iint_{Q_R} x_1 B_3[\mathbf{M}, z](x_1, x_2) dx_1 dx_2.$$

Now, observing that $Q'_R = Q_R$ and applying Equation (4) we get

$$\iint_{Q_R} x_2 B_3[\mathbf{m}, z](x_1, x_2) dx_1 dx_2 = \frac{\mu_0}{2} \langle M_1 \rangle + \frac{3\mu_0}{\pi R \sqrt{2}} (\langle t_1 M_3 \rangle + \langle t_3 M_1 \rangle - z \langle M_1 \rangle) + \mathcal{O}\left(\frac{1}{R^3}\right).$$

We conclude by remarking that $\langle M_1 \rangle = \langle m_2 \rangle$, $\langle t_1 M_3 \rangle = \langle t_2 m_3 \rangle$ and $\langle t_3 M_1 \rangle = \langle t_3 m_2 \rangle$ thanks to Equations (28), (29) with $\theta(t_3) = 1$ and Equation (28) with $\theta(t_3) = t_3 \cdot \chi_{[0, z]}(t_3)$.

5.3 Proofs of Equations (7), (8) and (9)

We define Ψ as the rotation of angle $\pi/4$ and we apply Equation (26) to $Q_{\frac{\sqrt{2}}{2}R}$, so that $Q'_{\frac{\sqrt{2}}{2}R} = S_R$.

In this context, and according to Equations (4) and (5), Equation (26) becomes

$$\begin{pmatrix} \iint_{S_R} x_1 B_3[\mathbf{m}, z](x_1, x_2) dx_1 dx_2 \\ \iint_{S_R} x_2 B_3[\mathbf{m}, z](x_1, x_2) dx_1 dx_2 \end{pmatrix} = \Psi \begin{pmatrix} \frac{\mu_0}{2} \langle M_1 \rangle + \frac{3\mu_0}{\pi R} (\langle t_1 M_3 \rangle + \langle t_3 M_1 \rangle - z \langle M_1 \rangle) + \mathcal{O}\left(\frac{1}{R^3}\right) \\ \frac{\mu_0}{2} \langle M_2 \rangle + \frac{3\mu_0}{\pi R} (\langle t_2 M_3 \rangle + \langle t_3 M_2 \rangle - z \langle M_2 \rangle) + \mathcal{O}\left(\frac{1}{R^3}\right) \end{pmatrix}.$$

Using the linearity of Ψ , the right hand side of the above expression becomes

$$\frac{\mu_0}{2} \Psi \begin{pmatrix} \langle M_1 \rangle \\ \langle M_2 \rangle \end{pmatrix} + \frac{3\mu_0}{\pi R} \left(\Psi \begin{pmatrix} \langle t_1 M_3 \rangle \\ \langle t_2 M_3 \rangle \end{pmatrix} + \Psi \begin{pmatrix} \langle t_3 M_1 \rangle \\ \langle t_3 M_2 \rangle \end{pmatrix} - z \Psi \begin{pmatrix} \langle M_1 \rangle \\ \langle M_2 \rangle \end{pmatrix} \right) + \mathcal{O}\left(\frac{1}{R^3}\right),$$

which directly gives Equations (7) and (8) thanks to Equations (28) and (29) with $\theta(t_3) = 1$ and Equation (28) with $\theta(t_3) = t_3 \cdot \chi_{[0,s]}(t_3)$.

Now, together with Equation (6), Equation (27) gives us

$$\begin{aligned} \iint_{S_R} R B_3[\mathbf{m}, z](x_1, x_2) dx_1 dx_2 &= \frac{2\mu_0}{\pi} \langle M_3 \rangle \\ &+ \frac{5\mu_0}{2\pi R^2} (-2z^2 \langle M_3 \rangle + z(4\langle t_3 M_3 \rangle - 2\langle t_1 M_1 \rangle - 2\langle t_2 M_2 \rangle)) \\ &+ \langle t_1^2 M_3 \rangle + \langle t_2^2 M_3 \rangle - 2\langle t_3^2 M_3 \rangle + 2\langle t_1 t_3 M_1 \rangle + 2\langle t_2 t_3 M_2 \rangle + \mathcal{O}\left(\frac{1}{R^3}\right). \end{aligned}$$

