



**HAL**  
open science

# Stable perfectly matched layers for a class of anisotropic dispersive models. part II: stability in corners and energy estimates

Maryna Kachanovska

## ► To cite this version:

Maryna Kachanovska. Stable perfectly matched layers for a class of anisotropic dispersive models. part II: stability in corners and energy estimates . 2016. hal-01419682v1

**HAL Id: hal-01419682**

**<https://inria.hal.science/hal-01419682v1>**

Preprint submitted on 19 Dec 2016 (v1), last revised 2 Mar 2017 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# STABLE PERFECTLY MATCHED LAYERS FOR A CLASS OF ANISOTROPIC DISPERSIVE MODELS. PART II: STABILITY IN CORNERS AND ENERGY ESTIMATES \*

MARYNA KACHANOVSKA<sup>1</sup>

**Abstract.** This article continues the stability analysis of the generalized perfectly matched layers for 2D anisotropic dispersive models studied in Part I of the work. We consider two questions: the stability of the PMLs in corners and the energy estimates for the PML systems in the time domain. Based on the Fourier analysis, we deduce necessary and sufficient conditions of stability of the newly constructed PMLs in corners. The obtained results are confirmed with the help of the numerical experiments. In order to obtain explicit energy estimates for the PML system in the time domain, we make use of the ideas stemming from the analysis of the associated sesquilinear form in the Laplace domain. This analysis is based on the introduction of a particular set of auxiliary unknowns related to the PML, which simplifies the derivation of the energy estimates for the resulting system.

## 1. INTRODUCTION

The method of the perfectly matched layers, introduced by Bérenger [1,2], is used ubiquitously in engineering and physics communities to compute a solution to a problem posed in an unbounded domain. However, it is well-known that for some classes of problems (e.g. the wave propagation in anisotropic or dispersive media) the classical PML method may result in instabilities [3,4].

In this work we continue the analysis of the stable perfectly matched layers constructed for a class of anisotropic dispersive models in [5]. This class of models describes the wave propagation in metamaterials and plasmas in two dimensions, as well as includes some non-hyperbolic problems. In the frequency domain, the model under consideration can be written in the form of an anisotropic dispersive wave equation:

$$\varepsilon_1(\omega)^{-1}\partial_x^2 u + \varepsilon_2(\omega)^{-1}\partial_y^2 u - \omega^2\mu(\omega)u = 0, \quad \omega \in \mathbb{R},$$

where  $\varepsilon_1(\omega)$ ,  $\varepsilon_2(\omega)$  have a meaning of dielectric permittivity and  $\mu(\omega)$  is a magnetic permeability, are frequency-dependent, and satisfy  $\text{Im}(\omega\varepsilon_j(\omega)) > 0$ ,  $\text{Im}(\omega\mu(\omega)) > 0$  for  $\text{Im}\omega > 0$ , cf. [6]. In [5] a method of construction of stable perfectly matched layers for this class of models was suggested, based on the ideas of Bécache et al. [4] for isotropic metamaterials, which generalized the work of Cummer [7] for the Drude model. However, the analysis in the aforementioned works does not fully cover the following (almost independent) questions, which we aim to answer in the present article:

- stability of the perfectly matched layers constructed in [5] in corners;

---

\* This work was supported by a public grant as part of the Investissement d'avenir project, reference ANR-11-LABX-0056-LMH, LabEx LMH, as well as a co-financing program PRESTIGE

<sup>1</sup> Laboratoire Poems (UMR 7231 CNRS/Inria/ENSTA ParisTech), ENSTA ParisTech, 828 Boulevard des Maréchaux 91120 Palaiseau, France, E-mail: maryna.kachanovska@ensta.fr

- decay of an energy associated to the newly constructed PML systems.

The second question has been answered for the Bérenger's PML in two dimensions for isotropic, nondispersive Maxwell equations by Bécache and Joly [8]. In general, the derivation of energy estimates for PML systems does not seem trivial. For example, in the work [8] the authors have obtained energy estimates for the PML for isotropic Maxwell equations written in the Zhao-Cangellaris formulation [9]. The estimates on the norms of the split fields in the Bérenger's split formulation were derived, however, based on the Fourier, rather than energy, analysis. Our approach is different from that of [8], where the PML system was given a priori. We suggest to look for a formulation that would be equivalent to the original system and allows to obtain the energy in a simpler way. We make use of the stability analysis in the Laplace domain, as it was done in the work [5], which provides intuition on derivation of explicit (without appeal to the Plancherel's theorem) energy estimates.

An alternative way to obtain energy estimates is offered by the use of the method of Hagstrom and Appelö [10]. However, even in the case of Bérenger's PML it provides estimates on the spatial derivatives of unknowns [11], while most of the estimates we obtain involve the original unknowns. Moreover, the application of this approach to Lorentz models does not seem trivial, due to the need to rewrite the original problem in the form of a single equation, rather than a system of equations, and then apply the method of [10] to the resulting problem.

Let us remark that of the limitations of our analysis is the assumption that the absorption parameter of the PML is constant, similarly to many other works on the PMLs, cf. e.g. [4, 8, 11]. This analysis can be viewed as a first step to deal with more complicated problems with non-constant absorption parameters.

This work is organized as follows. In Section 1 we introduce the problem under consideration in more detail, as well as provide succinctly some background on the stability of the perfectly matched layers based on [5]. We will assume a familiarity of the reader with cartesian PMLs and the PML geometrical setting.

In the first part of the article we concentrate on so-called generalized Lorentz materials [4]. In Section 2.2 we consider the question of the stability of perfectly matched layers in corners. We demonstrate the necessity of a fairly rigid condition of the stability of the corner PMLs that belong to a class introduced in [4]. In Section 3 we study the questions of the construction of the time domain PML system and the derivation of the corresponding energy estimates.

In the second part of the article we generalize the energy estimates of the first part of the work to a more general class of materials, including e.g. models with absorption or with memory, see Section 4.

We finish the exposition by a discussion of open problems in Section 5. The questions related to the discretization are out of the scope of the present work, and can be dealt with as in [8].

## 1.1. Problem Setting

The wave propagation in 2D dispersive media is described by the Maxwell system:

$$\begin{aligned}\partial_t D_x &= \partial_y H_z, \\ \partial_t D_y &= -\partial_x H_z, \\ \partial_t B_z &= -\partial_x E_y + \partial_y E_x.\end{aligned}\tag{1}$$

In the above we use the scaling  $c = \varepsilon_0 = \mu_0 = 1$ . The relations between the electric field and the electric displacement field, as well as between the magnetic field and the magnetizing field are given in the Laplace domain. Denoting by  $s$  the Laplace variable, we set  $\hat{u}(s) = (\mathcal{L}u)(s)$  the Laplace transform of  $u(t)$  and introduce  $\mathbb{C}_+ = \{s \in \mathbb{C} : \operatorname{Re} s > 0\}$ . Following [5], for a function  $f(s) : \mathbb{C} \rightarrow \mathbb{C}$ , we define  $\tilde{f}$  as follows:

$$\tilde{f}(\omega) = f(-i\omega), \quad \omega \in \mathbb{C}.\tag{2}$$

Thus, the constitutive relations  $\hat{\mathbf{D}} = \varepsilon(s)\hat{\mathbf{E}}$ ,  $\hat{H}_z = \mu^{-1}(s)\hat{B}_z$ . We will assume

$$\varepsilon(s) = \begin{pmatrix} \varepsilon_x(s) & 0 \\ 0 & \varepsilon_y(s) \end{pmatrix}$$

In this case the system (1) in the Laplace domain can be rewritten as an anisotropic dispersive wave equation:

$$s^2 \mu(s) \hat{H}_z - \varepsilon_y(s)^{-1} \partial_{xx} \hat{H}_z - \varepsilon_x(s)^{-1} \partial_{yy} \hat{H}_z = 0. \quad (3)$$

All over the article will assume that  $\varepsilon_x(s)$ ,  $\varepsilon_y(s)$  and  $\mu(s)$  are passive, as defined below.

**Definition 1.1** ([5]). A function  $c : \mathbb{C}_+ \rightarrow \mathbb{C}$  is passive if it is analytic in  $\mathbb{C}_+$  and satisfies  $\operatorname{Re}(sc(s)) > 0$  there.

One of the examples of such passive parameters is offered by so-called generalized Lorentz materials [4]:

$$\begin{aligned} \varepsilon_\alpha(s) &= 1 + \sum_{\ell=0}^{n_\alpha} \frac{\varepsilon_{\alpha\ell}}{s^2 + \omega_{\alpha\ell}^2}, & \varepsilon_{\alpha\ell} > 0, \omega_{\alpha\ell} \in \mathbb{R}, & \ell = 0, \dots, n_\alpha, \alpha \in \{x, y\}, \\ \mu(s) &= 1 + \sum_{\ell=0}^{n_\mu} \frac{\mu_\ell}{s^2 + \omega_{\mu\ell}^2}, & \mu_\ell > 0, \omega_{\mu\ell} \in \mathbb{R}, & \ell = 0, \dots, n_\mu. \end{aligned} \quad (4)$$

The above includes as a special case Drude materials [4] and a 2D uniaxial cold plasma [12, 13]. The assumption of the passivity covers nonhyperbolic models as well (the diffusion equation, with  $\varepsilon_x(s) = \varepsilon_y(s) = 1$ ,  $\mu(s) = s^{-1}$ ).

## 1.2. Construction and Stability of Perfectly Matched Layers

### 1.2.1. Construction of PMLs

There are several ways to write the perfectly matched system corresponding to (1), in particular, via splitting as per Bérenger [1, 2], or with the help of the change of variables as per Chew et al. [14]. For an introduction and overview of these techniques, we address an interested reader to [15].

Consider the system (1) written in the Laplace domain (3). To construct a PML in the region  $x \geq 0$ , one assumes that (3) is valid for  $x = \tilde{x} \in \mathbb{C}$ . The corresponding analytic continuation of  $\hat{H}_z$ , denoted by  $\hat{\mathcal{H}}_z$ , solves (3), with  $x = \tilde{x}$ . In [4] it is proposed to choose  $\tilde{x}$  as follows, for some analytic function  $\psi(s)$ ,

$$\tilde{x} = \begin{cases} x + s^{-1} \psi(s) \int_0^x \sigma(x') dx', & x \geq 0, \\ x, & x < 0. \end{cases} \quad (5)$$

The Bérenger's PML can be obtained taking  $\psi(s) = 1$ , while, in general, for dispersive materials other  $\psi$  should be found [4]. With this change of variables, (3) remains of the same form for  $x < 0$ , however, for  $x > 0$ , it reads

$$\varepsilon_y^{-1} (1 + \sigma s^{-1} \psi(s))^{-1} \partial_x \left( (1 + \sigma s^{-1} \psi(s))^{-1} \partial_x \hat{\mathcal{H}}_z \right) + \varepsilon_x^{-1} \partial_{yy} \hat{\mathcal{H}}_z - s^2 \mu \hat{\mathcal{H}}_z = 0. \quad (6)$$

At the interface  $x = 0$ , transmission conditions are used, see [15]. By analytic continuation,  $\hat{\mathcal{H}}_z(x, y) = \hat{H}_z(x, y)$  for  $x < 0$ . It remains to rewrite this system in the time domain, however, for the moment we do not address this question, postponing it to future sections.

To construct the PML in other directions (e.g. for  $y \geq 0$ ), the same procedure applies. In corners the change both of variables  $x$  and  $y$  are used. In order to truncate the perfectly matched layer at some  $x = L > 0$  in one uses zero Dirichlet or Neumann boundary conditions on the exterior boundary of the layer. In the experiments in this article we use vanishing Dirichlet BCs.

### 1.2.2. Existing Results on the Stability of PMLs

In this section we provide known results on the stability of the time-domain system obtained after the PML change of variables (5) applied to the problem in question, most of which can be found in [5]. For results on

the stability of the PMLs for other classes of models see [4] or [3]. The statements in this section are valid in the case of a constant absorption parameter  $\sigma(x) = \sigma$ , and the PML change of variables in  $\mathbb{R}$ :

$$\tilde{x} = x(1 + s^{-1}\psi(s)\sigma), \quad \sigma \geq 0. \quad (7)$$

Let us introduce the sesquilinear form corresponding to the operator (3):

$$A(u, v) := \varepsilon_y(s)^{-1}(\partial_x u, \partial_x v) + \varepsilon_x(s)^{-1}(\partial_y u, \partial_y v) + s^2\mu(s)(u, v), \quad u, v \in H^1(\mathbb{R}^2). \quad (8)$$

Here  $(u, v) = \int_{\mathbb{R}^2} u\bar{v}dx$ . This scalar product induces a duality pairing  $\langle f, u \rangle$ , with  $f \in H^{-1}(\mathbb{R}^2)$ ,  $u \in H^1(\mathbb{R}^2)$ , where  $H^{-1}(\mathbb{R}^2)$  contains antilinear functionals on  $H^1(\mathbb{R}^2)$ . Moreover, all over the article we will denote by  $\|\cdot\|$  an  $L^2(\mathbb{R}^2)$ -norm, and by  $\|\cdot\|_{H^1}$  a norm in  $H^1(\mathbb{R}^3)$ .

**Definition 1.2.** We will say that a sesquilinear form (8) is passive if  $\varepsilon_x(s)$ ,  $\varepsilon_y(s)$  and  $\mu(s)$  are passive.

For passive sesquilinear forms the following result holds true.

**Proposition 1.3.** *Let the sesquilinear form  $A(u, v)$  given by (8) be passive. Then:*

(1) *for all  $G \in H^{-1}(\mathbb{R}^2)$ ,  $s \in \mathbb{C}_+$  there exists a unique solution  $U(s) \in H^1(\mathbb{R}^2)$  to the variational formulation*

$$A(U(s), v) = \langle G, v \rangle, \quad v \in H^1(\mathbb{R}^2). \quad (9)$$

(2) *let  $G(s) \in H^{-1}(\mathbb{R}^2)$  be the Laplace transform of a causal  $C^2$ -function  $g(t) : \mathbb{R} \rightarrow H^{-1}(\mathbb{R}^2)$ , whose 3rd derivative in the sense of distributions  $g^{(3)} \in L_1(\mathbb{R}, H^{-1}(\mathbb{R}^2))$ . Then the solution  $U(s)$ ,  $s \in \mathbb{C}_+$ , of (9) with  $G = G(s)$  is a Laplace transform of a continuous causal function  $u(t) : \mathbb{R} \rightarrow H^1(\mathbb{R}^2)$ , and*

$$\|u(t)\|_{H^1} \leq \alpha \max(t^3, 1) \int_0^t \|(1 + \partial_\tau)^3 g(\tau)\|_{H^{-1}} d\tau, \quad t \geq 0, \quad \alpha > 0.$$

**Remark 1.4.** The idea of the proof of the first part of Proposition 1.3 will be of use later, when deriving energy estimates. Let us briefly comment on it, referring the interested reader to [5] for more details. To show the solvability of (9), one uses the Lax-Milgram lemma. In order to apply it, one demonstrates the following frequency-dependent bounds valid for all  $s \in \mathbb{C}_+$ :

$$\|A(u, v)\| \leq C(s)\|u\|_{H^1}\|v\|_{H^1}, \quad \operatorname{Re} A(u, su) \geq c(s)\|u\|_{H^1}^2, \quad s \in \mathbb{C}_+.$$

In particular, the first bound is obtained trivially, since the passive functions and their reciprocals are analytic in  $\mathbb{C}_+$ , and hence do not blow up. The coercivity follows by a direct computation

$$\operatorname{Re} A(u, su) = \operatorname{Re} (\bar{s}\varepsilon_y^{-1}) \|\partial_x u\|^2 + \operatorname{Re} (\bar{s}\varepsilon_x^{-1}) \|\partial_y u\|^2 + \operatorname{Re} (|s|^2 s\mu) \|u\|^2, \quad s \in \mathbb{C}_+.$$

It remains to notice that  $\operatorname{Re} (\bar{s}\varepsilon_y^{-1}) = \operatorname{Re} (|s|^2 s^{-1}\varepsilon_y^{-1}) > 0$  for  $s \in \mathbb{C}_+$ , due to passivity  $\operatorname{Re} (s\varepsilon_y) > 0$  in  $\mathbb{C}_+$ . Hence,  $\operatorname{Re} (\bar{s}\varepsilon_y^{-1}) > c_0(s)$ , for some  $c_0(s) > 0$  whenever  $s \in \mathbb{C}_+$ . Similarly one can show that  $\operatorname{Re} (\bar{s}\varepsilon_x^{-1}) > c_1(s)$  in  $\mathbb{C}_+$ . Finally, the lower bound  $\operatorname{Re} (|s|^2 s\mu) > c_2(s)$  in  $\mathbb{C}_+$  again follows from passivity of  $\mu$ .

Proposition 1.3 shows that a passive sesquilinear form corresponds to a well-posed stable in the time domain system (see e.g. Section 3.1). By stable we mean that provided a sufficiently smooth right hand side data which decays sufficiently fast at infinity (e.g. compactly supported), the solution of the system is bounded polynomially in time. The corresponding time domain system will be analyzed in Section 3.

The results of Proposition 1.3 can be applied to analyze the stability of the PMLs. To construct a PML stable in the direction  $x$  it is sufficient to find  $\psi(s)$  that ensures the passivity of the sesquilinear form

$$A_\sigma(u, v) = \varepsilon_y(s)^{-1} (1 + s^{-1}\psi(s)\sigma)^{-2} (\partial_x u, \partial_x v) + \varepsilon_x(s)^{-1} (\partial_y u, \partial_y v) + s^2 \mu(s)(u, v), \quad u, v \in H^1(\mathbb{R}^2), \quad (10)$$

which corresponds to the equation (3) with the PML change of variables, namely (6). The following result shows how to construct a PML, stable in the directions  $x, y$ , as well as in the corners, for any passive system.

