



Trefftz-DG Approximation for the Elasto-Acoustics

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Trefftz-DG Approximation for the Elasto-Acoustics

DIP workshop

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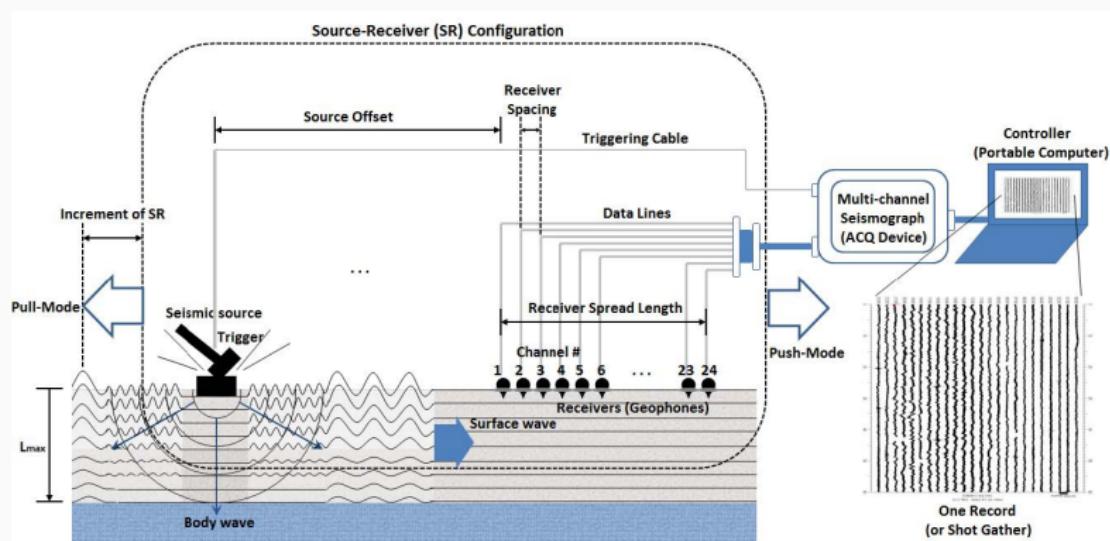
December 14, 2016

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Abstract

Seismic survey

Figure 1: Schematic of overall field setup for a seismic survey¹.



¹ Park Seismic LLC. <http://www.parkseismic.com>. Internet resource.

Basic numerical methods

Table 1: Generic properties of the most widely used numerical methods².

Numerical method	Complex geometries	High-order accuracy and hp -adaptivity	Explicit semi-discrete form	Conservation laws	Elliptic problems
FDM	●	●	●	●	●
FVM	●	●	●	●	○
FEM	●	●	●	○	●
DG-FEM	●	●	●	●	○

² J.S.Hesthaven T.Warburton. Nodal discontinuous galerkin methods. Algorithms, analysis, and applications. Texts in Applied Mathematics, (54):1-370, 2007.

DG-FEM :



- Adapted to the complex geometries ;
- High-order accuracy and hp -adaptivity ;
- Explicit semi-discrete form ;
- Conservation laws ;



Higher number of degrees of freedom
comparing to the methods with continuous approximation.

Trefftz method³:

Given a region of an Euclidean space of some partitions of that region, a **Trefftz Method** is any procedure for solving boundary value problems of partial differential equations or systems of such equations, on such region, **using solutions of that differential equation or its adjoint**, defined in its subregions.

³ I.Herrera. Trefftz method: a general theory. Instituto de Investigaciones en Matematicas Aplicadas y en Sistemas (IIMAS), pages 562-580, 2000.

Time-harmonic formulations :

O.Cassenat, B.Despres (1998);

C.Farhat, I.Harari, U.Hetmaniuk (2003); G.Gabard (2007);

T.Huttunen, P.Monk, J.P.Kaipo (2002);

R.Tezaur, C.Farhat (2006);

Time-domain formulations :

F.Kretzschmar, A.Moiola, I.Perugia (2015);

F.Kretzschmar, S.M.Schnepp, I.Tsukerman, T.Weiland(2014);

H.Egger, F.Kretzschmar, S.M.Schnepp, T.Weiland (2014).

Expected advantages of Trefftz method :



- Better orders of convergence ;
- Flexibility in the choice of basis functions ;
- Low dispersion ;
- Incorporation of wave propagation directions in the discrete space ;
- Adaptivity and local space-time mesh refinement.

³ I.Herrera. Trefftz method: a general theory. Instituto de Investigaciones en Matematicas Aplicadas y en Sistemas (IIMAS), pages 562-580, 2000.

Mathematical formulation.

Fluid case

Acoustic system. Problem equation

$\Omega_F \subset R^n$ - space domain ;

$I = (0, T)$ - time domain ;

$Q_F := \Omega_F \times I$;

$n_{Q_F} = (n_{Q_F}^x, n_{Q_F}^t)$ - o.p. unit normal vector on ∂Q_F ;

Acoustic system

$$\frac{1}{c_f^2 \rho_f} \frac{\partial p}{\partial t} + \operatorname{div} v_f = f, \text{ in } Q_F;$$

$$\rho_f \frac{\partial v_f}{\partial t} + \nabla p = 0, \text{ in } Q_F;$$

$$v_f(\cdot, 0) = v_{f0}, \quad p(\cdot, 0) = p_0, \quad \text{on } \Omega_F;$$

$$v_f = g_f^D, \quad \text{on } \partial\Omega_F \times I.$$

Acoustic system. Variational formulation

$K_F \subset Q_F$ ($c_f, \rho_f \equiv \text{const.}$ in K_F) ;

$n_{K_F} = (n_{K_F}^x, n_{K_F}^t)$ - o.p. normal vector on ∂K_F ;

$v_f, p \in H^1(K_F)$;

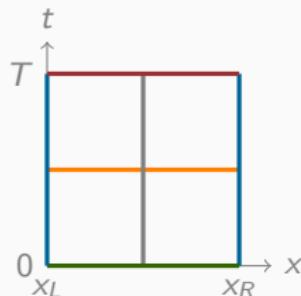
$\omega_f, q \in H^1(K_F)$;

Space-time integration

$$\begin{aligned} & - \int_{K_F} \left(v_f \cdot \left(\rho_f \frac{\partial \omega_f}{\partial t} + \nabla q \right) + p \left(\frac{1}{c_f^2 \rho_f} \frac{\partial q}{\partial t} + \operatorname{div} \omega_f \right) \right) dv + \\ & + \int_{\partial K_F} \left((p \omega_f + v_f q) \cdot n_{K_F}^x + \left(\frac{1}{c_f^2 \rho_f} pq + \rho_f v_f \cdot \omega_f \right) n_{K_F}^t \right) ds = \iint_{K_F} f q dv. \end{aligned}$$

Acoustic system. Space-time DG discretisation

Mesh and DG-notation



\mathcal{T}_h - mesh on $Q_F := \Omega_F \times I$

$\mathcal{F}_h = \cup_{K_F \in \mathcal{T}_h} \partial K_F$ - mesh skeleton :

$\mathcal{F}_h^{\Omega_F}$:= union of the internal Ω -like faces, ($t \equiv \text{const.}$) ;

$\mathcal{F}_h^{I_F}$:= union of the internal I -like faces, ($x \equiv \text{const.}$) ;

$\mathcal{F}_h^{0_F}$:= $\Omega_F \times \{0\}$;

$\mathcal{F}_h^{T_F}$:= $\Omega_F \times \{T\}$;

$\mathcal{F}_h^{D_F}$:= $\partial \Omega_F \times [0, T]$ - union of the Dirichlet boundary faces.

