Abstract
We are interested in the formal specification of safety properties of robot manipulators down to the mathematical physics. To this end, we have been developing a formalization of the mathematics of rigid body transformations in the Coq proof assistant. It can be used to address the forward kinematics problem, i.e., the computation of the position and orientation of the end-effector of a robot manipulator in terms of the link and joint parameters. Our formalization starts by extending the Mathematical Components library with a new theory for angles and by developing three-dimensional geometry. We use these theories to formalize the foundations of robotics. First, we formalize a comprehensive theory of three-dimensional rotations, including exponentials of skew-symmetric matrices and quaternions. Then, we provide a formalization of the various representations of rigid body transformations: isometries, homogeneous representation, the Denavit-Hartenberg convention, and screw motions. These ingredients make it possible to formalize robot manipulators: we illustrate this aspect by an application to the SCARA robot manipulator.

Categories and Subject Descriptors F.4.1 [Mathematical Logic]: Mechanical theorem proving; I.1.1 [Expressions and Their Representation]: Representations (general and polynomial); I.2.9 [Robotics]: Manipulators; I.3.5 [Computational Geometry and Object Modeling]: Hierarchy and geometric transformations

Keywords Formal verification, Proof-assistant, Coq, Robot manipulator, 3D Geometry

1. Towards a Formal Theory of Robotics
Our ultimate goal is the formal verification of safety properties of robots. Robots have already made their way in manufacturing plants into the form of robot manipulators on assembly lines. Even though industrial robots operate in a controlled environment, their safety is already subject to several standards (e.g., ISO 10218). Safety concerns will increase furthermore now that robots are considered for life-critical missions such as rescue and health care. It is therefore questionable whether testing can achieve a satisfactory level of safety. For this reason, formal methods are now being considered to improve the rigor of the safety analysis of robots (see Sect. 11 for examples).

In this paper, we focus on the formal verification of the theoretical foundations of robot manipulators. Robot manipulators are an interesting target because they are already pervasive in the industry and because advanced robots such as humanoid robots can be understood as made of several robot manipulators. The theoretical foundations of robot manipulators is a matter of three-dimensional geometry. This suggests the use of a proof-assistant to perform formal verification. Indeed, proof-assistants excel at the symbolic manipulation of formal algebra, in which geometry can be represented. We use the Coq proof-assistant (The Coq Development Team 1999–2016) and the Mathematical Components library. The latter is a library for formal algebra that was developed to formalize the odd order theorem, an important theorem in group theory (Gonthier et al. 2013). This library features in particular a formalization of linear algebra that we use as a starting point. We discuss in Sect. 12 the consequences of this choice.

Let us give a concrete idea of the kind of three-dimensional geometry robotics is dealing with. A robot manipulator consists of (rigid) links connected by joints. It is modeled by the relative positioning of frames attached to its links. Figure 1 is a schematic representation of the SCARA robot manipulator for pick-and-place operations. It consists of fives links (from the base link 0 to the end-effector—link 4) connected by four joints. The first two joints and the last one
are revolute joints: they can rotate around one axis (here, the vertical axis). The third joint is a prismatic joint: it can translate along one axis (here, the vertical axis). The rotation angles and the translation distance are called the joint parameters.

Reasoning about a robot manipulator amounts to determining the relative positions of links. To do so, one attaches frames to links, often near a joint, as illustrated in Fig. 2.

The orientation of a frame w.r.t. another is expressed by a rotation. For example, the orientation of frame 1 w.r.t. frame 0 can be expressed by the following matrix \( R^0 = \begin{bmatrix} \cos \theta_1 & -\sin \theta_1 & 0 \\ \sin \theta_1 & \cos \theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \), where \( \theta_1 \) is the angle of the rotation of the first joint. The relative position of a frame w.r.t. another is expressed by a rigid body transformation. For example, the coordinate of the origin of the frame 1 w.r.t. frame 0 is \( P^0 = [a_1 \cos \theta_1; a_1 \sin \theta_1; 0]^T \). The relative position of frame 1 w.r.t. frame 0 is \( A^0 = \begin{bmatrix} 1^T \end{bmatrix} \) (row-vector convention).

Once the relative positions of links are determined, we can address the forward kinematics problem, i.e., determining the position and orientation of the end-effector of a robot manipulator given the link and joint parameters. For our example, given the relative positions \( A^1 \) of the consecutive frames, the position and orientation of the end-effector are given by the matrix product \( A^3 A^2 A^1 A^0 \). We further develop this example in Sections 8.3, 9.2, and 10.3.

The main contribution of this paper is to provide a formalization of angles and trigonometric functions on top of real closed fields \( \mathbb{R}(\mathbb{R}) \). We assume some familiarity with the Coq proof-assistant. We display formal definitions verbatim, using only a few standard non-ASCII characters to ease reading. We explain notations when they are introduced. Let us recall some notations of the Mathematical Components library for convenience:

- \( a \times v \) \( v \) scaled by \( a \)
- \( x++ n \) \( n \) times \( x \)
- \( x^{n} \) to the \( n \)th power
- \( \|x\| \) norm of \( x \)
- \( x \{A \} \) satisfies the predicate \( A \)
- \( n\%\cdot R \) notation for \( \lfloor n \rceil \)
- \( x == y \) boolean equality between \( x \) and \( y \)

### 2. Angles and Trigonometric Functions

We start with an overview of a formalization of angles and trigonometric functions that we have been developing in particular to define formally rotations (in Sect. 4). Due to technical reasons explained in Sect. 12, we formalize angles and trigonometric functions on top of real closed fields from the Mathematical Components library (Cohen et al., 2012b).

#### 2.1 Formalization of Angles using Complex Numbers

Angles are defined using algebraic complex numbers of unit norm. In Mathematical Components, the type of complex numbers is \( \mathbb{R}^1 \), where \( \mathbb{R} \) is a real closed field (rcfType). An angle is defined as a complex number with norm 1:

\[
\begin{align*}
\theta &\in \mathbb{R}^1, \\
\theta &\sim \{a, b, c, d\}.
\end{align*}
\]
The foundations of robot manipulators make use of a number by convention.

### 2.2 Trigonometric Functions and Trigonometric Laws

Angles, the functions \(\cos\), \(\sin\), \(\tan\), and \(\cot\) need to be equipped with many trigonometric laws. We formalized in particular the cancellation laws between each trigonometric function and its inverse, and many trigonometric identities (Pythagorean identity, double-angle and half-angle formulas, law of cosines and of sines, etc.).

<table>
<thead>
<tr>
<th>function</th>
<th>formalization in CoQ</th>
<th>pencil-and-paper definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\arcsin(x))</td>
<td>(\arg \left(\sqrt{1 - x^2} + ix\right))</td>
<td>(\arg\sqrt{1 - x^2 + ix})</td>
</tr>
<tr>
<td>(\arccos(x))</td>
<td>(\arg \left(x + i\sqrt{1 - x^2}\right))</td>
<td>(\arg(x + i\sqrt{1 - x^2}))</td>
</tr>
<tr>
<td>(\arctan(x))</td>
<td>(\text{if } x == 0 \text{ then } 0 \text{ else } \arg((x^-1 + i) = \text{sgn}(x)))</td>
<td>(\arg(\text{sgn}(x)(\frac{1}{2} + i))) (0 if (x = 0))</td>
</tr>
</tbody>
</table>

Table 1: Formalization of inverse trigonometric functions

Variable \(R : \text{rcfType}\).

Record angle := Angle { expi : R[1]
_ : 1 | expi | == 1 }.