Notice that (using again the change of variable $\mathbf{t} = \bar{\Psi}^{-1}(\mathbf{t}')$ and the fact that $\bar{\Psi}$ is a linear isometry)

$$\begin{aligned} \langle t_1^2 M_3 \rangle + \langle t_2^2 M_3 \rangle + \langle t_3^2 M_3 \rangle &= \langle (t_1^2 + t_2^2 + t_3^2) M_3 \rangle \\ &= \iiint_{\mathbb{R}^3} \|\mathbf{t}\|^2 \tilde{m}_3(\bar{\Psi}(\mathbf{t})) d\mathbf{t} \\ &= \iiint_{\mathbb{R}^3} \|\bar{\Psi}^{-1}(\mathbf{t}')\|^2 \tilde{m}_3(\mathbf{t}') d\mathbf{t}' \\ &= \iiint_{\mathbb{R}^3} \|\mathbf{t}'\|^2 \tilde{m}_3(\mathbf{t}') d\mathbf{t}' = \langle t_1^2 m_3 \rangle + \langle t_2^2 m_3 \rangle + \langle t_3^2 m_3 \rangle. \end{aligned}$$

Combining this with Equation (28) where we set $\theta(t_3) = t_3^2 \cdot \chi_{[0,s]}(t_3)$, we get

$$\langle t_1^2 M_3 \rangle + \langle t_2^2 M_3 \rangle - 2\langle t_3^2 M_3 \rangle = \langle t_1^2 m_3 \rangle + \langle t_2^2 m_3 \rangle - 2\langle t_3^2 m_3 \rangle.$$

Also, if we denote by A the matrix of Ψ^{-1} in the canonical basis and by C_M (resp. C_m) the 3×3 matrix whose element (i, j) is $\langle t_i \theta(t_3) M_j \rangle$ (resp. $\langle t_i \theta(t_3) m_j \rangle$), we observe that

$$C_M = A C_m A^T. \quad (30)$$

Indeed, if i and j belong to $\{1, 2, 3\}$, then upon using the same change of variable we get that

$$\langle t_i \theta(t_3) M_j \rangle = \iiint_{\mathbb{R}^3} t_i \theta(t_3) \bar{\Psi}_j^{-1}(\tilde{\mathbf{m}}(\bar{\Psi}(\mathbf{t}))) d\mathbf{t} = \iiint_{\mathbb{R}^3} \bar{\Psi}_i^{-1}(\mathbf{t}') \theta(t_3') \bar{\Psi}_j^{-1}(\tilde{\mathbf{m}}(\mathbf{t}')) d\mathbf{t}',$$

Denoting by A_i and A_j the i -th and j -th row of A respectively, we see that

$$\bar{\Psi}_i^{-1}(\mathbf{t}') \bar{\Psi}_j^{-1}(\tilde{\mathbf{m}}(\mathbf{t}')) = (A_i \mathbf{t}')(A_j \tilde{\mathbf{m}}(\mathbf{t}')) = (A_i \mathbf{t}')(A_j \tilde{\mathbf{m}}(\mathbf{t}'))^T = A_i (\mathbf{t}' \tilde{\mathbf{m}}(\mathbf{t}')^T) A_j^T.$$

Therefore, $\langle t_i \theta(t_3) M_j \rangle = A_i C_m A_j^T$ whence Equation (30) holds. Now, Ψ^{-1} being an isometry we have $A^T = A^{-1}$, and therefore

$$\begin{aligned} \langle t_1 \theta(t_3) M_1 \rangle + \langle t_2 \theta(t_3) M_2 \rangle + \langle t_3 \theta(t_3) M_3 \rangle &= \text{tr}(A C_m A^T) \\ &= \text{tr}(A^T A C_m) \\ &= \text{tr}(C_m) \\ &= \langle t_1 \theta(t_3) m_1 \rangle + \langle t_2 \theta(t_3) m_2 \rangle + \langle t_3 \theta(t_3) m_3 \rangle. \end{aligned}$$

Together with Equation (29), this shows that $\langle t_1 \theta(t_3) M_1 \rangle + \langle t_2 \theta(t_3) M_2 \rangle = \langle t_1 \theta(t_3) m_1 \rangle + \langle t_2 \theta(t_3) m_2 \rangle$. We conclude the proof, using that last result with $\theta(t_3) = 1$ and $\theta(t_3) = t_3 \chi_{[0,s]}(t_3)$.

6 Comments, discussion

The main results (4) to (9) of Section 3 can be restated on the measurement area $M_R \in \{Q_R, S_R\}$ as:

$$\iint_{M_R} x_i B_3[\mathbf{m}, z](x_1, x_2) dx_1 dx_2 = c_{0,i}^{(1)}(M_R) \langle m_i \rangle + \frac{c_{1,i}^{(1)}(M_R)}{R} (\langle t_i m_3 \rangle + \langle (t_3 - z) m_i \rangle) + \mathcal{O}\left(\frac{1}{R^3}\right),$$

for $i = 1, 2$ and:

$$\begin{aligned} \iint_{M_R} B_3[\mathbf{m}, z](x_1, x_2) dx_1 dx_2 = & \frac{c_{1,3}^{(0)}(M_R)}{R} \langle m_3 \rangle \\ & + \frac{c_{3,3}^{(0)}(M_R)}{R^3} \left(\langle (t_1^2 + t_2^2 - 2(t_3 - z)^2) m_3 \rangle - 2 \langle (t_3 - z) (t_1 m_1 + t_2 m_2) \rangle \right) + \mathcal{O}\left(\frac{1}{R^4}\right), \end{aligned}$$