Acoustic system. Space-time DG formulation

Seek $(v_{fh}, p_h) \subset \mathbf{V}(\mathcal{T}_h) \subset H^1(\mathcal{T}_h)^2$, s.t. for all $K_F \in \mathcal{T}_h$, $(\omega_f, q) \subset \mathbf{V}(\mathcal{T}_h)$ it holds:

DG formulation

$$\begin{aligned} & - \int_{K_F} \left[p_h \left(\frac{1}{c_f^2 \rho_f} \frac{\partial q}{\partial t} + \operatorname{div} \omega_f \right) + v_{fh} \cdot \left(\rho_f \frac{\partial \omega_f}{\partial t} + \nabla q \right) \right] dv \\ & + \int_{\partial K_F} \left[\left(\frac{1}{c_f^2 \rho_f} \hat{p}_h q + \rho_f \hat{v}_{fh} \cdot \omega_f \right) n_{K_F}^t + (\hat{p}_h \omega_f + \hat{v}_{fh} q) \cdot n_{K_F}^x \right] ds = \int_{K_F} f q dv. \end{aligned}$$

Trefftz space

$$\begin{aligned} \mathbf{T}_F(\mathcal{T}_h) := \Big\{ & (\omega_f, q) \subset H^1(\mathcal{T}_h)^2, \text{ s. t. in all } K_F \in \mathcal{T}_h \\ & \rho_f \frac{\partial \omega_f}{\partial t} + \nabla q = 0, \quad \frac{1}{c_f^2 \rho_f} \frac{\partial q}{\partial t} + \operatorname{div} \omega_f = 0 \Big\}. \end{aligned}$$

Acoustic system. Space-time Trefftz-DG formulation

Seek $(v_{fh}, p_h) \subset \mathbf{V}(\mathcal{T}_h) \subset H^1(\mathcal{T}_h)^2$, s.t. for all $K_F \in \mathcal{T}_h$, $(\omega_f, q) \subset \mathbf{V}(\mathcal{T}_h)$ it holds:

Trefftz-DG formulation ($f \equiv 0$)

$$\int_{\partial K_F} \left[\left(\frac{1}{c_f^2 \rho_f} \hat{p}_h q + \rho_f \hat{v}_{fh} \cdot \omega_f \right) n_{K_F}^t + (\hat{p}_h \omega_f + \hat{v}_{fh} q) \cdot n_{K_F}^x \right] ds = 0$$

Acoustic system. Space-time Trefftz-DG formulation

Numerical flux $\hat{v}_{f,h}$ and \hat{p}_h :

on $\mathcal{F}_h^{I_F}$:
$$\begin{pmatrix} \hat{v}_{f,h} \\ \hat{p}_h \end{pmatrix} := \begin{pmatrix} \{v_{f,h}\} + \beta [\![p_h]\!]_x \\ \{p_h\} + \alpha [\![v_{f,h}]\!]_x \end{pmatrix},$$

on $\mathcal{F}_h^{\Omega_F}$:
$$\begin{pmatrix} \hat{v}_{f,h} \\ \hat{p}_h \end{pmatrix} := \begin{pmatrix} v_{f,h}^- \\ p_h^- \end{pmatrix},$$

on $\mathcal{F}_h^{T_F}$:
$$\begin{pmatrix} \hat{v}_{f,h} \\ \hat{p}_h \end{pmatrix} := \begin{pmatrix} v_{f,h} \\ p_h \end{pmatrix},$$

on $\mathcal{F}_h^{0_F}$:
$$\begin{pmatrix} \hat{v}_{f,h} \\ \hat{p}_h \end{pmatrix} := \begin{pmatrix} v_{f,0} \\ p_0 \end{pmatrix},$$

on $\mathcal{F}_h^{D_F}$:
$$\begin{pmatrix} \hat{v}_{f,h} \\ \hat{p}_h \end{pmatrix} := \begin{pmatrix} \hat{g}_{D_F} \\ p_h + \alpha(v_{f,h} - g_{D_F}) \cdot n_{K_F}^x \end{pmatrix},$$

($\alpha \in L^\infty(\mathcal{F}_h^{I_F} \cup \mathcal{F}_h^{D_F})$, $\beta \in L^\infty(\mathcal{F}_h^{I_F})$ - positive flux parameters)

Acoustic system. Trefftz-DG formulation

Trefftz-DG formulation :

$$\begin{aligned} & \int_{\mathcal{F}_h^{\Omega_F}} \left[\frac{1}{c_f^2 \rho_f} p_h^- \llbracket q \rrbracket_t + \rho_f v_{f,h}^- \llbracket \omega_f \rrbracket_t \right] ds \\ & + \int_{\mathcal{F}_h^{I_F}} \left[\{p_h\} \llbracket \omega_f \rrbracket_x + \{v_{f,h}\} \llbracket q \rrbracket_x + \alpha \llbracket v_{f,h} \rrbracket_x \llbracket \omega_f \rrbracket_x + \beta \llbracket p_h \rrbracket_x \llbracket q \rrbracket_x \right] ds \\ & + \int_{\mathcal{F}_h^{T_F}} \left[\frac{1}{c_f^2 \rho_f} p_h q + \rho_f v_{f,h} \cdot \omega_f \right] ds - \frac{1}{2} \int_{\mathcal{F}_h^{0_F}} \left[\frac{1}{c_f^2 \rho_f} p_h q + \rho_f v_{f,h} \cdot \omega_f \right] ds \\ & + \int_{\mathcal{F}_h^{D_F}} \left[\omega_f \cdot (p n_{K_F}^x + \alpha v_{f,h}) \right] ds = \\ & \frac{1}{2} \int_{\mathcal{F}_h^{0_F}} \left[\frac{1}{c_f^2 \rho_f} p_h q + \rho_f v_{f,h} \cdot \omega_f \right] ds + \int_{\mathcal{F}_h^{D_F}} \left[g_{D_F} (\alpha \omega_f - q \cdot n_{K_F}^x) \right] ds. \end{aligned}$$

Acoustic system. Trefftz-DG formulation

Seek $(v_{fh}, p_h) \subset \mathbf{V}(\mathcal{T}_h) \subset \mathbf{T}_F(\mathcal{T}_h)$ s.t. for all $(\omega_f, q) \subset \mathbf{V}(\mathcal{T}_h)$ it holds:

Trefftz-DG formulation

$$\mathcal{A}_{TDG_F}((v_{fh}, p_h); (\omega_f, q)) = \ell_{TDG_F}(\omega_f, q).$$

Acoustic system. Analysis of Trefftz-DG formulation

Trefftz-DG formulation in terms of $L^2(\mathcal{T}_h)$ -norms

$$\mathcal{A}_{TDGF}((\omega_f, q); (\omega_f, q)) :=$$

$$\begin{aligned} & \frac{1}{2} \left\| \left(\frac{1}{c_f^2 \rho_f} \right)^{1/2} [\![q]\!]_t \right\|_{L^2(\mathcal{F}_h^{\Omega_F})}^2 + \frac{1}{2} \left\| \rho_f^{1/2} [\![\omega_f]\!]_t \right\|_{L^2(\mathcal{F}_h^{\Omega_F})}^2 \\ & + \left\| \alpha^{1/2} [\![\omega_f]\!]_x \right\|_{L^2(\mathcal{F}_h^{I_F})}^2 + \left\| \beta^{1/2} [\![q]\!]_x \right\|_{L^2(\mathcal{F}_h^{I_F})}^2 \\ & + \frac{1}{2} \left\| \left(\frac{1}{c_f^2 \rho_f} \right)^{1/2} q \right\|_{L^2(\mathcal{F}_h^{T_F})}^2 + \frac{1}{2} \left\| \rho_f^{1/2} \omega_f \right\|_{L^2(\mathcal{F}_h^{T_F})}^2 \\ & + \left\| \alpha^{1/2} \omega_f \right\|_{L^2(\mathcal{F}_h^{D_F})}^2. \end{aligned}$$