In particular, the argument of a complex number defines an angle:

**Definition** \(\text{arg} : (x : \text{R[1]}) \rightarrow \text{insubd angle0 (x \div 1 \times 1)}\).

(angle0 is the 0 angle corresponding to the complex number 1; insubd is a Mathematical Components operator that tries to coerce to a sub-type and returns the given default value if it fails; division by 0 in Mathematical Components outputs 0 by convention.)

We show that angles form an additive group (so that we can scale angles to obtain, e.g., double-angles in Sect. 6.2 lemma \(\text{quat_rot_isRot}\)) and we define half-angles (used, e.g., in Sect. 10.2 lemma \(\text{etwist_is_onto_SE}\)). Specific angles are defined as the arguments of specific complex numbers. For example, we define \(\pi\) as the argument of \(-1\):

**Definition** \(\pi := \text{arg} (-1)\).

#### 2.2.1 Variable R : rcfType.

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**Definition** \(\pi := \text{arg} (-1)\).

#### 2.2.4 Technical definition:

**Definition** \(\cos(a) := \text{Re (expi a)}\)

**Definition** \(\sin(a) := \text{Im (expi a)}\)

**Definition** \(\tan(a) := \sin a / \cos a\)

**Definition** \(\cot(a) := (\tan a)^{-1}\)

Inverse trigonometric functions are formalized using their definition in terms of complex numbers (see Table 1).

To be useful, our formalization of trigonometric functions needs to be equipped with many trigonometric laws. We formalized in particular the cancellation laws between each trigonometric function and its inverse, and many trigonometric identities (Pythagorean identity, double-angle and half-angle formulas, law of cosines and of sines, etc.).

#### 3. Basics of Three-dimensional Geometry

##### 3.1 Extended Support to Reason with Vectors

Vectors are provided by the Mathematical Components library in the form of a type denoted by \(\text{v}[\text{R}][n]\) whose elements are row-vectors of length \(n\) with components of type \(R\).

We distinguish a few specific vectors to simplify notations. In particular, we denote by \(\text{e}_0\), \(\text{e}_1\), and \(\text{e}_2\) the vectors of the canonical basis. Concretely, \(\text{e}_k\) is formalized by the row-vector \([\delta_{0k}; \delta_{1k}; \delta_{2k}]\), where \(\delta\) is Kronecker symbol. Also, to construct a row-vector from its components, we resort to the \(\text{row}\) constructor \((a; b; c)\).

##### 3.1.1 The Dot-Product

We denote by \(u \ast v\) the dot-product of the vectors \(u\) and \(v\). It is formally defined by the only component of the \(1 \times 1\)-matrix \(u \ast v\), \(v^\top\), where \(\ast\) is matrix multiplication and \(^\top\) is matrix transpose. The dot-product is used to define a number of geometrical constructs. Let us introduce the formalization of a few of them that we will use in the forthcoming sections.

The norm of a vector \(u\) is defined using the square root of the dot-product \((u \cdot u)\):

**Definition** \(\text{norm} u := \text{Num.sqrt} (u \ast u)\).

A vector is normalized by scaling it to be of unit norm:

**Definition** \(\text{normalize} v := (\text{norm}\ v)\sqrt{-1 \ast v}\).

The normal component of a vector is defined using the dot-product (see Fig. 3):

**Definition** \(\text{normalcomp} v u := v - \text{normalize} u \ast u \ast v \ast \text{normalize} u\).

Figure 3: The normal component of \(v\) w.r.t. \(u\)

##### 3.1.2 The Cross-Product

The cross-product of two vectors \(u\) and \(v\) can be defined using determinants:

\[
\mathbf{u} \times \mathbf{v} = \begin{vmatrix}
1 & 0 & 0 \\
v_0 & v_1 & v_2 \\
0 & 1 & 0 \\
v_0 & v_1 & v_2 \\
0 & 0 & 1 \\
v_0 & v_1 & v_2 \\
\end{vmatrix}
\]

\[\mathbf{e}_0 + \mathbf{e}_1 + \mathbf{e}_2\]
We use the determinant \( \det \) of the Mathematical Components library to formalize the above formula:

**Definition** crossmul \( u, v \) := \\
row(\langle k < 3 \rangle) \det(c_{1,3} \; e_k \; u \; v).

\( e_k \) are the vectors of the canonical basis introduced in Sect. 3.1. \( c_{1,3} \) is the concatenation of three row-vectors into a \( 3 \times 3 \)-matrix. We denote \( \text{crossmul} \; u \; v \) by \( u \ast_v v \).

The cross-product is useful to define a number of geometrical constructs, such as colinearity:

**Definition** colinear \( u \; v \) := \( u \ast_v v = 0 \).

The cross-product is also used to build frames (see Sect. 3.4).

The pervasive usage of the cross-product in robotics naturally calls for a rich library. We formalized the expected standard lemmas. For illustration, the double cross-product is a technical but useful lemma about the cross-product and the standard lemmas. For illustration, the double cross-product is used in the proof of Rodrigues\(^\text{P}\) in Sect. 5.1 and to deal with quaternions in Sect. 6.2:

**Lemma** double_crossmul \( u \; v \; w \) := \\
\( u \ast_v (v \ast_w w) := (u \ast_{v \; w} w) \ast_v v - (u \ast_{v \; w} w) \ast_v w \).

### 3.2 Formalization of Lines

**Definition** colinear \( p \in \) vectors are colinear, perpendicular \( p \in \) vectors

\( p \in \) vectors is \( 0 \) means that the dot-product of the latter case, there is a constructive proof to find out the intersection point:

**Definition** is_interpoint \( p \in \) l1 \( \cap \) l2 := \\
(\langle \text{line} \rangle (\langle l1 \rangle ) \quad \& \quad (\langle \text{line} \rangle (\langle l2 \rangle ))

\( \langle \text{line} \rangle \) is defined as follows:

**Lemma** intersects_interpoint \( p \in \) l1 \( \cap \) l2 := \\
(\langle \text{line} \rangle (\langle l1 \rangle ) \quad \& \quad (\langle \text{line} \rangle (\langle l2 \rangle ))

### 3.3 Orthogonal and Rotation Matrices

A matrix \( M \) such that \( MMM^T = 1 \) is orthogonal. An orthogonal \( 3 \times 3 \)-matrix is a rotation matrix when \( \det(M) = 1 \). Let us denote by \( '0'[R]_n \) the set of orthogonal matrices and by \( 'SO'[R]_3 \) the set of rotation matrices (formal definition omitted here).

For example, \( R_x(a) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(a) & \sin(a) \\ 0 & -\sin(a) & \cos(a) \end{bmatrix} \) is a rotation matrix. It is meant to represent a rotation of \( \alpha \) around the canonical vector \( [1 \; 0 \; 0] \) (see Sect. 4). It is formally defined as follows (\( R_y(a) \) and \( R_z(a) \) are defined similarly):

**Definition** \( R_x a := c_{1,3} \)

\( e_0 \) (row3 \( [0 \; \cos(a) \; \sin(a)] \)) (row3 \( [- \; \sin(a) \; \cos(a)] \)).

### 3.4 Formalization of Frames

As highlighted in the introduction, frames are important to model robot manipulators. We define a non-oriented frame with an orthogonal matrix (Sect. 3.3):

**(* Module NOFrame *)**

\( t := \text{mk} \{ \text{M} := 'M'[R]_3 ; \text{MO} : \text{M} \is '0'[R]_3 \} \).

