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How Fuzzy FCA and Pattern Structures are connected?

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Abstract. FCA is a mathematical formalism having many applications in data mining and knowledge discovery. Originally it deals with binary data tables. However, there is a number of extensions that enrich standard FCA. In this paper we consider two important extensions: fuzzy FCA and pattern structures, and discuss the relation between them. In particular we introduce a scaling procedure that enables representing a fuzzy context as a pattern structure.

Keywords: fuzzy FCA, pattern structures, scaling

1 Introduction

In this paper we deal with Formal Concept Analysis (FCA) and its extensions. FCA is a mathematical formalism having many applications in data mining and knowledge discovery. It starts from a binary table, a so-called formal context (G, M, I), where G is the set of objects, M is the set of attributes, and $I \subseteq G \times M$ is a relation between G and M, and proceeds to a lattice of formal concepts [1]. Fuzzy FCA is an extension of standard FCA that allows for fuzzy sets of objects and attributes in order to express uncertainty.

Pattern structures is another extension of FCA that allows processing complex data, e.g., graph or sequence datasets. It is a quite general framework and the question if fuzzy FCA can be represented within Pattern Structures and vice versa is still open. In this paper we make a step in this direction and study the connections between pattern structures and fuzzy FCA.

We show how a fuzzy context can be scaled to a "Minimum Pattern Structure" (MnPS), a special kind of pattern structures, that is close to interval pattern structures when considering numerical data. A scaling is needed, since pattern structures deal with crisp sets of objects and, thus, fuzzy extents cannot be expressed within the formalism of pattern structures. For such a kind of scaling we add new objects to the fuzzy context that express objects with uncertain membership in fuzzy sets, allowing expressing fuzzy sets of objects in the formalism of pattern structures. The resulting context is processed by MnPS. This kind of scaling is applicable to fuzzy FCA based on residuated lattices, a special kind of lattices expressing uncertain membership degrees in fuzzy sets.

Table 1: A toy dataset of transactions for a supermaket and the related similarity matrix.

(a) A dataset with 5 transactions.

	i_1	i_2	i_3	i_4	i_5	i_6	i_7
t_1	x	x			x	x	x
t_2	x	x		x		x	x
t_3	x	x			x	x	
t_4		x	x				x
t_5	x			x	x	x	

(b) Similarity matrix.

	t_1	t_2	t_3	t_4	t_5
t_1	1.000	0.714	0.857	0.429	0.429
t_2	0.714	1.000	0.571	0.429	0.429
t_3	0.857	0.571	1.000	0.286	0.714
t_4	0.429	0.429	0.286	1.000	0.000
t_5	0.429	0.429	0.714	0.000	1.000

The rest of the paper is organized as follows. Section 2 describes a running example. Later, in Section 3 we introduce main definitions of fuzzy FCA and pattern structures. The main contribution of this paper is located in Section 4, where we introduce and discuss the scaling procedure of fuzzy FCA to pattern structures. Finally, at the end of the paper we discuss some related works.

2 Running Example

Let us consider a toy dataset of transactions within a supermarket. It is shown in Table 1a. Every row corresponds to a basket bought by a customer and every attribute corresponds to an item that can be bought in the supermarket. A cross in a cell (i, j) means that in the basket i there is the item j.

For making the example concrete, let us consider a clustering task. When dealing with clustering one typically needs a similarity or a distance measure. Such distance and similarity measures for the purpose of this example could be the fraction of different items shared by two baskets $\texttt{Dist}(t_1, t_2) = \frac{|t_1' + t_2'|}{|M|}$ and $\texttt{Sim}(t_1, t_2) = 1 - \texttt{Dist}(t_1, t_2)$, where operation '+' between sets is an exclusive OR (a so-called XOR or the symmetric difference, i.e., $A + B = (A \setminus B) \cup (B \setminus A)$). The similarity measure for any pair of transactions is shown in Table 1b. For example, similarity between t_1 and t_2 is equal 0.714. These baskets are different in two items i_4 and i_5. Thus $\texttt{Dist}(t_1, t_2) = \frac{2}{7} = 0.286$, where 7 is the number of items in the supermarket, and $\texttt{Sim}(t_1, t_2) = 1 - \texttt{Dist}(t_1, t_2) = 0.714$.