Acoustic system. Analysis of Trefftz-DG formulation

Norm $\|\cdot\|_{TDG_F}$ in $\mathbf{T}_F(\mathcal{T}_h)$

$$\begin{aligned} \|\cdot\|_{TDG_F}^2 := & \frac{1}{2} \left\| \left(\frac{1}{c_f^2 \rho_f} \right)^{1/2} [\![q]\!]_t \right\|_{L^2(\mathcal{F}_h^{\Omega_F})}^2 + \frac{1}{2} \left\| \rho_f^{1/2} [\![\omega_f]\!]_t \right\|_{L^2(\mathcal{F}_h^{\Omega_F})}^2 \\ & + \left\| \alpha^{1/2} [\![\omega_f]\!]_x \right\|_{L^2(\mathcal{F}_h^{I_F})}^2 + \left\| \beta^{1/2} [\![q]\!]_x \right\|_{L^2(\mathcal{F}_h^{I_F})}^2 \\ & + \frac{1}{2} \left\| \left(\frac{1}{c_f^2 \rho_f} \right)^{1/2} q \right\|_{L^2(\mathcal{F}_h^{T_F})}^2 + \frac{1}{2} \left\| \rho_f^{1/2} \omega_f \right\|_{L^2(\mathcal{F}_h^{T_F})}^2 \\ & + \left\| \alpha^{1/2} \omega_f \right\|_{L^2(\mathcal{F}_h^{D_F})}^2. \end{aligned}$$

Coercivity of Trefftz-DG formulation

$$\mathcal{A}_{TDG_F}((\omega_f, q); (\omega_f, q)) = |||(\omega_f, q)|||_{TDG_F}^2, \quad \forall (\omega_f, q) \in \mathbf{T}_F(\mathcal{T}_h).$$

Acoustic system. Analysis of Trefftz-DG formulation

Add-on norm $||| \cdot |||_{TDG_F^*}$ **in** $\mathbf{T}_F(\mathcal{T}_h)$

$$\begin{aligned} |||(\omega_f, q)|||_{TDG_F^*}^2 := & |||(\omega_f, q)|||_{TDG_F}^2 \\ & + \|\rho_f^{1/2} \omega_f^- \|_{L^2(\mathcal{F}_h^{\Omega_F})}^2 + \left\| \left(\frac{1}{c_f^2 \rho_f} \right)^{1/2} q^- \right\|_{L^2(\mathcal{F}_h^{\Omega_F})}^2 \\ & + \|\beta^{-1/2} \{\omega_f\}\|_{L^2(\mathcal{F}_h^{I_F})}^2 + \|\alpha^{-1/2} \{q\}\|_{L^2(\mathcal{F}_h^{I_F})}^2 \\ & + \|\alpha^{-1/2} \omega_f\|_{L^2(\mathcal{F}_h^{D_F})}^2. \end{aligned}$$

Continuity of Trefftz-DG formulation

$$|\mathcal{A}_{TDG_F}((v_f, p); (\omega_f, q))| \leq 2 ||| (v_f, p) |||_{TDG_F^*} ||| (\omega_f, q) |||_{TDG_F},$$

$$|\ell_{TDG_F}(\omega_f, q)| \leq \sqrt{2} \left[\|\rho_f^{1/2} v_{f0}\|_{L^2(\mathcal{F}_h^{0F})}^2 + \left\| \left(\frac{1}{c_f^2 \rho_f} \right)^{1/2} p_0 \right\|_{L^2(\mathcal{F}_h^{0F})}^2 \right]^{1/2}.$$

Well-posedness

$$|||(\nu_f - \nu_{f,h}, p - p_h)|||_{TDG_F} \leq 3 \inf_{(\omega_f, q) \in \mathbf{V}(\mathcal{T}_h)} |||(\nu_f - \omega_f, p - q)|||_{TDG_F^*}.$$

Mathematical formulation.

Solid case

Elastodynamic system. Problem equation

$\Omega_S \subset R^n$ - space domain ;

$I = (0, T)$ - time domain ;

$Q_S := \Omega_S \times I$;

$n_{Q_S} = (n_{Q_S}^x, n_{Q_S}^t)$ - o.p. unit normal vector on ∂Q_S ;

Elastodynamic system

$$\frac{\partial \underline{\underline{\sigma}}}{\partial t} - \underline{\underline{C}} \underline{\underline{\varepsilon}}(\underline{\underline{v}_s}) = 0 \text{ in } Q_S;$$

$$\rho_s \frac{\partial \underline{\underline{v}_s}}{\partial t} - \text{div} \underline{\underline{\sigma}} = 0 \text{ in } Q_S;$$

$$v_s(\cdot, 0) = v_{s0}, \quad \underline{\underline{\sigma}}(\cdot, 0) = \underline{\underline{\sigma}}_0 \text{ on } \Omega_S;$$

$$v_s = g_{D_S} \text{ on } \partial \Omega_S \times I.$$

Elastodynamic system. Problem equation

$\Omega_S \subset R^n$ - space domain ;

$I = (0, T)$ - time domain ;

$Q_S := \Omega_S \times I$;

$n_{Q_S} = (n_{Q_S}^x, n_{Q_S}^t)$ - o.p. unit normal vector on ∂Q_S ;

Elastodynamic system

$$A \frac{\partial \underline{\underline{\sigma}}}{\partial t} - \underline{\underline{\varepsilon}}(\underline{v}_s) = 0 \text{ in } Q_S;$$

$$\rho_s \frac{\partial \underline{v}_s}{\partial t} - \text{div} \underline{\underline{\sigma}} = 0 \text{ in } Q_S;$$

$$\underline{v}_s(\cdot, 0) = \underline{v}_{s0}, \quad \underline{\underline{\sigma}}(\cdot, 0) = \underline{\underline{\sigma}}_0 \text{ on } \Omega_S;$$

$$\underline{v}_s = g_{D_S} \text{ on } \partial \Omega_S \times I.$$

Elastodynamic system. Variational formulation

$K_S \subset Q_S$ (\underline{A} , $\rho_s \equiv \text{const.}$ in K_S) ;

$n_{K_S} = (n_{K_S}^x, n_{K_S}^t)$ - o.p. unit normal vector on ∂K_S ;

$v_s, \underline{\underline{\sigma}} \in H^1(K_S)$;

$\omega_s, \underline{\underline{\xi}} \in H^1(K_S)$;

Space-time integration

$$\begin{aligned} & - \int_{K_S} \left[\underline{\underline{\sigma}} : (\underline{A} \frac{\partial \underline{\underline{\xi}}}{\partial t} - \underline{\underline{\varepsilon}}(\omega_s)) + v_s \cdot (\rho_s \frac{\partial \omega_s}{\partial t} - \text{div} \underline{\underline{\xi}}) \right] dv \\ & + \int_{\partial K_S} \left[(\underline{A} \underline{\underline{\sigma}} : \underline{\underline{\xi}} + \rho_s v_s \cdot \omega_s) \cdot n_{K_S}^t - (v_s \cdot \underline{\underline{\xi}} + \underline{\underline{\sigma}} \cdot \omega_s) \cdot n_{K_S}^x \right] ds = 0. \end{aligned}$$

Trefftz space

$$\begin{aligned}\mathbf{T}_S(\mathcal{T}_h) := \Big\{ & (\omega_s, \underline{\xi}) \subset H^1(\mathcal{T}_h)^2, \text{ s. t. in all } K_S \in \mathcal{T}_h \\ & \rho_s \frac{\partial \omega_s}{\partial t} - \operatorname{div} \underline{\xi} = 0, \quad A \frac{\partial \underline{\xi}}{\partial t} - \underline{\varepsilon}(\omega_s) = 0 \Big\}. \end{aligned}$$