with

$$c_{0,i}^{(1)}(S_R) = \frac{\mu_0}{2}, \quad c_{1,i}^{(1)}(S_R) = \frac{3\mu_0}{\pi}, \quad c_{1,3}^{(0)}(S_R) = \frac{2\mu_0}{\pi}, \quad c_{3,3}^{(0)}(S_R) = \frac{5\mu_0}{2\pi},$$

and, for $j = 0, 1$,

$$c_{k,i}^{(j)}(Q_R) = \frac{c_{k,i}^{(j)}(S_R)}{2^{k/2}} = \frac{c_{k,i}^{(j)}(S_R)}{\sqrt{2^k}} \quad (\text{while } c_{2,i}^{(1)}(M_R) = c_{0,3}^{(0)}(M_R) = c_{2,3}^{(0)}(M_R) = 0).$$

Actually, $\sqrt{2}$ appears here because it is the ratio between the perimeters of the measurements areas Q_R and S_R . Similar properties appear to be true for circular shapes as well [9, Part III], and most probably also for more general geometries. Recall that $B_3[\mathbf{m}, z]$ has vanishing mean value on \mathbb{R}^2 by Green's theorem, which is consistent with the last equality above and with Equations (6) and (9).

This allows one to algebraically combine between the two families of expressions obtained for two different shapes in order to refine the estimates of $\langle m_i \rangle$ that could be obtained using a single shape. For instance, we get for $i = 1, 2$:

$$\begin{aligned} \frac{(\sqrt{2} - 1)\mu_0}{2} \langle m_i \rangle &= \sqrt{2} \iint_{Q_R} x_i B_3[\mathbf{m}, z](x_1, x_2) dx_1 dx_2 - \iint_{S_R} x_i B_3[\mathbf{m}, z](x_1, x_2) dx_1 dx_2 + \mathcal{O}\left(\frac{1}{R^3}\right), \\ \frac{2\mu_0}{\pi R} \langle m_3 \rangle &= 2\sqrt{2} \iint_{Q_R} B_3[\mathbf{m}, z](x_1, x_2) dx_1 dx_2 - \iint_{S_R} B_3[\mathbf{m}, z](x_1, x_2) dx_1 dx_2 + \mathcal{O}\left(\frac{1}{R^4}\right). \end{aligned}$$

Whenever R is such that Q_R and S_R are subsets of the actual measurement area, these furnish constructive approximations of $\langle m_i \rangle$. The precision of these estimations is driven by the size R of the measurement area, but also by the heights s, r, z . Indeed, the error terms decrease to 0 not only when R increases but also when s, r, z (and $r - z$) decrease, as the analysis of the bounds in the preceding section and in the appendix shows.

7 Conclusion

We took in this paper a few steps towards developing a “numerical magnetometer” which is adapted to planar normal field measurements as obtained in scanning magnetic macroscopy. Several issues of course remain to be addressed. For instance, the fact that measurements consist

of pointwise values leads one to approximate the integrals in Equations (4), (5), (6), (7), (8), and (9) by Riemann sums which makes for another approximation step. Moreover, the measurements themselves are corrupted with noise whose effect has to be analyzed somewhat systematically. For instance, it should be observed that our estimates of $\langle m_1 \rangle$, $\langle m_2 \rangle$ are unbiased, while it is the linear combination of equations proposed in Section 6 that can get us rid of the bias in our estimates of $\langle m_3 \rangle$. Nevertheless, we feel the formulas that we derived are a necessary step in magnetometric studies and lie at the basis of net moment estimation for planar measurement geometries.

Appendix: proof of Lemma 1

We first establish three lemmas about Taylor expansions of order 2 that will be used for the proof of Lemma 1.

As before in Section 4, we consider two positive constants ω_s and ω_z such that $\omega_s < 1$ and we define C as in Equation (13). In the sequel, we will extensively use the notations defined in Equations (14) and (16), together with their practical properties, as given by Equations (15) and (17).

Put $\mathcal{D} = \mathcal{D}(s, z, C) = [-s, s]^2 \times (0, z] \times [0, 1/C]$. The assumption $(\gamma, \xi) \in \mathcal{D}$ actually means $\gamma \in [-s, s]^2 \times (0, z]$, $\xi \in [0, 1/C]$.

The following lemma explains how to compose Taylor expansions of order 2 with rigorous bounds.