3 Definitions

Formal concept analysis (FCA) is a formalism for dealing with data mining and knowledge discovery tasks. It starts from a binary context (G, M, I), where G is the set of objects, M is the set of attributes and $I \subseteq G \times M$ is a relation between G and M. There are a number of extensions of Formal Concept Analysis (FCA) for dealing with complexity of descriptions, e.g., pattern structures [2], and with uncertainty, e.g., fuzzy FCA [3]. Below we give definitions of this two directions

without many details. The interested reader can address the original works on pattern structures and fuzzy FCA for more details and examples.

3.1 Fuzzy FCA

Fuzzy FCA works with fuzzy logic instead of crisp-logic, used in standard FCA. There are several generalizations of FCA to the fuzzy case [3]. Here the approach of Belohlavek is considered [4]. In fuzzy logic formulas can be valid up to a certain degree. It means that the formula can be completely valid, completely invalid, or between these two states. This fuzziness in fuzzy FCA is represented by a so-called residuated lattice, where the top of the lattice $\top$ corresponds to "completely valid" state of the logic and the bottom $\bot$ corresponds to "completely invalid" state.

Definition 1. *A Residuated Lattice is an algebra* $\boldsymbol{L} = \langle L, \vee, \wedge, \otimes, \rightsquigarrow, 0, 1\rangle$*, where* $\langle L, \vee, \wedge, 0, 1\rangle$ *is a complete lattice;* $\langle L, \otimes, 1\rangle$ *is a commutative monoid, i.e.* $\otimes$ *is commutative, associative, and* $\forall a(a \otimes 1 = 1 \otimes a = a)$*;* $\rightsquigarrow$ *and* $\otimes$ *form an adjiont pair, i.e.,* $a \otimes b \leq c \Leftrightarrow a \leq b \rightsquigarrow c$*.*

For the following, L refers to the set of elements of some residuated lattice and $\mathbf{L}$ for the residuated lattice itself.

An important residuated lattice based on a linearly ordered set is Göodel residuated lattice, which is used in examples of this paper. In Göodel residuated lattices the fuzzzy implication is defied as following:

$$a \rightsquigarrow b = \begin{cases} \top & a \leq b \\ b & a > b \end{cases} \tag{1}$$

In the crisp logic the implication $\top \to \bot$ is not valid, i.e., $\top \to \bot = \bot$, while other three possible implications are valid, i.e., $\top \to \top = \top$, $\bot \to \top = \top$, and $\bot \to \bot = \top$. The formula (1) generalizes this behavior. If the premise is less certain than the conclusion, then the implication is valid ($\top$), otherwise the validity of the implication is equal to the certainty of the conclusion.

In Definition 1 it is required that the fuzzy implication is *adjoint* (related) with an $\otimes$-operation. For Göodel residuated lattices the fuzzy implication is adjoint with $a \otimes b = \min(a, b)$.

A fuzzy dataset is encoded by means of a fuzzy context as defined below.

Definition 2. *A Fuzzy Relation between two sets* X *and* Y *is a function* $I : X \times Y \to L$*, for some residuated lattice* $\boldsymbol{L}$*.*

Definition 3. *A Fuzzy Context is a triple* (X, Y, I) *where* X *is a set of objects,* Y *is a set of attributes,* I *is a fuzzy relation,* $I : X \times Y \to L$*.*

Let us now define what is a fuzzy set, the next building block of fuzzy FCA.

Definition 4. *Given a crisp set* X*, a fuzzy set* A *is a function* $A : X \to L$*, mapping each element of the crisp set to an element of the residuated lattice. A fuzzy set is denoted as* $\{{}^{l_i \in L}/_{x_i \in X}\}$*, where* $\bigcup x_i = X$*, and for simplicity elements* $A(x \in X) = \bot$ *are omitted.*

In the fuzzy case of FCA one also defines Galois connections between a fuzzy set of objects $A : X \to L$ and a fuzzy set of attributes $B : Y \to L$.

