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Abstract

Abstract— Robustness with respect to delays is discussed for homogeneous systems with negative degree. It is shown that if homogeneous system with negative degree is globally asymptotically stable at the origin in the delay-free case then the system is globally asymptotically stable with respect to a compact set containing the origin independently of delay. The possibility of applying the result for local analysis of stability for not necessary homogeneous systems is analyzed. The theoretical results are supported by numerical examples.

1 Introduction

The time-delay dynamical systems (whose models are represented by functional differential equations) appear in many areas of science and technology, like e.g. systems biology and networked/distributed systems [9], [17]. Then influence of delays on the system stability and performance is critical for many natural and human-developed systems [19], [20], [23]. Synthesis of control and estimation algorithms, which are robust with respect to uncertain and time-varying delays, is an important and quickly developing topic of the modern control theory [18]. Despite of variety of methods solving this problem, most of them deal with linear time-delay models, which is originated by complexity of stability analysis for the nonlinear case (design of a Lyapunov-Krasovskii functional or a Lyapunov-Razumikhin function is a complex problem), and that constructive and computationally tractable conditions exist for linear systems only [30].

The theory of homogeneous dynamical systems has been established and well explored for ordinary differential equations [4], [7], [22], [33] and differential inclusions [24], [25], [27], [26], [6]. Linear systems form a subclass of homogeneous ones, moreover the main feature of a homogeneous nonlinear system is that its local behavior is the same as the global one [4], as in the linear case. In addition, the homogeneous stable/unstable systems admit homogeneous Lyapunov/Chetaev functions [33], [31], [16], [13]. Since the subclass of nonlinear systems, having a global behavior, is rather small, the concept of local homogeneity has been introduced [33], [2], [16]. Attempts to apply and extend the homogeneity theory for nonlinear functional differential equations, in order to simplify their stability analysis and design in presence of constant delays, have been performed in [12], [14], [15]. Applications of the conventional homogeneity framework for analysis of time-delay systems (considering delay as a kind of perturbation, for instance) have been carried out even earlier in [1], [3], [8], [10], [28].

In [15] it has been shown that homogeneous time-delay systems have certain stability robustness with respect to delays, e.g. if they are globally asymptotically stable for some delay, then they preserve this property independently of delay (IOD). For the case of nonnegative degree it has been proven that global asymptotic stability in the delay-free case implies local asymptotic stability for a sufficiently small delay (similarly to linear systems, i.e. a case of zero degree). The main goal of this work is to de-

⋆ This work was supported in part by the Government of Russian Federation (Grant 074-U01) and the Ministry of Education and Science of Russian Federation (Project 14.Z50.31.0031).

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Preprint submitted to Automatica *23 November 2016*

velop that result for the case of negative degree, such an extension is not trivial since the proof of [15] was heavily based on Lipschitz continuity property of the system, but this property is not satisfied for a negative degree, then a proof based on completely different arguments is presented in this work. It will be shown that in the case of negative degree, if the system is globally asymptotically stable in the delay-free case, then for any delay it is globally asymptotically stable with respect to a compact set containing the origin. Next, if the system can be approximated for big amplitudes of the state norm by a homogeneous system with negative degree which is asymptotically stable in delay-free case, then the system has bounded trajectories IOD. Similar problems have been partially considered in the papers [24], [25], [27], [26]. In comparison with these works present paper provides the complete proof based on conventional technique (Lyapunov function method). Also the paper extends the results for the case of local homogeneous approximations.

The outline of this paper is as follows. The preliminary definitions and the homogeneity for time-delay systems are given in Section 2. The main result is presented in Section 3. Two examples are considered in Section 4.

2 Preliminaries

Consider an autonomous functional differential equation of retarded type [23]:

$$dx(t)/dt = f(x_t),\ t \geq 0, \tag{1}$$

where $x \in \mathbb{R}^n$ and $x_t \in C_{[-\tau,0]}$ is the state function, $x_t(s) = x(t+s)$, $-\tau \leq s \leq 0$ (we denote by $C_{[-\tau,0]}$ the Banach space of continuous functions $\phi : [-\tau, 0] \to \mathbb{R}^n$ with the uniform norm $||\phi|| = \sup_{-\tau \leq \varsigma \leq 0} |\phi(\varsigma)|$, where $|\cdot|$ is the standard Euclidean norm); $f : C_{[-\tau,0]} \to \mathbb{R}^n$ is continuous and ensures existence and uniqueness [23] of solutions in forward time, $f(0) = 0$. The representation (1) includes pointwise or distributed time-delay systems. We assume that solutions of the system (1) satisfy the initial functional condition $x_0 \in C_{[-\tau,0]}$ for which the system (1) has a unique solution $x(t, x_0)$ and $x_{t,x_0}(s) = x(t+s, x_0)$ for $-\tau \leq s \leq 0$, which is defined on some finite time interval $[-\tau, T)$ (we will use the notation $x(t)$ to reference $x(t, x_0)$ if the origin of x_0 is clear from the context).

A continuous function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{K}$ if it is strictly increasing and $\sigma(0) = 0$; it belongs to class $\mathcal{K}_\infty$ if it is also radially unbounded. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{KL}$ if $\beta(\cdot, r) \in \mathcal{K}$ and $\beta(r, \cdot)$ is decreasing to zero for any fixed $r \in \mathbb{R}_+$. The symbol $\overline{1, m}$ is used to denote a sequence of integers $1, ..., m$.

2.1 Stability definitions

Let Ω be a neighborhood of the origin in $C_{[-\tau,0]}$.

Definition 1 [29] The system (1) is said to be

(a) stable at the origin in Ω if there is $\sigma \in \mathcal{K}$ such that for any $x_0 \in \Omega$ the solutions are defined for all $t \geq 0$ and $|x(t, x_0)| \leq \sigma(||x_0||)$ for all $t \geq 0$;

(b) asymptotically stable at the origin in Ω if it is stable in Ω and $\lim_{t\to+\infty} |x(t, x_0)| = 0$ for any $x_0 \in \Omega$;

(c) finite-time stable at the origin in Ω if it is stable in Ω and for any $x_0 \in \Omega$ there exists $0 \leq T^{x_0} < +\infty$ such that $x(t, x_0) = 0$ for all $t \geq T^{x_0}$. The functional $T_0(x_0) = \inf\{T^{x_0} \geq 0 : x(t, x_0) = 0\ \forall t \geq T^{x_0}\}$ is called the settling time of the system (1).

