Finding Optimal Formulae for Bilinear Maps

Razvan Barbulescu, Jérémie Detrey, Nicolas Estibals, Paul Zimmermann

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Finding Optimal Formulae for Bilinear Maps

(slides courtesy of Nicolas Estibals)

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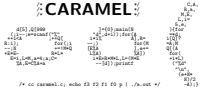

A bit of history

- Multiplication is an expensive arithmetic operation
- Well-studied problem
  - Karatsuba (1962)
  - Toom–Cook (1963), evaluation-interpolation schemes
  - CRT-based algorithms
  - Schönhage–Strassen algorithm (1971)
  - ...
- *Five-, six-, and seven-term Karatsuba-like formulae*, P. Montgomery (2005)
  - ad-hoc formulae
  - exhaustive search for five-term multiplication
  - non-exhaustive search for six- and seven-term multiplications
  - (January 2011) start a task group to reproduce this work

Outline of the talk

- Formulae for polynomial multiplication
- Enumerating formulae
- Further improvements and heuristics
- Some results

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Example: a 2-term polynomial times a 3-term one

- We want to compute

$$\begin{aligned} C(X) &= (a_1 \cdot X + a_0) \times (b_2 \cdot X^2 + b_1 \cdot X + b_0) \\ &= a_1 b_2 \cdot X^3 + (a_1 b_1 + a_0 b_2) \cdot X^2 + (a_0 b_1 + a_1 b_0) \cdot X + a_0 b_0 \end{aligned}$$

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- Only 5 products required instead of 6
  - use Karatsuba's trick

$$C(X) = a_1 b_2 \cdot X^3 + (a_1 b_1 + a_0 b_2) \cdot X^2 + ((a_0 + a_1)(b_0 + b_1) - a_1 b_1 - a_0 b_0) \cdot X + a_0 b_0$$

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  - compute the products

$$\begin{aligned} g_0 &= a_0 \cdot b_0, \\ g_1 &= a_0 \cdot b_2, \\ g_2 &= a_1 \cdot b_1, \\ g_3 &= a_1 \cdot b_2, \text{ and} \\ g_4 &= (a_0 + a_1) \cdot (b_0 + b_1). \end{aligned}$$

  - reconstruct the result

$$C(X) = g_3 \cdot X^3 + (g_1 + g_2) \cdot X^2 + (g_4 - g_2 - g_0) \cdot X + g_0$$

General form of a multiplication formula

- We want to compute, over a given field K (or any K-algebra $\mathbf{K}$),
  $(a_{n-1} \cdot X^{n-1} + \cdots + a_0) \times (b_{m-1} \cdot X^{m-1} + \cdots + b_0) = c_{n+m-2} \cdot X^{n+m-2} + \cdots + c_0$

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- This is also valid for any bilinear map
  $$\begin{array}{ccccccc} F: & K^n & \times & K^m & \longrightarrow & K^\ell \\ & ((a_0, \ldots, a_{n-1}) & , & (b_0, \ldots, b_{m-1})) & \longmapsto & (c_0, \ldots, c_{\ell-1}) \end{array}$$

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$$\mathcal{G} = \{(\sum \alpha_i a_i) \cdot (\sum \beta_j b_j) \mid \forall i,\ \alpha_i \in K \text{ and } \forall j,\ \beta_j \in K\} \setminus \{0\}$$

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  We only consider products modulo a nonzero scalar factor

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  where $g \sim g'$ when $\exists \lambda \in K,\ \lambda \neq 0$ such that $g = \lambda g'$

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- The set of generators contains 21 products:

$$\begin{aligned}
\mathcal{G} = \{\ & a_0 \cdot b_0, && a_1 \cdot b_0, && (a_0 + a_1) \cdot b_0, \\
& a_0 \cdot b_1, && a_1 \cdot b_1, && (a_0 + a_1) \cdot b_1, \\
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& a_0 \cdot (b_0 + b_1 + b_2), && a_1 \cdot (b_0 + b_1 + b_2), && (a_0 + a_1) \cdot (b_0 + b_1 + b_2)\ \}
\end{aligned}$$

Naive algorithm

- Goal: find the optimal formulae (*i.e.*, with a minimum number of products)
  - enumerate the subsets $\mathcal{W} \subset \mathcal{G}$ of exactly k products which yield a valid formula
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- Drawback: Distinct subsets may span the same subspace