Lemma 2. *We consider an interval I , a function $v : I \rightarrow \mathbb{R}$ and a function $u : \mathcal{D} \rightarrow I$ and we suppose that they are of the form*

$$\begin{cases} v(y) = b_0 + b_1 y + b_2 y^2 + \varepsilon_v(y) & \text{with } \forall y \in I, |\varepsilon_v(y)| \leq B_3 |y|^3, \\ u(\gamma, \xi) = a_1(\gamma)\xi + a_2(\gamma)\xi^2 + \varepsilon_u(\gamma, \xi) \\ \text{with } \forall (\gamma, \xi) \in \mathcal{D}, |\varepsilon_u(\gamma, \xi)| \leq A_3(s, z)\xi^3, \end{cases}$$

where $b_0, b_1, b_2, B_3 \in \mathbb{R}$, and $a_1(\gamma)$, $a_2(\gamma)$, $A_3(s, z)$ are homogeneous polynomials of degree 1, 2, 3, respectively.

Then, we have $(v \circ u)(\gamma, \xi) = c_0 + c_1(\gamma)\xi + c_2(\gamma)\xi^2 + \varepsilon(\gamma, \xi)$ with $\forall (\gamma, \xi) \in \mathcal{D}$, $|\varepsilon(\gamma, \xi)| \leq C_3(s, z)\xi^3$, where $c_0 = b_0$, $c_1(\gamma) = b_1 a_1(\gamma)$, $c_2(\gamma) = b_1 a_2(\gamma) + b_2 a_1^2(\gamma)$, and $C_3(s, z)$ is a homogeneous polynomial of degree 3.

Proof. Let $(\gamma, \xi) \in \mathcal{D}$. From the definitions of u and v , we get

$$\varepsilon(\gamma, \xi) = b_1 \varepsilon_u(\gamma, \xi) + b_2 \left(2a_1(\gamma)\xi + a_2(\gamma)\xi^2 + \varepsilon_u(\gamma, \xi) \right) \left(a_2(\gamma)\xi^2 + \varepsilon_u(\gamma, \xi) \right) + \varepsilon_v(u(\gamma, \xi)).$$

The bound then follows from the triangle inequality, with

$$\begin{aligned} C_3(s, z) &= |b_1| A_3(s, z) + |b_2| \left(2A_1(s, z) + A_{2,1}(s, z) + A_{3,2}(s, z) \right) \left(A_2(s, z) + A_{3,1}(s, z) \right) \\ &\quad + B_3 \left(A_1(s, z) + A_{2,1}(s, z) + A_{3,2}(s, z) \right)^3, \end{aligned}$$

and the definitions of A_1 , A_2 , $A_{2,1}$, $A_{3,1}$, and $A_{3,2}$ from $a_1(\gamma)$, $a_2(\gamma)$, and $A_3(s, z)$, by means of Equations (14) and (16), together with the corresponding properties expressed in Equations (15) and (17). \square

In particular, notice that $c_1(\gamma)$, $c_2(\gamma)$ are homogeneous polynomials of degree 1 and 2, respectively, as in the following lemma which, in the same spirit, shows how to multiply Taylor expansions of order 2 with rigorous bounds.

Lemma 3. *We consider two functions u and $v : \mathcal{D} \rightarrow \mathbb{R}$ and we suppose that they are of the form*

$$\begin{cases} u(\gamma, \xi) = 1 + a_1(\gamma)\xi + a_2(\gamma)\xi^2 + \varepsilon_u(\gamma, \xi) \\ v(\gamma, \xi) = 1 + b_1(\gamma)\xi + b_2(\gamma)\xi^2 + \varepsilon_v(\gamma, \xi) \end{cases}$$

with $\forall(\gamma, \xi) \in \mathcal{D}$, $|\varepsilon_u(\gamma, \xi)| \leq A_3(s, z)\xi^3$ and $|\varepsilon_v(\gamma, \xi)| \leq B_3(s, z)\xi^3$, where $a_1(\gamma)$, $b_1(\gamma)$, $a_2(\gamma)$, $b_2(\gamma)$, and $A_3(s, z)$, $B_3(s, z)$ are homogeneous polynomials of degree 1, 2, and 3, respectively.

Then, we have

$$u(\gamma, \xi)v(\gamma, \xi) = 1 + c_1(\gamma)\xi + c_2(\gamma)\xi^2 + \varepsilon(\gamma, \xi)$$

with $\forall(\gamma, \xi) \in \mathcal{D}$, $|\varepsilon(\gamma, \xi)| \leq C_3(s, z)\xi^3$, where $c_1(\gamma) = a_1(\gamma) + b_1(\gamma)$, $c_2(\gamma) = a_1(\gamma)b_1(\gamma) + a_2(\gamma) + b_2(\gamma)$, and $C_3(s, z)$ is a homogeneous polynomial of degree 3.