Seek $(v_{sh}, \underline{\underline{\sigma}}_h) \subset \mathbf{V}(\mathcal{T}_h) \subset H^1(\mathcal{T}_h)^2$, s.t. for all $K_S \in \mathcal{T}_h$, $(\omega_s, \underline{\xi}) \subset \mathbf{V}(\mathcal{T}_h)$ it holds:

Space-time Trefftz-DG formulation

$$\int_{\partial K_S} \left[(\underline{\underline{A}} \hat{\underline{\underline{\sigma}}}_h : \underline{\xi} + \rho_s \hat{v}_{sh} \cdot w_s) \cdot n_K^t - (\hat{v}_{sh} \underline{\xi} + \hat{\underline{\sigma}}_h \omega_s) \cdot n_K^x \right] ds = 0$$

Trefftz-DG formulation

$$\begin{aligned}
& \int_{\mathcal{F}_h^{\Omega_S}} \left[\underline{\underline{A}} \underline{\underline{\sigma}}_h : [\![\underline{\xi}]\!]_t + \rho_s v_{sh}^- [\![\omega_s]\!]_t \right] ds \\
& - \int_{\mathcal{F}_h^{IS}} \left[\{ \underline{\underline{\sigma}}_h \} [\![\omega_s]\!]_x + \{ v_{sh} \} [\![\underline{\xi}]\!]_x - \gamma [\![v_{sh}]\!]_x [\![\omega_s]\!]_x - \delta [\![\underline{\underline{\sigma}}_h]\!]_x [\![\underline{\xi}]\!]_x \right] ds \\
& + \int_{\mathcal{F}_h^{TS}} \left[\underline{\underline{A}} \underline{\underline{\sigma}}_h : \underline{\xi} + \rho_s v_{sh} \cdot \omega_s \right] ds - \frac{1}{2} \int_{\mathcal{F}_h^{0S}} \left[\underline{\underline{A}} \underline{\underline{\sigma}}_h : \underline{\xi} + \rho_s v_{sh} \cdot \omega_s \right] ds \\
& - \int_{\mathcal{F}_h^{DS}} \left[\underline{\xi} (v_{sh} \cdot n_{K_S}^x - \delta \underline{\underline{\sigma}}_h) \right] ds = \\
& \frac{1}{2} \int_{\mathcal{F}_h^{0S}} \left[\underline{\underline{A}} \underline{\underline{\sigma}}_h : \underline{\xi} + \rho_s v_{sh} \cdot \omega_s \right] ds + \int_{\mathcal{F}_h^{DS}} \left[g_{DS} (\omega_s \cdot n_{K_S}^x + \delta \underline{\xi}) \right] ds.
\end{aligned}$$

Seek $(v_{sh}, \underline{\underline{\sigma}}_h) \subset \mathbf{V}(\mathcal{T}_h) \subset \mathbf{T}_S(\mathcal{T}_h)$ s.t. for all $(\omega_s, \underline{\underline{\xi}}) \subset \mathbf{V}(\mathcal{T}_h)$ it holds:

Trefftz-DG formulation

$$\mathcal{A}_{TDG_S}((v_{fh}, \underline{\underline{\sigma}}_h); (\omega_s, \underline{\underline{\xi}})) = \ell_{TDG_S}(\omega_s, \underline{\underline{\xi}}).$$

Norms in $\mathbf{T}_S(\mathcal{T}_h)$

$$\|(\omega_s, \underline{\xi})\|_{TDG_S}^2 := \frac{1}{2} \left\| (\underline{\underline{A}})^{1/2} [\underline{\underline{\xi}}]_t \right\|_{L^2(\mathcal{F}_h^{\Omega_S})}^2 + \frac{1}{2} \left\| \rho_s^{1/2} [\omega_s]_t \right\|_{L^2(\mathcal{F}_h^{\Omega_S})}^2 \\ + \left\| \gamma^{1/2} [\omega_s]_x \right\|_{L^2(\mathcal{F}_h^{IS})}^2 + \left\| \delta^{1/2} [\underline{\underline{\xi}}]_x \right\|_{L^2(\mathcal{F}_h^{IS})}^2 \\ + \frac{1}{2} \left\| (\underline{\underline{A}})^{1/2} \underline{\underline{\xi}} \right\|_{L^2(\mathcal{F}_h^{TS})}^2 + \frac{1}{2} \left\| \rho_s^{1/2} \omega_s \right\|_{L^2(\mathcal{F}_h^{TS})}^2 \\ + \left\| \delta^{1/2} \underline{\underline{\xi}} \right\|_{L^2(\mathcal{F}_h^{DS})}^2.$$

Norms in $\mathbf{T}_S(\mathcal{T}_h)$

$$\begin{aligned} |||(\omega_s, \underline{\underline{\xi}})|||_{TDG_S^*}^2 := & |||(\omega_s, \underline{\underline{\xi}})|||_{TDG_S}^2 \\ & + \|\rho_s^{1/2} \omega_s^- \|_{L^2(\mathcal{F}_h^{\Omega_S})}^2 + \|(\underline{\underline{A}})^{1/2} \underline{\underline{\xi}}^- \|_{L^2(\mathcal{F}_h^{\Omega_S})}^2 \\ & + \|\delta^{-1/2} \{\omega_s\} \|_{L^2(\mathcal{F}_h^{IS})}^2 + \|\gamma^{-1/2} \{\underline{\underline{\xi}}\} \|_{L^2(\mathcal{F}_h^{IS})}^2 \\ & + \|\delta^{-1/2} \underline{\underline{\xi}} \|_{L^2(\mathcal{F}_h^{DS})}^2. \end{aligned}$$

Coercivity of Trefftz-DG formulation

$$\mathcal{A}_{TDG_S}((\omega_s, \underline{\xi}); (\omega_s, \underline{\xi})) = |||(\omega_s, \underline{\xi})|||_{TDG_S}^2, \quad \forall (\omega_s, \underline{\xi}) \in \mathbf{T}_S(\mathcal{T}_h).$$

Continuity of Trefftz-DG formulation

$$|\mathcal{A}_{TDG_S}((v_s, \underline{\underline{\sigma}}); (\omega_s, \underline{\xi}))| \leq 2 ||| (v_s, \underline{\underline{\sigma}}) |||_{TDG_S^*} ||| (\omega_s, \underline{\xi}) |||_{TDG_S},$$

$$|\ell_{TDG_S}(\omega_s, \underline{\xi})| \leq \sqrt{2} \left[\|\rho_s^{1/2} v_{s0}\|_{L^2(\mathcal{F}_h^{0s})}^2 + \|\underline{\underline{A}}_0^{1/2} \underline{\underline{\sigma}}_0\|_{L^2(\mathcal{F}_h^{0s})}^2 \right]^{1/2}.$$

Well-posedness

$$|||(\nu_s - \nu_{sh}, \underline{\underline{\sigma}} - \underline{\underline{\sigma}}_h)|||_{TDG_S} \leq 3 \inf_{(\omega_s, \underline{\underline{\xi}}) \in \mathbf{V}(\mathcal{T}_h)} |||(\nu_s - \omega_s, \underline{\underline{\sigma}} - \underline{\underline{\xi}})|||_{TDG_S^*}.$$

Mathematical formulation. Fluid-solid case

Coupled system

Representation of the **acoustic medium** as a limit case of an
elastic isotropic medium with shear modulus μ
tending or equal to 0 ?

Advantages of the numerical coupling:



Computing 1 unknown scalar pressure instead of 6 components of stress tensor ;

Avoiding presence of the numerical artifacts caused of the slow S -waves appearance.