Improved algorithm: subspaces instead of subsets

- Look instead for subspaces W of V such that
  - W can be generated by products only: $\operatorname{Span}(W \cap \mathcal{G}) = W$
  - only k products are required: $\dim W = k$
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- Algorithm:

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- Several formulae can correspond to a single solution subspace W
  - each basis of W comprising only elements of $\mathcal{G}$ gives a k-product formula

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- Complexity now depends on

$$\binom{\#\mathcal{G}}{k - \operatorname{rk} \mathcal{T}}$$

Example (cont'd)

2×3-term polynomial product in $\mathbb{F}_2[X]$

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- Attempt with $k = 4$:
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  - $\mathrm{rk}(T \cap \mathcal{G}) = 3 < k$
  - no solutions with $k = 4$ products only

Example (cont'd)

2×3-term polynomial product in $\mathbb{F}_2[X]$

- Targets: $\mathcal{T} = \{a_1 b_2, a_1 b_1 + a_0 b_2, a_0 b_1 + a_1 b_0, a_0 b_0\}$
  - at least $\mathrm{rk}(\mathcal{T}) = 4$ products required
- Attempt with $k = 4$:
  - $W = T$
  - $T \cap \mathcal{G} = \{\ a_0 \cdot b_0,\ \ a_1 \cdot b_2,\ \ (a_0 + a_1) \cdot (b_0 + b_1 + b_2)\ \}$
  - $\mathrm{rk}(T \cap \mathcal{G}) = 3 < k$
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- Attempt with $k = 5$:
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  - there are 3 such solution spaces which yield a total of 162 formulae

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  - example: squaring of a 2-term polynomial in $\mathbb{F}_3[X]$

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Outline of the talk

- Formulae for polynomial multiplication
- Enumerating formulae
- Further improvements and heuristics
- Some results

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  - same number of a_i's and b_j's: $F : K^n \times K^n \to K^\ell$
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- Going further: instead of a single $\sigma \in \operatorname{Aut}(V)$, take a whole subgroup $S \subset \operatorname{Aut}(V)$

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  - $\sigma_1 : a_i b_j \mapsto \left(\sum_{k=i}^{1} \binom{k}{i} a_k \right) \cdot \left(\sum_{k=j}^{2} \binom{k}{j} b_k \right)$ (*i.e.*, replace X by $1 - X$)
- Remark that $\#S = 6$ (hurray for non-commutativity!)
- $\mathcal{G}$ can be partitioned into
  - 3 orbits of size 3:

$$
\begin{aligned}
\overline{g_0} &= \{\ a_0 \cdot b_0, \quad a_1 \cdot b_2, \quad (a_0 + a_1) \cdot (b_0 + b_1 + b_2)\ \}, \\
\overline{g_1} &= \{\ a_0 \cdot b_1, \quad a_1 \cdot b_1, \quad (a_0 + a_1) \cdot b_1\ \}, \\
\overline{g_2} &= \{\ a_0 \cdot (b_0 + b_1), \ a_1 \cdot (b_1 + b_2), \ (a_0 + a_1) \cdot (b_0 + b_2)\ \}, \text{ and}
\end{aligned}
$$

Orbits of a group action: Example

2×3-term polynomial product in $\mathbb{F}_2[X]$

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\end{aligned}$$

  - 2 orbits of size 6:

$$\begin{aligned}
\overline{g_3} = \{\ & a_0 \cdot b_2, & a_0 \cdot (b_0 + b_1 + b_2), & (a_0 + a_1) \cdot b_0, \\
& a_1 \cdot b_0, & a_1 \cdot (b_0 + b_1 + b_2), & (a_0 + a_1) \cdot b_2\ \}, \\
\overline{g_4} = \{\ & a_0 \cdot (b_0 + b_2), & a_0 \cdot (b_1 + b_2), & (a_0 + a_1) \cdot (b_0 + b_1), \\
& a_1 \cdot (b_0 + b_2), & a_1 \cdot (b_0 + b_1), & (a_0 + a_1) \cdot (b_1 + b_2)\ \}
\end{aligned}$$