Proof. Let $(\gamma, \xi) \in \mathcal{D}$. From the definitions of u and v , we get

$$\begin{aligned} \varepsilon(\gamma, \xi) &= \varepsilon_u(\gamma, \xi) + \varepsilon_v(\gamma, \xi) \\ &+ \left(a_1(\gamma)\xi + \frac{1}{2}a_2(\gamma)\xi^2 + \frac{1}{2}\varepsilon_u(\gamma, \xi) \right) \left(b_2(\gamma)\xi^2 + \varepsilon_v(\gamma, \xi) \right) \\ &+ \left(b_1(\gamma)\xi + \frac{1}{2}b_2(\gamma)\xi^2 + \frac{1}{2}\varepsilon_v(\gamma, \xi) \right) \left(a_2(\gamma)\xi^2 + \varepsilon_u(\gamma, \xi) \right). \end{aligned}$$

The bound then again follows from the triangle inequality, with

$$\begin{aligned} C_3(s, z) &= A_3(s, z) + B_3(s, z) \\ &+ \left(A_1(s, z) + \frac{1}{2}A_{2,1}(s, z) + \frac{1}{2}A_{3,2}(s, z) \right) \left(B_2(s, z) + B_{3,1}(s, z) \right) \\ &+ \left(B_1(s, z) + \frac{1}{2}B_{2,1}(s, z) + \frac{1}{2}B_{3,2}(s, z) \right) \left(A_2(s, z) + A_{3,1}(s, z) \right), \end{aligned}$$

and the definitions of A_1 , A_2 , $A_{2,1}$, $A_{3,1}$, $A_{3,2}$, B_1 , B_2 , $B_{2,1}$, $B_{3,1}$, and $B_{3,2}$ from $a_1(\gamma)$, $a_2(\gamma)$, $A_3(s, z)$, $b_1(\gamma)$, $b_2(\gamma)$, and $B_3(s, z)$, by means of Equations (14) and (16), together with the corresponding properties expressed in Equations (15) and (17). \square

The following lemma collects some Taylor expansions with controlled bounds on their remainders. They will be used in the proof of Lemma 1.

Lemma 4. *We define the functions v_1 to v_5 by $v_1(y) = \frac{1}{\sqrt{1+y}}$, $v_2(y) = \frac{1}{1+y}$, $v_3(y) = \sqrt{1+y}$, $v_4(y) = \arctan(y)$ and $v_5(y) = \operatorname{arcsinh}(1+y)$. There exist constants $B^{(1)}, \dots, B^{(5)}$, depending on ω_s and ω_z only, such that*

$$\forall y \in [-1 + (1 - \omega_s)^2, 2\omega_s + \omega_s^2 + \omega_z^2],$$

$$v_1(y) = 1 - \frac{y}{2} + \frac{3y^2}{8} + \varepsilon_1(y) \quad \text{with} \quad |\varepsilon_1(y)| \leq B^{(1)}|y|^3, \quad (31)$$

$$v_2(y) = 1 - y + y^2 + \varepsilon_2(y) \quad \text{with} \quad |\varepsilon_2(y)| \leq B^{(2)}|y|^3, \quad (32)$$

$$v_3(y) = 1 + \frac{y}{2} - \frac{y^2}{8} + \varepsilon_3(y) \quad \text{with} \quad |\varepsilon_3(y)| \leq B^{(3)}|y|^3, \quad (33)$$

$$\forall y \in \mathbb{R},$$

$$v_4(y) = y + \varepsilon_4(y) \quad \text{with} \quad |\varepsilon_4(y)| \leq B^{(4)}|y|^3, \quad (34)$$

$$v_5(y) = \operatorname{arcsinh}(1) + \frac{y}{\sqrt{2}} - \frac{y^2}{4\sqrt{2}} + \varepsilon_5(y) \quad \text{with} \quad |\varepsilon_5(y)| \leq B^{(5)}|y|^3. \quad (35)$$

Proof. If $i \in \{1, 2, 3\}$, since $\omega_s < 1$, the function v_i is infinitely differentiable on the given interval and the existence of $B^{(i)}$ simply follows from Taylor's theorem at 0. If $i \in \{4, 5\}$, the same argument applied on the interval $[-1, 1]$ ensures the existence of a constant $B'^{(i)}$ such that Equations (34) and (35) hold true for $y \in [-1, 1]$. Besides, we observe that the function $y \mapsto \varepsilon_i(y)/y^3$ is continuous on $(-\infty, -1] \cup [1, +\infty)$ and tends to 0 at $\pm\infty$. Therefore, there exists a constant $B''^{(i)}$ such that Equations (34) and (35) hold true for $y \in (-\infty, -1] \cup [1, +\infty)$. Then $B^{(i)} = \max\{B'^{(i)}, B''^{(i)}\}$ satisfies the requirement. \square