If $\Omega = C_{[-\tau,0]}$, then the corresponding properties are called global stability/asymptotic stability/finite-time stability.

Theorem 1 *(Lyapunov–Razumikhin Theorem, [19]). Suppose $f : C_{[-\tau,0]} \to \mathbb{R}^n$ maps bounded sets in $C_{[-\tau,0]}$ into bounded sets of $\mathbb{R}^n$ and that $u, v, w : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous nondecreasing functions $u(s)$ and $v(s)$ are positive for $s > 0$ and $u(0) = v(0) = 0$, v is strictly increasing. The trivial solution of (1) is uniformly stable if there exists a differentiable function $V : \mathbb{R}^n \to \mathbb{R}^+$, which is positive-definite, i.e.*

$$u(|x|) \leq V(x) \leq v(|x|),$$

and such that the derivative of V along the solution $x(t)$ of (1) satisfies

$$\begin{aligned} &\tfrac{\partial V}{\partial x}\ f(x_t) \leq -w(|x(t)|) \\ &\quad \text{if}\ \ V(x_t(\theta)) \leq V(x(t))\ \ \forall \theta \in [-\tau, 0]. \end{aligned} \tag{2}$$

If, in addition, $w(s) > 0$ for $s > 0$, and there exists a continuous nondecreasing function $p(s) > s$ for $s > 0$ such that condition (2) is strengthened to

$$\begin{aligned} &\tfrac{\partial V}{\partial x} f(x_t)\ \leq -w(|x(t)|) \\ &\qquad \text{if}\ \ V(x_t(\theta)) \leq p(V(x(t)))\ \ \forall \theta \in [-\tau, 0], \end{aligned}$$

then the trivial solution of (1) is uniformly asymptotically stable. If, in addition, $\lim_{s\to\infty} u(s) = \infty$, then it is globally uniformly asymptotically stable.

2.2 Homogeneity

For any $r_i > 0$, $i = \overline{1, n}$ and $\lambda > 0$, define the dilation matrix $\Lambda_r(\lambda) = \text{diag}\{\lambda^{r_i}\}_{i=1}^n$ and the vector of weights $r = [r_1, ..., r_n]^T$.

For any $r_i > 0$, $i = \overline{1,n}$ and $x \in \mathbb{R}^n$ the homogeneous norm can be defined as follows

$$|x|_r = \left(\sum_{i=1}^{n} |x_i|^{\rho/r_i} \right)^{1/\rho}, \ \rho \geq \max_{1 \leq i \leq n} r_i.$$

For all $x \in \mathbb{R}^n$, its Euclidean norm $|x|$ is related with the homogeneous one:

$$\underline{\sigma}_r(|x|_r) \leq |x| \leq \bar{\sigma}_r(|x|_r)$$

for some $\underline{\sigma}_r, \bar{\sigma}_r \in \mathcal{K}_\infty$. The homogeneous norm has an important property that is $|\Lambda_r(\lambda)x|_r = \lambda |x|_r$ for all $x \in \mathbb{R}^n$. Define $\mathbb{S}_r = \{x \in \mathbb{R}^n : |x|_r = 1\}$.

For any $r_i > 0$, $i = \overline{1,n}$ and $\phi \in C_{[-\tau,0]}$ the homogeneous norm can be defined as follows

$$||\phi||_r = \left(\sum_{i=1}^{n} ||\phi_i||^{\rho/r_i} \right)^{1/\rho}, \ \rho \geq \max_{1 \leq i \leq n} r_i.$$

There exist two functions $\underline{\rho}_r, \bar{\rho}_r \in \mathcal{K}_\infty$ such that for all $\phi \in C_{[-\tau,0]}$ [14]:

$$\underline{\rho}_r(||\phi||_r) \leq ||\phi|| \leq \bar{\rho}_r(||\phi||_r). \tag{3}$$

The homogeneous norm in the Banach space has the same important property that $||\Lambda_r(\lambda)\phi||_r = \lambda ||\phi||_r$ for all $\phi \in C_{[-\tau,0]}$. In $C_{[-\tau,0]}$ the corresponding unit sphere $\mathcal{S}_r = \{\phi \in C_{[-\tau,0]} : ||\phi||_r = 1\}$. Define $B_\rho^\tau = \{\phi \in C_{[-\tau,0]} : ||\phi||_r \leq \rho\}$ as a closed ball of radius $\rho > 0$ in $C_{[-\tau,0]}$.

Definition 2 [12] The function $g : C_{[-\tau,0]} \to \mathbb{R}$ is called r-homogeneous ($r_i > 0$, $i = \overline{1,n}$), if for any $\phi \in C_{[-\tau,0]}$ the relation

$$g(\Lambda_r(\lambda)\phi) = \lambda^d g(\phi)$$

holds for some $d \in \mathbb{R}$ and all $\lambda > 0$.

The vector field $f : C_{[-\tau,0]} \to \mathbb{R}^n$ is called r-homogeneous ($r_i > 0$, $i = \overline{1,n}$), if for any $\phi \in C_{[-\tau,0]}$ the relation

$$f(\Lambda_r(\lambda)\phi) = \lambda^d \Lambda_r(\lambda) f(\phi)$$

holds for some $d \geq -\min_{1 \leq i \leq n} r_i$ and all $\lambda > 0$.

In both cases, the constant d is called the degree of homogeneity. The system (1) is called r-homogeneous of degree d if the function f admits this property.

The introduced notion of weighted homogeneity in $C_{[-\tau,0]}$ is reduced to the standard one in $\mathbb{R}^n$ if $\tau = 0$.