Coupled system. Problem equation

Acoustic system

$$\begin{aligned} \frac{1}{c_f^2 \rho_f} \frac{\partial p}{\partial t} + \operatorname{div} v_f &= f, \text{ in } Q_F; \\ \rho_f \frac{\partial v_f}{\partial t} + \nabla p &= 0, \text{ in } Q_F; \\ v_f &= g_{D_F}, \text{ on } \partial \Omega_F \times I; \end{aligned}$$

Elastodynamic system

$$\begin{aligned} A \frac{\partial \underline{\underline{\sigma}}}{\partial t} - \underline{\underline{\varepsilon}}(v_s) &= 0 \text{ in } Q_S; \\ \rho_s \frac{\partial v_s}{\partial t} - \operatorname{div} \underline{\underline{\sigma}} &= 0 \text{ in } Q_S; \\ v_s &= g_{D_S} \text{ on } \partial \Omega_S \times I; \end{aligned}$$

Coupled system. Problem equation

Fluid-solid transmission conditions through the Γ_{FS}

$$v_f \cdot n_\Omega^x = v_s \cdot n_\Omega^x \text{ on } \Gamma_{FS};$$

$$\underline{\underline{\sigma}} n_\Omega^x = -p n_\Omega^x \text{ on } \Gamma_{FS}.$$

Coupled system. Space-time Trefftz-DG formulation

Trefftz space

$$\begin{aligned} \mathbf{T}(\mathcal{T}_h) := \Big\{ & (\omega_f, q, \omega_s, \underline{\xi}) \subset H^1(\mathcal{T}_h)^4, \text{ s.t.} \\ & \rho_f \frac{\partial \omega_f}{\partial t} + \nabla q = 0, \quad \frac{1}{c_f^2 \rho_f} \frac{\partial q}{\partial t} + \operatorname{div} \omega_f = 0, \quad \forall K_F \in \mathcal{T}_h, \\ & \rho_s \frac{\partial \omega_s}{\partial t} - \operatorname{div} \underline{\xi} = 0, \quad \frac{\partial \underline{\xi}}{\partial t} - \underline{\xi}(\omega_s) = 0, \quad \forall K_S \in \mathcal{T}_h \Big\}. \end{aligned}$$

Coupled system. Space-time Trefftz-DG formulation

Seek $(v_{fh}, p_h, v_{sh}, \underline{\underline{\sigma}}_h) \subset \mathbf{V}(\mathcal{T}_h) \subset \mathbf{T}(\mathcal{T}_h)$, s. t. for all $K_F, K_S \in \mathcal{T}_h$,
 $(\omega_f, q, \omega_s, \underline{\underline{\xi}}) \subset \mathbf{V}(\mathcal{T}_h)$ it holds :

Trefftz-DG formulation ($f \equiv 0$)

$$\int_{\partial K_F} \left[\left(\frac{1}{c_f^2 \rho_f} \hat{p}_h q + \rho_f \hat{v}_{fh} \cdot \omega_f \right) n_{K_F}^t + (\hat{p}_h \omega_f + \hat{v}_{fh} q) \cdot n_{K_F}^x \right] ds = 0,$$

$$\int_{\partial K_S} \left[\left(\underline{\underline{A}} \underline{\underline{\sigma}}_h : \underline{\underline{\xi}} + \rho_s \hat{v}_{sh} \cdot \omega_s \right) \cdot n_K^t - (\hat{v}_{sh} \underline{\underline{\xi}} + \underline{\underline{\sigma}}_h \omega_s) \cdot n_K^x \right] ds = 0$$

Coupled system. Space-time DG formulation

Numerical flux $\hat{v}_f h$, \hat{v}_{sh} , \hat{p}_h and $\hat{\underline{\sigma}}_h$ through \mathcal{F}_h^{FS} :

on \mathcal{F}_h^{FS} :

$$\begin{pmatrix} \hat{v}_f h \cdot n_{K_F}^x \\ \hat{p}_h \\ \hat{v}_{sh} \\ \hat{\underline{\sigma}}_h \cdot n_{K_S}^x \end{pmatrix} := \begin{pmatrix} v_{sh} \cdot n_{K_F}^x \\ p + \alpha(v_f h - v_{sh}) \cdot n_{K_F}^x \\ v_{sh} - \delta(\underline{\sigma}_h - p_h) \cdot n_{K_S}^x \\ -p_h n_{K_S}^x \end{pmatrix}$$

Coupled system. Space-time Trefftz-DG formulation

Seek $(v_{fh}, p_h, v_{sh}, \underline{\underline{\sigma}}_h) \subset \mathbf{V}(\mathcal{T}_h) \subset \mathbf{T}(\mathcal{T}_h)$ s.t. $\forall (\omega_f, q, \omega_s, \underline{\xi}) \subset \mathbf{V}(\mathcal{T}_h)$ it holds:

Trefftz-DG formulation

$$\mathcal{A}_{TDG}((v_{fh}, p_h, v_{sh}, \underline{\underline{\sigma}}_h); (\omega_f, q, \omega_s, \underline{\xi})) = \ell_{TDG}(\omega_f, q, \omega_s, \underline{\xi})$$

Coupled system. Analysis of Trefftz-DG formulation

Norms in $\mathbf{T}(\mathcal{T}_h)$

$$|||(\omega_f, q, \omega_s, \underline{\xi}))|||_{TDG}^2 := |||(\omega_f, q)|||_{TDG_F}^2 + |||(\omega_s, \underline{\xi})|||_{TDG_S}^2 + 2\|\delta^{1/2}\underline{\xi}\|_{L^2(\mathcal{F}_h^{FS})}^2$$

$$|||(\omega_f, q, \omega_s, \underline{\xi})|||_{TDG^*}^2 := |||(\omega_f, q)|||_{TDG_F^*}^2 + |||(\omega_s, \underline{\xi})|||_{TDG_S^*}^2 + \frac{1}{2}\|\delta^{-1/2}\underline{\xi}\|_{L^2(\mathcal{F}_h^{FS})}^2$$

Coupled system. Analysis of Trefftz-DG formulation

Coercivity of Trefftz-DG formulation

$$\mathcal{A}_{TDG}((\omega_f, q, \omega_s, \underline{\xi}); (\omega_f, q, \omega_s, \underline{\xi})) = |||(\omega_f, q, \omega_s, \underline{\xi})|||_{TDG}^2$$

Continuity of Trefftz-DG formulation

$$|\mathcal{A}_{TDG}((v_f, p, v_s, \underline{\sigma}); (\omega_f, q, \omega_s, \underline{\xi}))| \leq 2 |||(\omega_f, q, \omega_s, \underline{\xi})|||_{TDG^*} |||(\omega_f, q, \omega_s, \underline{\xi})|||_{TDG},$$

$$\begin{aligned} |\ell_{TDG}(\omega_f, q, \omega_s, \underline{\xi})| &\leq \sqrt{2} \left[\|\rho_f^{1/2} v_{f0}\|_{L^2(\mathcal{F}_h^{0F})}^2 + \left\| \left(\frac{1}{c_f^2 \rho_f} \right)^{1/2} p_0 \right\|_{L^2(\mathcal{F}_h^{0F})}^2 \right. \\ &\quad \left. + \|\rho_s^{1/2} v_{s0}\|_{L^2(\mathcal{F}_h^{0S})}^2 + \|\underline{A}^{1/2} \underline{\sigma}_0\|_{L^2(\mathcal{F}_h^{0S})}^2 \right]^{1/2}. \end{aligned}$$