Definition 5 (Derivation Operators). *Given a fuzzy context (X, Y, I), a fuzzy set of objects $A : X \to L$, a fuzzy set of attributes $B : Y \to L$, the fuzzy membership for object $x \in X$ and for attribute $y \in Y$ in the corresponding sets $A^{\uparrow}$ and $B^{\downarrow}$ are as follows:*

$$A^{\uparrow}(y) = \bigwedge_{\forall x \in X} (A(x) \rightsquigarrow I(x, y))$$
$$B^{\downarrow}(x) = \bigwedge_{\forall y \in Y} (B(y) \rightsquigarrow I(x, y))$$

Definition 6. *A fuzzy concept is a pair (A, B), where A is a fuzzy set of objects, $A : X \to L$ and B is a fuzzy set of attributes $B : Y \to L$, such that $A^{\uparrow} = B$ and $A = B^{\downarrow}$.*

In particular there is the following fuzzy concept.

$$\left(\{{}^{1.0}/_{t_1}, {}^{1.0}/_{t_2}, {}^{0.286}/_{t_3}\}, \{{}^{0.714}/_{t_1}, {}^{0.714}/_{t_2}, {}^{0.571}/_{t_3}, {}^{0.429}/_{t_4}, {}^{0.429}/_{t_5}\}\right). \quad (2)$$

The set of fuzzy concepts is ordered such that $(A, B) \leq (X, Y)$ iff $A \subseteq X$ (or dually $B \supseteq Y$) forming a complete lattice, called *fuzzy concept lattice*.

3.2 Pattern Structures

A concept lattice $\mathfrak{L}(G, M, I)$ is constructed from a (binary) formal context (G, M, I) [1]. For non-binary data, such as sequences or graphs, lattices can be constructed in the same way using pattern structures [2].

Definition 7. *A pattern structure $\mathbb{P}$ is a triple $(G, (D, \sqcap), \delta)$, where G, D are sets, called the set of objects and the set of descriptions, and $\delta : G \to D$ maps an object to a description. Respectively, $(D, \sqcap)$ is a meet-semilattice on D w.r.t. $\sqcap$, called similarity operation such that $\delta(G) := \{\delta(g) \mid g \in G\}$ generates a complete subsemilattice $(D_\delta, \sqcap)$ of $(D, \sqcap)$.*

Derivation operator for a pattern structure $(G, (D, \sqcap), \delta)$, relating sets of objects and descriptions, is defined as follows:

$$A^{\diamond} := \bigsqcap_{g \in A} \delta(g), \qquad \text{for } A \subseteq G$$
$$d^{\diamond} := \{g \in G \mid d \sqsubseteq \delta(g)\}, \qquad \text{for } d \in D$$

Given a subset of objects A, $A^{\diamond}$ returns the description which is common to all objects in A. Given a description d, $d^{\diamond}$ is the set of all objects whose description subsumes d. The natural partial order (or subsumption order between descriptions) $\sqsubseteq$ on D is defined w.r.t. the similarity operation $\sqcap$: $c \sqsubseteq d \Leftrightarrow c \sqcap d = c$ (in this case we say that c is subsumed by d).

Definition 8. *A pattern concept of a pattern structure* $(G, (D, \sqcap), \delta)$ *is a pair* (A, d)*, where* $A \subseteq G$ *and* $d \in D$ *such that* $A^\diamond = d$ *and* $d^\diamond = A$*;* A *is called the pattern extent and* d *is called the pattern intent.*

The set of all pattern concepts is partially ordered w.r.t. inclusion of extents or, dually, w.r.t. subsumption of pattern intents within a concept lattice, these two anti-isomorphic orders form a lattice, called pattern lattice.

Let us return to the example in Table 1b. Let us consider a special case of pattern structures, a so-called Minimum Pattern Structure (MnPS), that is close to interval pattern structures [5]. MnPS is based on the minimum of two numbers as the similarity operation rather than on the convex hull of two intervals. We will show that MnPS is well adapted for formalizing fuzzy FCA within the framework of pattern structures.

In Table 1b we have the set G as both, a set of objects and a set of attributes. Let us first consider only one attribute. Then the set of descriptions D is just the interval $[0, 1]$ of real numbers and the similarity operation between two descriptions (numbers) is the minimum. When there are several attributes, the set of descriptions is just an element of $\mathbb{R}^{|N|}$, where $\mathbb{R}$ is the set of real numbers and N is the set of numerical attributes.