Lemma 1 *Let $f : C_{[-\tau,0]} \to \mathbb{R}^n$ be locally bounded and r-homogeneous with degree d, then there exists $k > 0$ such that*

$$||f(x)||_r \leq k \max_{1 \leq i \leq n} ||x||_r^{1+d/r_i} \quad \forall x \in C_{[-\tau,0]}.$$

Proof. Take $x \in C_{[-\tau,0]}$, then there exists $\xi \in \mathcal{S}_r$ such that $x = \Lambda_r(\lambda)\xi$ for $\lambda = ||x||_r$. By definition of homogeneity we obtain:

$$\begin{aligned} ||f(x)||_r &= ||f(\Lambda_r(\lambda)\xi)||_r \leq \max\nolimits_{1 \leq i \leq n} \lambda^{1+d/r_i} ||f(\xi)||_r \\ &\leq k \max\nolimits_{1 \leq i \leq n} ||x||_r^{1+d/r_i} \end{aligned}$$

for $k = \sup_{\xi \in \mathcal{S}_r} ||f(\xi)||_r$. Note that $d > -\min_{1 \leq i \leq n} r_i$ for a continuous f, then $1 + d/r_i > 0$. □

This lemma implies that if a continuous $f : C_{[-\tau,0]} \to \mathbb{R}^n$ is r-homogeneous with degree d, then it admits a kind of Hölder continuity at the origin (i.e. $\lambda < 1$) for the homogeneous norm of exponent $1 + d/\max_{1 \leq i \leq n} r_i$ if $d \geq 0$ or $1 + d/\min_{1 \leq i \leq n} r_i$ if $d < 0$, and taking into account the properties of the functions $\underline{\rho}_r, \bar{\rho}_r$ a similar conclusion holds for the conventional norm also.

Lemma 2 *Let $f : C_{[-\tau,0]} \to \mathbb{R}^n$ be r-homogeneous with degree d and uniformly continuous in B_ρ^τ* [1] *for some $\rho > 0$, then for any $\eta > 0$ there exists $k > 0$ such that*

$$||f(x) - f(z)||_r \leq \max\{k \max_{1 \leq i \leq n} ||x - z||_r^{1+\frac{d}{r_i}}, \eta\} \ \forall x, z \in B_\rho^\tau.$$

Proof. Since f is uniformly continuous in B_ρ^τ, then for $\eta > 0$ there is $\delta_\eta > 0$ such that $||f(x) - f(z)||_r < \eta$ for all $x, z \in B_\rho^\tau$ with $||x - z||_r < \delta_\eta$. Take $e = x - z \in C_{[-\tau,0]}$, then there exists $\epsilon \in \mathcal{S}_r$ such that $e = \Lambda_r(\lambda)\epsilon$ with $\lambda = ||e||_r$. For $\lambda \geq \delta_\eta$ define $z = \Lambda_r(\lambda)\zeta$ for some $\zeta \in C_{[-\tau,0]}$, then

$$\begin{aligned} ||f(x) - f(z)||_r &= ||f(z + e) - f(z)||_r \\ &= ||\lambda^d \Lambda_r(\lambda)[f(\zeta + \epsilon) - f(\zeta)]||_r \\ &\leq \max\nolimits_{1 \leq i \leq n} \lambda^{1+d/r_i} ||f(\zeta + \epsilon) - f(\zeta)||_r. \end{aligned}$$

Since $||z||_r \leq \rho$, then $||\zeta||_r \leq \delta_\eta^{-1}\rho$ and

$$||f(\zeta + \epsilon) - f(\zeta)||_r \leq k$$

[1] Function $f : C_{[-\tau,0]} \to \mathbb{R}^n$ is called uniformly continuous in B_ρ^τ if for any $\varepsilon > 0$ there is a $\delta > 0$ such that for all $x, z \in B_\rho^\tau$ satisfying $||x - z||_r < \delta$, the inequality $||f(x) - f(z)||_r < \varepsilon$ holds (the homogeneous norm is used for simplicity of notation).

for some $k > 0$ dependent on ρ and η for any $x, z \in B_\rho^\tau$ with $||x - z||_r \geq \delta_\eta$. The claim follows by combining these both cases. $\square$

Corollary 1 *Let $f : C_{[-\tau,0]} \to \mathbb{R}^n$ be r-homogeneous with degree $d < 0$ and uniformly continuous in B_ρ^τ for some $\rho > 0$, then for any $\eta > 0$ there exist $k' > 0$ such that*

$$||f(x) - f(z)||_r \leq \max\{k'||x - z||_r, \eta\} \quad \forall x, z \in B_\rho^\tau.$$

Proof. The result follows Lemma 2 noticing that for $d < 0$ we have $0 < 1 + d/r_i < 1$ for all $1 \leq i \leq n$, then for any $\eta > 0$ and $\rho > 0$ there exists $\tilde{k} > 0$ such that

$$\max_{1\leq i\leq n} ||x - z||_r^{1+d/r_i} \leq \max\{\tilde{k}||x - z||_r, \frac{\eta}{k}\} \quad \forall x, z \in B_\rho^\tau.$$

$\square$

An advantage of homogeneous systems described by ordinary differential equations is that any of its solutions can be obtained from another solution under the dilation rescaling and a suitable time re-parametrization. A similar property holds for functional homogeneous systems:

Proposition 1 *[14] Let $x : \mathbb{R}_+ \to \mathbb{R}^n$ be a solution of the r-homogeneous system (1) with the degree $d = 0$ for an initial condition $x_0 \in C_{[-\tau,0]}$. For any $\lambda > 0$ define $y(t) = \Lambda_r(\lambda)x(t)$ for all $t \geq 0$, then $y(t)$ is also a solution of (1) with the initial condition $y_0 = \Lambda_r(\lambda)x_0$.*

Proposition 2 *[15] Let $x(t, x_0)$ be a solution of the r-homogeneous system (1) with the degree $d \neq 0$ for an initial condition $x_0 \in C_{[-\tau,0]}$, $\tau \in (0, +\infty)$. For any $\lambda > 0$ the functional differential equation*

$$dy(t)/dt = f(y_t),\ t \geq 0 \tag{4}$$

with $y_t \in C_{[-\lambda^{-d}\tau,0]}$, has a solution $y(t, y_0) = \Lambda_r(\lambda)x(\lambda^d t, x_0)$ [2)] for all $t \geq 0$ with the initial condition $y_0 \in C_{[-\lambda^{-d}\tau,0]}$, $y_0(s) = \Lambda_r(\lambda)x_0(\lambda^d s)$ for $s \in [-\lambda^{-d}\tau, 0]$.