Well-posedness

$$|||(\nu_f - \nu_{fh}, p - p_h, \nu_s - \nu_{sh}, \underline{\underline{\sigma}} - \underline{\underline{\sigma}}_h)|||_{TDG}$$

$$\leq 3 \inf_{\substack{(\omega_f, q, \omega_s, \xi) \in \mathbf{V}(\mathcal{T}_h)}} |||(\nu_f - \omega_f, p - q, \nu_s - \omega_s, \underline{\underline{\sigma}} - \underline{\underline{\xi}})|||_{TDG^*}.$$

Implementation of the algorithm. Acoustic system

Implementation of the algorithm

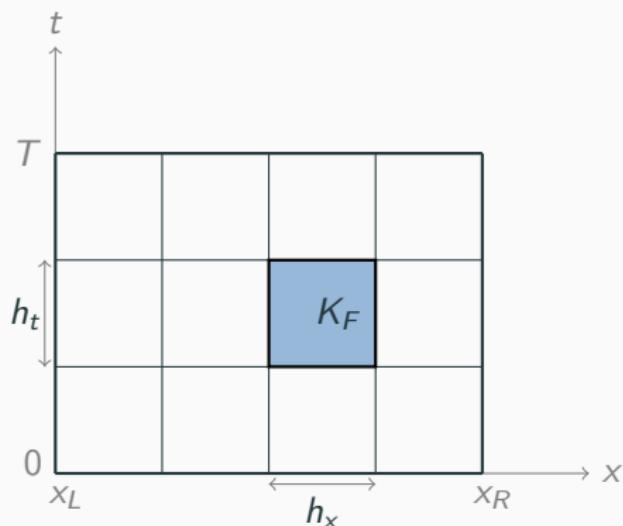


Figure 2: Rectangular mesh on $Q_F := [x_L, x_R] \times [0, T]$.

Implementation of the algorithm

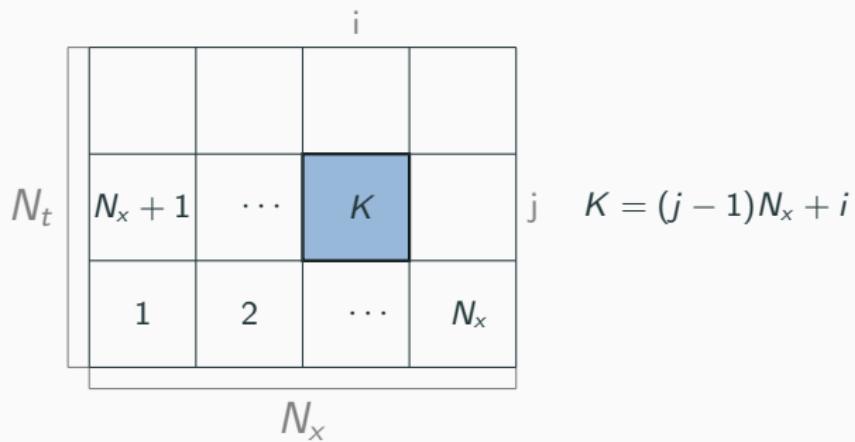


Figure 3: Element numbering.

Implementation of the algorithm

Seek $(v_{fh}, p_h) \subset \mathbf{V}(\mathcal{T}_h) \subset \mathbf{T}_F(\mathcal{T}_h)$ s.t. for all $(\omega_f, q) \subset \mathbf{V}(\mathcal{T}_h)$ it holds:

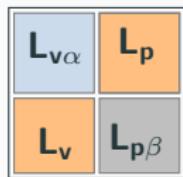
Treftz-DG formulation

$$\begin{aligned} & \int_{\mathcal{F}_h^{\Omega_F}} \left[\frac{1}{c_f^2 \rho_f} p_h^- [q]_t + \rho_f v_{fh}^- [\omega_f]_t \right] ds \\ & + \int_{\mathcal{F}_h^{I_F}} \left[\{p_h\} [\omega_f]_x + \{v_{fh}\} [q]_x + \alpha [v_{fh}]_x [\omega_f]_x + \beta [p_h]_x [q]_x \right] ds \\ & + \int_{\mathcal{F}_h^{T_F}} \left[\frac{1}{c_f^2 \rho_f} p_h q + \rho_f v_{fh} \cdot \omega_f \right] ds = \int_{\mathcal{F}_h^{0_F}} \left[\frac{1}{c_f^2 \rho_f} p_0 q + \rho_f v_{f0} \cdot \omega_f \right] ds. \end{aligned}$$

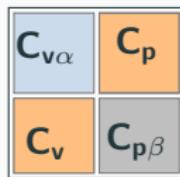
Implementation of the algorithm

Treftz-DG formulation. 1st layer

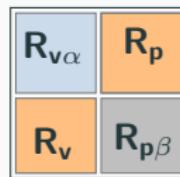
$$\begin{aligned} & [v_{K-1} \cdot L_{v\alpha} + p_{K-1} \cdot L_p] + [v_K \cdot C_{v\alpha} + p_K \cdot C_p] + [v_{K+1} \cdot R_{v\alpha} + p_{K+1} \cdot R_p] \\ & [v_{K-1} \cdot L_v + p_{K-1} \cdot L_{p\beta}] + [v_K \cdot C_v + p_K \cdot C_{p\beta}] + [v_{K+1} \cdot R_v + p_{K+1} \cdot R_{p\beta}] \end{aligned}$$



"Left" block



"Central" block



"Right" block

Implementation of the algorithm

Treftz-DG formulation. 1st layer

$$\begin{aligned} & [v_{K-1} \cdot \mathbf{L}_{v\alpha} + p_{K-1} \cdot \mathbf{L}_p] + [v_K \cdot \mathbf{C}_{v\alpha} + p_K \cdot \mathbf{C}_p] + [v_{K+1} \cdot \mathbf{R}_{v\alpha} + p_{K+1} \cdot \mathbf{R}_p] \\ & [v_{K-1} \cdot \mathbf{L}_v + p_{K-1} \cdot \mathbf{L}_{p\beta}] + [v_K \cdot \mathbf{C}_v + p_K \cdot \mathbf{C}_{p\beta}] + [v_{K+1} \cdot \mathbf{R}_v + p_{K+1} \cdot \mathbf{R}_{p\beta}] \end{aligned}$$



"Left" block



"Central" block



"Right" block

Implementation of the algorithm

$$\begin{bmatrix} \text{C} & \text{R} \\ \text{L} & \text{C} & \text{R} \\ \text{L} & \text{C} & \text{R} \\ \vdots & & \ddots & & \vdots \\ & & \text{L} & \text{C} & \text{R} \\ & & \text{L} & \text{C} \end{bmatrix} \cdot \begin{bmatrix} \text{U}_1 \\ \text{U}_2 \\ \text{U}_3 \\ \vdots \\ \text{U}_{N-2} \\ \text{U}_{N-1} \end{bmatrix} = \begin{bmatrix} \text{U}_1^0 \\ \text{U}_2^0 \\ \text{U}_3^0 \\ \vdots \\ \text{U}_{N-2}^0 \\ \text{U}_{N-1}^0 \end{bmatrix}$$

Implementation of the algorithm

$$\begin{bmatrix} \text{M} \\ 32N_x \times 32N_x \end{bmatrix} \cdot \begin{bmatrix} \text{U} \\ 1 \times 32N_x \end{bmatrix} = \begin{bmatrix} \text{U}^0 \\ 1 \times 32N_x \end{bmatrix}$$

Implementation of the algorithm

1. Initialization

$$\begin{bmatrix} M \end{bmatrix} ; \begin{bmatrix} \text{invM} \end{bmatrix} ; \begin{bmatrix} U^0 \end{bmatrix} ;$$

2. Propagation

For $t = 1 : N_t$

$$\begin{bmatrix} U \end{bmatrix} = \begin{bmatrix} \text{invM} \end{bmatrix} \cdot \begin{bmatrix} U^0 \end{bmatrix} ;$$

$$\begin{bmatrix} U^0 \end{bmatrix} = \begin{bmatrix} U \end{bmatrix} ;$$

end

Implementation of the algorithm

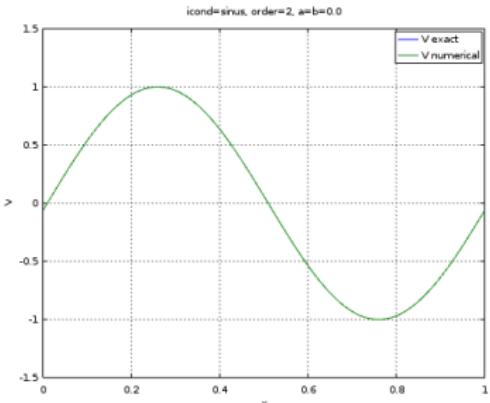


Figure 4: Exact and approximate solution for velocity $v_f(x, t)$, $t = \Delta t \times 1$, on the first time-layer ($\alpha = \beta = 0.5$).