In particular, in our example the set of objects is G. The set D of descriptions is $\mathbb{R}^5$, since we have 5 numerical attributes. The mapping function δ is given in Table 1b, e.g., $\delta(t_2) = \langle 0.714, 1, 0.571, 0.429, 0.429\rangle$. The similarity operation is the component-wise minimum, e.g., the similarity between descriptions of t_2 and t_3 is given by

$$\begin{aligned}
&\{t_2\}^\diamond \sqcap \{t_3\}^\diamond = \\
&\quad = \langle 0.714, 1, 0.571, 0.429, 0.429\rangle \sqcap \langle 0.857, 0.571, 1, 0.286, 0.714\rangle = \\
&\quad = \langle \min(0.714, 0.857), \min(1, 0.571), \\
&\qquad \min(0.571, 1), \min(0.429, 0.286), \min(0.429, 0.714)\rangle \\
&\quad = \langle 0.714, 0.571, 0.571, 0.286, 0.429\rangle
\end{aligned}$$

4 From Fuzzy FCA to Pattern Structures with Scaling

Let us now discuss a possible connection between fuzzy FCA and pattern structures. A certain connection was already proposed in [6]. In particular, every *crisply* closed subset of objects is an extent of an interval pattern structure. Here, *crisply closed subset of objects* means that the fuzzy closure of this set contains no additional objects g with a membership degree coinciding with the top of the residuated lattice, i.e., $A(g) = \top$.

Here, we discuss a loss-less scaling from a fuzzy formal context (X, Y, I) to a pattern structure, that allows a more efficient processing than the loss-less scaling to crisp formal context and highlights another connection between fuzzy FCA and pattern structures.

Table 2: Scaling of the fuzzy context from table Table 1b to a number-minimum pattern structure.

	t_1	t_2	t_3	t_4	t_5
$\langle t_1, 1.000\rangle$	1.000	0.714	0.857	0.429	0.429
$\langle t_1, 0.857\rangle$	1.000	0.714	1.000	0.429	0.429
$\langle t_1, 0.714\rangle$	1.000	1.000	1.000	0.429	0.429
$\cdots$					
$\langle t_2, 1.000\rangle$	0.714	1.000	0.571	0.429	0.429
$\cdots$					
$\langle t_2, 0.571\rangle$	1.000	1.000	1.000	0.429	0.429
$\cdots$					
$\langle t_3, 1.000\rangle$	0.857	0.571	1.000	0.286	0.714
$\cdots$					

	t_1	t_2	t_3	t_4	t_5
$\cdots$					
$\langle t_3, 0.429\rangle$	1.000	1.000	1.000	0.286	1.000
$\cdots$					
$\langle t_4, 1.000\rangle$	0.429	0.429	0.286	1.000	0.000
$\cdots$					
$\langle t_4, 0.286\rangle$	1.000	1.000	1.000	1.000	0.000
$\cdots$					
$\langle t_5, 1.000\rangle$	0.429	0.429	0.714	0.000	1.000
$\cdots$					
$\langle t_5, 0.286\rangle$	1.000	1.000	1.000	1.000	0.000

Since pattern structures can deal with any kind of descriptions, they should take into account fuzziness on the intent side. However, for the extent side it is not so straightforward, since pattern structures deal only with crisp sets of objects. Accordingly, we should somehow "scale" object sets, in order to express fuzzy sets of objects.

4.1 On expressing Fuzziness on the Extent Side of Pattern Structures

A natural way is to scale the object set X from a fuzzy context (X, Y, I) by substituting it with the direct product of the crisp set of objects and the residuated lattice (the degrees of confidence), $X \times L$. For every scaled object from this new set, we should compute a description. Let us consider the scaled description for the pair $\langle x, l\rangle$, where $x \in X$ is an object and $l \in L$ is the membership degree of this object. The description of this element should correspond to the description of the fuzzy set $\{^l/_x\}$, since $\langle x, l\rangle$ is "a model of" this fuzzy set.

The derivation operator $\{^l/_x\}^{\uparrow}(y \in Y) = \{^l/_x\}(x) \rightsquigarrow I(x, y) = l \rightsquigarrow I(x, y)$ gives the description of the element $\langle x, l\rangle$ and allows computing the fuzzy relation $\tilde{I}$ between $X \times L$ and Y.

Let us return to our example. Let T be the set of transaction IDs. The scaled fuzzy context is partially shown in Table 2. It consist of $|T| \cdot |L| = 5 \cdot 7 = 35$ objects, 5 attributes and the fuzzy relation between them. Every subset of objects corresponds to a fuzzy set of objects by joining corresponding fuzzy representation for every object. This is made precise in the next subsection.