Orbits of a group action: Example

2×3-term polynomial product in $\mathbb{F}_2[X]$

- First, consider adding to T a generator from orbit $\overline{g_1}$:

$$\sigma_0 \circlearrowright (a_0 + a_1) \cdot b_1 \overset{\sigma_1}{\leftrightarrows} a_0 \cdot b_1 \overset{\sigma_0}{\leftrightarrows} a_1 \cdot b_1 \circlearrowleft \sigma_1$$

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T

Orbits of a group action: Example

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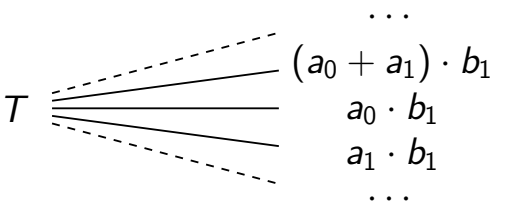

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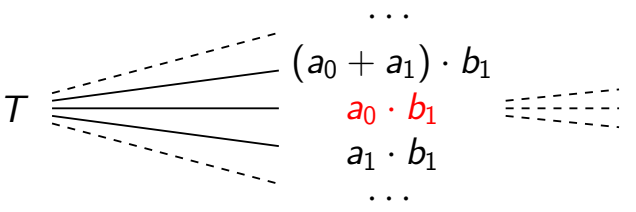

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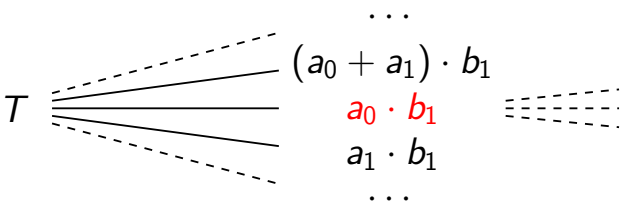
- For any solution of the form $W = T \oplus \mathrm{Span}(a_0 \cdot b_1) \oplus \mathrm{Span}\, \mathcal{W}'$, with $\mathcal{W}' \subset \mathcal{G}$:



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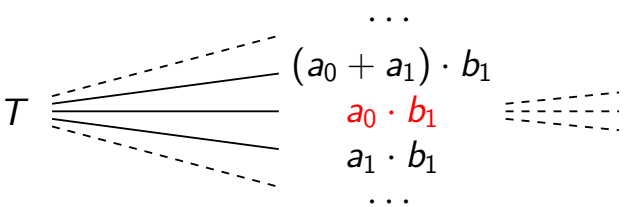
  - $W^{\sigma_1} = T \oplus \mathrm{Span}((a_0 + a_1) \cdot b_1) \oplus \mathcal{W}'^{\sigma_1}$, with $\mathcal{W}'^{\sigma_1} \subset \mathcal{G}$, is also a solution



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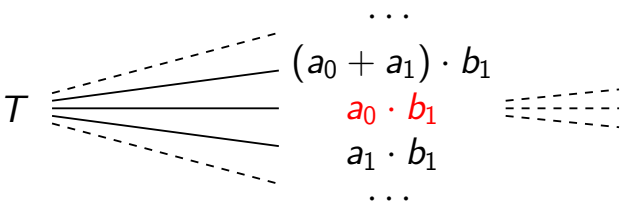
  - thus, no need to enumerate both $(a_0 + a_1) \cdot b_1$ and $a_0 \cdot b_1$



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2×3-term polynomial product in $\mathbb{F}_2[X]$

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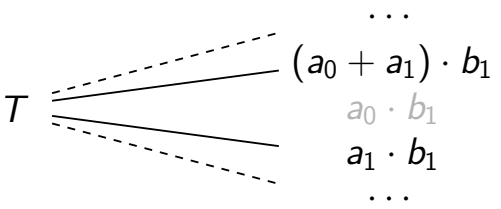