Let \( f \) be a non-oriented frame. We denote by \( i \) (resp. \( j, k \)) the first (resp. second, third) row of the matrix \( (M \; f) \). The vectors \( i, j, \) and \( k \) are pairwise orthogonal unit vectors and we regard them as forming the frame \( f \). The sign of a non-oriented frame \( f \) is defined by \( \det(M \; f) \) (which is equivalent to \( i \ast_{v\; w} (j \ast_{v\; w} k) \) and can only be \( 1 \) or \( -1 \). When it is \( 1 \), the frame is positive (or right-handed):

**(* Module Frame *)**

\( t := \text{mk} \{ \text{noframe}_\text{of} := \text{NOFrame.r} \; \text{R} ; \text{MO} : \text{NOFrame} \; \text{M} \is '0'[R]_3 \} \).

In particular, we denote by \( \text{can}_\text{frame} \; R \) the (positive) frame consisting of the canonical vectors \( e_0, e_1, \) and \( e_2 \). We may add an origin to positive frames:

**(* Module TFrame *)**

\( t := \text{mk} \{ \text{tFrame}_\text{of} := \text{Frame.r} \; \text{T} ; \text{tFrame} := \text{Frame} \; \text{M} \} \).

A frame with an origin defines three line-axes. The \( x \)-axis of a frame is defined as follows (and similarly for other axes):

**Definition** xaxis \( R \; f := \text{Line.mk} \; (\text{TFrame.o} \; f) \; \text{NOFrame.i} \; f \).

**Frame Built from a Non-Zero Vector** We will need to build positive frames given one non-zero vector (for example to define rotations in Sect. 3.3). For this purpose, we provide a module \( \text{Base} \) that features two functions \( j \) and \( k \) such that, given a unit vector \( i \), the vectors \( i, j, \) and \( k \) form a positive frame. Concretely, the function \( j \) is defined as follows:

**(* Module Base *)**

\( j := \text{if} \; \text{colinear} \; i \; \text{then} \; e_1 \; \text{else} \; \text{normalize} \; (\text{normalcomp} \; e_0) \).
If $i$ is colinear with $e_0$ then the desired frame can be built using the vectors of the canonical basis. Otherwise, we compute the normal component of $e_0$ w.r.t. $i$ and we normalize to a unit vector. As for the function $k$ and $i$, it is simply defined by the cross-product of $i$ and $j$: this guarantees that the frame is positive.

**Rotation Between Frames** A rotation matrix can be regarded as a transformation from one frame to another. For example, multiplication by the matrix $R_x(a)$ (Sect. 3.3) transforms a vector whose coordinates are expressed in the frame $\langle 1;0;0 \rangle$, $[0;\cos(a);\sin(a)], [0; -\sin(a); \cos(a)]$ to the same vector but with coordinates expressed in the canonical frame. More generally, let us denote by $A \cdot R \cdot B$ the rotation matrix that transforms a vector expressed in frame $A$ into a vector expressed in frame $B$; it is defined as follows:

**Definition** FromTo $R \cdot (A \cdot B) := \cdot row \cdot i \cdot (A \cdot B)$.

The matrix $A \cdot R \cdot B$ is a rotation matrix and represents the vectors of the frame $A$ in terms of the vectors of the frame $B$ (or, in other words, the relative position of the frame $A$ w.r.t. the frame $B$, providing their origins coincide).

### 4. Rotations and Rotation Matrices

In this section, we formally define three-dimensional rotations and show the equivalence with rotation matrices.

#### 4.1 Definition of Rotations

A rotation of angle $a$ around vector $u$ is a linear function $f$ such that $f(u) = u$ (in other words, the vector-axis is invariant by rotation), $f(j) = \cos(a)j + \sin(a)k$, and $f(k) = -\sin(a)j + \cos(a)k$, where the vectors $\frac{1}{||u||}u$, $j$, and $k$ form a positive frame.

We build the positive frame from $u$ by means of the functions $Base.j$ and $Base.k$ explained in Sect. 3.4 Using these functions, the formal definition of a rotation becomes:

**Definition** $isRot \cdot a \cdot u \cdot f \cdot \langle \text{linear \ 'rV_3' \ \rightarrow \ 'rV_3' \rangle \rangle$:

\[
\begin{align*}
&\text{let } j := \text{Base.j} \quad \text{u in let } k := \text{Base.k} \quad \text{u in} \\
&\quad &\text{[& u \equiv u, } \\
&\quad &f \ j \equiv \cos \ a \ = \ j + \sin \ a \ := \ k \ & \text{f k} \equiv -\sin \ a \ = \ j + \cos \ a \ := \ k].
\end{align*}
\]

#### 4.2 Equivalence between Rotations and Rotation Matrices

We show that any rotation (in the sense of Sect. 4.1) can be represented by a rotation matrix:

**Lemma** $\text{isRot} \cdot a \cdot u \cdot f \cdot u := 0 \rightarrow \text{isRot} \cdot a \cdot f \rightarrow \text{lin1mxf} \ \langle \text{is 'SO[R_3]' } \rangle$.

\((\text{lin1mxf} \ f \) is the matrix corresponding to the linear application $f$.)

**Proof:** The idea of the proof is to express $\text{lin1mxf}$ as the product $P^{-1} \cdot R_x(a) \cdot P$. $R_x(a)$ was defined in Sect. 3.3; $P$ is built using the vectors of the frame (normalize $u, j, k$) (see Sect. 3.4) expressed in the canonical basis.

Conversely, given a rotation matrix, one can always find a vector-axis and an angle such that the corresponding linear application is a rotation. The vector-axis in question is provided by Euler’s theorem:

**Lemma** $euler \cdot M := \langle M[R_3] \rangle : \ \langle \text{is 'SO[R_3]' } \rangle \rightarrow \langle x := \text{rV[R_3]} \ (x \ != \ 0) \ & \& (x \ * \ m \ = \ x) \rangle$.

Since the proof of Euler’s theorem is constructive, we can use it to define the vector-axis $\text{vaxis euler} \cdot M$ of a rotation matrix $M$.

Finally, we prove the existence of an angle such that any rotation matrix is a rotation in the sense of Sect. 4.1:

**Lemma** $\text{SO} \_3 \_isRot \_M := \langle M \ \langle \text{is 'SO[R_3]' } \rangle \rightarrow \exists \ a \ . \ \text{isRot} \_a \ (\text{vaxis euler} \_M) \ (\text{mxlin1} \ M)$.

(\text{mxlin1} \ turns \ a \ matrix \ into \ the \ corresponding \ linear \ function.)

**Proof:** Let $i$ be $\text{normalize (vaxis euler} \_M)$. We use $i$ to build a positive frame formed by the vectors $i$, $j$, and $k$ (see Sect. 3.4). We then inspect the result of the action of the matrix $M$ on the frame vectors. Concretely, it happens that $jM$ and $kM$ can be written as follows: $jM = \cos(a)j + \sin(a)k$ and $kM = -\sin(a)j + \cos(a)k$. $M$ is thus a rotation of angle $a$ around $i$.

### 5. Exponential Coordinates for Rotations

Rotation matrices can be represented using exponentials of skew-symmetric matrices. This is the representation used when dealing with rigid body transformations represented as screw motions (Sect. 10). In general, the exponential of a matrix is a power series. For skew-symmetric matrices, there is a closed expression: $e^{\text{SO}(w)} = 1 + [\sin(a)]S(w) + (1 - \cos(a))S(w)^2$, where $a$ is an angle and $S(w)$ is the skew-symmetric matrix corresponding to vector $w$. In this section, we show that rotations can be represented by this formula.