4.2 Relation between fuzzy and pattern extents and intents

Let (X, Y, I) be a fuzzy context with a residuated lattice $\mathbf{L}$ and (G, D, δ) be a pattern structure, where G is the scaled set of objects $G = X \times L$. Let us formally define the correspondence between fuzzy sets of objects and scaled sets of objects.

Definition 9 (Object sets equivalence). *A fuzzy object set $A : X \to L$ is equivalent to a scaled object set $N \subseteq G$, denoted as $A \sim N$, when*

$$(\forall \langle x, l\rangle \in G)(A(x) \geq l \Leftrightarrow \langle x, l\rangle \in N)$$

Then object sets are equivalent when all scaled objects with membership degree smaller than or equal to $A(x)$ (w.r.t. the residuated lattice) are present in the scaled object set. For example, the fuzzy set $\{^{0.286}/_{t_1}, ^{0.429}/_{t_4}\}$ is equivalent to the scaled set $\{\langle t_1, 0.286\rangle, \langle t_4, 0.429\rangle, \langle t_4, 0.286\rangle\}$[3], where $\langle g, l\rangle \in X \times L$ is an element of the direct product of the set of objects and the residuated lattice.

Given a scaled fuzzy context $(X \times L, Y, \tilde{I})$ we can process it as a minimum pattern structure $(X \times L, D, \delta)$, where $D = L^{|Y|}$ is a tuple of elements from the residuated lattice $\mathbf{L}$ and the semilattice operation is given by the component-wise infimum of $\mathbf{L}$. In particular, we have discussed that for the numerical case, the similarity operation is the component-wise minimum. Indeed, fuzziness on the extent side is expressed by means of scaled object sets, and fuzziness on the intent side is directly processed by the pattern structure. Let us discuss the correspondence between fuzzy intents and patterns.

Definition 10. *A fuzzy attribute set $B : Y \to L$ is equivalent to a pattern $d \in D$, written as $B \sim d$, iff $(\forall y \in Y)(B(y) = d(y))$, where $d(y)$ is the value of the tuple d corresponding to the attribute y.*

A fuzzy attribute set B is equivalent to a pattern d iff for any attribute $y \in Y$, the membership degree $B(y)$ in the fuzzy set is equal to the value in the pattern tuple in the position corresponding to the attribute y, e.g., the pattern $\langle 0.5, 0.7\rangle$ corresponds to the fuzzy set $\{^{0.5}/_{y_1}, ^{0.7}/_{y_2}\}$.

It should be noticed that the definition of equality between fuzzy sets of attributes and patterns is a bijection, while there are scaled sets of objects that have no equivalent fuzzy set of objects. Indeed, there is no equivalent fuzzy set to the scaled set $\{\langle t_1, 0.286\rangle, \langle t_4, 0.429\rangle\}$, since according to Definition 9 all $\langle x, l\rangle$ such that $A(x) \geq l$ should be in this set. And since we have $\langle t_4, 0.429\rangle$ in this set, we should also have $\langle t_4, 0.286\rangle$ in the set. We can notice here that in our particular example the residuated lattice has only the element 0.286 that is smaller than 0.429. By contrast, if we take the real interval $[0, 1]$, then all points smaller than 0.429 should be added to the scaled set.

Let us define equivalence classes of scaled sets of objects in order to have a bijection between the equivalence classes and the fuzzy sets of objects.

Definition 11. *A scaled object set $N \subseteq G$ is* complete *iff a scaled object $\langle x \in X, l \in L\rangle$ belongs to N, then $(\forall l^* \in L, l^* \leq l)\langle x, l^*\rangle \in N$.*

It can be checked that for any scaled object set $N \subseteq G$ there is only one minimal complete superset of N. Let us denote this complete set by $\phi(N)$.

For example, the set $N = \{\langle t_1, 0.286\rangle, \langle t_4, 0.429\rangle\}$ is not complete, since the scaled object $\langle t_4, 0.286\rangle$ is not in N.

By contrast, $N_c = \phi(N) = \{\langle t_1, 0.286\rangle, \langle t_4, 0.429\rangle, \langle t_4, 0.286\rangle\}$ is complete. Moreover, it can be seen that this set is equivalent to $\{^{0.286}/_{t_1}, ^{0.429}/_{t_4}\}$ according to Definition 9. Furthermore, it can be checked, that any complete scaled set of objects is equivalent to a fuzzy set and accordingly the function $\phi(\cdot)$ defines the required equivalence classes.