Proof of Lemma 1. Let us define

$$u_1(\gamma, \xi) = -(\gamma_1 + \gamma_2)\xi + \frac{\gamma_1^2 + \gamma_2^2 + \gamma_3^2}{2} \xi^2 = -1 + \frac{\gamma_3^2 \xi^2}{2} + \frac{(1 - \gamma_1 \xi)^2}{2} + \frac{(1 - \gamma_2 \xi)^2}{2}.$$

According to Equation (13), observe that for $(\gamma, \xi) \in \mathcal{D}$, $\gamma_1 \xi \leq |\gamma_1 \xi| \leq s\xi \leq s/C \leq \omega_s < 1$. Therefore $(1 - \gamma_1 \xi) \geq (1 - \omega_s) > 0$, hence $(1 - \gamma_1 \xi)^2 \geq (1 - \omega_s)^2$. Similarly, $(1 - \gamma_2 \xi)^2 \geq (1 - \omega_s)^2$. This shows that

$$u_1(\gamma, \xi) \geq -1 + \frac{\gamma_3^2 \xi^2}{2} + (1 - \omega_s)^2 \geq -1 + (1 - \omega_s)^2. \quad (36)$$

Besides, from the definition of u_1 , the triangle inequality and Equation (13) we obtain

$$u_1(\gamma, \xi) \leq |u_1(\gamma, \xi)| \leq \frac{2s}{C} + \frac{2s^2 + z^2}{2C^2} \leq 2\omega_s + \omega_s^2 + \frac{\omega_z^2}{2}. \quad (37)$$

We also define

$$u_2(\gamma, \xi) = -2\gamma_1 \xi + (\gamma_1^2 + \gamma_3^2) \xi^2 = -1 + \gamma_3^2 \xi^2 + (1 - \gamma_1 \xi)^2.$$

Reasoning as we did for u_1 , we see that for $(\gamma, \xi) \in \mathcal{D}$,

$$u_2(\gamma, \xi) \geq -1 + (1 - \omega_s)^2, \quad (38)$$

$$u_2(\gamma, \xi) \leq |u_2(\gamma, \xi)| \leq 2\omega_s + \omega_s^2 + \omega_z^2. \quad (39)$$

Finally, we define $u_3(\gamma, \xi) = -(\gamma_1 + \gamma_2)\xi + \gamma_1 \gamma_2 \xi^2 = -1 + (1 - \gamma_1 \xi)(1 - \gamma_2 \xi)$, for which, accordingly,

$$u_3(\gamma, \xi) \geq -1 + (1 - \omega_s)^2, \quad (40)$$

$$u_3(\gamma, \xi) \leq |u_3(\gamma, \xi)| \leq 2\omega_s + \omega_s^2. \quad (41)$$

Now, for any $t \in \mathcal{A}$ and any $R \geq C$, we have

$$\begin{aligned} (R - t_1)^2 + (R - t_2)^2 + (z - t_3)^2 &= 2R^2 (1 + u_1(\gamma, \xi)), \\ (R - t_1)^2 + (z - t_3)^2 &= R^2 (1 + u_2(\gamma, \xi)), \\ (R - t_1)(R - t_2) &= R^2 (1 + u_3(\gamma, \xi)), \end{aligned}$$

where $\gamma = (t_1, t_2, (z - t_3))$ and $\xi = 1/R$. Therefore, the functions F_t , G_t , K_t , and L_t can be expressed by means of v_1 to v_5 and u_1 to u_3 . Namely (for K_t we use the identity $\arctan(\alpha) = \frac{\pi}{2} - \arctan(1/\alpha)$ which holds for $\alpha > 0$):

$$\begin{aligned} F_t(R) &= \frac{\xi^2}{\sqrt{2}} (1 - \gamma_2 \xi) v_2(u_2(\gamma, \xi)) v_1(u_1(\gamma, \xi)), \\ G_t(R) &= \frac{-\xi}{\sqrt{2}} v_1(u_1(\gamma, \xi)), \\ K_t(R) &= \frac{\pi}{2\gamma_3} - \frac{1}{\gamma_3} v_4 \left(\gamma_3 \xi \sqrt{2} v_3(u_1(\gamma, \xi)) v_2(u_3(\gamma, \xi)) \right), \\ L_t(R) &= -v_5 (-1 + (1 - \gamma_2 \xi) v_1(u_2(\gamma, \xi))). \end{aligned}$$