**Proposition 1.5** ([5]). *Let the sesquilinear form (8) be passive. Then the sesquilinear form obtained by the PML change of variables  $x \rightarrow x \left(1 + \frac{\sigma_x \psi_x(s)}{s}\right)$ , with  $\psi_x(s) = \varepsilon_y(s)^{-1}$  and  $y \rightarrow y \left(1 + \frac{\sigma_y \psi_y(s)}{s}\right)$ , with  $\psi_y(s) = \varepsilon_x(s)^{-1}$ , namely*

$$A_{\sigma_x, \sigma_y}(u, v) = \varepsilon_y^{-1} \left(1 + \frac{\sigma_x \varepsilon_y^{-1}}{s}\right)^{-2} (\partial_x u, \partial_x v) + \varepsilon_x^{-1} \left(1 + \frac{\sigma_y \varepsilon_x^{-1}}{s}\right)^{-2} (\partial_y u, \partial_y v) + s^2 \mu(u, v), \quad u, v \in H^1(\mathbb{R}^2),$$

is passive for any  $\sigma_x, \sigma_y \geq 0$ .

However, as argued in [5], the above change of variables might be non-optimal, i.e. require the introduction of too many auxiliary unknowns associated to the time-domain PML system. Other possible changes of variables are proposed in the next section.

## 2. PMLs FOR GENERALIZED LORENTZ MATERIALS: STABILITY IN CORNERS

In this section we will concentrate on generalized Lorentz materials (4). The motivation, as well as existing results on the stability of the PMLs are given in Section 2.1. The stability of the PMLs in corners is a subject of Sections 2.2.1 and 2.2.2. Section 3 is dedicated to the derivation of the energy estimates for the PML system in time.

### 2.1. Existing Results on Construction and Stability of PMLs

For the moment, let us simply assume that  $\varepsilon_x, \varepsilon_y, \mu$  satisfy the following assumption of [4, 5].

**Assumption 2.1.** *A function  $r(s)$  is a rational function  $r(s) = 1 + \frac{p_r(s^2)}{q_r(s^2)}$ , where the polynomials  $p_r$  and  $q_r$  have real coefficients, no common roots, and also  $\deg p_r < \deg q_r$ . All its zeros and poles lie in  $i\mathbb{R}$ .*

With any such a function  $\tilde{r}(\omega)$ ,  $\omega \in \mathbb{R}$ , we will associate the set  $\mathcal{D}_r$  of its poles and zeros. A class of passive functions satisfying the above condition is fairly narrow, see [16, Theorem 5], [17, Corollary 10.1].

**Theorem 2.2.** *A function  $r(s)$  satisfying Assumption 2.1 is passive if and only if  $r(s) = 1 + \sum_{\ell=0}^n \frac{r_\ell}{s^2 + \omega_\ell^2}$ , where  $\omega_\ell \in \mathbb{R}$  and  $r_\ell > 0$  for all  $\ell = 0, \dots, n$ .*

Thus, passive functions that satisfy Assumption 2.1 correspond to Lorentz materials (4). It is natural to consider a subclass of passive sesquilinear forms.

**Assumption 2.3.** *The sesquilinear form (8) is passive and the functions  $\varepsilon_x(s), \varepsilon_y(s), \mu(s)$  satisfy additionally Assumption 2.1.*

To explain the stability condition of the PML, let us introduce two sets:

$$\Omega_p^{\tilde{\varepsilon}_y \tilde{\mu} > 0} = \{\omega \in \mathbb{R} \setminus (\mathcal{D}_{\varepsilon_y} \cup \mathcal{D}_\mu) : \tilde{\varepsilon}_x(\omega) \tilde{\mu}(\omega) > 0\}, \quad \Omega_p^{\tilde{\varepsilon}_y \tilde{\varepsilon}_x < 0} = \{\omega \in \mathbb{R} \setminus (\mathcal{D}_{\varepsilon_x} \cup \mathcal{D}_{\varepsilon_y}) : \tilde{\varepsilon}_y(\omega) \tilde{\varepsilon}_x(\omega) < 0\}.$$

Under assumption that  $\psi(s)$  in (7) satisfies Assumption 2.3, the authors of [4] have shown that for the stability of the PML it is necessary that  $\psi(s)^{-1}$  is passive. The second necessary stability condition of the PML, obtained in [4] (and reformulated as (NSC1) below), is equivalent to the sufficient stability condition (NSC2).

**Theorem 2.4.** *Let the sesquilinear form (8) satisfy Assumption 2.3. Let the function  $\psi(s)$  satisfy Assumption 2.1 and  $\operatorname{Re}(\bar{s}\psi(s)) > 0$ ,  $s \in \mathbb{C}_+$ . Then the following two conditions are equivalent:*

(NSC1) *for all  $\omega \in \Omega_p^{\tilde{\varepsilon}_y \tilde{\varepsilon}_x < 0} \cup \Omega_p^{\tilde{\varepsilon}_y \tilde{\mu} > 0}$ , it holds that  $\tilde{\psi}(\omega) \tilde{\varepsilon}_x(\omega) \geq 0$ .*

(NSC2) *for any  $\sigma \geq 0$ , the sesquilinear form  $\tilde{A}_\sigma(u, v)$ ,  $u, v \in H^1(\mathbb{R}^2)$ , defined by*

$$\tilde{A}_\sigma(u, v) = \psi(s) \varepsilon_y(s) A_\sigma(u, v) = \psi(1 + s^{-1} \psi \sigma)^{-2} (\partial_x u, \partial_x v) + \psi \varepsilon_y \varepsilon_x^{-1} (\partial_y u, \partial_y v) + s^2 \psi \varepsilon_y \mu(u, v) \quad (11)$$

*is passive (thus satisfies Assumption 2.3).*

**Remark 2.5.** Indeed, the stability of a PML in the direction  $y$  can be ensured by the change of variables (5) with  $\psi(s)$  which would satisfy the following condition: for all  $\omega \in \Omega_p^{\tilde{\varepsilon}_x \tilde{\varepsilon}_y < 0} \cup \Omega_p^{\tilde{\varepsilon}_x \tilde{\mu} > 0}$ , it holds that  $\tilde{\psi}(\omega) \tilde{\varepsilon}_x(\omega) \geq 0$ .

One readily notices that the above result combined with Proposition 1.3 shows the stability of a PML in one direction only, and it is not immediately clear whether it can be applied to the PML in corners. Even if one argues that the PML change of variables in directions  $x$  and  $y$  that satisfies the conditions of the above theorem may not necessarily lead to a passive sesquilinear form, it is still not clear whether the passivity of a sesquilinear form is *necessary* for the stability, see also the related discussion in [18]. We discuss the problem of the stability of such perfectly matched layers in corners in the following section.

## 2.2. Stability in Corners

### 2.2.1. Applicability of Theorem 2.4 in Corners

Here we will concentrate on the studies of the stability of corner PMLs for passive systems that stem from the change of variables (7) with  $\psi(s)$  satisfying Assumption 2.3. While Theorem 2.4 provides necessary and sufficient conditions of the stability of the PMLs in one direction, a stricter condition is required to ensure the stability in corners. The following numerical example provides a more precise explanation.

**Example 2.6** (Motivation). Consider the problem (1) with the following parameters:

$$\tilde{\varepsilon}_y(\omega) = \frac{(\omega^2 - 4^2)(\omega^2 - 8^2)}{(\omega^2 - 2^2)(\omega^2 - 6^2)}, \quad \tilde{\varepsilon}_x(\omega) = \frac{(\omega^2 - 4^2)(\omega^2 - 8^2)}{\omega^2(\omega^2 - 6^2)}, \quad \tilde{\mu}(\omega) = \frac{(\omega^2 - 2^2)(\omega^2 - 6^2)}{\omega^2(\omega^2 - 4^2)}. \quad (12)$$

One can show that  $\varepsilon_x$ ,  $\varepsilon_y$  and  $\mu$  are passive using Lemma 4.8 of [5]. To apply Theorem 2.4, we define:

$$\Omega_p^{\tilde{\varepsilon}_x \tilde{\mu} > 0} = \{\omega \in \mathbb{R} : 0 < |\omega| < 2 \text{ or } |\omega| > 8\}, \quad \Omega_p^{\tilde{\varepsilon}_x \tilde{\varepsilon}_y < 0} = \Omega_p^{\tilde{\varepsilon}_x \tilde{\mu} > 0}, \quad \Omega_p^{\tilde{\varepsilon}_y \tilde{\mu} > 0} = \{\omega \in \mathbb{R} : |\omega| > 8\}.$$

The condition (NSC1) of Theorem 2.4 requires that in the direction  $x$  we choose the PML change of variables (7) with  $\psi(s) = \psi_x(s)$ , and in  $y$  with  $\psi(s) = \psi_y(s)$  (see also Remark 2.5), with

$$\tilde{\psi}_x(\omega) > 0, \quad \tilde{\psi}_y(\omega) < 0 \text{ for } |\omega| \in (0, 2), \text{ and } \tilde{\psi}_x(\omega) > 0, \quad \tilde{\psi}_y(\omega) > 0 \text{ for } |\omega| > 8.$$

The conditions of Theorem 2.4 are satisfied by  $\tilde{\psi}_x(\omega) = 1$ , and  $\tilde{\psi}_y(\omega) = \frac{\omega^2}{\omega^2 - 4^2}$ . We conduct three numerical experiments: 1) with the PML in  $y$ -direction  $\sigma_x \equiv 0$ ,  $\sigma_y \neq 0$ ; 2) with the PML in  $x$ -direction  $\sigma_x \neq 0$ ,  $\sigma_y \equiv 0$ ; 3) with the PML in both directions  $\sigma_x \sigma_y \neq 0$ . The numerical experiment are done with the help of the Yee scheme [19], where for the dispersive terms we use an implicit trapezoid rule. In Figures 1, 2 we show the results of this experiment. The PML is stable in each of the directions separately, however, is unstable in corners. Explaining this instability is the subject of the next section.

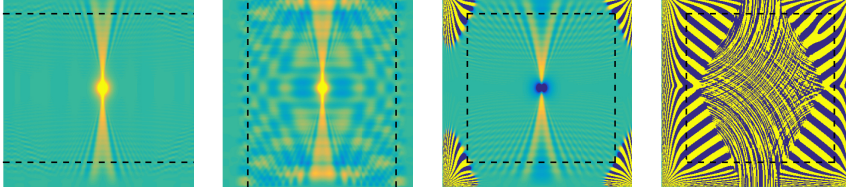


FIGURE 1. The results of the experiment of Example 2.6. From left to right:  $H_z$  computed with the PML in  $y$ -direction, at  $t = 150$ ; with the PML in  $x$ -direction,  $t = 150$ ; with the PML in both directions, at  $t = 100$  and  $t = 150$ . The boundary between the physical domain and the perfectly matched layer is marked by a black dashed line.

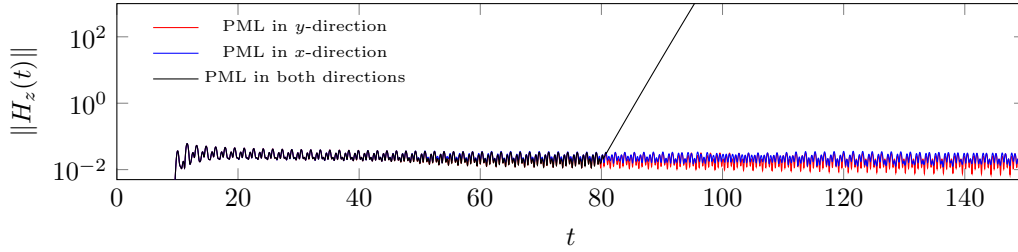


FIGURE 2. The  $L^2$ -norm of the solution  $H_z(t)$  of Example 2.6 depending on time.

### 2.2.2. Necessary and Sufficient Conditions for Stability of PMLs in Corners

To study the PML in corners, let us fix a choice of a stable PML in one direction. By scaling the sesquilinear form (10) can be reduced to the form (11), which can be viewed as an application of the PML of Proposition 1.5 to a system defined by the scaled sesquilinear form:

$$\tilde{A}(u, v) = \psi(s)(\partial_x u, \partial_x v) + \varepsilon_x(s)^{-1} \psi(s) \varepsilon_y(s) (\partial_y u, \partial_y v) + s^2 \mu(s) \psi(s) \varepsilon_y(s) (u, v).$$

Moreover, as shown in Theorem 2.4, the above sesquilinear form is passive. Therefore, we can fix the PML change of variables in the direction  $x$ , and examine what happens in the direction  $y$ . Hence, we apply the PML change of variables (7) in a corner

$$x \rightarrow x(1 + \sigma_x \varepsilon_y(s)^{-1} s^{-1}), \quad y \rightarrow y(1 + \sigma_y \psi(s) s^{-1}), \quad \sigma_{x,y} \geq 0. \quad (13)$$

Since Laplace-based techniques are more suitable for the derivation of sufficient stability conditions, and at the moment we are interested in a failure of a *necessary* stability condition, we will make use of Fourier-based techniques, see e.g. the stability theory of [20, 21] as well as works on the stability of PMLs [3, 4, 8]. In order to use the stability condition of these works, let us assume that  $\psi(s)^{-1}$  is passive and satisfies Assumption 2.1. Then the corresponding time domain system can be written, cf. [5] for the PML in one direction, so that the plane-wave solutions of this system  $\mathbf{H}e^{st+i\mathbf{k}\mathbf{x}}$ , with  $\mathbf{H}$  being a  $\mathbb{C}^m$ -vector, satisfy the following:

- (1) either  $s$  is a constant from a half-plane  $\mathbb{C} \setminus \mathbb{C}_+$ ;
- (2) or it solves the dispersion relation

$$F(k_x, k_y, \sigma_x, \sigma_y, s) = 0, \quad (14)$$



with  $F : \mathbb{R}^2 \times \mathbb{R}_{\geq 0}^2 \times \mathbb{C} \rightarrow \mathbb{C}$ , defined by the identity

$$F(k_x, k_y, \sigma_x, \sigma_y, s) = \frac{s^{-1}\varepsilon_y(s)^{-1}}{(1 + \sigma_x\varepsilon_y(s)^{-1}s^{-1})^2}k_x^2 + \frac{s^{-1}\varepsilon_x(s)^{-1}}{(1 + \sigma_y\psi(s)s^{-1})^2}k_y^2 + s\mu(s).$$

In [4] the authors have introduced the notion of a uniform stability, which we use as well.

**Definition 2.7.** A PML system corresponding to (1) with the change of variables (13) is called uniformly stable if for all  $\sigma_x, \sigma_y \geq 0$ , the solutions  $s(\mathbf{k})$ ,  $\mathbf{k} \in \mathbb{R}^2$ , of (14) are outside of  $\mathbb{C}_+$ , i.e.  $s(\mathbf{k}) \in \mathbb{C} \setminus \mathbb{C}_+$ ,  $\mathbf{k} \in \mathbb{R}^2$ .

More precisely, uniformly stable systems do not admit exponentially growing plane-wave solutions for any non-negative values of absorption parameters  $\sigma_{x,y} \geq 0$ . It appears that the system stemming from (14) with the PML (13) may have such unstable plane-wave solutions, even if  $\psi(s)$  is chosen so that the stability condition of Theorem 2.4 in the direction  $y$  is satisfied. This is a subject of the following result.