The following result has also been obtained in [15]:

Lemma 3 *[15] Let $f(x_\tau) = F[x(t), x(t - \tau)]$ in (1) and the system (1) be r-homogeneous with degree $d \geq 0$ and globally asymptotically stable at the origin for $\tau = 0$, then for any $\rho > 0$ there is $0 < \tau_0 < +\infty$ such that (1) is asymptotically stable at the origin in B_ρ^τ for any delay $0 \leq \tau \leq \tau_0$.*

[2)] If time is scaled $t \to \lambda^d t$ then the argument of $f : C_{[-\tau,0]} \to \mathbb{R}^n$ in (1) is also scaled to $f : C_{[-\lambda^{-d}\tau,0]} \to \mathbb{R}^n$ in (4).

Thus, (1) is locally robustly stable with respect to a sufficiently small delay if it is r-homogeneous with a nonnegative degree and stable in the delay-free case.

2.3 Local homogeneity

A disadvantage of the global homogeneity introduced so far is that such systems possess the same behavior "globally" [12], [14].

Definition 3 [12] The function $g : C_{[-\tau,0]} \to \mathbb{R}$ is called (r,λ_0,g_0)-homogeneous ($r_i > 0$, $i = \overline{1,n}$; $\lambda_0 \in \mathbb{R}_+ \cup \{+\infty\}$; $g_0 : C_{[-\tau,0]} \to \mathbb{R}$) with degree d_0 if for any $\phi \in \mathcal{S}_r$ the relation

$$\lim_{\lambda\to\lambda_0} \lambda^{-d_0} g(\Lambda_r(\lambda)\phi) - g_0(\phi) = 0$$

is satisfied (uniformly on $\mathcal{S}_r$ for $\lambda_0 \in \{0, +\infty\}$) for some $d_0 \in \mathbb{R}_+$.

The function $f : C_{[-\tau,0]} \to \mathbb{R}^n$ is called (r,λ_0,f_0)-homogeneous ($r_i > 0$, $i = \overline{1,n}$; $\lambda_0 \in \mathbb{R}_+ \cup \{+\infty\}$; $f_0 : C_{[-\tau,0]} \to \mathbb{R}^n$) with degree d_0 if for any $\phi \in \mathcal{S}_r$ the relation

$$\lim_{\lambda\to\lambda_0} \lambda^{-d_0}\Lambda_r^{-1}(\lambda)f(\Lambda_r(\lambda)\phi) - f_0(\phi) = 0$$

is satisfied (uniformly on $\mathcal{S}_r$ for $\lambda_0 \in \{0, +\infty\}$) for some $d_0 \geq -\min_{1\leq i\leq n} r_i$.

In the paper [2] the definition of local homogeneity has been introduced for the cases $\lambda = 0$ and $\lambda = +\infty$. If the function g (system (1)) is simultaneously $(r, 0, g_0)$-homogeneous and $(r, +\infty, g_0)$-homogeneous, then it is called homogeneous in the bi-limit. Note, that the system (1) can be also homogeneous in more than two limits (see, for example, [11]).

For a given λ_0, g_0 and f_0 are called approximating functions.

For any $0 < \lambda_0 < +\infty$ the following formulas give an example of r-homogeneous approximating functions g_0 and f_0:

$$g_0(\phi) = ||\phi||_r^{d_0}\lambda_0^{-d_0}g(\Lambda_r(\lambda_0)\Lambda_r^{-1}(||\phi||_r)\phi),\ d_0 \geq 0,$$
$$f_0(\phi)=||\phi||_r^{d_0}\lambda_0^{-d_0}\Lambda_r(||\phi||_r)\Lambda_r^{-1}(\lambda_0)f(\Lambda_r(\lambda_0)\Lambda_r^{-1}(||\phi||_r)\phi),$$

$d_0 \geq -\min_{1\leq i\leq n} r_i$. This property allows us to analyze local stability/instability of the system (1) on the basis of a simplified system

$$dy(t)/dt = f_0[y_\tau(t)], t \geq 0, \tag{5}$$

called the *local approximating dynamics* for (1).

Theorem 2 *[12] Let the system $\underline{(1)}$ be (r,λ_0,f_0)-homogeneous for some $r_i > 0$, $i = \overline{1,n}$, the function f_0 be continuous and r-homogeneous with the degree d_0. Suppose there exists a differentiable r-homogeneous Lyapunov-Razumikhin function $V_0 : \mathbb{R}^n \to \mathbb{R}_+$ with the degree $\nu_0 > \max\{0, -d_0\}$,*

$$\alpha_1(|x|) \le V_0(x) \le \alpha_2(|x|)$$

for all $x \in \mathbb{R}^n$ and some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that:

(i) there exist functions $\alpha, \gamma \in \mathcal{K}$ such that for all $\varphi \in \mathcal{S}_r$

$$\max_{\theta\in[-\tau,0]} V_0[\varphi(\theta)] < \gamma\{V_0[\varphi(0)]\} \Rightarrow \frac{\partial V_0(\varphi(0))}{\partial \varphi(0)} f_0(\varphi) \le -\alpha(|\varphi(0)|);$$

(ii) there exists a function $\tilde{\gamma} \in \mathcal{K}$ such that $\lambda s < \tilde{\gamma}(\lambda s) \le \lambda\gamma(s)$ for all $s, \lambda \in \mathbb{R}_+ \setminus \{0\}$.