[3] We notice that $\langle t_4, 0.429\rangle$ and $\langle t_4, 0.286\rangle$ are two different scaled objects.

4.3 Isomorphism of fuzzy and pattern lattices

In this subsection we show that our scaling procedure is correct. And the resulting pattern lattice and the fuzzy lattice are isomorphic. Moreover, the extents and intents of these lattices are connected by means of Definitions 9 and 10. The first lemma (a standard property of residuated lattices) shows that fuzzy implications are related if their premises are comparable.

Lemma 1 *If there are $l_1, l_2, l \in L$ such that $l_1 \leq l_2$ then*

$$l_1 \rightsquigarrow l \geq l_2 \rightsquigarrow l$$

Proof. Let $l_2 \rightsquigarrow l = r$ then according to Def. 1:

$$(\forall f \in \mathbf{L}, f \leq r)(f \otimes l_2 \leq l) \Leftrightarrow (\forall f \leq r)(l_2 \otimes f \leq l)$$
$$\Leftrightarrow (\forall f \leq r)(f \rightsquigarrow l \geq l_2 \geq l_1) \Leftrightarrow (\forall f \leq r)(l_1 \rightsquigarrow l \geq f) \Rightarrow l_1 \rightsquigarrow l \geq r.$$

Let us now show that starting from two (fuzzy and scaled) equivalent sets of objects the resulting descriptions are also equivalent.

Lemma 2 *Given a fuzzy set of objects $A : X \to L$ and a scaled set of objects $N \subseteq G$, such that $A \sim \phi(N)$, we have $A^{\uparrow} \sim N^{\diamond}$.*

Proof. Consider the value of the pattern tuple $N^{\diamond}$ corresponding to an attribute y: $N^{\diamond}(y) = \big(\bigsqcap_{\forall g \in N} \delta(g)\big)(y)$. The semilattice operation of the minimum pattern structure corresponds to the infimum in the residuated lattice:

$$N^{\diamond}(y) = \bigwedge_{\forall \langle x,l \rangle \in N} \tilde{I}(\langle x, l \rangle, y) = \bigwedge_{\forall \langle x,l \rangle \in N} l \rightsquigarrow I(x,y) =$$
$$= \Big(\bigwedge_{\forall x \in X} A(x) \rightsquigarrow I(x,y) \Big) \wedge \Big(\bigwedge_{\forall \langle x,l \rangle \in N : l < A(x)} l \rightsquigarrow I(x,y) \Big) =$$
$$= {}_{\text{Lemma 1}} \Big(\bigwedge_{\forall x \in X} A(x) \rightsquigarrow I(x,y) \Big) = A^{\uparrow}(y).$$

Finally let us show, that starting from equivalent fuzzy set of attributes and pattern, the sets of objects given by the derivation operators are also equivalent.

Lemma 3 *Given a fuzzy set of attributes $B : Y \to L$ and a pattern $d \in D$, such that $B \sim d$, we have $B^{\downarrow} \sim d^{\diamond}$.*

Proof. Let us study when object $\langle x \in X, l \in L \rangle$ can be included into $d^{\diamond}$.

$$\langle x \in X, l \in L \rangle \in d^{\diamond} \Leftrightarrow \delta(\langle x, l \rangle) \sqsupseteq d \Leftrightarrow (\forall y \in Y)(\delta(\langle x, l \rangle)(y) \geq d(y))$$
$$\Leftrightarrow (\forall y \in Y)(l \rightsquigarrow I(x,y) \geq d(y))$$
$$\Leftrightarrow (\forall y \in Y)(d(y) \otimes l \leq I(x,y)) \Leftrightarrow (\forall y \in Y)(l \otimes d(y) \leq I(x,y))$$
$$\Leftrightarrow (\forall y \in Y)(d(y) \rightsquigarrow I(x,y) \geq l)$$
$$\Leftrightarrow (\forall y \in Y)(B(y) \rightsquigarrow I(x,y) \geq l)$$
$$\Leftrightarrow l \leq \bigwedge_{\forall y \in Y} B(y) \rightsquigarrow I(x,y)$$
$$\Leftrightarrow l \leq B^{\downarrow}(x)$$

Thus an object $\langle x, l\rangle \in G$ is included in $d^{\diamond}$ iff $B^{\downarrow}(x) \geq l$ which is the definition of the equality of a fuzzy set of objects and a scaled set of objects.