**Theorem 2.8.** *Let the Assumption 2.3 hold true, and let  $\psi^{-1}$  be passive and satisfy Assumption 2.1. Then the equation (14) has no solutions  $s \in \mathbb{C}_+$  for any  $k_x, k_y \in \mathbb{R}$ ,  $\sigma_x, \sigma_y \in \mathbb{R}_{\geq 0}$ , if and only if  $\psi(s) = \varepsilon_x(s)^{-1}$ .*

To prove the above theorem we will need several technical results. Let us introduce  $\mathcal{D} = \mathcal{D}_{\varepsilon_x} \cup \mathcal{D}_{\varepsilon_y} \cup \mathcal{D}_\mu \cup \mathcal{D}_\psi \cup \{0\}$ . Also, let  $h_x = -is^{-1}\varepsilon_y(s)^{-1}$ ,  $h_y = -is^{-1}\varepsilon_x(s)^{-1}$ ,  $h_\mu = is\mu(s)$  and  $h_\psi(s) = -is^{-1}\psi(s)$ . Then

$$F(k_x, k_y, \sigma_x, \sigma_y, s) = \frac{ih_x(s)}{(1 + i\sigma_x h_x(s))^2}k_x^2 + \frac{ih_y(s)}{(1 + i\sigma_y h_y(s))^2}k_y^2 - ih_\mu(s). \quad (15)$$

**Lemma 2.9** (Crossing the imaginary axis). *Let  $\mu, \varepsilon_x, \varepsilon_y, \psi$  satisfy Assumption 2.1. Assume that  $\omega_0 \in \mathbb{R} \setminus \mathcal{D}$ . The following conditions are equivalent:*

- (1) *the dispersion relation (14) holds for some  $(k_x, k_y, \sigma_x, \sigma_y) \in \mathbb{R}^2 \times \mathbb{R}_{> 0}^2$  and  $s = -i\omega_0$ .*
- (2) *the following inequality is valid in  $\omega_0$ :  $\tilde{\psi}(\omega_0)\tilde{\varepsilon}_x(\omega_0) < 0$ .*

*Proof.* Proof of (1)  $\implies$  (2). Let the equation (14) have a solution  $s = -i\omega_0$ ,  $\omega_0 \in \mathbb{R} \setminus \mathcal{D}$ . We define

$$\begin{aligned} h_{y0} = \tilde{h}_y(\omega_0) &= \frac{\tilde{\varepsilon}_x(\omega_0)}{\omega_0}, & h_{x0} = \tilde{h}_x(\omega_0) &= \frac{\tilde{\varepsilon}_y(\omega_0)}{\omega_0}, & h_{\psi0} = \tilde{h}_\psi(\omega_0) &= \frac{\tilde{\psi}(\omega_0)}{\omega_0}, & h_{\mu0} = \tilde{h}_\mu(\omega_0) &= \frac{\tilde{\mu}(\omega_0)}{\omega_0}, \\ \gamma_x = 1 + \sigma_x i h_x, & & \gamma_{x0} = 1 + \sigma_x i \tilde{h}_{x0}, & & \gamma_\psi = 1 + \sigma_y i h_\psi, & & \gamma_{\psi0} = 1 + \sigma_y i \tilde{h}_{\psi0}. \end{aligned} \quad (16)$$

Notice that  $h_{\alpha0} \in \mathbb{R}$ ,  $\alpha \in \{x, y, \mu, \psi\}$ . Due to (14),  $\text{Re } F(k_x, k_y, \sigma_x, \sigma_y, s)$  vanishes at  $s = -i\omega_0$ , more precisely

$$\text{Re} (ih_{x0}\tilde{\gamma}_{x0}^2|\gamma_{x0}|^{-4}k_x^2 + ih_{y0}\tilde{\gamma}_{y0}^2|\gamma_{y0}|^{-4}k_y^2 - ih_{\mu0}) = 0, \quad (17)$$

from which, using the explicit expressions (16) for  $\gamma_{x0}$ ,  $\gamma_{\psi0}$ , we get

$$\sigma_x h_{x0}^2 |\gamma_{x0}|^{-4} k_x^2 + \sigma_y h_{y0} h_{\psi0} |\gamma_{\psi0}|^{-4} k_y^2 = 0. \quad (18)$$

The above holds for some  $k_x, k_y \in \mathbb{R}$  and  $\sigma_x, \sigma_y > 0$  if and only if  $h_{y0} h_{\psi0} \leq 0$ . Using the definition of  $h_y, h_\psi$ , this condition reads  $(\omega_0 \tilde{\varepsilon}_x(\omega_0))^{-1} (\omega_0 \tilde{\psi}(\omega_0))^{-1} < 0$ , or, alternatively,  $\tilde{\varepsilon}_x(\omega_0) \tilde{\psi}(\omega_0) < 0$ .

Proof of (2)  $\implies$  (1). Reciprocally, assume that  $\omega_0$  is s.t.  $\tilde{\psi}(\omega_0)\tilde{\varepsilon}_x(\omega_0) < 0$ . It suffices to show that there exist  $k_x, k_y \in \mathbb{R}$ ,  $\sigma_x, \sigma_y \geq 0$  s.t. (14) holds true. More precisely, the imaginary and the real parts of  $F(k_x, k_y, \sigma_x, \sigma_y, s)$  vanish in  $s = -i\omega_0$ . In particular, from (18) we obtain

$$k_x^2 = -k_y^2 \sigma_y \sigma_x^{-1} h_{y0} h_{\psi0} h_{x0}^{-2} |\gamma_{\psi0}|^{-4} |\gamma_{x0}|^4. \quad (19)$$

The condition on the vanishing imaginary part of  $F(k_x, k_y, \sigma_x, \sigma_y, s)$ , cf. (17), reads

$$h_{x0} (1 - \sigma_x^2 h_{x0}^2) |\gamma_{x0}|^{-4} k_x^2 + h_{y0} (1 - \sigma_y^2 h_{y0}^2) |\gamma_{y0}|^{-4} k_y^2 - h_{\mu 0} = 0,$$

or, with (19),

$$-\sigma_y \sigma_x^{-1} h_{y0} h_{\psi 0} h_{x0}^{-1} |\gamma_{\psi 0}|^{-4} (1 - \sigma_x^2 h_{x0}^2) k_y^2 + h_{y0} (1 - \sigma_y^2 h_{\psi 0}^2) |\gamma_{\psi 0}|^{-4} k_y^2 - h_{\mu 0} = 0,$$

which results in the following condition on  $k_y$ :

$$k_y^2 = h_{\mu 0} |\gamma_{\psi 0}|^4 h_{y0}^{-1} (1 - \sigma_y^2 h_{\psi 0}^2 - \sigma_y \sigma_x^{-1} h_{\psi 0} h_{x0}^{-1} (1 - \sigma_x^2 h_{x0}^2))^{-1}. \quad (20)$$

Introducing  $\nu_\psi = \sigma_y h_{\psi 0}$  and  $\nu_x = \sigma_x h_{x0}$ , we obtain

$$k_y^2 = h_{\mu 0} |\gamma_{\psi 0}|^4 h_{y0}^{-1} (1 - \nu_\psi^2 - \nu_\psi \nu_x^{-1} (1 - \nu_x^2))^{-1} = -h_{\mu 0} |\gamma_{\psi 0}|^4 h_{y0}^{-1} (\nu_\psi - \nu_x)^{-1} (\nu_\psi + \nu_x^{-1})^{-1}. \quad (21)$$

To ensure that the right hand side of the above is positive, we must verify that if  $h_{\mu 0} h_{y0} > 0$ , necessarily  $\nu_\psi \in [\min(-\nu_x^{-1}, \nu_x), \max(-\nu_x^{-1}, \nu_x)]$ , for which it is sufficient that  $|\nu_\psi| < \min(|\nu_x|^{-1}, |\nu_x|)$ . This can be always ensured by a proper choice of  $\sigma_y, \sigma_x > 0$ . If  $h_{\mu 0} h_{y0} < 0$ , it is sufficient that  $|\nu_\psi| > \max(|\nu_x|^{-1}, |\nu_x|)$ , which again can be ensured by choosing  $\sigma_x, \sigma_y > 0$ . Hence, if  $\tilde{\psi}(\omega_0) \tilde{\varepsilon}_x(\omega_0) < 0$ , the equation (14) necessarily has a purely imaginary solution  $s = -i\omega_0$ .  $\square$

As we have seen in the proof of the previous lemma, for any  $\omega_0$  satisfying the assumptions of the lemma, there exists a subset of  $\mathbb{R}^2 \times \mathbb{R}_{>0}^2$  such that for all  $\mathbf{p}$  in this set,  $F(\mathbf{p}, -i\omega_0) = 0$ . It will be seen from the proof of Theorem 2.8, that it is sufficient to consider  $\omega_0$  that satisfy additional constraints (in the form of inequalities) compared to Lemma 2.9, as in the corollary below. Physically, this would mean  $\omega_0$  is outside of a so-called propagative set of the system without the PML, i.e. such  $s_0 = -i\omega_0$  cannot solve the corresponding dispersion relation, namely  $F(k_x, k_y, 0, 0, -i\omega_0) \neq 0$  for any  $\mathbf{k} \in \mathbb{R}^2$  (see [5]). However, such a solution is indeed included into the propagative set of the system with the PML.

**Corollary 2.10.** *Assume that  $\omega_0 \in \mathbb{R} \setminus \mathcal{D}$  is s.t.  $\tilde{\varepsilon}_x(\omega_0) \tilde{\psi}(\omega_0) < 0$ ,  $\tilde{\mu}(\omega_0) \tilde{\varepsilon}_x(\omega_0) < 0$  and  $\tilde{\varepsilon}_y(\omega_0) \tilde{\varepsilon}_x(\omega_0) > 0$ . Then for all  $\nu \in \mathcal{V}_{\omega_0}$ , where*

$$\mathcal{V}_{\omega_0} = \{\nu \in \mathbb{R} : \nu h_{x0} > 0, |\nu| > 1\}, \quad (22)$$

it holds

$$F(\mathbf{p}(\nu), -i\omega_0) = 0, \text{ where } \mathbf{p}(\nu) = (k_{x0}(\nu), k_{y0}(\nu), \sigma_{x0}(\nu), \sigma_{y0}(\nu)), \quad (23)$$

$$k_{x0}(\nu) = (1 + \nu^2) \left( \frac{h_{\mu 0}}{2h_{x0}(1 - \nu^2)} \right)^{\frac{1}{2}}, \quad k_{y0}(\nu) = (1 + \nu^2) \left( \frac{h_{\mu 0}}{2h_{y0}(1 - \nu^2)} \right)^{\frac{1}{2}}, \quad (24)$$

$$\sigma_{x0}(\nu) = \nu h_{x0}^{-1}, \quad \sigma_{y0}(\nu) = -\nu h_{\psi 0}^{-1},$$

where  $h_{\mu 0}, h_{x0}, h_{y0}, h_{\psi 0}$  are defined in (16).

*Proof.* From the proof of Lemma 2.9 we saw that provided  $\omega_0$  as in the condition of the corollary, the dispersion relation  $F(k_x, k_y, \sigma_x, \sigma_y, -i\omega_0) = 0$  for any  $k_x, k_y$  expressed via  $\sigma_x, \sigma_y$  with the help of the relations (19, 20), assuming an extra bound on the values of  $|\sigma_x|, |\sigma_y|$ , see the end of the proof of Lemma 2.9. In particular, consider (21), with  $h_{\mu 0} h_{y0} < 0$ . Let us choose  $\sigma_x > 0$  so that  $\nu_x = \nu = \sigma_x h_{x0}$  satisfies  $|\nu| > 1$ , and  $\sigma_y > 0$  so that  $\nu_\psi = \sigma_y h_{\psi 0} = -\nu$ . The inequality  $\tilde{\psi}_0 \tilde{\varepsilon}_{x0} < 0$  is equivalent to  $h_{y0} h_{\psi 0} < 0$ , and, according to the condition of the corollary,  $h_{x0} h_{y0} > 0$ , and thus  $h_{x0} h_{\psi 0} < 0$ , therefore such  $\sigma_y > 0$  always exists. In this case

$\gamma_{\psi 0} = \bar{\gamma}_{x 0} = 1 - i\nu$ . It is not difficult to verify that  $k_y^2$  as in (21) is indeed positive with the above choice of  $\sigma_x, \sigma_y$ :

$$k_y^2 = \frac{h_{\mu 0} h_{y 0}^{-1} (1 + \nu^2)^2}{2(1 - \nu^2)} > 0,$$

since  $h_{\mu 0} h_{y 0} < 0$  according to the condition of the corollary. Similarly, we obtain from (19) the required expression for  $k_x^2$ :

$$k_x^2 = k_y^2 h_{y 0} h_{x 0}^{-1} = \frac{h_{\mu 0} h_{x 0}^{-1} (1 + \nu^2)^2}{2(1 - \nu^2)}.$$

The right hand side of the above is again positive, due to  $h_{y 0} h_{x 0} > 0$ .  $\square$

The above result strengthens the statement of Lemma 2.9: in fact, if  $s_0 = -i\omega_0$  belongs to a non-propagative set of the original system, and  $\tilde{\psi}(\omega_0) \tilde{\varepsilon}_x(\omega_0) < 0$ , there are infinitely many points  $\mathbf{p} = (k_x, k_y, \sigma_x, \sigma_y)$  for which  $s = -i\omega_0$  solves the dispersion relation with the PML. The statement that follows will serve as a basis for the application of the implicit function theorem in the proof of Theorem 2.8, and contains is a condition on  $\text{Re}(\partial_{\sigma_y} F(\partial_s F)^{-1})$  in the point where the solution of the dispersion relation  $s$  crosses the imaginary axis. More precisely, it is always possible to choose such  $\sigma_y > 0$  that the above expression does not vanish.

**Lemma 2.11.** *Assume that  $\omega_0$  is as in Corollary 2.10. There exists  $\nu \in \mathcal{V}_{\omega_0}$ , where  $\mathcal{V}_{\omega_0}$  is defined in (22), s.t. (23) is satisfied, and the function  $F(.,.) : (\mathbb{R}^2 \times \mathbb{R}_{\geq 0}) \times \mathbb{C} \rightarrow \mathbb{C}$  defined in (15) is differentiable in a vicinity of  $(\mathbf{p}(\nu), -i\omega_0)$ , provided  $\mathbf{p}(\nu)$  is as in (23, 24), with*

$$\partial_s F|_{(\mathbf{p}(\nu), -i\omega_0)} \neq 0, \quad \text{Re} \left( (\partial_s F)^{-1} (\partial_{\sigma_y} F) \right) \Big|_{(\mathbf{p}(\nu), -i\omega_0)} \neq 0. \quad (25)$$

*Proof.* We start with the evaluation of the partial derivatives of the almost everywhere smooth function  $F$ :

$$\begin{aligned} \partial_s F(k_x, k_y, \sigma_x, \sigma_y, s) &= i \frac{k_x^2}{\gamma_x^3} (h'_x \gamma_x - 2\sigma_x i h'_x h_x) + i \frac{k_y^2}{\gamma_\psi^3} (h'_y \gamma_\psi - 2\sigma_y i h'_\psi h_y) - i h'_\mu \\ &= i k_x^2 \gamma_x^{-3} h'_x (1 - \sigma_x i h_x) + k_y^2 \gamma_\psi^{-3} i (h'_y \gamma_\psi - 2\sigma_y i h'_\psi h_y) - i h'_\mu, \\ \partial_{\sigma_y} F(k_x, k_y, \sigma_x, \sigma_y, s) &= 2h_y h_\psi k_y^2 \gamma_\psi^{-3}. \end{aligned}$$

Using an identity  $v'(-i\omega_0) = i\tilde{v}'(\omega_0)$ , and evaluating the above derivatives at  $s = -i\omega_0$ ,  $\mathbf{p} = \mathbf{p}(\nu)$  given by (24), we obtain, with  $\mathcal{R}_s(\nu) = \partial_s F|_{(\mathbf{p}(\nu), -i\omega_0)}$ :

$$\mathcal{R}_s(\nu) = -k_{x 0}^2 \gamma_{x 0}^{-3} h'_{x 0} (1 - \sigma_{x 0} i h_{x 0}) + k_{y 0}^2 \gamma_{\psi 0}^{-3} i (i h'_{y 0} \gamma_{\psi 0} + 2\sigma_y h'_{\psi 0} h_{y 0}) + h'_{\mu 0},$$

where  $h'_{\alpha 0} = \tilde{h}'_{\alpha}(\omega_0)$ , with  $\alpha \in \{x, y, \mu, \psi\}$ . Now we are going to express  $\mathcal{R}_s$  explicitly as a function of  $\nu$ . Recall that  $\gamma = \gamma_{x 0} = 1 + i\nu = \bar{\gamma}_{\psi 0}$ , and  $k_{x 0}, k_{y 0}$  are as in (24):

$$\mathcal{R}_s(\nu) = -h_{\mu 0} h_{x 0}^{-1} h'_{x 0} \frac{(1 + \nu^2)^2 \bar{\gamma}}{2(1 - \nu^2) \gamma^3} + h_{\mu 0} h_{y 0}^{-1} \frac{(1 + \nu^2)^2}{2(1 - \nu^2) \bar{\gamma}^3} (-h'_{y 0} \bar{\gamma} + 2\sigma_y i h'_{\psi 0} h_{y 0}) + h'_{\mu 0}.$$

Similarly,

$$\mathcal{R}_{\sigma_y}(\nu) = \partial_{\sigma_y} F|_{(\mathbf{p}(\nu), -i\omega_0)} = 2h_{y 0} h_{\psi 0} k_{y 0}^2 \gamma_{\psi 0}^{-3} = h_{\mu 0} (1 + \nu^2)^2 (1 - \nu^2)^{-1} \bar{\gamma}^{-3}.$$

Thanks to the fact that  $\operatorname{Re} \bar{\gamma} = 1$ , we can verify that  $\mathcal{R}_{\sigma_y}(\nu) \neq 0$ ,  $\mathcal{R}_{\sigma_y}^{-1}(\nu) \neq 0$  for any  $\nu \neq \pm 1$ . Therefore, if we verify that for some  $\nu_* \in \mathcal{V}_{\omega_0}$ ,

$$\mathcal{R}(\nu_*) = h_{\mu 0}(1 + \nu_*^2)^2(1 - \nu_*^2)^{-1} \operatorname{Re} \left( \mathcal{R}_{\sigma_y}^{-1}(\nu_*) \mathcal{R}_s(\nu_*) \right) = (\bar{\gamma}^3(\nu_*) \mathcal{R}_s(\nu_*)) \neq 0,$$

this would immediately imply that  $\mathcal{R}_s(\nu_*) \neq 0$  and additionally that  $\operatorname{Re} \left( \mathcal{R}_{\sigma_y}(\nu_*) \mathcal{R}_s^{-1}(\nu_*) \right) \neq 0$ , i.e. (25). Let us thus compute

$$\mathcal{R}(\nu) = -h_{\mu 0} h_{x 0}^{-1} h'_{x 0} \frac{(1 + \nu^2)^2}{2(1 - \nu^2)} \operatorname{Re} (\gamma^{-3} \bar{\gamma}^4) - h'_{y 0} h_{\mu 0} h_{y 0}^{-1} \frac{(1 + \nu^2)^2}{2(1 - \nu^2)} \operatorname{Re} \bar{\gamma} + h'_{\mu 0} \operatorname{Re} \bar{\gamma}^3,$$

where we used  $\operatorname{Re} (i h'_{\psi 0} h_{y 0}) = 0$ . Thus,

$$\begin{aligned} \mathcal{R}(\nu) &= -h_{\mu 0} h_{x 0}^{-1} h'_{x 0} \frac{1 - 21\nu^2 + 35\nu^4 - 7\nu^6}{2(1 + \nu^2)(1 - \nu^2)} - h'_{y 0} h_{\mu 0} h_{y 0}^{-1} \frac{(1 + \nu^2)^2}{2(1 - \nu^2)} + h'_{\mu 0}(1 - 3\nu^2) \\ &= \frac{1}{2(1 - \nu^4)} \left( -h_{\mu 0} h_{x 0}^{-1} h'_{x 0} (1 - 21\nu^2 + 35\nu^4 - 7\nu^6) - h'_{y 0} h_{\mu 0} h_{y 0}^{-1} (1 + \nu^2)^3 + 2h'_{\mu 0}(1 - 3\nu^2)(1 - \nu^4) \right). \end{aligned}$$

The above expression is a rational in  $\nu$  function, hence it vanishes for all  $\nu \in \mathcal{V}_{\omega_0}$  if and only if all of its coefficients vanish; one can verify that this is impossible by checking the corresponding conditions on the coefficients near the powers of  $\nu$  in the numerator. In particular, consider the coefficient near  $\nu^4$ , which equals  $c = -35h_{\mu 0} h_{x 0}^{-1} h'_{x 0} - 3h_{\mu 0} h_{y 0}^{-1} h'_{y 0} - 2h'_{\mu 0}$ . Thanks to the so-called growing property, see Lemmas 4.7, 4.8 in [5],  $h'_{\mu 0} > 0$  and  $h'_{y 0}, h'_{x 0} < 0$ . On the other hand, by the condition of lemma,  $h_{\mu 0} h_{x 0} < 0$ ,  $h_{\mu 0} h_{y 0} < 0$ . From this it follows that  $c < 0$ , and therefore the numerator does not vanish identically. Hence there exists  $\nu = \nu_* \in \mathcal{V}_{\omega_0}$ , s.t.  $\mathcal{R}(\nu_*) \neq 0$ . As argued before, from this (25) follows.  $\square$

Now we have all the means to prove Theorem 2.8.