#### 5.1 Skew-Symmetric Matrices and their Exponentials

We first formalize the exponential of a skew-symmetric matrix. Let $\text{so[R]} \_n$ be the set of skew-symmetric matrices, i.e., the matrices $M$ such that $M = -M^T$. To a vector $[w_x; w_y; w_z]$ corresponds the (row-vector convention) skew-symmetric matrix $[\begin{bmatrix} 0 & -w_z & w_y \\ w_z & 0 & -w_x \\ -w_y & w_x & 0 \end{bmatrix}]$. Hereafter, we denote the skew-symmetric matrix corresponding to the vector $w$ by $\S(w)$. Let us define the following function (where $a$ is an angle and $M$ is a $3 \times 3$-matrix):

**Definition** $\text{emx3} \ a \ := \ I + \sin \ a \ := \ M + (1 - \cos \ a) \ := \ M^{\times} + 2$.

We are interested in the case where the matrix $M$ is $\S(w)$ for some $w$. We denote such a matrix by $e^a(a, w)$ and call $a$ and $w$ the exponential coordinates.
**Rodrigues’ Formula**  Rodrigues’ formula is a classical result that provides an alternative expression for the multiplication by a matrix \( e^\theta(a, w) \). It transforms a vector \( v \) according to the angle \( a \) and the vector \( w \) as follows:

\[
\text{Definition } \text{rodrigues } u \ w := \cos a \ v + (1 - \cos a) \ (u \times w) \ v + \sin a \ (w \times u).
\]

It performs the same transformation as multiplication by \( e^\theta(a, w) \):

\[
\text{Lemma } \text{rodriguesP } u \ w : \text{norm } w = 1 \rightarrow \text{rodrigues } u \ w = u + m \ e^\theta(a, w).
\]

**Proof:** The proof is by appealing to the properties of skew-symmetric matrices and of the cross-product, in particular the double cross-product (see Sect. 5.1.2).

### 5.2 Equivalence between Rotations and Exponential Coordinates

Any rotation matrix can be represented by its exponential coordinates: \( e^\theta(a, \ w) \). This definition is justified by the following lemma that shows the relation between the trace \( \text{tr} \) of a rotation and the cosine of the rotation angle:

\[
\text{Lemma } \text{mxtrace_isRot } a \ u \ M : u != 0 \rightarrow \text{isRot } a \ u \ (\text{mx_lin1 } M) \ \text{tr } M = 1 + \cos a *+ 2.
\]

**Proof:** The proof is by expressing \( M \) by means of a rotation around the canonical vector \( e_0 \) and by appealing to the properties of the trace.

The vector-axis of the angle-axis representation of \( M \) is defined using the axial vector (row-vector convention):

\[
w = [M_{1,2} - M_{2,1}; M_{2,0} - M_{0,2}; M_{0,1} - M_{1,0}].
\]

Given an axial vector \( \mathbf{w} \), the vector-axis of the angle-axis representation is defined by \( 1 / (\sin(a)) \ \mathbf{w} \), where \( a \) is the angle of the angle-axis representation defined just above. We extend this definition to the case \( a = 0 \) by resorting to the vector-axis defined in Sect. 4.2. When \( a = 0 \), the rotation is the identity and the vector-axis matters less:

\[
(* \text{ Module Aa } *) \quad \text{Definition } \text{vaxis} \ M : \text{vaxis } \mathbf{R}[3] \rightarrow \text{let } a := \text{angle } M \text{ in } \begin{cases} \text{if } a \neq \pi \text{ then } \text{vaxis_euler } M \\ \text{else } 1/((\sin(a)) *+ 2) *: \text{axial_vec } M. \end{cases}
\]

We can prove that the above definitions are correct, in the sense that a rotation matrix \( M \) indeed represents a rotation of angle \( \text{Aa} . \angle M \) around the vector \( \text{Aa} . \text{vaxis } M \):

\[
\text{Lemma } \text{angle_axis_eskew } M : M \ \text{is } \text{SO}[3] \rightarrow M = e^\theta(\text{Aa } M, \text{normalize } (\text{vaxis_euler } M)).
\]

**Proof:** When \( \text{Aa} . \angle M = 0 \), \( M = 1 \), and so is \( e^\theta(a, w) \) for any \( w \). When \( \text{Aa} . \angle M = \pi \), \( \text{Aa} . \text{vaxis } M = u = \text{vaxis_euler } M \) and both the lhs and the rhs can be proven to be \( u^T *+ _m u *+ 2 - 1 \). Otherwise, we distinguish whether the angle of the rotation is \( \text{Aa} . \angle M \) or \( -\text{Aa} . \angle M \). In the former case, normalize \((\text{vaxis_euler } M)\) is

\[
(1 / (\sin(\text{Aa} . \angle M)) *+ 2)) *: \text{axial_vec } M
\]

and the lhs and the rhs can be proven to be the same rotation. The latter case is similar.

### 6. Other Representations of Rotations

For the sake of completeness, we discuss the formalization of other representations of rotations that are commonly used in robotics. This demonstrates the usefulness of the formal theories we have introduced so far.

#### 6.1 Angle-Axis Representation

The angle-axis representation of a rotation matrix \( M \) is a pair of an angle and a vector that are computed directly from \( M \). In this section, we formalize the definition of the angle-axis representation and prove its correctness.

The angle of the angle-axis representation of \( M \) is defined as follows:

\[
(* \text{ Module Aa } *) \quad \text{Definition } \text{angle} M := \text{acos } (\text{tr } M - 1) / 2 \% : \text{R}.
\]

This definition is justified by the following lemma that shows the relation between the trace \( \text{tr} \) of a rotation and the cosine of the rotation angle:

\[
\text{Lemma } \text{mxtrace_isRot } a \ u \ M : u != 0 \rightarrow \text{isRot } a \ u \ (\text{mx_lin1 } M) \ \text{tr } M = 1 + \cos a *+ 2.
\]

**Proof:** The proof is by expressing \( M \) by means of a rotation around the canonical vector \( e_0 \) and by appealing to the properties of the trace.

The vector-axis of the angle-axis representation of \( M \) is defined using the axial vector (row-vector convention):

\[
w = [M_{1,2} - M_{2,1}; M_{2,0} - M_{0,2}; M_{0,1} - M_{1,0}].
\]

Given an axial vector \( w \), the vector-axis of the angle-axis representation is defined by \( 1 / (\sin(a)) w \), where \( a \) is the angle of the angle-axis representation defined just above. We extend this definition to the case \( a = 0 \) by resorting to the vector-axis defined in Sect. 4.2. When \( a = 0 \), the rotation is the identity and the vector-axis matters less:

\[
(* \text{ Module Aa } *) \quad \text{Definition } \text{vaxis} M : \text{vaxis } \mathbf{R}[3] \rightarrow \text{let } a := \text{angle } M \text{ in } \begin{cases} \text{if } a \neq \pi \text{ then } \text{vaxis_euler } M \\ \text{else } 1/((\sin(a)) *+ 2) *: \text{axial_vec } M. \end{cases}
\]

We can prove that the above definitions are correct, in the sense that a rotation matrix \( M \) indeed represents a rotation of angle \( \text{Aa} . \angle M \) around the vector \( \text{Aa} . \text{vaxis } M \):

\[
\text{Lemma } \text{angle_axis_eskew } M : M \ \text{is } \text{SO}[3] \rightarrow M = e^\theta(\text{Aa } M, \text{normalize } (\text{vaxis_euler } M)).
\]

**Proof:** When \( \text{Aa} . \angle M = 0 \), \( M = 1 \), and so is \( e^\theta(a, w) \) for any \( w \). When \( \text{Aa} . \angle M = \pi \), \( \text{Aa} . \text{vaxis } M = u = \text{vaxis_euler } M \) and both the lhs and the rhs can be proven to be \( u^T *+ _m u *+ 2 - 1 \). Otherwise, we distinguish whether the angle of the rotation is \( \text{Aa} . \angle M \) or \( -\text{Aa} . \angle M \). In the former case, normalize \((\text{vaxis_euler } M)\) is

\[
(1 / (\sin(\text{Aa} . \angle M)) *+ 2)) *: \text{axial_vec } M
\]

and the lhs and the rhs can be proven to be the same rotation. The latter case is similar.