Theorem 1 *The fuzzy lattice $\mathfrak{L}_f$ corresponding to the context (X, Y, I) and the pattern lattice $\mathfrak{L}_p$ corresponding to the pattern structure $(G, \underline{D}, \delta)$, where $G = X \times L$, $D = L^{|Y|}$ with component-wise minimum as the semilattice operation, and $\delta(\langle x \in X, l \in L\rangle)(y) = l \rightsquigarrow I(x, y)$ are isomorphic. The extents and intents of the corresponding concepts are equivalent.*

Proof. Let us show, that for any concept in one lattice there is a concept in the other lattice with equivalent extents and intents. Lemmas 2 and 3 are symmetric w.r.t. the type of extents and intents. Accordingly, we can just denote by $\mathfrak{L}_1$ and $\mathfrak{L}_2$ fuzzy and pattern lattices and prove the theorem in both directions. If we take an intent i_1 from $\mathfrak{L}_1$, we can always find an equivalent pattern p (for simplicity, fuzzy set of attributes is also referred as a pattern). Applying appropriate derivation operators to i_1 and p we get equivalent sets of objects according to Lemma 3. Both sets are closed and are extents of $\mathfrak{L}_1$ and $\mathfrak{L}_2$. Applying derivation operators to the extents we get equivalent intents according to Lemma 2. Thus, for any concept of $\mathfrak{L}_1$ there is an equivalent concept in $\mathfrak{L}_2$ and *vice versa*.

4.4 Application of the Theorem

Let us demonstrate how the theorem works in the running example. The proof is based on search of concepts with equivalent extents and intents. Let us find the scaled concept corresponding to the fuzzy concept (2). In the theorem we start from the intent. It can be seen that

$$\{{}^{0.714}/_{t_1}, {}^{0.714}/_{t_2}, {}^{0.571}/_{t_3}, {}^{0.429}/_{t_4}, {}^{0.429}/_{t_5}\} \sim \langle 0.714, 0.714, 0.571, 0.429, 0.429\rangle. \quad (3)$$

For the moment we are not sure that the pattern on the right side is an intent. Accordingly we apply derivation operators to the left and right hand sides and according to Lemma 3 the resulting object sets should be equivalent. Indeed,

$$\{{}^{1}/_{t_1}, {}^{1}/_{t_2}, {}^{0.286}/_{t_3}\} \sim \\ \sim \{\langle t_1, 1\rangle, \langle t_1, 0.857\rangle, \ldots, \langle t_1, 0.286\rangle, \langle t_2, 1\rangle, \ldots \langle t_2, 0.286\rangle, \langle t_3, 0.286\rangle\}.$$

On the left side we have the extent of the concept, while on the right side we have a closed scaled set of objects, since the result of the derivation operator is always closed. If we apply the derivation operators to these two sets of objects, we have equivalent patterns according to Lemma 2. In fact we have exactly the patterns from (3). Thus, we have found the scaled concept corresponding to the fuzzy concept. Similarly, we can start from a scaled concept and find the corresponding fuzzy concept.

5 Discussion and Conclusion

In this paper we highlighted the relation between fuzzy FCA and pattern structures. Our result is related to the work of [6]. Indeed, the authors have shown that extents of crisply closed fuzzy concepts are also closed in the interval pattern structure. In our work, we used the Minimum Pattern Structure that can be considered as a projection of the interval pattern structure. Indeed, let us consider the following component-wise projection. If $[a, b]$ is an interval, than the projection $\psi([a, b]) = [a, +\inf]$ changes the IPS to the MnPS. Accordingly the set of extents of the MnPS is the subset of the extents of the IPS. However, in our work we have shown, that the MnPS lattice is isomorphic to a fuzzy lattice under the scaling. It can be seen, that if we do not apply the scaling we generate exactly the lattice of the crisply generated fuzzy concepts. And this set of concepts is the subset of the concepts of the corresponding IPS.

The introduced scaling procedure can be useful, first, for migrating results between pattern structure community and fuzzy FCA community, and, second, for efficient implementation of software dealing with both pattern structures and fuzzy FCA at the same time.

Finally, we notice that such a work naturally raises (as it was already mentioned in [6]) the question of a "two-sided" pattern structure as a generalization of both pattern structures and fuzzy FCA. Some suggestions going in this directions can be found in the work of Soldano et al. [7], where the authors discussed projections applied to the extent side.

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