Then (the functions $\bar{\rho}_r$ and $\underline{\rho}_r$ have been defined in (3))

1) if $\lambda_0 = 0$, then there exists $0 < \bar{\lambda}_\varepsilon$ such that the system (1) is locally asymptotically stable at the origin with the domain of attraction containing the set

$$X_0 = \{\phi \in C_{[-\tau,0]} : ||\phi|| \le \alpha_1^{-1} \circ \alpha_2 \circ \bar{\rho}_r(\bar{\lambda}_\varepsilon)\};$$

2) if $\lambda_0 = +\infty$, then there exists $0 < \underline{\lambda}_\varepsilon < +\infty$ such that the system (1) is globally asymptotically stable with respect to forward invariant set

$$X_\infty = \{\phi \in C_{[-\tau,0]} : ||\phi|| \le \alpha_1^{-1} \circ \alpha_2 \circ \underline{\rho}_r(\underline{\lambda}_\varepsilon)\};$$

3) if $0 < \lambda_0 < +\infty$, then there exist $0 < \underline{\lambda}_\varepsilon \le \lambda_0 \le \bar{\lambda}_\varepsilon < +\infty$ such that the system (1) is asymptotically stable with respect to the forward invariant set X_∞ with region of attraction

$$X = \{\phi \in C_{[-\tau,0]} : \alpha_1^{-1} \circ \alpha_2 \circ \underline{\rho}_r(\underline{\lambda}_\varepsilon) < ||\phi|| < \alpha_1^{-1} \circ \alpha_2 \circ \bar{\rho}_r(\bar{\lambda}_\varepsilon)\}$$

provided that the set X is connected and nonempty.

In [14] analysis of the input-to-state stability property for the system (1) has been presented using the homogeneity theory.

3 Main results

In this work we propose the following extension of Lemma 3 for the case of negative degree.

Lemma 4 *Let $f(x_t) = F[x(t), x(t-\tau)]$ in (1) be uniformly continuous and the system (1) be r-homogeneous with degree $d < 0$ and globally asymptotically stable at the origin for $\tau = 0$, then for any $\varepsilon > 0$ there is $0 < \tau_0 < +\infty$ such that (1) is globally asymptotically stable with respect to B_ε^τ for any delay $0 \le \tau \le \tau_0$* [3)] .

Proof. Fix some $\rho > 0$ and $\tau > 0$, and let us consider $x_0 \in B_\rho^\tau$, then

$$x(t, x_0) = x_0(0) + \int_0^t F[x(s, x_0), x(s-\tau, x_0)]ds$$

is a solution of (1) if it is defined on the interval $[0, t]$ for $t \ge 0$ [20], $||x_0|| \le \overline{\rho}_r(\rho)$ by (3). For all $||\phi|| \le 2\overline{\rho}_r(\rho)$ according to Lemma 1 there exists $k > 0$ such that $|F[\phi(0), \phi(-\tau)]|_r \le k \max_{1\le i\le n} ||\phi||_r^{1+d/r_i}$, then (using also (3))

$$\begin{aligned}
|F[\phi(0), \phi(-\tau)]| &\le \bar{\sigma}_r(|F[\phi(0), \phi(-\tau)]|_r) \\
&\le \bar{\sigma}_r(k \max_{1\le i\le n} ||\phi||_r^{1+d/r_i}) \\
&\le \bar{\sigma}_r(k \max_{1\le i\le n} [\underline{\rho}_r^{-1}(||\phi||)]^{1+d/r_i}) \\
&= \iota_k(||\phi||)
\end{aligned}$$

where $\iota_k(s) = \bar{\sigma}_r(k \max_{1\le i\le n} [\underline{\rho}_r^{-1}(s)]^{1+d/r_i})$ is a function of class $\mathcal{K}_\infty$ (note that by definition $d > -\min_{1\le i\le n} r_i$ for continuous F and ι_k is well defined). Select $0 < \tau < \frac{\overline{\rho}_r(\rho)}{\iota_k(2\overline{\rho}_r(\rho))}$ (the denominator is separated from zero for any $\rho > 0$) and let us show that under this restriction $|x(t, x_0)| \le 2\overline{\rho}_r(\rho)$ for $0 \le t \le \tau$. To this end, suppose that $||x_s|| < 2\overline{\rho}_r(\rho)$ for all $s \in [0, t')$ for some $t' < \tau$ and $||x_s|| \ge 2\overline{\rho}_r(\rho)$ for $s \in [t', \tau]$, then we obtain

$$\begin{aligned}
|x(t', x_0)| &\le |x_0(0)| + \textstyle\int_0^{t'} |F[x(s, x_0), x(s-\tau, x_0)]|ds \\
&\le |x_0(0)| + \textstyle\int_0^{t'} \iota_k(||x_s||)ds \\
&\le |x_0(0)| + t' \sup_{0\le s<t'} \iota_k(||x_s||) \\
&< \overline{\rho}_r(\rho) + t'\iota_k(2\overline{\rho}_r(\rho)) \\
&< 2\overline{\rho}_r(\rho)
\end{aligned}$$

that is a contradiction, thus $||x_s|| \le 2\overline{\rho}_r(\rho)$ for $s \in [0, \tau]$. Therefore, for any $\rho > 0$ there exists $\tau > 0$ such that for $x_0 \in B_\rho^\tau$ we have $x_t \in B_{\rho'}^\tau$ for $t \in [0, \tau]$ and $\rho' = \underline{\rho}_r^{-1}[2\overline{\rho}_r(\rho)]$ by (3).