#### 6.2 Rotations Using Quaternions

Quaternions provide a concise (4 elements as opposed to the 9 elements of a matrix) and computationally-efficient way to represent rotations. They can be thought of as a generalization of complex numbers. Indeed, they have a scalar part and a vector part:

\[
\text{Record } \text{quat} := \text{mkQuat } \{ \text{quat1 } : \text{R} ; \text{quat } : \text{vR}[3] \}.
\]

Let us denote the scalar (resp. vector) part of the quaternion \( a \) by \( a' \) (resp. \( a' \)).

Rotations are expressed by means of quaternions with norm 1 (i.e., unit quaternions). The norm of a quaternion is defined as follows:

\[
\text{Definition } \text{sqrq } a := a' *+ 2 + \text{norm } (a'1) *+ 2.
\]

\[
\text{Definition } \text{norm } a := \text{Num.sqr } (\text{sqrq } a).
\]

Let \( \text{quat} \) be the set of unit quaternions.
Quaternions are interesting in their own right as the first non-commutative algebra to be studied. Addition of quaternions is defined component-wise. Multiplication is defined using the dot-product and the cross-product (Sect. 2.1):

**Definition** \(\text{mulq } a \ b := \text{mkQuat} (a'0 \ b'0 - a'1 *a\ b'1)

\((a'0 : b'1 + b'0 : a'1 + a'1 * a\ b'1)\).

We denote by \(a^{-1}\) the conjugate of \(a,\) defined as follows:

**Definition** \(\text{conjq } a := \text{mkQuat} (a'0) (- a'1).\)

Inverse is defined using the norm and the conjugate:

**Definition** \(\text{invq } a := (1 / \text{sqrq } a) * (a^{-1}).\)

Using above operations, we showed that quaternions form a ring (ringType in the Mathematical Components library) with scaling (lmodType) and units (unitRingType).

Let \(a\) be a unit quaternion. It turns out (under some conditions) that the application \(v \mapsto ava^*\) provides us with rotation of (three-dimensional) vectors \(v:\)

**Definition** \(\text{quat_rot } a \ v := \text{quat} := (a : \text{quat}) \ast v \% v \ast a^{-1}.\)

\((v \% v\) is a quaternion with only the vector part \(v\).\) The angle and the vector-axis of this rotation come from the polar coordinates of a quaternion, they are defined as follows:

**Definition** \(\text{polar_of_quat } a :=\)

\((\text{normalize } a'1, \text{atan} (\text{norm } a'1 / a'0))).\)

(See Sect. 2.2 for the definition of \(\text{atan}\) and Sect. 3 for normalize and norm). Precisely, given a unit quaternion \(a\) with a non-zero scalar part (i.e., it is not \(\text{pure}\)) and polar coordinates \((u, \theta)\), we can show that the function \(v \mapsto ava^*\) is a rotation of angle \(2\theta\) around \(u:\)

**Lemma** \(\text{quat_rot_isRot } a : a \is \text{quat} \rightarrow \text{pureq } a \rightarrow\)

\(\text{let:} (u, \theta) := \text{polar_of_quat } a \in u := 0 \rightarrow\)

\(\text{isRot} (\theta \leftrightarrow 2) u \text{ (linear} (\text{quat_rot_is_linear } a)).\)

(\(\text{quat_rot_is_linear}\) is a proof that the function \(\text{fun } v \mapsto\)

\((\text{quat_rot v q v}) \ast \text{is linear}.)\)

7. **Formalization of Rigid Body Transformations**

In this section, we show that we can formalize a rigid body transformation (hereafter, RBT) as a direct isometry because the latter preserves orientation.

7.1 **Definition of Rigid Body Transformations**

A **RBT** is a mapping \(\mathbb{R}^3 \rightarrow \mathbb{R}^3\) that preserves lengths and orientation. A mapping that preserves lengths is called an isometry. We first formalize the type \(\text{Iso}[\mathbb{R}].\) of isometries:

\(\text{(* Module Iso *)}\)

**Record** \(t := \text{mk} \{\)

\(\text{f := } \text{r'V[R].} \rightarrow \text{r'V[R].}\)

\(\text{P := } [\text{mono } f : a \ b / \text{norm} (a - b)].\)
8. Rigid Body Transformation Using Homogeneous Representation

In Sect. 7, we showed that a RBT is a direct isometry. Alternatively, a RBT can be represented by an element of the special Euclidean group, i.e., a pair of a translation and a rotation:

\(*\) Module SE (*

Record t (R: rcfType): Type := mk {
  trans := 'rV[R]_3;
  rot := 'M[R]_3;
  rotP := rot \in 'SO[R]_3.
}

This view is interesting in practice because its elements have a representation in terms of $4 \times 4$-matrices that lend themselves well to computations. We formalize this homogeneous representation in Sect. 8.1 and show in Sect. 8.2 that an element of the special Euclidean group indeed defines a RBT, i.e., that it preserves lengths and orientation. We can use this formalization to address the forward kinematics problem for the SCARA robot manipulator (Sect. 8.3).

8.1 Homogeneous Representation

The homogeneous representation of an element of the special Euclidean group is a $4 \times 4$-matrix \([t \ 0]\) where \(r\) is a rotation and the vector \(t\) represents a translation. We define formally such matrices as follows:

**Definition** \(\text{hom}(r: 'M[R]_3)(t: 'rV[R]_3): 'M[R]_4 := \text{block}_m_x r \ t \ t\).

For example, the homogeneous representation \(\text{hom}(Rz(a))\) of the rotation of angle \(a\) around the “\(x\) axis” is defined by \(\text{hom}(Rz a) 0 (Rx a)\) (\(Rx\) was defined in Sect. 3.3). The homogeneous representation \(\text{hom}(T_x(d))\) of the translation of length \(d\) along the “\(x\) axis” is defined as \(\text{hom}\ 1 (row3 d 0 0)\). Rotations and translations w.r.t. to the \(y\) and \(z\) axes are defined similarly.

We denote by \(\text{SE}(R)\) the set of matrices of the shape \([t \ 0\]

8.2 Elements of the Special Euclidean Group Are Rigid Body Transformations

We can now define application of an element of the special Euclidean group to a (three-dimensional) point or vector. The homogeneous point (resp. vector) corresponding to the three-dimensional row-vector \(x\) is \(\text{to}_p\) (resp. \(\text{to}_v\)). Define \(\text{from}_h\) is the projection that cancels \(\text{to}_p\) and \(\text{to}_v\). We use these functions to embed points and vectors into their homogeneous representation and recover points and vectors after having applied \(\text{hom}\):

**Definition** \(\text{ap}_p(t: \text{SE} t R): \text{from}_h(\text{hom}_a p (to\_p p))\).

**Definition** \(\text{ap}_v(t: \text{SE} t R): \text{from}_h(\text{hom}_a p (to\_v v))\).

We can show that the application of an element of the special Euclidean group to a point using \(\text{ap}_p\) preserves length between points:

**Lemma** \(\text{SE}\_\text{preserves}\_\text{length}(t: \text{SE} t R): (\text{mono}(\text{ap}_p T): \text{a b l norm}(\text{a - b}))\).