Let us recall the statement that we wish to prove, namely that for any $R \geq C$ and $\mathbf{t} \in \mathcal{A}$ we have:

$$\begin{aligned} F_{\mathbf{t}}(R) &= \frac{1}{\sqrt{2}} \cdot \frac{1}{R^2} + \frac{5t_1 - t_2}{2\sqrt{2}} \cdot \frac{1}{R^3} + \frac{33t_1^2 - 3t_2^2 - 6t_1t_2 - 10t_3^2 + 20zt_3 - 10z^2}{8\sqrt{2}} \cdot \frac{1}{R^4} + \frac{\delta_1(\mathbf{t}, z, R)}{R^5}, \\ G_{\mathbf{t}}(R) &= \frac{-1}{\sqrt{2}} \cdot \frac{1}{R} - \frac{t_1 + t_2}{2\sqrt{2}} \cdot \frac{1}{R^2} - \frac{t_1^2 + t_2^2 + 6t_1t_2 - 2t_3^2 + 4zt_3 - 2z^2}{8\sqrt{2}} \cdot \frac{1}{R^3} + \frac{\delta_2(\mathbf{t}, z, R)}{R^4}, \\ K_{\mathbf{t}}(R) &= \frac{\pi}{2(z - t_3)} - \sqrt{2} \frac{1}{R} - \frac{\sqrt{2}(t_1 + t_2)}{2} \cdot \frac{1}{R^2} + \frac{\delta_3(\mathbf{t}, z, R)}{R^3}, \\ L_{\mathbf{t}}(R) &= -\operatorname{arcsinh}(1) + \frac{t_2 - t_1}{\sqrt{2}} \cdot \frac{1}{R} - \frac{3t_1^2 - t_2^2 - 2t_1t_2 - 2t_3^2 + 4zt_3 - 2z^2}{4\sqrt{2}} \cdot \frac{1}{R^2} + \frac{\delta_4(\mathbf{t}, z, R)}{R^3}, \end{aligned}$$

where $|\delta_1(\mathbf{t}, z, R)| \leq \Delta_3^{(1)}(s, z)$, $|\delta_2(\mathbf{t}, z, R)| \leq \Delta_3^{(2)}(s, z)$, $|\delta_3(\mathbf{t}, z, R)| \leq \Delta_2^{(3)}(s, z)$, and $|\delta_4(\mathbf{t}, z, R)| \leq \Delta_3^{(4)}(s, z)$, for some homogeneous polynomials $\Delta_3^{(1)}$, $\Delta_3^{(2)}$, $\Delta_2^{(3)}$, and $\Delta_3^{(4)}$, of degrees 3, 3, 2, and 3, respectively. Except for $K_{\mathbf{t}}$ (which requires some explanation), the statement is easily deduced from our previous results, using Lemma 2 with $I = [-1 + (1 - \omega_s)^2, 2\omega_s + \omega_s^2 + \omega_z^2]$ for the compositions involving v_1 , v_2 , and v_3 , and $I = (-\infty, +\infty)$ for the compositions involving v_4 and v_5 , together with Lemmas 3 and 4.

In the case of $K_{\mathbf{t}}$ a difficulty arises from the division by $\gamma_3 = z - t_3$ which casts doubt on whether $\Delta_2^{(3)}(s, z)$ can be chosen as a polynomial. Let us put $d(\gamma, \xi) = \sqrt{2} v_3(u_1(\gamma, \xi)) v_2(u_3(\gamma, \xi))$ whence $K_{\mathbf{t}}(R) = \frac{\pi}{2\gamma_3} - \frac{1}{\gamma_3} v_4(\gamma_3 \xi d(\gamma, \xi))$. Thanks to our lemmas, we easily obtain that

$$d(\gamma, \xi) = \sqrt{2} + d_1(\gamma)\xi + d_2(\gamma)\xi^2 + \varepsilon_d(\gamma, \xi),$$

where d_1 and d_2 are homogeneous polynomials of degree 1 and 2 respectively, and $|\varepsilon_d(\gamma, \xi)| \leq D_3(s, z)\xi^3$ where D_3 is a homogeneous polynomial of degree 3. Now, from Equation (34) we see that $K_{\mathbf{t}}$ has the prescribed form with the remainder $\delta_3(\mathbf{t}, z, R)/R^3$ being given by

$$d_2(\gamma)\xi^3 + \varepsilon_d(\gamma, \xi)\xi + \frac{1}{\gamma_3} \varepsilon_4(\gamma_3 \xi d(\gamma, \xi)),$$

with, as before, $\gamma = (t_1, t_2, (z - t_3))$ and $\xi = 1/R$. Now, observe that $|d_2(\gamma)\xi^3| \leq D_2(s, z)\xi^3$, $|\varepsilon_d(\gamma, \xi)\xi| \leq D_{3,1}(s, z)\xi^3$ and

$$\left| \frac{1}{\gamma_3} \varepsilon_4(\gamma_3 \xi d(\gamma, \xi)) \right| \leq \frac{1}{|\gamma_3|} B^{(4)} |\gamma_3|^3 |\xi|^3 d(\gamma, \xi)^3 \leq B^{(4)} z^2 (\sqrt{2} + D_{1,1} + D_{2,2} + D_{3,3})^3$$

which proves the claim. \square

Remark 2. For the purpose of proving our result in the most general framework, we introduced the constants ω_s and ω_z , so as to determine the constants usually hidden behind the \mathcal{O} notation and make these constants explicit as functions depending only on s and z but not on \mathbf{t} and R . This allows us to integrate these bounds with respect to variable \mathbf{t} and obtain upper bounds for δ_{18} and δ_{23} (at the end of Sections 4.2, 4.3, respectively), therefore proving rigorous asymptotic formulas that give approximate identities more and more accurate as R goes large.