Orbits of a group action: Example

2×3-term polynomial product in $\mathbb{F}_2[X]$

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- For any solution of the form $W = T \oplus \mathrm{Span}(a_0 \cdot b_1) \oplus \mathrm{Span}\, \mathcal{W}'$, with $\mathcal{W}' \subset \mathcal{G}$:
  - $W^{\sigma_1} = T \oplus \mathrm{Span}((a_0 + a_1) \cdot b_1) \oplus \mathcal{W}'^{\sigma_1}$, with $\mathcal{W}'^{\sigma_1} \subset \mathcal{G}$, is also a solution
  - thus, no need to enumerate both $(a_0 + a_1) \cdot b_1$ and $a_0 \cdot b_1$
- Similarly, for any solution $W = T \oplus \mathrm{Span}(a_1 \cdot b_1) \oplus \mathrm{Span}\, \mathcal{W}'$, with $\mathcal{W}' \subset \mathcal{G}$:
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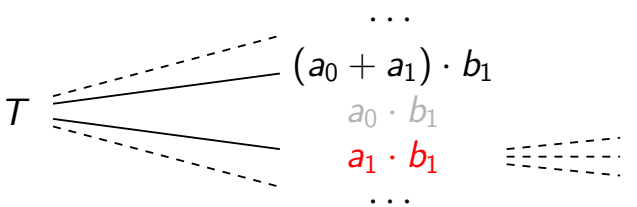
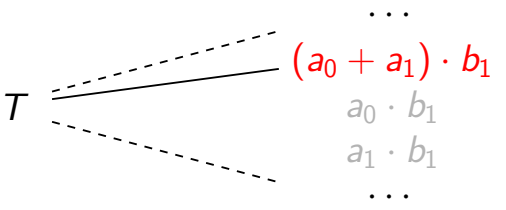


Orbits of a group action: Example

2×3-term polynomial product in $\mathbb{F}_2[X]$

- So we add generator $(a_0 + a_1) \cdot b_1$, fixed by σ_0



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- So we add generator $(a_0 + a_1) \cdot b_1$, fixed by σ_0
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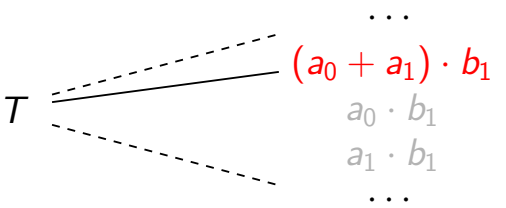
$$\begin{array}{ccccc} a_0 \cdot b_2 & \overset{\sigma_0}{\leftrightarrows} & a_1 \cdot b_0 & \overset{\sigma_1}{\leftrightarrows} & a_1 \cdot (b_0 + b_1 + b_2) \\ \sigma_1 \uparrow\downarrow & & & & \uparrow\downarrow \sigma_0 \\ (a_0 + a_1) \cdot b_2 & \overset{\sigma_0}{\leftrightarrows} & (a_0 + a_1) \cdot b_0 & \overset{\sigma_1}{\leftrightarrows} & a_0 \cdot (b_0 + b_1 + b_2) \end{array}$$



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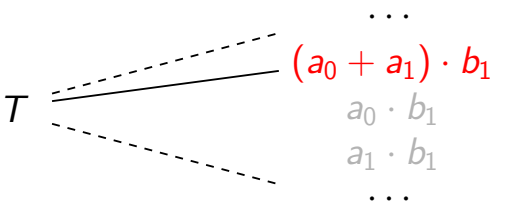

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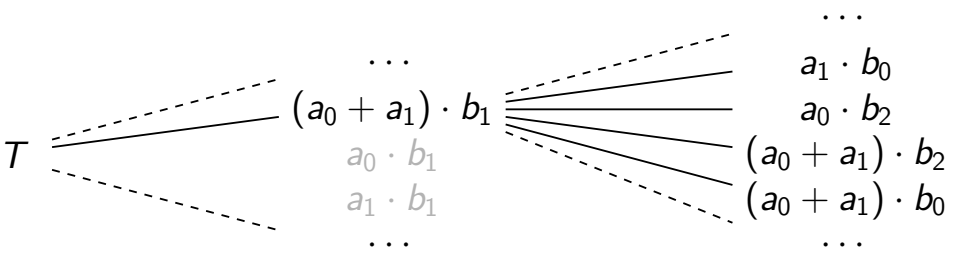