The conditions of this lemma mean that $F[\Lambda_r(\lambda)x, \Lambda_r(\lambda)z] = \lambda^d \Lambda_r(\lambda)F[x, z]$ for any $x, z \in \mathbb{R}^n$ and $\lambda \in (0, +\infty)$. In

[3)] In this case for any $0 \le \tau \le \tau_0$, any $\epsilon > 0$ and $\kappa \ge \varepsilon$ there is $0 \le T_{\kappa,\tau}^\epsilon < +\infty$ such that $||x_{t,x_0}||_r \le \varepsilon + \epsilon$ for all $t \ge T_{\kappa,\tau}^\epsilon$ for any $x_0 \in B_\kappa^\tau$, and $|x(t, x_0)|_r \le \sigma_\tau(||x_0||_r)$ for all $t \ge 0$ for some function $\sigma_\tau \in \mathcal{K}_\infty$ for all $x_0 \notin B_\varepsilon^\tau$.

addition, $x = 0$ for the system

$$\dot{x} = F(x, x) \tag{6}$$

is globally asymptotically stable. Note that the system (6) is also homogeneous of degree $d < 0$ ($F[\Lambda_r(\lambda)x, \Lambda_r(\lambda)x] = \lambda^d \Lambda_r(\lambda) F[x, x]$ for any $x \in \mathbb{R}^n$ and $\lambda \in (0, +\infty)$), then according to [33], [31] there is a differentiable and r-homogeneous Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_+$ of degree $v > -d$ such that

$$\begin{aligned}
&a = -\sup_{\xi \in \mathbb{S}_r} \tfrac{\partial V}{\partial \xi} F(\xi, \xi) > 0, \\
&0 < b = \sup_{|\xi|_r \leq 1} \left| \tfrac{\partial V(\xi)}{\partial \xi} \right| < +\infty, \\
&c_1 = \inf_{\xi \in \mathbb{S}_r} V(\xi), \; c_2 = \sup_{\xi \in \mathbb{S}_r} V(\xi), \\
&c_1 |x|_r^v \leq V(x) \leq c_2 |x|_r^v \quad \forall x \in \mathbb{R}^n.
\end{aligned}$$

Consider derivative of this function calculated for the system (1) for some $\phi \in C_{[-\tau,0]}$. To this end define $\lambda_1 = |\phi(0)|_r$ and $\lambda_2 = ||\phi||_r$ with $\phi(0) = \Lambda_r(\lambda_1)\xi$ for some $\xi \in \mathbb{S}_r$ and $\phi = \Lambda_r(\lambda_2)\varphi$ for some $\varphi \in \mathcal{S}_r \subset C_{[-\tau,0]}$ respectively, $\lambda_2 \geq \lambda_1$ by definition of the norms and $|\varphi(0)|_r \leq |\xi|_r = 1$ (a necessity to define two dilation transformations with λ_1 and λ_2 is originated by the fact that the state vectors of (1) and (6) belong to different spaces, $\mathbb{R}^n$ and $C_{[-\tau,0]}$):

$$\begin{aligned}
&\tfrac{\partial V(\phi(0))}{\partial \phi(0)} F[\phi(0), \phi(-\tau)] = \tfrac{\partial V(\phi(0))}{\partial \phi(0)} F[\phi(0), \phi(0)] \\
&+ \tfrac{\partial V(\phi(0))}{\partial \phi(0)} \{F[\phi(0), \phi(-\tau)] - F[\phi(0), \phi(0)]\} \\
&= \lambda_1^{d+v} \tfrac{\partial V}{\partial \xi} F[\xi, \xi] + \lambda_2^{d+v} \tfrac{\partial V(\varphi(0))}{\partial \varphi(0)} \{F[\varphi(0), \varphi(-\tau)] \\
&- F[\varphi(0), \varphi(0)]\}.
\end{aligned}$$

By Lemma 2 for all $\varphi \in \mathcal{S}_r$ and for some $\eta > 0$ (which will be defined later) there exists $L > 0$ (dependent on η) such that

$$\begin{aligned}
&|F[\varphi(0), \varphi(-\tau)] - F[\varphi(0), \varphi(0)]|_r \\
&\leq \max\{L \max_{1 \leq i \leq n} |\varphi(0) - \varphi(-\tau)|_r^{1+d/r_i}, \eta\}.
\end{aligned}$$

Let us consider the solutions of (1) for $t \geq \tau$ (we have shown above that for $x_0 \in B_\rho^\tau$ we have $x_t \in B_{\rho'}^\tau$ for $t \in [0, \tau]$ for any $\rho > 0$, $\rho' = \underline{\rho}_r^{-1}[2\overline{\rho}_r(\rho)]$ and a properly selected $\tau < \frac{\overline{\rho}_r(\rho)}{\iota_k(2\overline{\rho}_r(\rho))}$), then $|\phi(0) - \phi(-\tau)| \leq M\tau$, where $M = \sup_{|z|_r \leq \rho', |y|_r \leq \rho'} |F[z, y]|$, and

$$\begin{aligned}
&||\phi||_r |\varphi(0) - \varphi(-\tau)|_r = |\phi(0) - \phi(-\tau)|_r \leq \underline{\sigma}_r^{-1}(M\tau). \\
&|F[\varphi(0), \varphi(-\tau)] - F[\varphi(0), \varphi(0)]| \\
&\leq \bar{\sigma}_r \circ \max\{L \max_{1 \leq i \leq n} ||\phi||_r^{-1-d/r_i} \underline{\sigma}_r^{-1}(M\tau)^{1+d/r_i}, \eta\}.
\end{aligned}$$

Finally, on solutions for $t \geq \tau$

$$|F[\varphi(0), \varphi(-\tau)] - F[\varphi(0), \varphi(0)| \leq \bar{\sigma}_r \circ \max\{\pi_1(\tau)\pi_2(\lambda_2), \eta\}$$

for

$$\begin{aligned}
&\pi_1(\tau) = L \max_{1 \leq i \leq n} \underline{\sigma}_r^{-1}(M\tau)^{1+d/r_i}, \\
&\pi_2(\lambda_2) = \begin{cases} \lambda_2^{-1-d/\min_{1 \leq i \leq n} r_i} & \text{if } \lambda_2 \geq 1 \\ \lambda_2^{-1-d/\max_{1 \leq i \leq n} r_i} & \text{if } \lambda_2 < 1 \end{cases}
\end{aligned}$$