However, the practice is usually completely different from this situation. One generally does not actually let R tends to $+\infty$, but one rather has some measurements on a square Q_R with

a given value R and one would like to get an approximation of the moments $\langle m_i \rangle$ ($i = 1, 2, 3$) using the asymptotic formulas, together with an estimate of the error contained in the remainder. In order to obtain a small bound for the remainder, it is clearly desirable to choose ω_s and ω_z as small as possible, so one practically chooses $\omega_s = s/R$ and $\omega_z = z/R$ (as soon as $R > s$). Therefore Equation (13) defines the constant C as being equal to R , see also Remark 1.

Furthermore, instead of using the interval $[-1 + (1 - \omega_s)^2, 2\omega_s + \omega_s^2 + \omega_z^2]$ in Lemma 4, one can use slightly tighter intervals since all the constants are known. For instance, Equations (36) and (37) can be reworked to show that $u_1(t_1, t_2, (z - t_3), 1/R)$ indeed ranges in the interval $[-1 + \frac{(z-r)^2}{2R^2} + (1 - \frac{s}{R})^2, \frac{2s}{R} + \frac{2s^2+z^2}{2R^2}]$ for any $t \in \mathcal{A}$. Also, notice that explicit tight and rigorous constants $B^{(1)}$ to $B^{(5)}$ for Lemma 4 can be automatically computed on demand for a given interval using rigorous arithmetic tools such as Taylor models [8]. Together with the present article, we provide a Maple script that explicitly computes all the presented bounds, for given values s , z and R . For the computation of the bounds of Lemma 4, we rest on a script run with the Sollya software tool [3] that provides rigorous and proven results, accounting for all possible roundoff errors in numeric computations.

References

- [1] L. Baratchart, D. P. Hardin, E. A. Lima, E. B. Saff, and B. P. Weiss. Characterizing kernels of operators related to thin-plate magnetizations via generalizations of hodge decompositions. *Inverse Problems*, 29(1), 2013. <http://dx.doi.org/10.1088/0266-5611/29/1/015004>.
- [2] Laurent Baratchart, Sylvain Chevillard, and Juliette Leblond. Silent and equivalent magnetic distributions on thin plates. To appear in a volume of the Theta Series in Advanced Mathematics (published by the Theta Foundation, distributed by the AMS), <https://hal.inria.fr/hal-01286117hal-01286117>, 2016.
- [3] S. Chevillard, M. Joldes, and C. Lauter. Sollya: An Environment for the Development of Numerical Codes. In K. Fukuda, J. van der Hoeven, M. Joswig, and N. Takayama, editors, *Mathematical Software - ICMS 2010*, volume 6327 of *Lecture Notes in Computer Science*, pages 28–31, Heidelberg, Germany, September 2010. Springer.
- [4] D. W. Collinson. *Methods in rock magnetism and paleomagnetism: techniques and instrumentations*. Chapman and Hall, New York, 1983.
- [5] John David Jackson. *Classical Electrodynamics*. J. Wiley & Sons, 3rd edition, 1998.
- [6] J. R. Kirtley and J. P. Wikswo. Scanning SQUID microscopy. *Annu. Rev. Mater. Sci.*, 29, 1999.
- [7] E. A. Lima, B. P. Weiss, L. Baratchart, D. P. Hardin, and E. B. Saff. Fast inversion of magnetic field maps of unidirectional planar geological magnetization. *Journal of Geophysical Research: Solid Earth*, 118(6):2723–2752, 2013. <http://dx.doi.org/10.1002/jgrb.50229>.
- [8] K. Makino and M. Berz. Taylor Models and Other Validated Functional Inclusion Methods. *International Journal of Pure and Applied Mathematics*, 4(4):379–456, 2003. <http://bt.pa.msu.edu/pub/papers/TMIJPAM03/TMIJPAM03.pdf>.
- [9] D. Ponomarev. *Some inverse problems with partial data*. PhD thesis, University Nice Sophia Antipolis, ED STIC, June 2016. <https://hal.archives-ouvertes.fr/tel-01400595>.

- [10] Elias M. Stein and Guido Weiss. *Introduction to Fourier analysis on Euclidean spaces*. Princeton Univeristy Press, 1971.
- [11] B. P. Weiss, E. A. Lima, L. E. Fong, and F. J. Baudenbacher. Paleomagnetic analysis using SQUID microscopy. *J. Geophys Res.*, 112, 2007.