Since they preserve lengths, the elements of the special Euclidean group can be used to build isometries. Moreover, an isometry built in this way preserves orientation:

**Lemma** \(\text{SE}\_\text{preserves}\_\text{orientation}(t: \text{SE} t R): \text{preserves}\_\text{orientation}(\text{Iso.mk}(\text{SE}\_\text{preserves}\_\text{length} T))\).

8.3 Forward Kinematics Problem for the SCARA Robot Manipulator

We now address the forward kinematics problem for the SCARA robot manipulator of Fig. 14. We first provide concrete expressions for the RBT’s between the frames of Fig. 2. For example, the RBT \(A_1\) (see Sect. 1) is the result of a rotation of angle \(\theta_1\) around the \(z\)-axis (i.e., \(Rz \theta_1\)) and of a translation of vector \(\text{row}_3 (a_1 \cos \theta_1) (a_1 \sin \theta_1) 0\). Expressions for the other RBT’s are similar:

**Definition** \(A10 := \text{hom}(Rz \theta_1) (\text{row}_3 (a_1 \cos \theta_1) (a_1 \sin \theta_1) 0)\).

**Definition** \(A21 := \text{hom}(Rz \theta_2) (\text{row}_3 (a_2 \cos \theta_2) (a_2 \sin \theta_2) 0)\).

**Definition** \(A32 := hTz d3\).

**Definition** \(A43 := \text{hom}(Rz \theta_3) (\text{row}_3 0 0 d4)\).

The position and orientation of the end-effector is expressed by the RBT corresponding to the product of the above matrices. It has the following closed expression (depending only on the link and joint parameters) and the proof is a matter of a few lines using the properties we saw so far about angles and about the homogeneous representation:

**Definition** \(\text{scara}_\text{rot} := Rz (\theta_1 + \theta_2 + \theta_3)\).

**Definition** \(\text{scara}_\text{trans} := \text{row}_3 (a_2 \cos (\theta_2 + \theta_1) + a_1 \cos \theta_1) (a_2 \sin (\theta_2 + \theta_1) + a_1 \sin \theta_1) (d4 + d3)\).

**Definition** \(\text{hom}_\text{SCARA}\_\text{forward} := A43 * A32 * A21 * A10 = \text{hom}_\text{scara}\_\text{rot} \text{scara}_\text{trans}\).

(For square matrices, multiplication can be denoted by * instead of \(*_m\).) Sections 9.2 and 10.3 complete this example.
9. The Denavit-Hartenberg Convention

The Denavit-Hartenberg convention is a way to position frames on a robot manipulator so as to reduce the number of the parameters necessary to specify (using a RBT) the relative position of two consecutive frames. In general, a RBT requires at least six parameters to be characterized. Using the Denavit-Hartenberg convention, only four parameters are needed: the joint angle \( \theta \), the link offset \( d \), the link length \( a \), and the link twist \( \alpha \). For example, the parameters that appear in Fig. 1 are the Denavit-Hartenberg parameters of the SCARA robot manipulator. In Sect. 9.2 we prove that only four parameters are required under the Denavit-Hartenberg convention. We illustrate the convention in Sect. 9.2.

9.1 The Denavit-Hartenberg Convention and Parameters

A robot manipulator satisfies the Denavit-Hartenberg convention when any two consecutive frames \(i\) and \(j\) are such that (1) the directions of \((o_j, x_j)\) and \((o_i, z_i)\) are perpendicular (condition \(DH1_{i,j}\)) and (2) \((o_j, x_j)\) intersects \((o_i, z_i)\) (condition \(DH2_{i,j}\)).

Let \(F_0\) and \(F_1\) be two frames of type \(\text{Frame}\_\text{R}\) (with origins \(\text{Frame}\_\text{O}\) and \(\text{Frame}_1\), see Sect. 3.4). The relative position of \(F_1\) w.r.t. \(F_0\) is \(\text{hom}(F_1^\text{R} F_0)\). The rotation matrix \(F_1^\text{R} F_0\) was explained in Sect. 3.4. \(\text{p}_1\_\text{in}_0\) is the position of the origin of \(F_1\) w.r.t. \(F_0\). It can be computed by converting the coordinates of the vector \(\text{Frame}_0 F_1 - \text{Frame}_0 F_0\) (which are expressed in the canonical frame) into coordinates expressed in the frame \(F_0\):

**Definition** \(\text{p}_1\_\text{in}_0 := (\text{Frame}_0 F_1 - \text{Frame}_0 F_0) \cdot \text{can}_\text{frame} R \cdot \text{R}^\text{F} F_0\).

Let us assume moreover that the frames \(F_0\) and \(F_1\) fulfill the conditions \(DH1_{0,1}\) and \(DH2_{0,1}\):

**Hypothesis** \(\text{dh}1\) : perpendicular (xaxis \(F_1\)) (zaxis \(F_0\)).

**Hypothesis** \(\text{dh}2\) : intersects (xaxis \(F_1\)) (zaxis \(F_0\)).

Under these hypotheses, we show that the RBT corresponding to the relative position of \(F_1\) w.r.t. \(F_0\) can be expressed by means of the four Denavit-Hartenberg parameters:

**Lemma** \(\text{dh} \_\text{mat} \_\text{correct} : exists \theta \_\text{d} \_\alpha\).

\[\text{hom}(F_1^\text{R} F_0) \_\text{p}_1\_\text{in}_0 = \text{dh} \_\text{mat} \_\theta \_\text{d} \_\alpha\].

where \(\text{dh} \_\text{mat}\) is the Denavit-Hartenberg matrix, i.e., the matrix \(h R_\alpha(h T_x(a)) h R_\theta(d) h R_z(\theta)\) (see Sect. 8.1) for the definitions of the \(h R_\alpha(h T_x(a)) h R_\theta(d) h R_z(\theta)\) matrices.

**Proof:** First, use the hypothesis \(DH1_{0,1}\) to show that \(F_1^\text{R} F_0\) can be expressed as the matrix \(\text{dh} \_\text{rot} \_\theta \_\alpha\) where \(\text{dh} \_\text{rot}\) is defined as follows

\[
\begin{bmatrix}
\cos(\theta) & -\cos(\alpha) \sin(\theta) & 0 \\
\sin(\theta) & \cos(\alpha) \cos(\theta) & \sin(\alpha) \\
\sin(\alpha) \sin(\theta) & -\sin(\alpha) \cos(\theta) & \cos(\alpha)
\end{bmatrix}.
\]

Second, use the hypothesis \(DH2_{0,1}\) to show that \(\text{p}_1\_\text{in}_0\) can be written as \(d : \_e_2 + a : \_\text{row} 3 (\cos(\theta))(\sin(\theta)) 0\) using a change of coordinates. This results in

Figure 4: As a result of a screw motion, \(p\) successively becomes \(p'\) and then \(p''\)

\[\text{hom}(F_1^\text{R} F_0) \_\text{p}_1\_\text{in}_0\] being equal to \(\begin{bmatrix}dh \_\text{rot}(\theta, \alpha) & 0 \\ \alpha \cos(\theta) : \alpha \sin(\theta) & d \end{bmatrix}\).

The latter is the result of performing the matrix product \(h R_x(a) h T_x(a) h T_z(d) h R_z(\theta)\) [Spong et al. 2006, Chapter 3].