*Proof of Theorem 2.8.* The sufficiency of the condition  $\psi(s) = \varepsilon_x(s)^{-1}$  follows from passivity of the corresponding sesquilinear form, see Proposition 1.5. The proof of the necessity is slightly more involved. We make use of one of the ideas of [4]. The equation (14) defines  $s$  as an implicit function of  $\mathbf{k} \in \mathbb{R}^2$ , and  $\sigma_x, \sigma_y \in \mathbb{R}_{\geq 0}$ ; moreover, due to the continuous dependence of  $s$  on the parameters, (14) holds true for  $s \in \mathbb{C}_+$  only if there exist  $\mathbf{p} = (k_x, k_y, \sigma_x, \sigma_y) \in \mathbb{R}^2 \times \mathbb{R}_{\geq 0}^2$ , s.t.  $s$  satisfying (14) crosses the imaginary axis. We will show that if  $\psi(s) \neq \varepsilon_y(s)^{-1}$ , there always exists such a  $\mathbf{p}$ . Next, we will fix the parameters  $k_x, k_y, \sigma_x$  and study the dependence of the solution of (14) on a perturbation of  $\sigma_y$  with the help of the implicit function theorem. Using Lemma 2.11, we will show that the real part of the corresponding solution is positive, and hence the result.

First of all, if  $\sigma_x = 0, \sigma_y \neq 0$ , the necessary condition of the stability of the PML reads, cf. Theorem 2.4,

$$\omega \in \Omega_p^{\tilde{\mu}\tilde{\varepsilon}_x > 0} \cup \Omega_p^{\tilde{\varepsilon}_y\tilde{\varepsilon}_x < 0} \implies \tilde{\varepsilon}_x(\omega)\tilde{\psi}(\omega) \geq 0. \quad (26)$$

Thus,  $\psi(s)$  must satisfy the above condition. We prove the desired result by contradiction. If  $\psi \neq \varepsilon_x^{-1}$ , there necessarily exists  $\omega_0 \in \mathbb{R} \setminus \mathcal{D}$ , s.t.  $\tilde{\psi}(\omega_0)\tilde{\varepsilon}_x(\omega_0) < 0$ . Let us fix such  $\omega_0$ . Due to Corollary 2.10,  $s = -i\omega_0$  is a solution of (14) for  $\mathbf{p}(\nu)$  as in (23, 24), and any  $\nu \in \mathcal{V}_{\omega_0}$ , see (22). As shown in Lemma 2.11, there exists  $\nu = \nu_* \in \mathcal{V}_{\omega_0}$ , s.t.  $\partial_s F|_{(\mathbf{p}(\nu_*), -i\omega_0)} \neq 0$ , and  $\operatorname{Re} \left( (\partial_s F)^{-1} (\partial_{\sigma_y} F) \right)|_{(\mathbf{p}(\nu_*), -i\omega_0)} \neq 0$ . Using the implicit function theorem, the solution  $s$  of  $F(k_{x 0}(\nu_*), k_{y 0}(\nu_*), \sigma_{x 0}(\nu_*), \sigma_{y 0}(\nu_*) + \varepsilon, s) = 0$  (notice that  $\sigma_{y 0}(\nu_*) > 0$ ), provided that  $\varepsilon$  is sufficiently small, can be represented as

$$s = -i\omega_0 - \left( (\partial_s F)^{-1} (\partial_{\sigma_y} F) \right)|_{(\mathbf{p}(\nu_*), -i\omega_0)} \varepsilon + o(\varepsilon).$$

For sufficiently small  $\varepsilon$ , the sign of the real part of  $s$  is defined by the sign of the real part of the first-order term in the above expansion:

$$\text{sign Re } s = -\text{sign Re} \left( (\partial_s F)^{-1} (\partial_{\sigma_y} F) \right) \Big|_{(\mathbf{p}(\nu_*), -i\omega_0)} \varepsilon.$$

Since  $\sigma_{y0}(\nu_*) > 0$ , we can always choose  $\varepsilon$  so that  $\sigma_{y0}(\nu_*) + \varepsilon > 0$  and  $\text{Re } s > 0$ , thus obtaining a solution  $s$  to (14) with a positive real part.  $\square$

This result shows that fixing  $\psi(s) = \psi_x(s)$  in (5) for a PML in the direction  $x$  immediately fixes a choice of  $\psi(s) = \varepsilon_y \psi_x \varepsilon_x^{-1}$  for the PML change of variables (5) in the direction  $y$ , see also the corollary below.

**Corollary 2.12.** *Let the sesquilinear form (8) satisfy Assumption 2.3. Let the functions  $\psi_x, \psi_y$  be s.t.  $\psi_x^{-1}, \psi_y^{-1}$  are passive, satisfy Assumption 2.1, and let  $\psi_x(s)$  satisfy the stability condition (NSC1) of Theorem 2.4. The scaled form  $\tilde{A}_{\sigma_x, \sigma_y}(u, v)$  defined as*

$$\begin{aligned} \tilde{A}_{\sigma_x, \sigma_y}(u, v) &= \psi_x(s) \varepsilon_y(s) A_{\sigma_x, \sigma_y}(u, v), \\ A_{\sigma_x, \sigma_y}(u, v) &= \varepsilon_y(s)^{-1} \left( 1 + \frac{\sigma_x \psi_x(s)}{s} \right)^{-2} (\partial_x u, \partial_x v) + \varepsilon_x(s)^{-1} \left( 1 + \frac{\sigma_y \psi_y(s)}{s} \right)^{-2} (\partial_y u, \partial_y v) + s^2 \mu(s)(u, v) \end{aligned}$$

is passive for all  $\sigma_x, \sigma_y \geq 0$  if and only if  $\psi_y = \varepsilon_x^{-1} \varepsilon_y \psi_x$ .

*Proof.* The sufficiency follows from Proposition 1.5. The necessity is a result of Theorem 2.8. Due to Theorem 2.4,  $\varepsilon_x \varepsilon_y^{-1} \psi_x^{-1}$  and  $\mu \varepsilon_y^{-1} \psi_x^{-1}$  (which are the coefficients of the sesquilinear form  $\tilde{A}_{\sigma_x, \sigma_y}$  for  $\sigma_x = \sigma_y = 0$ ) are passive. Suppose that  $\psi_y \neq \varepsilon_x \varepsilon_y^{-1} \psi_x^{-1}$ . The dispersion relation

$$\tilde{F}(k_x, k_y, \sigma_x, \sigma_y, s) = \frac{s^{-1} \psi_x}{(1 + \sigma_x \psi_x^{-1} s^{-1})^2} k_x^2 + \frac{s^{-1} \varepsilon_y \psi_x \varepsilon_x^{-1}}{(1 + \sigma_y \psi_y s^{-1})^2} k_y^2 + s \varepsilon_y \psi_x \mu = 0$$

has solutions  $s \in \mathbb{C}_+$ , per Theorem 2.8. This is a contradiction to passivity due to Lemma 2.14 in [5].  $\square$

**Example 2.13.** We consider the problem of Example 2.6, where we choose the PML in the direction  $x$  with  $\psi(s) = \psi_x(s) = 1$ . The stable PML in the direction  $y$  and in the corner is given by (5) with  $\psi_y(s) = \varepsilon_y(s) \psi_x(s) \varepsilon_x(s)^{-1} = s^2 (s^2 + 2^2)^{-1}$ . The rest of the parameters remain the same. The stability of such a PML in corners is demonstrated by the results in Figure 3.

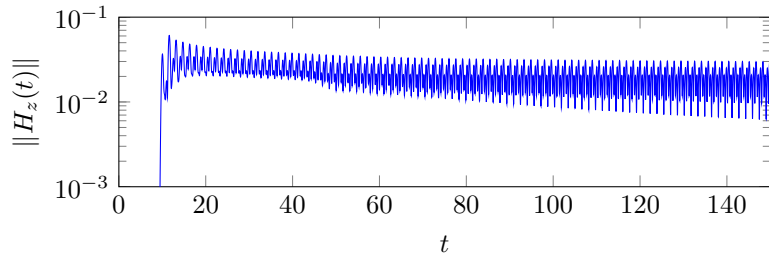
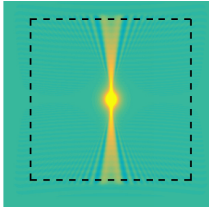


FIGURE 3. In the left plot we show the solution  $H_z(t)$  of Example 2.13 at  $t = 150$ . In the right plot we demonstrate  $\|H_z\|_{L^2}$  depending on time.

### 3. GENERALIZED LORENTZ MATERIALS: STABILITY OF THE PML VIA THE ENERGY TECHNIQUES

This section is dedicated to the derivation of the energy estimates for the PML system. Like before, we will consider the case of constant absorption parameters. In Section 3.1 we recall how the energy is derived for the system (1) without the PML. In Section 3.2 we show how a PML system can be constructed in order to simplify the derivation of the energy estimates. We apply these ideas in Section 3.3.

#### 3.1. Time-Domain System Without the PML

We consider (1) with (4), i.e.

$$\varepsilon_\alpha(s) = 1 + \sum_{\ell=0}^{n_\alpha} \frac{\varepsilon_{\alpha\ell}}{s^2 + \omega_{\alpha\ell}^2}, \quad \varepsilon_{\alpha\ell} > 0, \omega_{\alpha\ell} \in \mathbb{R}, \quad \ell = 0, \dots, n_\alpha, \alpha \in \{x, y\},$$

$$\mu(s) = 1 + \sum_{\ell=0}^{n_\mu} \frac{\mu_\ell}{s^2 + \omega_{\mu\ell}^2}, \quad \mu_\ell > 0, \omega_{\mu\ell} \in \mathbb{R}, \quad \ell = 0, \dots, n_\mu.$$

In the time domain, we obtain the Maxwell equations with currents, coupled to  $\mathbf{E}$ ,  $H_z$  via ODEs, cf. [4]:

$$\partial_t D_x = \partial_y H_z, \tag{27a}$$

$$\partial_t D_y = -\partial_x H_z, \tag{27b}$$

$$\partial_t B_z = \partial_y E_x - \partial_x E_y, \tag{27c}$$

$$\partial_t D_\alpha = \partial_t E_\alpha + \sum_{\ell=0}^{n_\alpha} \varepsilon_{\alpha\ell} \lambda_{\alpha\ell}, \tag{27d}$$

$$\partial_t \lambda_{\alpha\ell} + \omega_{\alpha\ell}^2 p_{\alpha\ell} = E_\alpha, \quad \partial_t p_{\alpha\ell} = \lambda_{\alpha\ell}, \quad \ell = 0, \dots, n_\alpha, \alpha \in \{x, y\}, \tag{27e}$$

$$\partial_t B_z = \partial_t H_z + \sum_{\ell=0}^{n_\mu} \mu_\ell \lambda_{\mu\ell}, \tag{27f}$$

$$\partial_t \lambda_{\mu\ell} + \omega_{\mu\ell}^2 p_{\mu\ell} = H_z, \quad \partial_t p_{\mu\ell} = \lambda_{\mu\ell}, \quad \ell = 0, \dots, n_\mu. \tag{27g}$$

Let us clarify the time-domain realization of  $\hat{B}_z = \mu(s)\hat{H}_z$  (the time-domain equivalent of  $\mathbf{D} = \underline{\underline{\varepsilon}}(s)\mathbf{E}$  can be obtained similarly). The latter equivalently reads  $s\hat{B}_z = s\mu(s)\hat{H}_z = \left(s + \sum_{\ell=0}^{n_\mu} \frac{\mu_\ell}{s + \omega_{\mu\ell}^2 s^{-1}}\right)\hat{H}_z$ . One defines

$$\hat{\lambda}_{\mu\ell} = \frac{1}{s + \omega_{\mu\ell}^2 s^{-1}} \hat{H}_z, \quad \text{or, equivalently,} \quad s\hat{\lambda}_{\mu\ell} + \omega_{\mu\ell}^2 \hat{p}_{\mu\ell} = \hat{H}_z, \quad s\hat{p}_{\mu\ell} = \hat{\lambda}_{\mu\ell}.$$

Provided that  $\lambda_{\mu\ell}|_{t=0} = 0 = p_{\mu\ell}|_{t=0}$ , the above coincides with (27g) in the time domain. Similarly, for  $B_z|_{t=0} = H_z|_{t=0}$ , we obtain (27f). We can immediately formulate the following well-known result, see also [4]. We provide its proof since it will be of use later.

**Proposition 3.1.** *An energy  $\mathcal{E}$  of (27), defined below, satisfies  $\frac{d}{dt}\mathcal{E} = 0$ . Here*

$$\mathcal{E} = \frac{1}{2} (\|E_x\|^2 + \|E_y\|^2 + \|H_z\|^2) + \mathcal{E}_x + \mathcal{E}_y + \mathcal{E}_z,$$

$$\mathcal{E}_\alpha = \frac{1}{2} \sum_{\ell=0}^{n_\alpha} \varepsilon_{\alpha\ell} (\|\lambda_{\alpha\ell}\|^2 + \omega_{\alpha\ell}^2 \|p_{\alpha\ell}\|^2), \quad \alpha \in \{x, y\}, \quad \mathcal{E}_z = \frac{1}{2} \sum_{\ell=0}^{n_\mu} \mu_\ell (\|\lambda_{\mu\ell}\|^2 + \omega_{\mu\ell}^2 \|p_{\mu\ell}\|^2).$$

*Proof.* Test the equation (27c) with  $H_z$ , to obtain

$$(\partial_t B_z, H_z) - (\partial_y E_x, H_z) + (\partial_x E_y, H_z) = (\partial_t B_z, H_z) + (E_x, \partial_y H_z) - (E_y, \partial_x H_z) = 0. \quad (28)$$

We consider the first term of the above expression; using (27f),

$$\begin{aligned} (\partial_t B_z, H_z) &= \frac{1}{2} \frac{d}{dt} \|H_z\|^2 + \sum_{\ell=0}^{n_\mu} \mu_\ell (\lambda_{\mu\ell}, H_z) \stackrel{(27g)}{=} \frac{1}{2} \frac{d}{dt} \|H_z\|^2 + \sum_{\ell=0}^{n_\mu} \mu_\ell (\lambda_{\mu\ell}, \partial_t \lambda_{\mu\ell} + \omega_{\mu\ell}^2 p_{\mu\ell}) \\ &\stackrel{(27g)}{=} \frac{1}{2} \frac{d}{dt} \|H_z\|^2 + \frac{1}{2} \frac{d}{dt} \sum_{\ell=0}^{n_\mu} \mu_\ell \|\lambda_{\mu\ell}\|^2 + \sum_{\ell=0}^{n_\mu} \mu_\ell \omega_{\mu\ell}^2 (\partial_t p_{\mu\ell}, p_{\mu\ell}) = \frac{1}{2} \frac{d}{dt} \|H_z\|^2 + \frac{d}{dt} \mathcal{E}_z. \end{aligned} \quad (29)$$

The term  $(E_x, \partial_y H_z)$  in (28) can be rewritten exactly like in the previous case:

$$(E_x, \partial_y H_z) \stackrel{(27a)}{=} (E_x, \partial_t D_x) \stackrel{(27d)}{=} \frac{1}{2} \frac{d}{dt} \|E_x\|^2 + \frac{1}{2} \frac{d}{dt} \sum_{\ell=0}^{n_\mu} \varepsilon_\ell (\|\lambda_{x\ell}\|^2 + \omega_{x\ell}^2 \|p_{x\ell}\|^2) = \frac{1}{2} \frac{d}{dt} \|E_x\|^2 + \frac{d}{dt} \mathcal{E}_x. \quad (30)$$