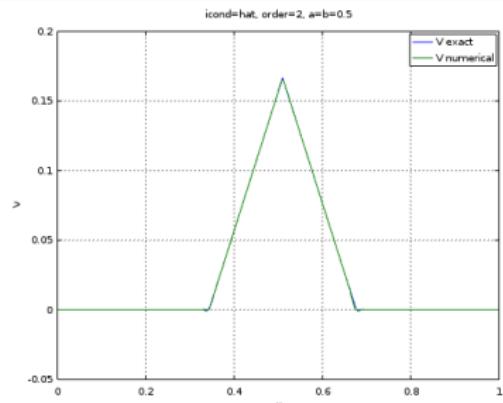


Figure 5: Exact and approximate solution for velocity $v_f(x, t)$, $t = \Delta t \times 1$, on the first time-layer ($\alpha = \beta = 0.5$).

Implementation of the algorithm

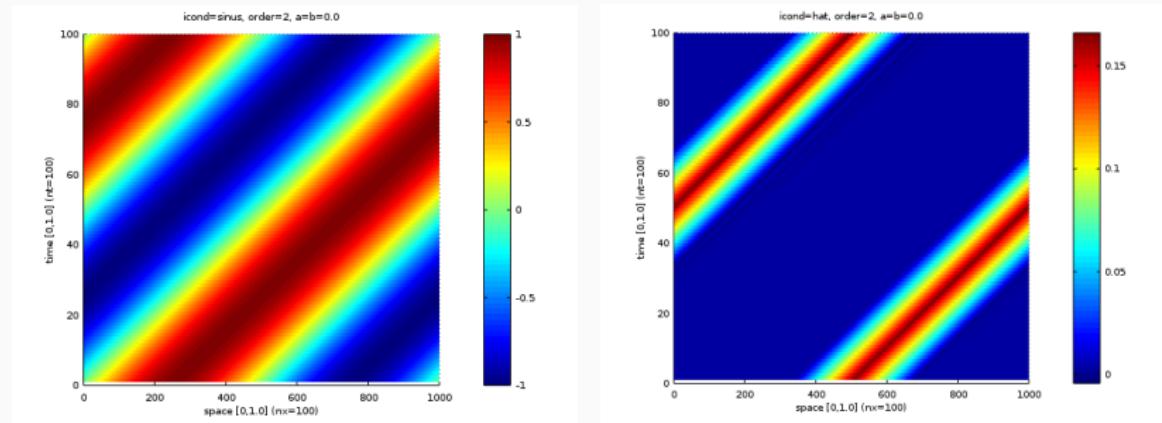


Figure 6: Time propagation of the approximate solution for velocity $v_f(x, t)$ ($\alpha = \beta = 0.5$).

Figure 7: Time propagation of the approximate solution for velocity $v_f(x, t)$ ($\alpha = \beta = 0.5$).

Implementation of the algorithm

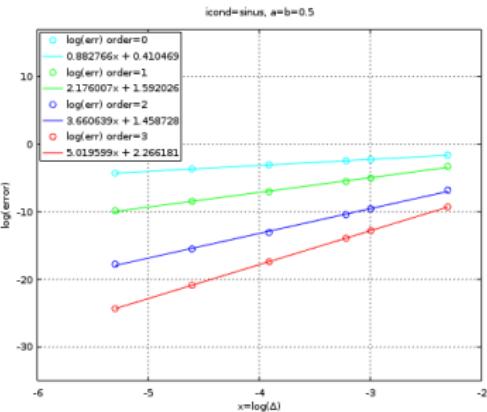


Figure 8: Convergence of numerical solution for velocity $v_f(x, t)$ in logarithmic scale ($\alpha = \beta = 0.5$).

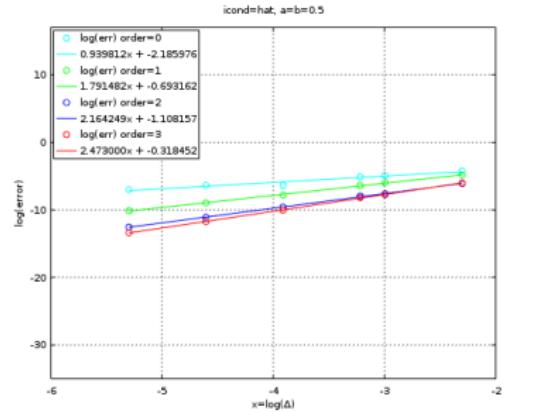


Figure 9: Convergence of numerical solution for velocity $v_f(x, t)$ in logarithmic scale ($\alpha = \beta = 0.5$).

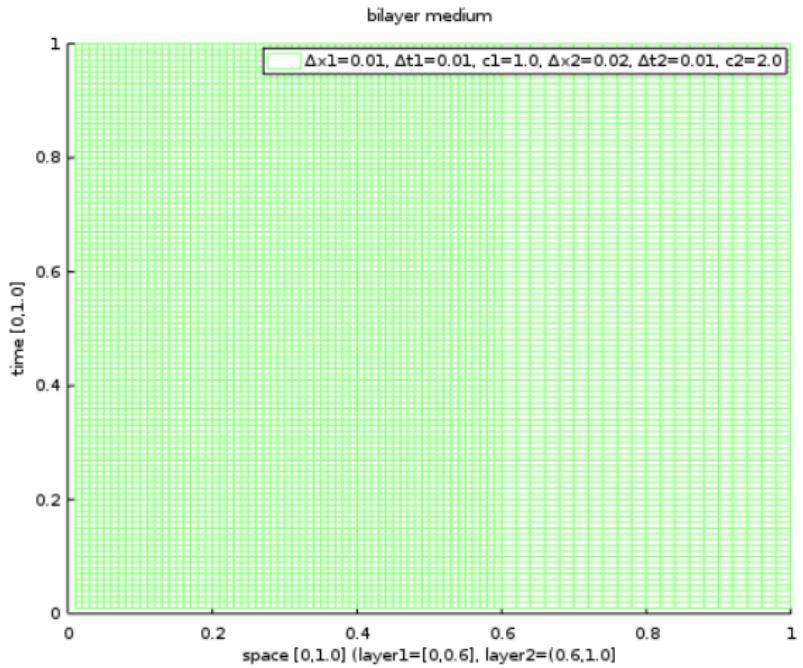


Figure 10: Mesh for heterogeneous case (bilayer medium).

$\Delta x_1 = 0.01, \Delta t_1 = 0.01, c_1 = 1.0, \Delta x_2 = 0.02, \Delta t_2 = 0.01, c_2 = 2.0$.