being functions from class $\mathcal{K}_\infty$. In order to prove asymptotic stability, the Lyapunov-Razumikhin approach will be applied [19]. To this end, assume for some $\gamma > 1$ that $c_1 \sup_{\theta \in [-\tau,0]} |\phi(\theta)|_r^v \leq \sup_{\theta \in [-\tau,0]} V[\phi(\theta)] < \gamma V[\phi(0)] \leq \gamma c_2 |\phi(0)|_r^v$, therefore $\sup_{\theta \in [-\tau,0]} |\phi(\theta)|_r \leq (\gamma c_1^{-1} c_2)^{1/v} |\phi(0)|_r = (\gamma c_1^{-1} c_2)^{1/v} \lambda_1$, then

$$\begin{aligned}
\lambda_2 &= ||\phi||_r = \left(\textstyle\sum_{i=1}^n \sup_{\theta \in [-\tau,0]} |\phi_i(\theta)|^{\rho/r_i} \right)^{1/\rho} \\
&\leq \left(\textstyle\sum_{i=1}^n \sup_{\theta \in [-\tau,0]} |\phi(\theta)|_r^\rho \right)^{1/\rho} \\
&= n^{1/\rho} \sup_{\theta \in [-\tau,0]} |\phi(\theta)|_r \leq R\lambda_1,
\end{aligned}$$

where $R = n^{1/\rho} (\gamma c_1^{-1} c_2)^{1/v}$. Finally,

$$\begin{aligned}
&\tfrac{\partial V(\phi(0))}{\partial \phi(0)} F[\phi(0), \phi(-\tau)] \\
&\leq b\bar{\sigma}_r \circ \max\{\pi_1(\tau)\pi_2(\lambda_2), \eta\} \lambda_2^{d+v} - aR^{-d-v} \lambda_2^{d+v} \\
&= ||\phi||_r^{d+v} (b\bar{\sigma}_r \circ \max\{\pi_1(\tau)\pi_2(||\phi||_r), \eta\} - aR^{-d-v}),
\end{aligned}$$

select $\eta = \frac{\bar{\sigma}_r^{-1}(aR^{-d-v} - \epsilon)}{b}$ (dependent on ρ and independent on τ) and $\rho > \varepsilon > 0$ then for all $0 \leq \tau \leq \tau_0 = \min\{\frac{\overline{\rho}_r(\rho)}{\iota_k(2\overline{\rho}_r(\rho))}, \frac{1}{\pi_2(\varepsilon)} \pi^{-1}(\frac{\bar{\sigma}_r^{-1}(aR^{-d-v} - \epsilon)}{b})\}$ for some $\epsilon \in (0, aR^{-d-v})$ we have

$$\begin{aligned}
\max_{\theta \in [-\tau,0]} V[\phi(\theta)] &< \gamma V[\phi(0)] \Rightarrow \tfrac{\partial V(\phi(0))}{\partial \phi(0)} F[\phi(0), \phi(-\tau)] \\
&\leq -\epsilon |\phi(0)|_r^{d+v}
\end{aligned}$$

for all $\phi \in B_{\rho'}^\tau \setminus B_\varepsilon^\tau$ that are solutions of (1) for $t \geq \tau$. Consequently [19], the trajectories of (1) are decreasing to B_ε^τ and stay bounded for $t \geq \tau$, but they also stay bounded for $0 \leq t \leq \tau$ with $||x_0||_r \leq \rho$, which gives asymptotic stability with respect to B_ε^τ in B_ρ^τ for any delay $0 \leq \tau \leq \tau_0$.

Finally, applying Proposition 2 for some scaling $\lambda_0 > 1$ we obtain sequences $\varepsilon_i = \lambda_0^i \varepsilon$ and $\rho_i = \lambda_0^i \rho$ for all $i \geq 0$ such that $\varepsilon_i < \rho_{i-1}$ for all $i \geq 1$ and (1) is asymptotically stable with respect to $B_{\varepsilon_i}^\tau$ in $B_{\rho_i}^\tau$ for any delay $0 \leq \tau \leq \tau_0$. Then inclusion property $\varepsilon_i < \rho_{i-1}$ for all $i \geq 1$ ensures global attractivity of B_ε^τ and stability is preserved, that gives the required result. $\square$

A version of Lemma 4 for differential inclusions is presented in the paper [27] and based on analog of theorem on continuous dependence of solutions on parameters.

In comparison with [27] Lemma 4 provides the complete proof based on Lyapunov function method.

Remark It is easy to see that the result of Lemma 4 is preserved for a time-varying delay, and for multiple delays as well.

Using Theorem 2, these results can be used for local analysis of stability for not necessary homogeneous systems.

Theorem 3 *Let system (1) be (r,$+\infty$,f_0)-homogeneous with degree $d_0 < 0$, $f_0(x_t) = F_0[x(t), x(t-\tau)]$ and the origin for the approximating system (5) be globally asymptotically stable for $\tau = 0$. Then (1) has bounded trajectories IOD.*

Proof. The proof of Theorem 3 is a direct consequence of Theorem 2 and Lemma 4. □

Roughly speaking the result of Theorem 3 says that if the approximating dynamics (5) is stable in the delay-free case, then the original system (1) has bounded trajectories for any delay.

4 Applications

4.1 Double integrator system

According to [4], the double integrator system

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = u \end{cases}$$

is finite-time stable and r-homogeneous of degree $\alpha - 1 < 0$ for $\alpha \in (0,1)$ with the vector of weights $r = \begin{bmatrix} 2-\alpha, & 1 \end{bmatrix}^T$ under the following control

$$u = -\lceil x_2 \rfloor^{\alpha} - \lceil \chi_\alpha \rfloor^{\frac{\alpha}{2-\alpha}},$$

where $\chi_\alpha = x_1 + \frac{1}{2-\alpha} sign(x_2)|x_2|^{2-\alpha}$ and $\lceil x \rfloor^{\beta} = |x|^{\beta}\mathrm{sign}(x)$.

As Lemma 4 can be easily extended to the case of multiple delays, let us assume that the state is available for measurements with delays $\tau_i \in (0, \tau_{\max})$, $i = \overline{1,2}$, $0 < \tau_{\max} \leq \tau_0 < +\infty$. Then the double integrator system is globally asymptotically stable with respect to $B_\varepsilon^{\tau_{\max}}$ for some $\varepsilon > 0$ dependent on the selected values of delays.