Similarly one shows  $-(E_y, \partial_x H_z) = \frac{1}{2} \frac{d}{dt} \|E_y\|^2 + \frac{d}{dt} \mathcal{E}_y$ .  $\square$

### 3.2. The Main Idea of the Construction of the PML System

It is well known that there exists many ways to write a PML system for a given problem, see for example [8] and references therein, or [4]. To be able to derive the energy conservation in an easy manner, it is crucial to introduce a very special set of auxiliary unknowns. One of the choices of such unknowns can be obtained by studying the coercivity of the corresponding sesquilinear form. Let us explain our idea based on a very simple example of the isotropic non-dispersive 2D wave equation written in the second order formulation, to which the Bérenger's PML is applied in one direction. In the Laplace domain it reads (where  $\hat{f}$  is the right hand side):

$$-(1 + s^{-1}\sigma)^{-1} \partial_x \left( (1 + s^{-1}\sigma)^{-1} \partial_x \hat{u} \right) - \partial_y^2 \hat{u} + s^2 \hat{u} = \hat{f}, \quad s \in \mathbb{C}_+. \quad (31)$$

The corresponding variational formulation is then

$$A_\sigma(\hat{u}, v) = (1 + s^{-1}\sigma)^{-2} (\partial_x \hat{u}, \partial_x v) + (\partial_y \hat{u}, \partial_y v) + s^2 (\hat{u}, v) = \langle \hat{f}, v \rangle, \quad \hat{u}, v \in H^1(\mathbb{R}^2).$$

To show the coercivity, one tests the above with  $v = s\hat{u}$  (and then takes the real part), cf. Remark 1.4,

$$A_\sigma(\hat{u}, s\hat{u}) = \operatorname{Re} \left( (1 + s^{-1}\sigma)^{-2} (\partial_x \hat{u}, s\partial_x \hat{u}) + (\partial_y \hat{u}, s\partial_y \hat{u}) + s(s\hat{u}, s\hat{u}) \right) = \operatorname{Re} \langle \hat{f}, s\hat{u} \rangle. \quad (32)$$

Testing with  $v = s\hat{u}$  corresponds in the time domain to testing the equation with  $\partial_t u$  (if  $u|_{t=0} = 0$ ), which is exactly what we do to obtain the energy estimates for the equation without the PML. On the other hand, one could obtain some energy identities directly from (32), with the help of the Plancherel's equality. To apply it, let us rewrite the first term of (32) in a more convenient form:

$$\begin{aligned} \left( (1 + s^{-1}\sigma)^{-2} \partial_x \hat{u}, s\partial_x \hat{u} \right) &= \left( \frac{\partial_x \hat{u}}{(1 + s^{-1}\sigma)^2}, \frac{s(1 + s^{-1}\sigma)^2}{(1 + s^{-1}\sigma)^2} \partial_x \hat{u} \right) = \left( \frac{\partial_x \hat{u}}{(1 + s^{-1}\sigma)^2}, \frac{s + 2\sigma + \sigma^2 s^{-1}}{(1 + s^{-1}\sigma)^2} \partial_x \hat{u} \right) \\ &= (\bar{s} + 2\sigma) \left\| \frac{\partial_x \hat{u}}{(1 + s^{-1}\sigma)^2} \right\|^2 + \sigma^2 s \left\| \frac{\partial_x \hat{u}}{s(1 + s^{-1}\sigma)^2} \right\|^2. \end{aligned}$$

Thus, (32) can be rewritten as follows:

$$\operatorname{Re}(s + 2\sigma) \left\| \frac{\partial_x \hat{u}}{(1 + s^{-1}\sigma)^2} \right\|^2 + \sigma^2 \operatorname{Re} s \left\| \frac{\partial_x \hat{u}}{s(1 + s^{-1}\sigma)^2} \right\|^2 + \operatorname{Re} s \|\partial_y \hat{u}\|^2 + \operatorname{Re} s \|s\hat{u}\|^2 = \operatorname{Re}\langle \hat{f}, s\hat{u} \rangle. \quad (33)$$

Recall the Plancherel's identity (here  $\eta > 0$ ):

$$\int_0^{+\infty} e^{-2\eta t} (u(t), v(t)) dt = \frac{1}{2\pi i} \int_{\eta+i\mathbb{R}} (\hat{u}(s), \hat{v}(s)) ds \implies \int_0^{+\infty} e^{-2\eta t} \operatorname{Re}(u(t), v(t)) dt = \frac{1}{2\pi i} \int_{\eta+i\mathbb{R}} \operatorname{Re}(\hat{u}(s), \hat{v}(s)) ds.$$

Before applying the last identity of the above to (33), let us introduce additionally auxiliary unknowns:

$$\hat{h}_u = \frac{\partial_x \hat{u}}{(1 + s^{-1}\sigma)^2}, \quad \hat{H}_u = \frac{\partial_x \hat{u}}{s(1 + s^{-1}\sigma)^2}, \quad (34)$$

and assume that  $u|_{t=0} = 0$ . This allows us to obtain a formal energy identity:

$$\int_0^{+\infty} e^{-2\eta t} ((\eta + 2\sigma)\|h_u\|^2 + \sigma^2\eta\|H_u\|^2 + \eta\|\partial_y u\|^2 + \eta\|\partial_t u\|^2) dt = \int_0^{+\infty} e^{-2\eta t} \langle f, \partial_t u \rangle dt.$$

Importantly, this time-domain equality involves newly introduced unknowns (34). This suggests that when writing the PML system corresponding to (31), it may be advantageous to introduce the unknowns (34). Let us rewrite (31) in time using this idea (now taking  $f = 0$  but assuming non-zero initial conditions):

$$\partial_t^2 u - \partial_x h_u - \partial_y^2 u = 0, \quad (35a)$$

$$\partial_t h_u + 2\sigma h_u + \sigma^2 H_u = \partial_x \partial_t u, \quad (35b)$$

$$\partial_t H_u = h_u. \quad (35c)$$

Testing (35a) with  $\partial_t u$ , we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\partial_t u\|^2 + \|\partial_y u\|^2) + (h_u, \partial_x \partial_t u) = 0, \\ & (h_u, \partial_x \partial_t u) \stackrel{(35b)}{=} (h_u, \partial_t h_u + 2\sigma h_u + \sigma^2 H_u) \stackrel{(35c)}{=} \frac{1}{2} \frac{d}{dt} \|h_u\|^2 + 2\sigma \|h_u\|^2 + \sigma^2 (\partial_t H_u, H_u) \\ & = \frac{1}{2} \frac{d}{dt} (\|h_u\|^2 + \|\sigma H_u\|^2) + 2\sigma \|h_u\|^2. \end{aligned}$$

Combining the above, we get the following energy decay law

$$\frac{d}{dt} \mathcal{E} = -2\sigma \|h_u\|^2, \quad \mathcal{E} = \frac{1}{2} (\|\partial_t u\|^2 + \|\partial_y u\|^2 + \|h_u\|^2 + \sigma^2 \|H_u\|^2). \quad (36)$$

Indeed, when  $\sigma \neq \text{const}$ , the formulation (35) is not perfectly matched. However, for  $\sigma = \text{const}$ , it is equivalent to another, perfectly matched formulation.

Let us now comment on how to derive the first order formulation. On one hand, we would like to introduce unknowns that would correspond to  $s^{-1}\hat{h}_u$ ,  $s^{-1}\hat{H}_u$  (morally, the energy of the first-order formulation for the wave equation coincides with the energy of the second-order formulation where  $u$  is replaced by  $\partial_t^{-1}u$ ). On the other hand, we would like to simplify the corresponding PML system of equations. First, let us write the system



of equations in the Laplace domain with the PML change of variables, and then comment on the introduction of the time-domain unknowns:

$$s\hat{u} = \frac{\partial_x \hat{v}_x}{1 + s^{-1}\sigma} + \partial_y \hat{v}_y, \quad (37a)$$

$$s\hat{v}_x = \frac{\partial_x \hat{u}}{1 + s^{-1}\sigma}, \quad (37b)$$

$$s\hat{v}_y = \partial_y \hat{u}. \quad (37c)$$

In this first-order formulation we suggest to introduce an auxiliary unknown  $v_x^*$  (it corresponds to  $s^{-1}\hat{H}_u$ ), s.t.

$$\hat{v}_x^* = \frac{\partial_x \hat{u}}{(s + \sigma)^2} \stackrel{(37b)}{=} \frac{\hat{v}_x}{s + \sigma}. \quad (38)$$

This choice is advantageous for two reasons:

- there is no need to introduce an unknown corresponding to  $s^{-1}\hat{h}_u$ , since it can be represented as a linear combination of the existing unknowns

$$s^{-1}\hat{h}_u = \frac{\partial_x \hat{u}}{(s + \sigma)(1 + s^{-1}\sigma)} \stackrel{(37b)}{=} \frac{\hat{v}_x}{1 + s^{-1}\sigma} = \left(1 - \frac{\sigma}{s + \sigma}\right) \hat{v}_x \stackrel{(38)}{=} \hat{v}_x - \sigma \hat{v}_x^*.$$

- the above linear combination can replace the first term in the right-hand side (37a), namely  $(1 + s^{-1}\sigma) \partial_x \hat{v}_x$  by  $\partial_x(\hat{v}_x - \sigma \hat{v}_x^*)$ . Here we make use of the assumption  $\sigma = \text{const}$ .

Therefore, (37) can be rewritten in the time domain as follows:

$$\partial_t u = \partial_x v_x - \sigma \partial_x v_x^* + \partial_y v_y, \quad (39a)$$

$$\partial_t v_x + \sigma v_x = \partial_x u, \quad (39b)$$

$$\partial_t v_y = \partial_y u, \quad (39c)$$

$$\partial_t v_x^* + \sigma v_x^* = v_x. \quad (39d)$$

Let us show that the energy is consistent with (36). We test the first equation above with  $u$ , to obtain

$$\partial_t (u, u) - (\partial_x v_x - \sigma \partial_x v_x^*, u) - (\partial_y v_y, u) = 0. \quad (40)$$

The second term in the above

$$\begin{aligned} -(\partial_x v_x - \sigma \partial_x v_x^*, u) &= (v_x - \sigma v_x^*, \partial_x u) \stackrel{(39b)}{=} (v_x - \sigma v_x^*, \partial_t v_x + \sigma v_x) = (v_x - \sigma v_x^*, \partial_t (v_x - \sigma v_x^*) + \sigma \partial_t v_x^* + \sigma v_x) \\ &= \frac{1}{2} \frac{d}{dt} \|v_x - \sigma v_x^*\|^2 + (v_x - \sigma v_x^*, \sigma \partial_t v_x^* + \sigma (\partial_t v_x^* + \sigma v_x^*)) \\ &= (v_x - \sigma v_x^*, 2\sigma \partial_t v_x^*) + (v_x - \sigma v_x^*, \sigma^2 v_x^*), \end{aligned}$$

with each of the last terms being rewritten thanks to (39d):

$$(v_x - \sigma v_x^*, 2\sigma \partial_t v_x^*) \stackrel{(39d)}{=} 2\sigma \|v_x - \sigma v_x^*\|^2, \quad (v_x - \sigma v_x^*, \sigma^2 v_x^*) \stackrel{(39d)}{=} (\partial_t v_x^*, \sigma^2 v_x^*) = \frac{\sigma^2}{2} \frac{d}{dt} \|v_x^*\|^2.$$

Finally, combining (40) and the latter considerations, we obtain:

$$\frac{d}{dt} \mathcal{E} = -2\sigma \|v_x - \sigma v_x^*\|^2, \quad \mathcal{E} = \|u\|^2 + \|v_y\|^2 + \|v_x - \sigma v_x^*\|^2 + \|\sigma v_x^*\|^2.$$

### 3.3. Energy Identities for the PML for Generalized Lorentz Materials

In this section we extend the ideas on the construction of the PML systems of Section 3.2 to derive the energy for the PML. Unlike in the previous case, we will present the first-order PML system as is, and will not refer to the Plancherel's identity (although it was used implicitly in the derivation of this system).

#### 3.3.1. Construction of a PML System Corresponding to Proposition 1.5

Let us write a PML system corresponding to (27) with the PML change of variables as per Proposition 1.5. We perform the PML change of variables in both directions, however, assume that  $\sigma_{x,y} \geq 0$ . Setting  $\sigma_y = 0$  (correspondingly,  $\sigma_x = 0$ ) and discarding the auxiliary unknowns associated to the PML in the direction  $y$  ( $x$ ), we will obtain the PML system in the direction  $x$  ( $y$ ), see Remark 3.3. Let us start with the first equation (27a), which in the Laplace domain with the PML change of variables of Proposition 1.5 and with the use of the constitutive relations reads

$$s\varepsilon_x \hat{E}_x = \left(1 + \frac{\sigma_y}{s\varepsilon_x}\right)^{-1} \partial_y \hat{H}_z.$$

Alternatively, in the time domain, this can be rewritten as  $\partial_t D_x + \sigma_y E_x = \partial_y H_z$ . The equation (27b) can be treated similarly. To deal with (27c), we suggest to employ the splitting idea of Bérenger [1]. Let us introduce split fields  $\hat{H}_{zx}$  and  $\hat{H}_{zy}$  so that they satisfy in the Laplace domain the following equations:

$$s\varepsilon_x(s) \hat{H}_{zy} = \partial_y \hat{E}_x, \quad s\varepsilon_y(s) \hat{H}_{zx} = -\partial_x \hat{E}_y, \quad s\hat{B}_z = \partial_y \hat{E}_x - \partial_x \hat{E}_y = s\varepsilon_x(s) \hat{H}_{zy} + s\varepsilon_y(s) \hat{H}_{zx}. \quad (41)$$

Applying the PML change of variables of Proposition 1.5 to the above equations, we obtain the following:

$$\begin{aligned} s\varepsilon_x \hat{H}_{zy} + \sigma_y \hat{H}_{zy} &= \partial_y \hat{E}_x, & s\varepsilon_y \hat{H}_{zx} + \sigma_x \hat{H}_{zx} &= -\partial_x \hat{E}_y, \\ s\hat{B}_z &= s\varepsilon_x(s) \hat{H}_{zy} + s\varepsilon_y(s) \hat{H}_{zx} = \partial_y \hat{E}_x - \partial_x \hat{E}_y - \sigma_y \hat{H}_{zy} - \sigma_x \hat{H}_{zx}. \end{aligned}$$

Finally, the time-domain expression of the operators  $s\varepsilon_x(s)$ ,  $s\varepsilon_y(s)$  is done similarly to the system (27), via the introduction of the auxiliary unknowns. After the PML change of variables, (27) becomes:

$$\partial_t D_x + \sigma_y E_x = \partial_y H_z, \quad (42a)$$

$$\partial_t D_y + \sigma_x E_y = -\partial_x H_z, \quad (42b)$$

$$\partial_t B_z + \sigma_y H_{zy} + \sigma_x H_{zx} = \partial_y E_x - \partial_x E_y, \quad (42c)$$

$$\partial_t H_{zx} + \sum_{\ell=0}^{n_y} \varepsilon_{y\ell} h_{x\ell} + \sigma_x H_{zx} = -\partial_x E_y, \quad (42d)$$

$$\partial_t H_{zy} + \sum_{\ell=0}^{n_x} \varepsilon_{x\ell} h_{y\ell} + \sigma_y H_{zy} = \partial_y E_x, \quad (42e)$$

$$\begin{aligned} \partial_t h_{\alpha\ell} + \omega_{\beta\ell}^2 b_{\alpha\ell} &= H_{z\alpha}, \quad \partial_t b_{\alpha\ell} = h_{\alpha\ell}, \quad \ell = 0, \dots, n_\beta, \quad (\alpha, \beta) \in \{(x, y), (y, x)\}, \\ &(27d - 27g). \end{aligned} \quad (42f)$$

In the derivation of the above system we did not use the assumption  $\sigma_x, \sigma_y = \text{const}$ , hence it is perfectly matched. However, obtaining the energy estimates for it directly is fairly complicated.