9.2 Denavit-Hartenberg Parameters for the SCARA Robot Manipulator

The SCARA robot manipulator of Fig. 1 can be succinctly specified by its Denavit-Hartenberg parameters:

<table>
<thead>
<tr>
<th>link</th>
<th>(a_i)</th>
<th>(a_i)</th>
<th>(d_i)</th>
<th>(\theta_i)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>(\theta_1)</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>(\theta_2)</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>(\theta_3)</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>(\theta_4)</td>
<td></td>
</tr>
</tbody>
</table>

In Sect. 9.1 we showed that the position of frame \(i\) w.r.t. frame \(i-1\) can be defined as \(h R_x(a_1) h T_x(a_1) h T_z(d_i) h R_z(\theta_1)\). This gives us the relative positions of the consecutive frames of the SCARA robot manipulator:

**Definition** \(B10 := h T x_1 a_1 h R z \_\theta_1\).

**Definition** \(B21 := h T x_2 a_2 h R z \_\theta_2\).

**Definition** \(B32 := h T z d_3\).

**Definition** \(B43 := h T z d_4 h R z \_\theta_4\).

We can use the lemma \(\text{dh} \_\text{mat} \_\text{correct}\) from the previous section to prove that their composition gives the same RBT as computed in Sect. 8.3.

10. Screw Motions

A screw motion is an alternative representation of a RBT that is reminiscent of the motion of a screw: it combines a rotation around an axis and a translation along the same axis. In Sect. 10.1 we define screws and screw motions. In Sect. 10.2 we formalize screw motions using exponential coordinates. In Sect. 10.3 we use screw motions to address the forward kinematics problem for the SCARA robot manipulator.

10.1 Screws and Screw Motions

A screw is defined by a line (the screw axis), an angle, and a pitch, put together in the type \(\text{Screw}\).

\(\ast\) Module Screw \(\ast\)

\(\text{Record t} := \text{mk} [1 : \text{Line}\_\text{R} ; a : \text{angle}\_\text{R} ; h : \text{R}]\).
It is possible to represent the RBT associated with a screw using a combination of rotation and translation. Figure 4 shows the effect of a screw motion on a point \( p \). The axis is \((q, w)\), the angle is \( a \), and the pitch (say, \( h \)) is the ratio of translation to rotation. Using the rotation of angle \( a \) around \( w \) and the translation of \( h a \) along \( w \), the RBT of Fig. 4 can be formalized as follows:\(^2\)

**Definition screw_motion** \( s : p \rightarrow \mathbb{R}^3 \)

\[
\text{let:} (l, a, h) := (\text{Screw.1} s, \text{Screw.a} s, \text{Screw.h} s) \text{ in} \\
\text{let} (q, w) := \langle \text{pt}(1), \text{vec}(1) \rangle \text{ in} \\
q + (p - q) + m \cdot e^a(w, a) + (h \cdot \text{Rad.f} a) \cdot w.
\]

The function \( \text{Rad.f} \) of type angle \( \mathbb{R} \rightarrow \mathbb{R} \) interprets an angle as its measure in radians (see (Affeldt and Cohen 2016, Appendix A) ).

10.2 Exponential Coordinates of Screw Motions

The preferred way to represent a screw motion is its exponential coordinates. This representation extends the exponential coordinates of rotation matrices from Sect. 5.2 by generalizing skew-symmetric matrices to twists. A twist \( \mathcal{T}(v, w) \) is essentially a pair of two vectors. Intuitively, the vector \( v \) represents the linear velocity of the motion and the vector \( w \) represents the angular velocity. The exponential of a twist \( \mathcal{T}(v, w) \) with angle \( a \) is defined as follows:\(^3\)

\[
\left[
\begin{array}{ccc}
I & 0 & 1 \\
0 & v & 1 \\
(\vec{w} \times v)(1-e^{-\hat{S}(w)}) & (a \cdot v)(e^\hat{w}w) & 0 \\
\end{array}
\right]
\quad \text{if } w = 0
\]

\[
\left[
\begin{array}{ccc}
I & 0 & 1 \\
0 & v & 1 \\
0 & (a \cdot v)(e^\hat{w}w) & 0 \\
\end{array}
\right]
\quad \text{if } w \neq 0.
\]

We denote the exponential of a twist \( t \) with angle \( a \) by \( \text{e}^t(a, t) \) and its formalization is direct using the formal definitions introduced so far:

**Definition** \( \text{hom_twist} \ t \ a \ e := \)

\[
\text{let} (v, w) := (\text{lv}(t), \text{lw}(t)) \text{ in} \\
\text{if } w == 0 \text{ then } \text{hom} l (a : v) \\
\text{else } \text{hom} e (\text{norm} w : S) := \\
((w \times v) + m((1 - e) + (a : v) + m(w \cdot \text{Rad.f.a} \cdot \text{lw}(t)))).
\]

**Definition** \( \text{etwist} t \ a \ e := \)

\[
\text{hom_twist} t (\text{Rad.f.a} \cdot e^a(w, \text{lw}(t))).
\]

We observe that the screw_motion function of Sect. 10.1 can be recovered using exponential coordinates. Let \((l, a, h)\) be a screw where the line 1 is formed by the point \( q \) and the vector \( w \). The exponential of the twist \( \mathcal{T}(w \times v, q + h \cdot w) \) is the (homogeneous representation of the) function screw_motion:

**Lemma** \( \text{hom_screw_motion} \mathcal{E} \times \mathcal{T} := \)

\[
\text{let} (l, a, h) := (\text{Screw.1} s, \text{Screw.a} s, \text{Screw.h} s) \text{ in} \\
\text{let} v := -w \times q + h \cdot w \text{ in} \\
\text{hom_screw_motion} s = \text{e}^t(a, \mathcal{T}(v, w)).
\]

In fact, any RBT can be represented by exponential coordinates:

**Lemma** \( \text{etwist} \mathcal{I} \rightarrow \mathcal{E} \mathcal{T} := \)

\[
\text{let} v := -w \times q + h \cdot w \text{ in} \\
\text{hom_screw_motion} s = \text{e}^t(a, \mathcal{T}(v, w)).
\]

**Proof:** Let us assume that \( f \) is of the form \([p, q]\). If \( f \) is a pure translation, then it suffices to choose the twist \( \mathcal{T} \) (normalize \( p \)) and the angle norm \( p \). Otherwise, let \( a \) and \( w \) be the exponential coordinates of \( r \) (obtained from lemma skew_is_onto_S0 in Sect. 5.2). Then it suffices to choose the twist \( \mathcal{T}(v, w) \) and the angle \( a \), where \( v \) is such that \( p = (\text{norm} w) \cdot \mathcal{T} \cdot (v * m, G) \) with \( G = \langle \text{S}(w), \mathcal{T} \cdot (1 - \mathcal{T}) \rangle + \text{Rad.f.a} \cdot (\mathcal{T} \cdot m, w) \). The matrix \( G \) is indeed invertible of inverse \( \frac{1}{2} \cdot (\frac{1}{a} - \frac{1}{a} \cdot \mu) \cdot \mathcal{T}(w)^2 \) (Park and Lynch 2012). This lemma is often referred to as (a constructive proof of) Chasles’ theorem. There is also a simpler presentation of Chasles’ theorem that just shows the existence of the screw axis (see (Affeldt and Cohen 2016, Appendix B) ).