The Fig. 1 shows the simulation results for $\alpha = 0.5$ and two different values of delays in the measurement channel: $\tau_1 = 0.2, \tau_2 = \tau_{\max} = 0.4$ and $\tau_1 = 0.2, \tau_2 = \tau_{\max} = 0.7$ correspondingly.

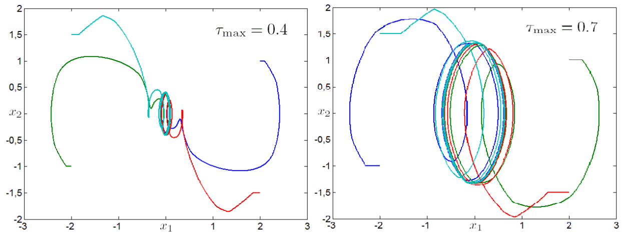

Fig. 1. The results of simulation for the double integrator system

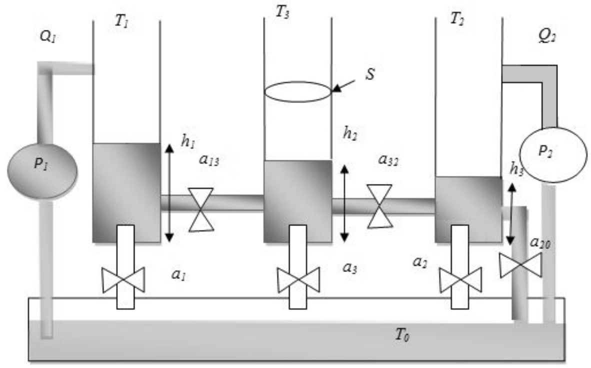

Fig. 2. The three tank system [32]

4.2 Three tank hydraulic system

The three tank hydraulic system depicted in Fig. 2 can be modeled as in [32], [21] by

$$\begin{cases} \dot{x}_1 = -\frac{a_{13}}{S} \lceil x_1 - x_3 \rfloor^{0.5} + \frac{1}{S} u_1, \\ \dot{x}_2 = \frac{a_{32}}{S} \lceil x_3 - x_2 \rfloor^{0.5} - \frac{a_{20}}{S} \lceil x_2 \rfloor^{0.5} + \frac{1}{S} u_2, \\ \dot{x}_3 = \frac{a_{13}}{S} \lceil x_1 - x_3 \rfloor^{0.5} - \frac{a_{32}}{S} \lceil x_3 - x_2 \rfloor^{0.5}, \end{cases} \tag{7}$$

where the liquid levels in tanks represent the state vector $x = [x_1, x_2, x_3]^T = [h_1, h_2, h_3]^T$, the input flows are control signals $u = [u_1, u_2]^T = [Q_1, Q_2]^T$.

The system (7) is globally asymptotically stable for $u_1 = u_2 = 0$ (to check this claim, since the state takes only positive values, we can consider Lyapunov function $V(x_1, x_2, x_3) = x_1 + x_2 + x_3$) and r-homogeneous with degree $d = -0.5 < 0$ for $r = [1\ 1\ 1]$, thus it is finite-time stable. Next, using the results of [5] we can show that the system is input-to-state stable with respect to inputs u_1 and u_2.

Let us assume that liquid flows from the tank 1 into the tank 3 and from the tank 3 into the tank 2 with some delays $\tau_i \in (0, \tau_{\max})$, $i = \overline{1,2}$, $0 < \tau_{\max} \leq \tau_0 < +\infty$ that corresponds to increased distances between the tanks

$$\begin{cases} \dot{x}_1(t) = -\frac{a_{13}}{S} \lceil x_1(t) - x_3(t-\tau_1) \rfloor^{0.5} + \frac{1}{S} u_1(t), \\ \dot{x}_2(t) = \frac{a_{32}}{S} \lceil x_3(t-\tau_2) - x_2(t) \rfloor^{0.5} - \frac{a_{20}}{S} \lceil x_2(t) \rfloor^{0.5} + \frac{1}{S} u_2(t), \\ \dot{x}_3(t) = \frac{a_{13}}{S} \lceil x_1(t-\tau_1) - x_3(t) \rfloor^{0.5} - \frac{a_{32}}{S} \lceil x_3(t) - x_2(t-\tau_2) \rfloor^{0.5}, \end{cases}$$

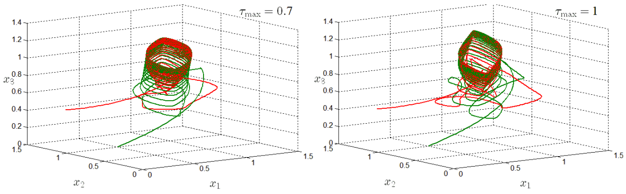

Fig. 3. The results of simulation for the three tank system

then by Lemma 4 the system (7) for $u_1 = u_2 = 0$ is globally asymptotically stable IOD with respect to $B_\varepsilon^{\tau_{\max}}$ for some $\varepsilon > 0$ dependent on $\tau_{\max}$. For any constant inputs u_1 and u_2 the three tank system has approximation at infinity corresponding to (7) with $u_1 = u_2 = 0$. Then by Theorem 3 the system (7) has bounded trajectories IOD. The simulation results for $S = 1$, $a_{13} = 3$, $a_{32} = 2$, $a_{20} = 1$, $u_1 = 0$, $u_2 = 1$ and two different values of delays ($\tau_1 = \tau_{\max} = 0.7, \tau_2 = 0.2$ and $\tau_1 = \tau_{\max} = 1, \tau_2 = 0.25$) are shown in Fig. 3.

5 Conclusions

For time-delay systems with negative degree it has been shown that if for zero delay the system is globally asymptotically stable, then for any delay it is converging to some compact set around the origin, that is an interesting robustness property of nonlinear homogeneous systems with respect to delays, which is not observed in the linear case. Efficiency of the proposed approach is demonstrated on example of three tank hydraulic system. Design of new stabilization and estimation algorithms, which preserve boundedness of the system trajectories for any delays, is a direction of future research.

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