We suggest to consider another formulation, which is equivalent (see Remark 3.2) to (42) for constant absorption parameters. It differs from (42) only by the treatment of the equation for  $B_z$  (27c). More precisely, we introduce  $\hat{E}_x^*$ ,  $\hat{E}_y^*$ , s.t.  $\hat{H}_{zy} = \partial_y \hat{E}_x^*$ ,  $\hat{H}_{zx} = -\partial_x \hat{E}_y^*$ , cf. (41). Then, formally, using  $\sigma_x = \text{const}$ ,  $\sigma_y = \text{const}$ ,

$$s\varepsilon_x \hat{E}_x^* + \sigma_y \hat{E}_x^* = \hat{E}_x, \quad s\varepsilon_y \hat{E}_y^* + \sigma_x \hat{E}_y^* = \hat{E}_y, \quad s\hat{B}_z = \partial_y \hat{E}_x - \partial_x \hat{E}_y - \sigma_y \partial_y \hat{E}_x^* + \sigma_x \partial_x \hat{E}_y^*. \quad (43)$$

The realization of the operators  $s\varepsilon_x$ ,  $s\varepsilon_y$  is done similarly to (27). Thus, we obtain the following system:

$$\partial_t D_x + \sigma_y E_x = \partial_y H_z, \quad (44a)$$

$$\partial_t D_y + \sigma_x E_y = -\partial_x H_z, \quad (44b)$$

$$\partial_t B_z + \sigma_y \partial_y E_x^* - \sigma_x \partial_x E_y^* = \partial_y E_x - \partial_x E_y, \quad (44c)$$

$$\partial_t E_x^* + \sum_{\ell=0}^{n_x} \varepsilon_{x\ell} \lambda_{x\ell}^* + \sigma_y E_x^* = E_x, \quad (44d)$$

$$\partial_t E_y^* + \sum_{\ell=0}^{n_y} \varepsilon_{y\ell} \lambda_{y\ell}^* + \sigma_x E_y^* = E_y, \quad (44e)$$

$$\partial_t \lambda_{\alpha\ell}^* + \omega_{\alpha\ell}^2 p_{\alpha\ell}^* = E_{\alpha}^*, \quad \partial_t p_{\alpha\ell}^* = \lambda_{\alpha\ell}^*, \quad \ell = 0, \dots, n_{\alpha}, \alpha \in \{x, y\}, \quad (44f)$$

$$\partial_t D_{\alpha} = \partial_t E_{\alpha} + \sum_{\ell=0}^{n_{\alpha}} \varepsilon_{\alpha\ell} \lambda_{\alpha\ell}, \quad (44g)$$

$$\partial_t \lambda_{\alpha\ell} + \omega_{\alpha\ell}^2 p_{\alpha\ell} = E_{\alpha}, \quad \partial_t p_{\alpha\ell} = \lambda_{\alpha\ell}, \quad \ell = 0, \dots, n_{\alpha}, \alpha \in \{x, y\}, \quad (44h)$$

$$\partial_t B_z = \partial_t H_z + \sum_{\ell=0}^{n_{\mu}} \mu_{\ell} \lambda_{\mu\ell}, \quad (44i)$$

$$\partial_t \lambda_{\mu\ell} + \omega_{\mu\ell}^2 p_{\mu\ell} = H_z, \quad \partial_t p_{\mu\ell} = \lambda_{\mu\ell}, \quad \ell = 0, \dots, n_{\mu}. \quad (44j)$$

In the above the PML related unknowns are  $E_x^*$ ,  $(\lambda_{x\ell}^*, p_{x\ell}^*)$ ,  $\ell = 0, \dots, n_x$  and  $E_y^*$ ,  $(\lambda_{y\ell}^*, p_{y\ell}^*)$ ,  $\ell = 0, \dots, n_y$ .

**Remark 3.2.** The equivalence of (42) and (44), in the sense that provided properly chosen initial data and for constant absorption parameters, the solution of (42) solves (44), and vice versa, can be shown similarly to [8].

**Remark 3.3.** In the system (44) the PML in the direction  $y$  is realized via the introduction of the auxiliary unknowns  $E_y^*$ ,  $\lambda_{y\ell}^*$ ,  $p_{y\ell}^*$ ,  $\ell = 0, \dots, n_y$ . To obtain the PML in the direction  $x$  only, one sets  $\sigma_y = 0$ . In this case the equations (44e) and the ODE system (44f) for  $\alpha = y$  are decoupled from the rest of the equations and can be discarded.

### 3.3.2. Stability for the PML System (44)

The energy non-growth result is summarized in the following theorem.

**Theorem 3.4.** *An energy associated to (44) defined as*

$$\begin{aligned} \mathcal{E} &= \frac{1}{2} (\|E_x - \sigma_y E_x^*\|^2 + \|\sigma_y E_x^*\|^2 + \|E_y - \sigma_x E_y^*\|^2 + \|\sigma_x E_y^*\|^2 + \|H_z\|^2) + \mathcal{E}_x^* + \mathcal{E}_y^* + \mathcal{E}_z, \quad (45) \\ \mathcal{E}_x^* &= \frac{1}{2} \sum_{\ell=0}^{n_x} \varepsilon_{x\ell} (\|\lambda_{x\ell} - \sigma_y \lambda_{x\ell}^*\|^2 + \omega_{x\ell}^2 \|p_{x\ell} - \sigma_y p_{x\ell}^*\|^2 + \|\sigma_y \lambda_{x\ell}^*\|^2 + \omega_{x\ell}^2 \|\sigma_y p_{x\ell}^*\|^2), \\ \mathcal{E}_y^* &= \frac{1}{2} \sum_{\ell=0}^{n_y} \varepsilon_{y\ell} (\|\lambda_{y\ell} - \sigma_x \lambda_{y\ell}^*\|^2 + \omega_{y\ell}^2 \|p_{y\ell} - \sigma_x p_{y\ell}^*\|^2 + \|\sigma_x \lambda_{y\ell}^*\|^2 + \omega_{y\ell}^2 \|\sigma_x p_{y\ell}^*\|^2), \\ \mathcal{E}_z &= \frac{1}{2} \sum_{\ell=0}^{n_{\mu}} \mu_{\ell} (\|\lambda_{\mu\ell}\|^2 + \omega_{\mu\ell}^2 \|p_{\mu\ell}\|^2). \end{aligned}$$

does not grow, provided any non-negative  $\sigma_x$ ,  $\sigma_y \geq 0$ :

$$\frac{d}{dt} \mathcal{E} = -2\sigma_y \|E_x - \sigma_y E_x^*\|^2 - 2\sigma_x \|E_y - \sigma_x E_y^*\|^2.$$

*Proof.* First, we test (44c) with  $H_z$  to obtain

$$(\partial_t B_z, H_z) - (\partial_y E_x - \sigma_y \partial_y E_x^*, H_z) + (\partial_x E_y - \sigma_x \partial_x E_y^*, H_z) = 0. \quad (46)$$

Let us consider each term of the above expression separately. The first term is exactly as in (29):

$$(\partial_t B_z, H_z) \stackrel{(29)}{=} \frac{d}{dt} \left( \frac{1}{2} \|H_z\|^2 + \mathcal{E}_z \right).$$

The second term, after the integration by parts, becomes

$$\begin{aligned} -(\partial_y E_x - \sigma_y \partial_y E_x^*, H_z) &= (E_x - \sigma_y E_x^*, \partial_y H_z) \stackrel{(44a)}{=} (E_x - \sigma_y E_x^*, \partial_t D_x + \sigma_y E_x) = (E_x - \sigma_y E_x^*, \partial_t D_x) \\ &\quad + \sigma_y (E_x - \sigma_y E_x^*, E_x - \sigma_y E_x^* + \sigma_y E_x^*) = (E_x - \sigma_y E_x^*, \partial_t D_x) + \sigma_y \|E_x - \sigma_y E_x^*\|^2 \\ &\quad + \sigma_y^2 (E_x - \sigma_y E_x^*, E_x^*) = \mathcal{T}_1 + \sigma_y \|E_x - \sigma_y E_x^*\|^2 + \mathcal{T}_2. \end{aligned} \quad (47)$$

Let us first consider

$$\begin{aligned} \mathcal{T}_1 &= (E_x - \sigma_y E_x^*, \partial_t D_x) \stackrel{(44g)}{=} \left( E_x - \sigma_y E_x^*, \partial_t E_x + \sum_{\ell=0}^{n_x} \varepsilon_{x\ell} \lambda_{x\ell} \right) = (E_x - \sigma_y E_x^*, \partial_t (E_x - \sigma_y E_x^*) + \sigma_y \partial_t E_x^*) \\ &\quad + \left( E_x - \sigma_y E_x^*, \sum_{\ell=0}^{n_x} \varepsilon_{x\ell} \lambda_{x\ell} \right) \stackrel{(44d)}{=} \frac{1}{2} \frac{d}{dt} \|E_x - \sigma_y E_x^*\|^2 + \sigma_y \left( E_x - \sigma_y E_x^*, E_x - \sigma_y E_x^* - \sum_{\ell=0}^{n_x} \varepsilon_{x\ell} \lambda_{x\ell}^* \right) \\ &\quad + \left( E_x - \sigma_y E_x^*, \sum_{\ell=0}^{n_x} \varepsilon_{x\ell} \lambda_{x\ell} \right) = \frac{1}{2} \frac{d}{dt} \|E_x - \sigma_y E_x^*\|^2 + \sigma_y \|E_x - \sigma_y E_x^*\|^2 + \sum_{\ell=0}^{n_x} \varepsilon_{x\ell} (E_x - \sigma_y E_x^*, \lambda_{x\ell} - \sigma_y \lambda_{x\ell}^*) \end{aligned}$$

The last term in the above corresponds to a weighted sum of norms of a linear combination of auxiliary unknowns:

$$\begin{aligned} \sum_{\ell=0}^{n_x} \varepsilon_{x\ell} (E_x - \sigma_y E_x^*, \lambda_{x\ell} - \sigma_y \lambda_{x\ell}^*) &\stackrel{(44h,44f)}{=} \sum_{\ell=0}^{n_x} \varepsilon_{x\ell} (\partial_t (\lambda_{x\ell} - \sigma_y \lambda_{x\ell}^*) + \omega_{x\ell}^2 (p_{x\ell} - \sigma_y p_{x\ell}^*), \lambda_{x\ell} - \sigma_y \lambda_{x\ell}^*) \\ &\stackrel{(44h,44f)}{=} \frac{1}{2} \frac{d}{dt} \sum_{\ell=0}^{n_x} \varepsilon_{x\ell} \|\lambda_{x\ell} - \sigma_y \lambda_{x\ell}^*\|^2 + \sum_{\ell=0}^{n_x} \varepsilon_{x\ell} \omega_{x\ell}^2 (p_{x\ell} - \sigma_y p_{x\ell}^*, \partial_t (p_{x\ell} - \sigma_y p_{x\ell}^*)) \\ &= \frac{1}{2} \frac{d}{dt} \sum_{\ell=0}^{n_x} \varepsilon_{x\ell} (\|\lambda_{x\ell} - \sigma_y \lambda_{x\ell}^*\|^2 + \omega_{x\ell}^2 \|p_{x\ell} - \sigma_y p_{x\ell}^*\|^2). \end{aligned} \quad (48)$$

Summarizing the above

$$\mathcal{T}_1 = \frac{1}{2} \frac{d}{dt} \|E_x - \sigma_y E_x^*\|^2 + \sigma_y \|E_x - \sigma_y E_x^*\|^2 + \frac{1}{2} \frac{d}{dt} \sum_{\ell=0}^{n_x} \varepsilon_{x\ell} (\|\lambda_{x\ell} - \sigma_y \lambda_{x\ell}^*\|^2 + \omega_{x\ell}^2 \|p_{x\ell} - \sigma_y p_{x\ell}^*\|^2).$$

It remains to obtain an explicit energy-like expression for the term  $\mathcal{T}_2$  in (47):

$$\mathcal{T}_2 = \sigma_y^2 (E_x - \sigma_y E_x^*, E_x^*) \stackrel{(44d)}{=} \sigma_y^2 \left( \partial_t E_x^* + \sum_{\ell=0}^{n_x} \varepsilon_{x\ell} \lambda_{x\ell}^*, E_x^* \right) \stackrel{(44f)}{=} \frac{\sigma_y^2}{2} \frac{d}{dt} \left( \|E_x^*\|^2 + \sum_{\ell=0}^{n_x} \varepsilon_{x\ell} (\|\lambda_{x\ell}^*\|^2 + \omega_{x\ell}^2 \|p_{x\ell}^*\|^2) \right),$$

where the derivation of the latter identity is done similarly as e.g. in (29) or (48). Summarizing the above, we obtain the following expression for the second term in (46):

$$-(\partial_y E_x - \sigma_y \partial_y E_x^*, H_z) = \frac{1}{2} \frac{d}{dt} (\|E_x - \sigma_y E_x^*\|^2 + \|\sigma_y E_x^*\|^2) + 2\sigma_y \|E_x - \sigma_y E_x^*\|^2 + \frac{d}{dt} \mathcal{E}_x^*.$$

Repeating almost verbatim the above computations, one can show that the third term in (46) reads:

$$(\partial_x E_y - \sigma_x \partial_x E_y^*, H_z) = \frac{1}{2} \frac{d}{dt} (\|E_y - \sigma_x E_y^*\|^2 + \|\sigma_x E_y^*\|^2) + 2\sigma_x \|E_y - \sigma_x E_y^*\|^2 + \frac{d}{dt} \mathcal{E}_y^*.$$

Summing up all the terms of (46) we obtain the desired statement.  $\square$

**Remark 3.5.** The above result can be generalized to the case of Lorentz materials with dissipation terms, namely,

$$\begin{aligned} \varepsilon_\alpha(s) &= 1 + \sum_{\ell=0}^{n_\alpha} \frac{\varepsilon_{\alpha\ell}}{s^2 + 2\nu_{\alpha\ell}s + \omega_{\alpha\ell}^2}, & \varepsilon_{\alpha\ell} > 0, \nu_{\alpha\ell} \geq 0, \omega_{\alpha\ell} \geq 0, & \ell = 0, \dots, n_\alpha, \alpha \in \{x, y\}, \\ \mu(s) &= 1 + \sum_{\ell=0}^{n_\mu} \frac{\mu_\ell}{s^2 + 2\nu_{\mu\ell}s + \omega_{\mu\ell}^2}, & \mu_\ell > 0, \nu_{\mu\ell} \geq 0, \omega_{\mu\ell} \geq 0, & \ell = 0, \dots, n_\mu. \end{aligned}$$

**Remark 3.6.** With the help of the Young's inequality, one can show that the norm of the field  $\mathbf{E}$  is controlled with the help of the stability result of Theorem 3.4. E.g. consider the first term in (45):

$$\begin{aligned} \|E_x - \sigma_y E_x^*\|^2 + \|\sigma_y E_x^*\|^2 &\geq \|E_x\|^2 + 2\|\sigma_y E_x^*\|^2 - 2\|E_x\| \|\sigma_y E_x^*\| \geq \|E_x\|^2 + 2\|\sigma_y E_x^*\|^2 - \frac{2}{3}\|E_x\| - \frac{3}{2}\|\sigma_y E_x^*\| \\ &= \frac{1}{3}\|E_x\|^2 + \frac{1}{2}\|\sigma_y E_x^*\|^2. \end{aligned}$$

**Example 3.7** (Maxwell Equations in Two Dimensions). Our goal is to compare the energy expression (45) with existing in the literature results. In particular, Bécache and Joly [8] obtained explicit estimates for the energy decay for the 2D PML Maxwell system written in the following form (Zhao-Cangellaris [9] formulation, with the PML in the direction  $x$ , i.e.  $\sigma_y = 0$ ):

$$\partial_t E_x - \partial_y H_z = 0, \tag{49a}$$

$$\partial_t \tilde{E}_y + \partial_x H_z = 0, \tag{49b}$$

$$\partial_t \tilde{E}_y = \partial_t E_y + \sigma_x E_y, \tag{49c}$$

$$\partial_t H_z + \sigma_x H_z + \partial_x E_y - \partial_y \bar{E}_x = 0, \tag{49d}$$

$$\partial_t E_x + \sigma_x E_x = \partial_t \bar{E}_x. \tag{49e}$$

They deduce

$$\begin{aligned} \frac{d}{dt} \mathcal{E}_m &= -2\sigma_x \|\partial_t E_y\|^2, & \mathcal{E}_m &= \frac{1}{2} \left( \|\partial_t \bar{E}_x\|^2 + \|\partial_t E_y\|^2 + \|\partial_t (E_y - \tilde{E}_y)\|^2 + \|\partial_t H_z + \sigma_x H_z\|^2 \right) \\ &= \frac{1}{2} \left( \|\partial_t E_x + \sigma_x E_x\|^2 + \|\partial_t E_y\|^2 + \|\sigma_x E_y\|^2 + \|\partial_t H_z + \sigma_x H_z\|^2 \right). \end{aligned} \tag{50}$$

On the other hand, the formulation (44) for  $\sigma_y = 0$  reads

$$\partial_t E_x - \partial_y H_z = 0, \quad (51a)$$

$$\partial_t E_y + \sigma_x E_y + \partial_x H_z = 0, \quad (51b)$$

$$\partial_t H_z = \partial_y E_x - \partial_x E_y + \sigma_x \partial_x E_y^*, \quad (51c)$$

$$\partial_t E_y^* + \sigma_x E_y^* = E_y. \quad (51d)$$

The corresponding energy of Theorem 3.4 obeys:

$$\frac{d}{dt} \mathcal{E}_n = -2\sigma_x \|E_y - \sigma_x E_y^*\|^2, \quad \mathcal{E}_n = \frac{1}{2} (\|E_x\|^2 + \|E_y - \sigma_x E_y^*\|^2 + \|\sigma_x E_y^*\|^2 + \|H_z\|^2). \quad (52)$$

For  $\sigma = \text{const}$  the formulations (49) and (51) give rise to the same solution  $E_x, E_y, H_z$ . This can be shown with the methods of [8], see also Appendix A.