**Example of Twist Computation** Since the proof of the lemma etwist_onto_SE above is constructive, one can use it to compute twists. For example, let us compute the twist corresponding to the motion of frame 2 w.r.t. frame 0 in Fig. 2. We denote the position of frame 2 w.r.t. frame 0 by \( \mathcal{T} \). The vector \( q \) is of \([a_1 + a_2 \cdot \text{cos} \theta_2; a_2 \cdot \text{sin} \theta_2; 0]\). The rotation \( \mathcal{T} \) is better thought in terms of its exponential coordinates (Sect. 5.2). They are the axis \( e_2 \) and the angle \( \theta_2 \). Let us denote by \( \mathcal{T}(v, w) \) the twist that we are looking for. From the exponential coordinates of the rotation \( \mathcal{T} \), we already know that \( w \) is \( e_2 \). According to the proof of etwist_onto_SE, \( v \) is given by \( 2 \mathcal{T} \cdot G \), that is, after calculation, \( \left[ \begin{array}{c} a_1 + a_2 \cdot \text{sin} \theta_2 \\ -a_1 \cdot \text{cos} \theta_2 \\ 2 \end{array} \right] \).

10.3 Forward Kinematics Problem for the SCARA Robot Manipulator with Screw Motions

We address the forward kinematics problem for the SCARA robot manipulator of Fig. 1 using screw motions and show that we recover the same results as in Sect. 8.3.

First, we provide the twists corresponding to revolute and prismatic joints. They are computed with all other joint parameters held fixed at 0, all points and vectors being specified w.r.t. the base frame. For a revolute joint, the twist is \( \mathcal{T}(w \times q, w) \) where \( w \) is a unit vector in the direction of the twist axis and \( q \) is a point on the axis. For
a prismatic joint, the twist is T(v, 0) where v is a unit vector in the direction of the translation. Put formally:

Definition rjoint_twist w q := \( T(\cdot w, v, q, w) \).

Definition pjoint_twist v := \( T(v, 0) \).

The exponential of a twist represents a relative RBT. Let \( \theta_0 \) be the RBT corresponding to the position and orientation of the end-effector when the joint parameters are held fixed at 0. Then \( g_0 e^{\theta t} \) is the RBT corresponding to the position and orientation of the end-effector after the RBT with twist \( t \) and angle \( \theta \) has been applied. More generally, for a robot manipulator with \( n \) joints with twists \( t_i \) and joint parameters \( \theta_i \), the position and orientation of the end-effector is given by

\[
g_0 e^\theta_0 t_n \cdots e^\theta_1 t_1.
\]

Fig. 5 displays the screw axes for the SCARA robot manipulator. It has the following twists:

Definition t1 := rjoint_twist w1 q1.
Definition t2 := rjoint_twist w2 q2.
Definition t3 := pjoint_twist v3.
Definition t4 := rjoint_twist w4 q4.

where \( w_1, w_2, v_3, \) and \( w_4 \) are equal to \( e_2 \). The position and orientation of the end-effector is therefore:

Definition g0 := hom l (row3 (a1 + a2) 0 d4).
Definition g := g0 * `e$(\theta_2, t4) *  
  `e$(\text{Rad.angle_of}(d3, t3) + `e$(\theta_2, t2) + `e$(\theta_1, t1), t1).

We can show that \( g \) is actually the same RBT as the one computed using the Denavit-Hartenberg parameters in Sect. 9.2

Hypothesis Hd3 : d3 \in Rad.f_codom R.
Lemma screw_SCARA_forward : g = hom scara_rot scara_trans.

The hypothesis \( \text{Hd3} \) says that the parameter of the prismatic joint is scaled so that it can be measured in terms of radians.

11. Related Work

Walter et al. perform in Isabelle the verification of a collision avoidance algorithm for a vehicle moving in a plane (Walter et al. 2010). This algorithm computes safety zones as supersets of braking areas and therefore involves mathematics such as velocity calculations and computational geometry in a plane. This does not compare well to our work (except for the fact that our libraries generalize some geometry to three dimensions, but anyway we do not deal with velocity yet). However, the work by Walter et al. is a concrete evidence of the relevance of proof-assistants for robotics since it was used as part of a certification effort and received positive reviews with this respect. More generally, formal verification for mobile robots has been drawing much attention, for example with the verification in Coq of gathering algorithms for autonomous robots and impossibility results (Auger et al. 2013; Courtieu et al. 2015, 2016).

Farooq et al. propose a formalization of two-link planar manipulators in HOL-Light (Farooq et al. 2013) using the theory of Euclidean space formalized by Harrison (Harrison 2013). They apply their theories to the analysis of a two-dimensional biped walking robot. The restriction to two dimensions is a significant simplification, yet they tackle the inverse kinematics problem which is often harder than forward kinematics. Our libraries are a significant improvement. They deal with three dimensions and are not restricted to Cartesian coordinates: they provide several representations that are important to simplify the analysis of robots. Also, Farooq et al. mentioned as future work the SCARA robot manipulator and the Denavit-Hartenberg parameters that we have formalized in our work.

Anand et al. use Coq to write, verify, and execute programs to be used in concrete mobile robots moving in a plane (Anand and Knepper 2015). For that purpose, they provide an event-based programming framework dealing with time. They tackle the problem of verified robot software in a pragmatic way. In comparison, we are dealing with the complementary task of formalizing generic formal foundations.

Ma et al. propose a formalization of conformal geometric algebra in HOL-Light (Ma et al. 2016) based on Harrison’s formalization of Clifford theories (Harrison 2013). Conformal geometric algebra is used to define rigid body transformations and is applied to the verification of a robotic grasping algorithm. Conformal geometric algebra sometimes simplifies the analysis of robot manipulators but is not a mainstream approach. In our work, we stick to the standard foundations of robotics. We are nevertheless planning to explore conformal geometric algebra because it deals with multivectors: they generalize the cross-product and could therefore lead to simplifications in our formalization.

12. Conclusion, Discussions and Perspectives

This paper provides a formalization of the foundations of robot manipulators. Our formalization is already useful to compute basic elements of robot manipulators (e.g., angle-axis representation in Sect. 6.1) and to solve the forward kinematics problem for basic robot manipulators using standard representations (SCARA robot manipulator using homogeneous matrices in Sect. 8.3, with the Denavit-Hartenberg convention in Sect. 9.2 or with screw motions in Sect. 10.3).

Technical Improvements Except for the library on angles, our work is the result of an incremental formalization of reference books on the foundations of robotics (e.g., Siciliano and Khatib 2008; Sect. 1.2: Position and Orientation Representation), (Murray et al. 1994, Chapter 2: Rigid Body Motion) with, as usual, a huge work on filling up information that is kept implicit in most books. Now this code-base is available (see the breakdown in Table 2), we expect to reduce its size by using more linear algebra and spectral theory (development in progress) and by taking advantage of dependent types. We anticipate several ways to improve our formalization:

- the theory of lines and intersection can be rephrased into non-homogeneous systems represented by matrices,
Although using discrete real closed field forces us to redevelop all this material, this also excels at verification of numerical computations, using the CoQ effective algebra library CoqEAL. Compared to the very modular hierarchy from the Math Classes library (Spitters and van der Weegen 2011), we cannot select exactly the operations and axiomatic needed for each lemma, but this was not problematic in our case.

Towards Concrete Applications Our formalization of the theoretical foundations of robot manipulators is only the first step towards ensuring the correctness of robot software. Concrete applications will require us to deal with robot control, and in particular numerical computations. One way to tackle this problem is to annotate robot software with specifications expressed by means of three-dimensional geometry and verify that execution preserves invariants. This approach has been demonstrated in the broader context of control software using the PVS proof-assistant, but was hampered by the need to formalize much linear algebra (Herencia-Zapana et al. 2012). In comparison, the Mathematical Components library is richer. Moreover, like the abstract algebra we relied on in this paper, CoQ also excels at verification of numerical computations, using the CoQ effective algebra library CoqEAL. (Dénès et al. 2012) Cohen et al. 2013 (Dénès 2013). We thus have good reasons to think that our work could pave the way towards a comprehensive formal verification framework for robot manipulators.

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