Implementation of the algorithm

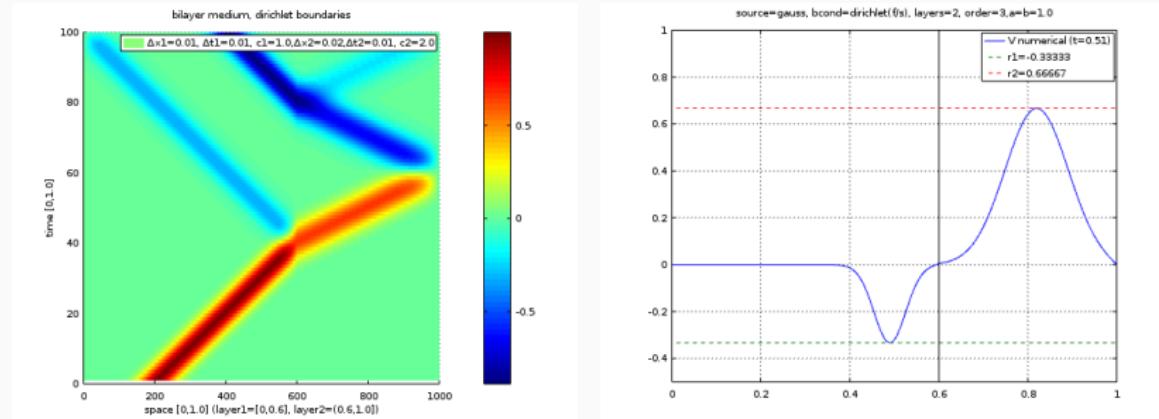


Figure 11: Approximate solution for velocity $v(x, t)$.

Figure 12: Approximate solution for velocity $v(x, t)$, $t = \Delta t \times 51$.

Conclusion

On-going work :



- Numerical implementation of the method for 1D AS ;
- Different choice of the basis functions ;
- Analysis of efficiency of the method / algorithm / code.

Perspectives :



- Numerical implementation of the method for ES 1D,2D,3D ;
- Numerical coupling 1D,2D,3D ;
- Coupling Trefftz-DG in solid with FVM in fluid,
integrating of the transmission layer.

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Thank you for your attention!

Questions?

Implementation of the algorithm

Polynomial waves basis functions

- Generating functions $g^v(x, t)$ and $g^p(x, t)$ - solution of wave equation if $a^2 = b^2$

$$g^v(a, b, x, t) = e^{i(ax+bc_f t)}, \quad g^p(a, b, x, t) = -c_f e^{i(bx+ac_f t)}.$$

- Taylor expansions for $g^v(x, t)$, $g^p(x, t)$ ($a^2 = b^2$):

$$e^{i(ax+bc_f t)} = \sum_{n=0}^{\infty} \sum_{k=0, k<2}^n Q_{n-k,k}^v(x, t) a^{n-k} b^k,$$

$$-c_f e^{i(bx+ac_f t)} = \sum_{n=0}^{\infty} \sum_{k=0, k<2}^n Q_{n-k,k}^p(x, t) a^{n-k} b^k.$$

$$R_{n-k,k}^v(x, t) = \Re(Q_{n-k,k}^v(x, t)), \quad I_{n-k,k}^v(x, t) = \Im(Q_{n-k,k}^v(x, t))$$

$$R_{n-k,k}^p(x, t) = \Re(Q_{n-k,k}^p(x, t)), \quad I_{n-k,k}^p(x, t) = \Im(Q_{n-k,k}^p(x, t)).$$

Implementation of the algorithm

Polynomial waves basis functions 1D

$$\phi_1^v = 0$$

$$\phi_5^v = -\frac{x^2}{2} - \frac{c_f^2 t^2}{2}$$

$$\phi_2^v = 1$$

$$\phi_6^v = -c_f x t$$

$$\phi_3^v = x$$

$$\phi_7^v = -\frac{x^3}{6} - \frac{x c_f^2 t^2}{2}$$

$$\phi_4^v = c_f t$$

$$\phi_8^v = -\frac{c_f^3 t^3}{6} - \frac{x^2 c_f t}{2}$$

$$\phi_1^p = -c_f$$

$$\phi_5^p = c_f^2 x t$$

$$\phi_2^p = 0$$

$$\phi_6^p = c_f \left(\frac{x^2}{2} + \frac{c_f^2 t^2}{2} \right)$$

$$\phi_3^p = -c_f^2 t$$

$$\phi_5^p = c_f \left(\frac{c_f^3 t^3}{6} + \frac{x^2 c_f t}{2} \right)$$

$$\phi_4^p = -c_f x$$

$$\phi_6^p = c_f \left(\frac{x^3}{6} + \frac{x c_f^2 t^2}{2} \right).$$

Implementation of the algorithm

Polynomial waves basis functions 2D

p = 0		
$\phi_1^{vx} = 0$ $\phi_2^{vx} = 1$	$\phi_1^{vy} = 0$ $\phi_2^{vy} = 1$	$\phi_1^p = -c_f$ $\phi_2^p = 0$
p = 1		
$\phi_3^{vx} = x$ $\phi_4^{vx} = c_f t$ $\phi_5^{vx} = y$	$\phi_3^{vy} = y$ $\phi_4^{vy} = x$ $\phi_5^{vy} = c_f t$	$\phi_3^p = -c_f(c_f t)$ $\phi_4^p = -c_f x$ $\phi_5^p = -c_f y$
p = 2		
$\phi_7^{vx} = -\frac{x^2}{2} - \frac{y^2}{2} - \frac{c_f^2 t^2}{2}$ $\phi_8^{vx} = -xy$ $\phi_9^{vx} = -c_f ty$ $\phi_{10}^{vx} = -c_f tx$	$\phi_7^{vy} = -xy$ $\phi_8^{vy} = -\frac{x^2}{2} - \frac{y^2}{2} - \frac{c_f^2 t^2}{2}$ $\phi_9^{vy} = -c_f tx$ $\phi_{10}^{vy} = -c_f ty$	$\phi_7^p = c_f(c_f tx)$ $\phi_8^p = c_f(c_f ty)$ $\phi_9^p = c_f(xy)$ $\phi_{10}^p = c_f(\frac{x^2}{2} + \frac{y^2}{2} + \frac{c_f^2 t^2}{2})$
p = 3		
$\phi_{11}^{vx} = -\frac{x^2 y}{2} - \frac{c_f^2 t^2 y}{2} - \frac{y^3}{6}$ $\phi_{12}^{vx} = -c_f txy$ $\phi_{13}^{vx} = -\frac{c_f^3 t^3}{6} - \frac{c_f t x^2}{2} - \frac{c_f t y^2}{2}$ $\phi_{14}^{vx} = -\frac{xy^2}{2} - \frac{c_f^2 t^2 x}{2} - \frac{x^3}{6}$	$\phi_{11}^{vy} = -\frac{xy^2}{2} - \frac{c_f^2 t^2 x}{2} - \frac{x^3}{6}$ $\phi_{12}^{vy} = -\frac{c_f^3 t^3}{6} - \frac{c_f t x^2}{2} - \frac{c_f t y^2}{2}$ $\phi_{13}^{vy} = -c_f ty$ $\phi_{14}^{vy} = -\frac{x^2 y}{2} - \frac{c_f^2 t^2 y}{2} - \frac{y^3}{6}$	$\phi_{11}^p = c_f(c_f txy)$ $\phi_{12}^p = c_f(\frac{x^2 y}{2} + \frac{c_f^2 t^2 y}{2} + \frac{y^3}{6})$ $\phi_{13}^p = c_f(\frac{xy^2}{2} + \frac{c_f^2 t^2 x}{2} + \frac{x^3}{6})$ $\phi_{14}^p = c_f(\frac{c_f^3 t^3}{6} + \frac{c_f t x^2}{2} + \frac{c_f t y^2}{2})$