If  $(E_x, E_y, H_z, E_y^*)$  solve (51), so do  $((\partial_t + \sigma_x)E_x, (\partial_t + \sigma_x)E_y, (\partial_t + \sigma_x)H_z, (\partial_t + \sigma_x)E_y^*)$ . Therefore, from (52) we obtain the following:

$$\begin{aligned} \mathcal{E}_n^m &= \frac{1}{2} (\|(\partial_t + \sigma_x)E_x\|^2 + \|(\partial_t + \sigma_x)E_y - \sigma_x(\partial_t + \sigma_x)E_y^*\|^2 + \|\sigma_x(\partial_t + \sigma_x)E_y^*\|^2 + \|(\partial_t + \sigma_x)H_z\|^2), \\ \frac{d}{dt} \mathcal{E}_n^m &= -2\sigma_x \|(\partial_t + \sigma_x)E_y - \sigma_x(\partial_t + \sigma_x)E_y^*\|^2. \end{aligned}$$

Using (51d), we notice in particular that  $(\partial_t + \sigma_x)(E_y - \sigma_x E_y^*) = (\partial_t + \sigma_x)\partial_t E_y^* = \partial_t E_y$ , and eliminate  $E_y^*$  from the above, to obtain the expression coinciding with (50):

$$\frac{d}{dt} \mathcal{E}_n^m = -2\sigma_x \|\partial_t E_y\|^2, \quad \mathcal{E}_n^m = \frac{1}{2} (\|(\partial_t + \sigma_x)E_x\|^2 + \|\partial_t E_y\|^2 + \|\sigma_x E_y\|^2 + \|(\partial_t + \sigma_x)H_z\|^2).$$

### 3.3.3. Stability of the PML of Corollary 2.12

An exact expression of the corresponding to the PML with the change of variables as in Corollary 2.12 is slightly more complicated. A particularity of this formulation is that the energy will be expressed not in terms of the original unknowns, but new, 'effective' quantities. Here for brevity we will only present the derivation of the system which is not perfectly matched for a non-constant absorption parameter, i.e. analogous to (44); the perfectly matched system can be derived similarly combining the ideas of Section 3.3.1 and the derivation below.

First of all, let us define the following quantities (in the Laplace domain):

$$\mu^e(s) := \mu(s)\varepsilon_y(s)\psi_x(s), \quad \varepsilon_x^e(s) := \varepsilon_x(s)(\varepsilon_y(s)\psi_x(s))^{-1} = \psi_y(s)^{-1}, \quad \varepsilon_y^e(s) := \psi_x(s)^{-1}, \quad s \in \mathbb{C}.$$

The index  $e$  stands for 'effective'. Thanks to Theorem 2.4, these quantities are passive. Therefore, they are of the following form, see Theorem 2.2:

$$\begin{aligned} \varepsilon_\alpha^e(s) &= 1 + \sum_{\ell=0}^{n_\alpha^e} \frac{\varepsilon_{\alpha\ell}^e}{s^2 + (\omega_{\alpha\ell}^e)^2}, \quad \varepsilon_{\alpha\ell}^e > 0, \quad \omega_{\alpha\ell}^e \in \mathbb{R}, \quad \ell = 0, \dots, n_\alpha^e, \quad \alpha \in \{x, y\}, \\ \mu(s) &= 1 + \sum_{\ell=0}^{n_\mu^e} \frac{\mu_\ell^e}{s^2 + (\omega_{\mu\ell}^e)^2}, \quad \mu_\ell^e > 0, \quad \omega_{\mu\ell}^e \in \mathbb{R}, \quad \ell = 0, \dots, n_\mu^e. \end{aligned}$$

Let us introduce two new unknowns:

$$\hat{H}_z^e = (\varepsilon_y(s)\psi_x(s))^{-1} \hat{H}_z, \quad \hat{D}_\alpha^e = \varepsilon_\alpha^e \hat{E}_\alpha, \quad \alpha \in \{x, y\}. \quad (53)$$

Now let us consider (27) written in the Laplace domain, to which we apply the change of variables of Corollary 2.12. First of all, we rewrite the equation (27a) with the PML change of variables in the Laplace domain:

$$s\varepsilon_x(s)\hat{E}_x = \left(1 + \frac{\sigma_y\varepsilon_y(s)\psi_x(s)}{s\varepsilon_x(s)}\right)^{-1} \partial_y\hat{H}_z = \left(1 + \frac{\sigma_y}{s\varepsilon_x^e(s)}\right)^{-1} \partial_y\hat{H}_z,$$

which, after multiplication by  $(\varepsilon_y(s)\psi_x(s))^{-1}$ , becomes

$$s\varepsilon_x^e(s)\hat{E}_x = \left(1 + \frac{\sigma_y}{s\varepsilon_x^e(s)}\right)^{-1} \partial_y\hat{H}_z^e.$$

In the time domain this reads  $\partial_t D_x^e + \sigma_y E_x = \partial_y H_z^e$ . Similarly, (27b) after the PML change of variables reads:

$$s\varepsilon_y(s)\hat{E}_y = -\left(1 + \frac{\sigma_x\psi_x(s)}{s}\right)^{-1} \partial_x\hat{H}_z = -\left(1 + \frac{\sigma_x}{s\varepsilon_y^e(s)}\right)^{-1} \partial_x\hat{H}_z,$$

and after the multiplication of the above expression with  $(\psi(s)\varepsilon_y(s))^{-1}$ , we obtain

$$s\varepsilon_y^e\hat{E}_y = -\left(1 + \frac{\sigma_x}{s\varepsilon_y^e(s)}\right)^{-1} \partial_x\hat{H}_z^e.$$

Or, in the time domain,  $\partial_t D_y + \sigma_x E_y = -\partial_x H_z^e$ . The equation (27c) for  $B_z$  is treated in a manner similar to (44). More precisely, we suggest to perform the following formal splitting  $\hat{B}_z = \varepsilon_x^e\hat{H}_{zy} + \varepsilon_y^e\hat{H}_{zx}$ , with  $s\varepsilon_x^e\hat{H}_{zy} = \partial_y\hat{E}_x$  and  $s\varepsilon_y^e\hat{H}_{zx} = \partial_x\hat{E}_y$ . Proceeding as before, we obtain an analogue of (43), with the new unknowns  $\hat{E}_x^*$  and  $\hat{E}_y^*$  (recall that  $\psi_x = (\varepsilon_y^e)^{-1}$  and  $\psi_y = (\varepsilon_x^e)^{-1}$ ):

$$s\varepsilon_x^e\hat{E}_x^* + \sigma_y\hat{E}_x^* = \hat{E}_x, \quad s\varepsilon_y^e\hat{E}_y^* + \sigma_x\hat{E}_y^* = \hat{E}_y, \quad s\hat{B}_z = \partial_y\hat{E}_x - \partial_x\hat{E}_y - \sigma_y\partial_y\hat{E}_x^* + \sigma_x\partial_x\hat{E}_y^*.$$

Finally, let us summarize the above into the PML system:

$$\partial_t D_x^e + \sigma_y E_x = \partial_y H_z^e, \quad \partial_t D_y^e + \sigma_x E_y = -\partial_x H_z^e, \quad (54a)$$

$$\partial_t B_z = \partial_y E_x - \sigma_y \partial_y E_x^* - \partial_x E_y + \sigma_x \partial_x E_y^*, \quad (54b)$$

$$\partial_t E_x^* + \sum_{\ell=0}^{n_x} \varepsilon_{x\ell}^e \lambda_{x\ell}^* + \sigma_y E_x^* = E_x, \quad \partial_t E_y^* + \sum_{\ell=0}^{n_y} \varepsilon_{y\ell}^e \lambda_{y\ell}^* + \sigma_x E_y^* = E_y, \quad (54c)$$

$$\partial_t \lambda_{\alpha\ell}^* + (\omega_{\alpha\ell}^e)^2 p_{\alpha\ell}^e = E_{\alpha}, \quad \partial_t p_{\alpha\ell}^* = \lambda_{\alpha\ell}^*, \quad \ell = 0, \dots, n_{\alpha}, \quad \alpha \in \{x, y\}, \quad (54d)$$

$$\partial_t D_{\alpha}^e = \partial_t E_{\alpha} + \sum_{\ell=0}^{n_{\alpha}} \varepsilon_{\alpha\ell}^e \lambda_{\alpha\ell}^e, \quad (54e)$$

$$\partial_t \lambda_{\alpha\ell}^e + (\omega_{\alpha\ell}^e)^2 p_{\alpha\ell}^e = E_{\alpha}, \quad \partial_t p_{\alpha\ell}^e = \lambda_{\alpha\ell}^e, \quad \ell = 0, \dots, n_{\alpha}^e, \quad \alpha \in \{x, y\}, \quad (54f)$$

$$\partial_t B_z = \partial_t H_z^e + \sum_{\ell=0}^{n_{\mu}} \mu_{\ell}^e \lambda_{\mu\ell}^e, \quad (54g)$$

$$\partial_t \lambda_{\mu\ell}^e + (\omega_{\mu\ell}^e)^2 p_{\mu\ell}^e = H_z^e, \quad \partial_t p_{\mu\ell}^e = \lambda_{\mu\ell}^e, \quad \ell = 0, \dots, n_{\mu}^e, \quad (54h)$$

$$\partial_t H_z^e + \sum_{\ell=0}^{n_{\mu}} \mu_{\ell} \lambda_{\mu\ell}^e = \partial_t H_z + \sum_{\ell=0}^{n_{\mu}} \mu_{\ell} \lambda_{\mu\ell}, \quad (54i)$$

$$\partial_t \lambda_{\mu\ell} + \omega_{\mu\ell}^2 p_{\mu\ell} = H_z, \quad \partial_t p_{\mu\ell} = \lambda_{\mu\ell}, \quad \ell = 0, \dots, n_{\mu}. \quad (54j)$$

We can see that the system (54) is exactly of the same form as (44), hence the energy associated to this system can be derived as in Theorem 3.4.

**Remark 3.8.** The only equations that differ are (54i) and (54j), which are decoupled from the rest of the system, however, show the connection of the unknown  $H_z$  (inside the physical domain), and the unknown  $H_z^e$  (inside the PML domain), via the identities  $s\hat{B}_z = \mu(s)\hat{H}_z = \mu^e(s)\hat{H}_z^e$ . In practical computations, one would write the perfectly matched analogue of the system (54) and couple it through the interface to the system without the PML (27), using the transmission conditions on  $\mathbf{E} \times \mathbf{n}$  and the trace of  $H_z$  (or  $B_z$ , which is connected to  $H_z$  via a system of ODEs).

We present the corresponding energy result without proof, since it effectively mimics Proposition 3.4.

**Theorem 3.9.** *An energy associated to (54) defined as*

$$\begin{aligned} \mathcal{E} &= \frac{1}{2} (\|E_x - \sigma_y E_x^*\|^2 + \|\sigma_y E_x^*\|^2 + \|E_y - \sigma_x E_y^*\|^2 + \|\sigma_x E_y^*\|^2 + \|H_z\|^2) + \mathcal{E}_x^{e,*} + \mathcal{E}_y^{e,*} + \mathcal{E}_z^e, \quad (55) \\ \mathcal{E}_x^{e,*} &= \frac{1}{2} \sum_{\ell=0}^{n_x} \varepsilon_{x\ell}^e (\|\lambda_{x\ell} - \sigma_y \lambda_{x\ell}^*\|^2 + (\omega_{x\ell}^e)^2 \|p_{x\ell} - \sigma_y p_{x\ell}^*\|^2 + \|\sigma_y \lambda_{x\ell}^*\|^2 + (\omega_{x\ell}^e)^2 \|\sigma_y p_{x\ell}^*\|^2), \\ \mathcal{E}_y^{e,*} &= \frac{1}{2} \sum_{\ell=0}^{n_y} \varepsilon_{y\ell}^e (\|\lambda_{y\ell} - \sigma_x \lambda_{y\ell}^*\|^2 + (\omega_{y\ell}^e)^2 \|p_{y\ell} - \sigma_x p_{y\ell}^*\|^2 + \|\sigma_x \lambda_{y\ell}^*\|^2 + (\omega_{y\ell}^e)^2 \|\sigma_x p_{y\ell}^*\|^2), \\ \mathcal{E}_z^e &= \frac{1}{2} \sum_{\ell=0}^{n_\mu} \mu_\ell^e (\|\lambda_{\mu\ell}\|^2 + (\omega_{\mu\ell}^e)^2 \|p_{\mu\ell}\|^2) \end{aligned}$$

does not grow, provided any non-negative  $\sigma_x, \sigma_y \geq 0$ :

$$\frac{d}{dt} \mathcal{E} = -2\sigma_y \|E_x - \sigma_y E_x^*\|^2 - 2\sigma_x \|E_y - \sigma_x E_y^*\|^2.$$

As for the field  $H_z$  in (54i), one can demonstrate that it grows at most linearly, see Appendix B.

## 4. NON-LORENTZ MATERIALS: STABILITY WITH ENERGY TECHNIQUES

In this short section we would like to summarize some results on the materials whose dielectric permittivity and magnetic permeability do not satisfy the conditions of Section 2.1. One of the simplest examples is given by Warbourg's law, when the dielectric permittivity  $\varepsilon(s) = 1 + \frac{1+a\sqrt{s}}{s(1+b\sqrt{s})}$ ,  $a, b > 0$ , see e.g. [22] for the corresponding numerical implementation.

The stability of the corresponding system without the PML was shown in [18] with the help of a Herglotz-Nevalinna representation formula. This integral representation allows to write any system (even dissipative) in the form (27), and thus, to associate to any system an energy that is conserved (while at the same time, there may exist another energy that decays). A conservative extension for a general class of dissipative systems had been constructed, though in a slightly different form, in the seminal work by Figotin and Schenker in [23].

Such an approximation of an arbitrary material with the help of Lorentz materials will be used to construct a PML system similar to (44), see Section 4.2.

### 4.1. Time-Domain System without the PML

The results of this section are due to [18], where it is suggested that passive functions corresponding to the Laplace transform of real-valued in the time domain distributions (i.e. the property  $\overline{f(s)} = f(\bar{s})$ ,  $s \in \mathbb{C}_+$ , holds in the Laplace domain) satisfy the following property.



**Lemma 4.1** (Herglotz-Nevalinna Representation). *The function  $f(s)$ , which satisfies  $\overline{f(s)} = f(\bar{s})$ , with  $s \in \mathbb{C}_+$ , is passive if and only if there exists a non-decreasing function of a bounded variation  $\nu_f(\xi)$ , for which the Stiltjes integral  $\int_{-\infty}^{\infty} \frac{d\nu_f(\xi)}{1+\xi^2}$  is finite and  $f(s)$  is represented as the following Stiltjes integral:*

$$f(s) = a + \int_{-\infty}^{\infty} \frac{d\nu_f(\xi)}{s^2 + \xi^2}, \quad a \geq 0. \quad (56)$$

Under the assumption that  $\nu_f(\xi)$  is left-continuous and provided a normalization condition  $\lim_{\xi \rightarrow -\infty} \nu_f(\xi) = 0$ , such a function  $\nu_f$  is unique.

In particular, to obtain the Lorentz magnetic permeability (4), we can take  $\nu_\mu(\xi) = \sum_{\ell=0}^{n_\mu} \mu_\ell H(\xi - \omega_{\mu\ell})$ , where  $H$  is the left-continuous Heaviside function. Following the above result, we set (without loss of generality assuming  $a = 1$  in (56))

$$\varepsilon_\alpha(s) = 1 + \int_{-\infty}^{+\infty} \frac{d\nu_\alpha}{s^2 + \xi^2}, \quad \alpha \in \{x, y\}, \quad \mu(s) = 1 + \int_{-\infty}^{+\infty} \frac{d\nu_\mu}{s^2 + \xi^2}, \quad s \in \mathbb{C}_+.$$

These identities are essentially the generalization of the corresponding definitions for the Lorentz materials (4), with the exception that the weighted sums are substituted by integrals with Borel measures. Our goal is to write the system (1) in the time domain, using the above representation of  $\underline{\varepsilon}$ ,  $\underline{\mu}$ . The difficulty indeed lies in the constitutive relations. Let us recall from [18] how the equation  $s\hat{D}_x = s\varepsilon_x(s)\hat{E}_x$  can be written in the time domain. Let us introduce  $\hat{\lambda}_x(\xi)$  and  $\hat{p}_x(\xi)$ , which map  $\xi \in \mathbb{R}$  into a set of analytic functions from  $s \in \mathbb{C}_+$  into  $L^2(\mathbb{R}^2)$ , so that they satisfy the following identities:

$$s\hat{\lambda}_x(\xi) + \xi^2\hat{p}_x(\xi) = \hat{E}_x, \quad s\hat{p}_x(\xi) = \hat{\lambda}_x(\xi), \quad \xi \in \mathbb{R}.$$

Then, in the time domain, we can rewrite (formally)  $s\hat{D}_x = s\varepsilon\hat{E}_x$  as:

$$\begin{aligned} \partial_t D_x &= \partial_t E_x + \int_{-\infty}^{\infty} \lambda_x(\xi) d\nu_x(\xi), \\ \partial_t \lambda_x(\xi) + \xi^2 p_x(\xi) &= E_x, \quad \partial_t p_x(\xi) = \lambda_x(\xi), \quad \xi \in \mathbb{R}. \end{aligned} \quad (57)$$

If the measure  $\nu_x$  has a density  $f_{\nu_x}$ , we can define the quantities  $\lambda_x(\xi)$ ,  $p_x(\xi)$  only for  $\xi \in \text{supp } f_{\nu_x}(\xi)$  (this is the case for the Lorentz materials, where the density has a discrete support). Thus, extending the above idea to the rest of the constitutive relations, we write (1) in the form (27):

$$\partial_t D_x = \partial_y H_z, \quad \partial_t D_y = -\partial_x H_z, \quad (58a)$$

$$\partial_t B_z = \partial_y E_x - \partial_x E_y, \quad (58b)$$

$$\partial_t D_\alpha = \partial_t E_\alpha + \int_{-\infty}^{\infty} \lambda_\alpha(\xi) d\nu_\alpha(\xi), \quad \partial_t \lambda_\alpha + \xi^2 p_\alpha = E_\alpha, \quad \partial_t p_\alpha = \lambda_\alpha, \quad \xi \in \mathbb{R}, \quad \alpha \in \{x, y\}, \quad (58c)$$

$$\partial_t B_z = \partial_t H_z + \int_{-\infty}^{\infty} \lambda_\mu(\xi) d\nu_\mu(\xi), \quad \partial_t \lambda_\mu + \xi^2 p_\mu = E_\mu, \quad \partial_t p_\mu = \lambda_\mu, \quad \xi \in \mathbb{R}. \quad (58d)$$

Here  $\lambda_\alpha$ ,  $\lambda_\mu$  are functions from  $\mathbb{R}$  to causal  $L^2(\mathbb{R}^2)$ -valued functions on  $\mathbb{R}$ . The difference between (58) and (27) is that the sums are substituted by the integrals. An associated energy is conserved [18].

**Proposition 4.2.** *An energy  $\mathcal{E}$  associated to (58), defined below, satisfies  $\frac{d}{dt}\mathcal{E} = 0$ . Here*

$$\begin{aligned}\mathcal{E} &= \frac{1}{2} (\|E_x\|^2 + \|E_y\|^2 + \|H_z\|^2) + \mathcal{E}_x + \mathcal{E}_y + \mathcal{E}_z, \\ \mathcal{E}_\alpha &= \frac{1}{2} \int_{-\infty}^{\infty} (\|\lambda_\alpha\|^2 + \xi^2 \|p_\alpha\|^2) d\nu_\alpha(\xi), \quad \alpha \in \{x, y\}, \quad \mathcal{E}_z = \frac{1}{2} \int_{-\infty}^{\infty} (\|\lambda_\mu\|^2 + \xi^2 \|p_\mu\|^2) d\nu_\mu(\xi).\end{aligned}$$

## 4.2. The PML System and Its Stability

Let us perform the PML change of variables for the system (58) as per Proposition 1.5. This results in the following analogue of the system (44) (where again the difference is that the sums are substituted by integrals). The derivation of this system follows the derivation of (44), combined with the use of (57).

$$\partial_t D_x + \sigma_y E_x = \partial_y H_z, \quad \partial_t D_y + \sigma_x E_y = -\partial_x H_z, \quad (59a)$$

$$\partial_t B_z + \sigma_y \partial_y E_x^* - \sigma_x \partial_x E_y^* = \partial_y E_x - \partial_x E_y, \quad (59b)$$

$$\partial_t D_\alpha = \partial_t E_\alpha + \int_{-\infty}^{\infty} \lambda_\alpha(\xi) d\nu_\alpha(\xi), \quad \partial_t \lambda_\alpha + \xi^2 p_\alpha = E_\alpha, \quad \partial_t p_\alpha = \lambda_\alpha, \quad \xi \in \mathbb{R}, \quad \alpha \in \{x, y\}, \quad (59c)$$

$$\partial_t E_x^* + \int_{-\infty}^{\infty} \lambda_x^*(\xi) d\nu_x(\xi) + \sigma_y E_x^* = E_x, \quad \partial_t E_y^* + \int_{-\infty}^{\infty} \lambda_y^*(\xi) d\nu_y(\xi) + \sigma_x E_y^* = E_y, \quad (59d)$$

$$\partial_t \lambda_\alpha^* + \xi^2 p_\alpha^* = E_\alpha^*, \quad \partial_t p_\alpha^* = \lambda_\alpha^*, \quad \xi \in \mathbb{R}, \quad \alpha \in \{x, y\}, \quad (59e)$$

$$\partial_t B_z = \partial_t H_z + \int_{-\infty}^{\infty} \lambda_\mu(\xi) d\nu_\mu(\xi), \quad \partial_t \lambda_\mu + \xi^2 p_\mu = E_\mu, \quad \partial_t p_\mu = \lambda_\mu, \quad \xi \in \mathbb{R}. \quad (59f)$$

We can formulate the following analogue of Theorem 3.4, with the proof almost verbatim the same.

**Theorem 4.3.** *An energy  $\mathcal{E}$  associated to (59), defined as*

$$\begin{aligned}\mathcal{E} &= \frac{1}{2} (\|E_x - \sigma_y E_x^*\|^2 + \|\sigma_y E_x^*\|^2 + \|E_y - \sigma_x E_y^*\|^2 + \|\sigma_x E_y^*\|^2 + \|H_z\|^2) + \mathcal{E}_x + \mathcal{E}_y + \mathcal{E}_z, \\ \mathcal{E}_x &= \frac{1}{2} \left( \int_{-\infty}^{+\infty} (\|\lambda_x - \sigma_y \lambda_x^*\|^2 + \xi^2 \|p_x - \sigma_y p_x^*\|^2 + \|\sigma_y \lambda_x^*\|^2 + \xi^2 \|\sigma_y p_x^*\|^2) d\nu_x \right), \\ \mathcal{E}_y &= \frac{1}{2} \left( \int_{-\infty}^{+\infty} (\|\lambda_y - \sigma_x \lambda_y^*\|^2 + \xi^2 \|p_y - \sigma_x p_y^*\|^2 + \|\sigma_x \lambda_y^*\|^2 + \xi^2 \|\sigma_x p_y^*\|^2) d\nu_y \right), \\ \mathcal{E}_z &= \frac{1}{2} \int_{-\infty}^{+\infty} (\|\lambda_\mu\|^2 + \xi^2 \|p_\mu\|^2) d\nu_\mu,\end{aligned}$$

does not grow:

$$\frac{d}{dt}\mathcal{E} = -2\sigma_x \|E_y - \sigma_x E_y^*\|^2 - 2\sigma_y \|E_x - \sigma_y E_x^*\|^2.$$

## 5. CONCLUSION

In this work we have examined two questions of the stability of the PMLs of [5], namely the stability in corners and the derivation of the energy estimates. When applied to the second order formulation of the Maxwell equations in two dimensions (written in the Laplace domain)

$$\varepsilon_y(s)^{-1}\partial_{xx}\hat{H}_z + \varepsilon_x(s)^{-1}\partial_{yy}\hat{H}_z - s^2\mu(s)\hat{H}_z = 0, \quad (60)$$

our results can be summarized as follows:

- for Lorentz materials without dissipation: provided a stable in one direction change of variables  $x \rightarrow x \left(1 + \frac{\sigma_x \psi}{s}\right)$  with  $\psi(s)^{-1}$  being passive and correspond to Lorentz materials without dissipation, the only change of variables in the direction  $y$  that would stabilize the corner PML is  $y \rightarrow y \left(1 + \frac{\sigma_y \psi \varepsilon_x}{\varepsilon_y}\right)^{-1}$ . However, it is possible to show that this is not the only stable PMLs in corners, as soon as one lifts the requirement that  $\psi(s)^{-1}$  is Lorentz;
- based on considerations coming from the well-posedness in the Laplace domain, we have suggested a strategy to construct a perfectly matched system that would allow for an easier derivation of the energy estimates for an arbitrary passive system of the form (60).

Here we omit the discussion of the problem of the derivation of the energy estimates for 3D systems, and concentrate on a relatively simple 2D system; in [24] we will demonstrate how the strategy suggested in this work will pay off when applied to more complicated systems.

## APPENDIX A. EQUIVALENCE OF FORMULATIONS (49) AND (51)

Let us show how (51) can be transformed into (49). First of all, the equations (49a) and (51a) are identical. The equation (51b) can be rewritten as (49b), if one introduces the unknown  $\bar{E}_y$  as in (49c).

To see how from (51c) one obtains (49d), we apply  $\partial_t + \sigma_x$  to (51c), which results in

$$\partial_t(\partial_t + \sigma_x)H_z = \partial_y(\partial_t + \sigma_x)E_x - \partial_x(\partial_t + \sigma_x)E_y + \sigma_x(\partial_t + \sigma_x)\partial_x E_y^*.$$

Using (51d), and introducing  $\bar{E}_x$  defined as in (49e), we obtain the following:

$$\partial_t \left( (\partial_t + \sigma_x)H_z - \partial_y \bar{E}_x - \partial_x E_y \right) = 0.$$

We thus get from (51) the following system of equations:

$$\begin{aligned} \partial_t E_x - \partial_y H_z &= 0, \\ \partial_t \tilde{E}_y + \partial_x H_z &= 0, \\ \partial_t \tilde{E}_y &= \partial_t E_y + \sigma_x E_y, \\ \partial_t \left( (\partial_t + \sigma_x)H_z - \partial_y \bar{E}_x - \partial_x E_y \right) &= 0, \\ \partial_t E_x + \sigma_x E_x &= \partial_t \bar{E}_x. \end{aligned}$$

Hence, if at  $t = 0$ , the initial data for  $\partial_t H_z$  is chosen so that the expression  $\partial_t H_z|_{t=0} + \sigma_x H_z|_{t=0} - \partial_y \bar{E}_x|_{t=0} - \partial_x E_y|_{t=0} = 0$ , and the initial data for the rest of unknowns coincides with the initial data for (49), the solution of the above system of equations solves (49).

APPENDIX B. BOUND ON THE FIELD  $H_z$  IN (54)

**Lemma B.1.** *Assume that  $H_z^e$ ,  $H_z$  satisfy the initial value problem for the (54), and the energy (55) at  $t = 0$  equals  $\mathcal{E}_0$ . Then, for all  $t \geq 0$ , it holds, with some  $C \geq 0$  independent of  $t$ :*

$$\tilde{\mathcal{E}}_z(t) := \frac{1}{2} \left( \|H_z(t) - H_z^e(t)\|^2 + \sum_{\ell=0}^{n_\mu} \mu_\ell (\|\lambda_{\mu\ell}(t)\|^2 + \omega_{\mu\ell}^2 \|p_{\mu\ell}(t)\|^2) \right) \leq 2\tilde{\mathcal{E}}_z(0) + C\mathcal{E}_0 t^2.$$

*Proof.* Testing (54i) with  $H_z - H_z^e$ , we obtain

$$(\partial_t(H_z - H_z^e), H_z - H_z^e) + \sum_{\ell=0}^{n_\mu} \mu_\ell (\lambda_{\mu\ell}, H_z) = \sum_{\ell=0}^{n_\mu} \mu_\ell (\lambda_{\mu\ell}, H_z^e) + \sum_{\ell=0}^{n_\mu^e} \mu_\ell^e (\lambda_{\mu\ell}^e, H_z - H_z^e). \quad (61)$$

The left-hand side of the above is  $\frac{d}{dt}\tilde{\mathcal{E}}$ , see also (29). As for the right-hand side, let us consider each term separately. The first term, with the use of the Cauchy-Schwartz formula, can be bounded

$$\left| \sum_{\ell=0}^{n_\mu} \mu_\ell (\lambda_{\mu\ell}, H_z^e) \right| \leq \|H_z^e\| \sum_{\ell=0}^{n_\mu} \mu_\ell \|\lambda_{\mu\ell}\| \leq C \|H_z^e\| \left( \sum_{\ell=0}^{n_\mu} \mu_\ell \|\lambda_{\mu\ell}\|^2 \right)^{\frac{1}{2}} \leq C_0 \sqrt{\mathcal{E}_0 \tilde{\mathcal{E}}_z}, \quad C_0 > 0,$$

where we used the energy identity of Theorem 3.9 and the definition of  $\tilde{\mathcal{E}}_z$ . Similarly, for some  $C', C_1 > 0$ , the second term of the right-hand side of (61),

$$\left| \sum_{\ell=0}^{n_\mu^e} \mu_\ell^e (\lambda_{\mu\ell}^e, H_z - H_z^e) \right| \leq C' \|H_z - H_z^e\| \sum_{\ell=0}^{n_\mu^e} \mu_\ell^e \|\lambda_{\mu\ell}^e\| \leq C_1 \sqrt{\mathcal{E}_0 \tilde{\mathcal{E}}_z},$$

see also Remark 3.6, which can be applied to obtain an explicit bound on  $\|\lambda_{\mu\ell}^e(t)\|$  from the bound of Theorem 3.9. Thus we get from (61) the following differential inequality:

$$\frac{d}{dt}\tilde{\mathcal{E}}_z \leq C \sqrt{\mathcal{E}_0 \tilde{\mathcal{E}}_z}, \quad C > 0,$$

or

$$\frac{1}{2} \frac{d}{dt} \sqrt{\tilde{\mathcal{E}}_z} \leq C \sqrt{\mathcal{E}_0}.$$

From the above we immediately obtain

$$\tilde{\mathcal{E}}_z(t) \leq \left( 2C \sqrt{\mathcal{E}_0} t + \sqrt{\tilde{\mathcal{E}}_z(0)} \right)^2,$$

from which the desired inequality follows. □

The above result shows that  $\|H_z(t) - H_z^e(t)\|$  grows at most linearly, and since  $\|H_z^e(t)\|$  is uniformly bounded in time, the growth of  $\|H_z(t)\|$  is at most linear.

REFERENCES

- [1] J.-P. Bérenger, A perfectly matched layer for the absorption of electromagnetic waves, *J. Comput. Phys.* 114 (2) (1994) 185–200.

- [2] J.-P. Bérenger, Three-dimensional perfectly matched layer for the absorption of electromagnetic waves, *J. Comput. Phys.* 127 (2) (1996) 363–379.
- [3] E. Bécache, S. Fauqueux, P. Joly, Stability of perfectly matched layers, group velocities and anisotropic waves, *J. Comput. Phys.* 188 (2) (2003) 399–433.
- [4] E. Bécache, P. Joly, V. Violes, On the analysis of perfectly matched layers for a class of dispersive media. application to negative index metamaterials. Submitted .  
URL <https://hal.inria.fr/hal-01327315>
- [5] E. Bécache, M. Kachanovska, Stable perfectly matched layers for a class of anisotropic dispersive models. part i: Necessary and sufficient conditions of stability. Submitted.  
URL <https://hal.archives-ouvertes.fr/hal-01356811>
- [6] A. Welters, Y. Avniel, S. G. Johnson, Speed-of-light limitations in passive linear media, *Phys. Rev. A* 90 (2014) 023847.
- [7] S. A. Cummer, Perfectly matched layer behavior in negative refractive index materials, *IEEE Antennas and Wireless Propagation Letters* 3 (1) (2004) 172–175.
- [8] E. Bécache, P. Joly, On the analysis of Bérenger’s perfectly matched layers for Maxwell’s equations, *M2AN Math. Model. Numer. Anal.* 36 (1) (2002) 87–119.
- [9] L. Zhao, A. C. Cangellaris, Gt-pml: generalized theory of perfectly matched layers and its application to the reflectionless truncation of finite-difference time-domain grids, *IEEE Transactions on Microwave Theory and Techniques* 44 (12) (1996) 2555–2563.
- [10] T. Hagstrom, D. Appelö, Automatic symmetrization and energy estimates using local operators for partial differential equations, *Comm. Partial Differential Equations* 32 (7-9) (2007) 1129–1145.
- [11] D. Appelö, T. Hagstrom, G. Kreiss, Perfectly matched layers for hyperbolic systems: general formulation, well-posedness, and stability, *SIAM J. Appl. Math.* 67 (1) (2006) 1–23.
- [12] É. Bécache, P. Joly, M. Kachanovska, V. Violes, Perfectly matched layers in negative index metamaterials and plasmas, in: *CANUM 2014—42e Congrès National d’Analyse Numérique*, Vol. 50 of *ESAIM Proc. Surveys, EDP Sci., Les Ulis*, 2015, pp. 113–132.
- [13] E. Bécache, P. Joly, M. Kachanovska, Stable perfectly matched layers for a cold plasma in a strong background magnetic field. Submitted.  
URL <https://hal.inria.fr/hal-01397581v1>
- [14] W. C. Chew, W. H. Weedon, A 3D perfectly matched medium from modified Maxwell’s equations with stretched coordinates, *Microwave and Optical Technology Letters* 7 (13) (1994) 599–604.
- [15] P. Joly, An elementary introduction to the construction and the analysis of perfectly matched layers for time domain wave propagation, *SéMA J.* (57) (2012) 5–48.
- [16] P. I. Richards, A special class of functions with positive real part in a half-plane, *Duke Math. J.* 14 (1947) 777–786.
- [17] O. Brune, Synthesis of a finite two-terminal network whose driving-point impedance is a prescribed function of frequency, *Journal of Mathematics and Physics* 10 (1-4) (1931) 191–236.
- [18] M. Cassier, P. Joly, M. Kachanovska, On properties of passive systems In preparation.
- [19] K. S. Yee, Numerical solution of initial boundary value problems involving Maxwell’s equations in isotropic media, *IEEE Trans. Antennas Propag* 14 (3) (1966) 302–307.
- [20] H.-O. Kreiss, H. U. Busenhardt, Time-dependent partial differential equations and their numerical solution, *Lectures in Mathematics ETH Zürich*, Birkhäuser Verlag, Basel, 2001.
- [21] H.-O. Kreiss, J. Lorenz, *Initial-Boundary Value Problems and the Navier-Stokes Equations*, Academic Press, Inc., 1989.
- [22] A. Bamberger, B. Cockburn, Y. Goldman, M. Kern, P. Joly, Numerical solution of Maxwell’s equations in a conductive and polarizable medium, in: *Proceedings of the Eighth International Conference on Computing Methods in Applied Sciences and Engineering (Versailles, 1987)*, Vol. 75, 1989, pp. 11–25.
- [23] A. Figotin, J. H. Schenker, Spectral theory of time dispersive and dissipative systems, *J. Stat. Phys.* 118 (1-2) (2005) 199–263.
- [24] E. Bécache, P. Joly, M. Kachanovska, Energy estimates for perfectly matched layers. Bérenger split system and Maxwell equations in three dimensions. In preparation.