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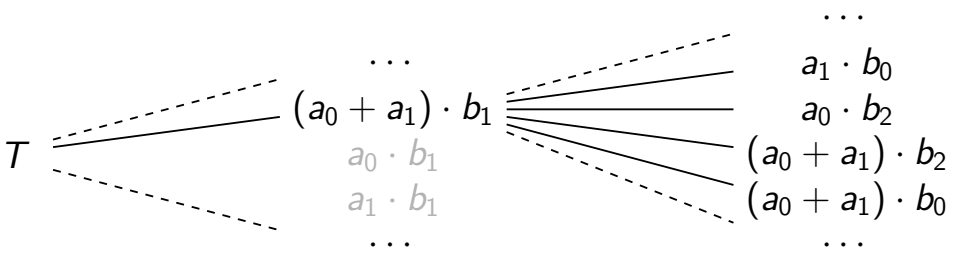

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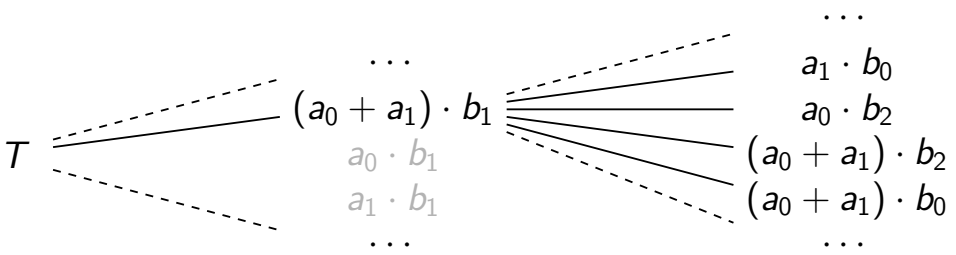

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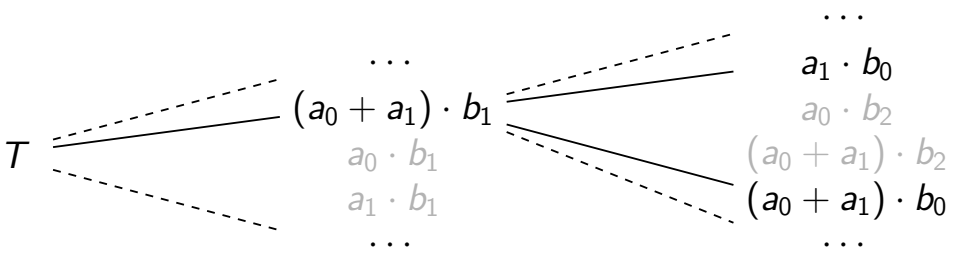

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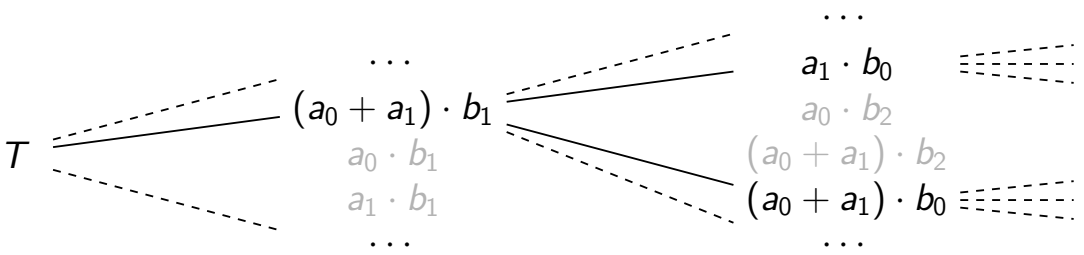

Outline of the talk

- Formulae for polynomial multiplication
- Enumerating formulae
- Further improvements and heuristics
- Some results

$n \times m$-term polynomial multiplication

Ring	$n \times m$	$\#\mathcal{G}$	k	# of tests	# of solutions	# of formulae	Calculation time [s]
$F_2[X]$	2×2	9	3	1	1	1	0.00
	3×3	49	6	9	3	9	0.00
	4×4	225	9	$6.60 \cdot 10^3$	4	4	0.03
	5×5	961	13	$9.65 \cdot 10^9$	27	27	$2.28 \cdot 10^5$
	6×6	3 969	14	$4.37 \cdot 10^9$	—	—	$6.03 \cdot 10^5$
		(Sym.) 63	17	$8.08 \cdot 10^6$	6	54	17.7
	7×7	(Sym.) 127	22	$3.38 \cdot 10^{12}$	2 618	19 550	$1.59 \cdot 10^7$
$F_3[X]$	2×2	16	3	1	1	4	0.00
	3×3	169	6	24	22	1 493	0.00
	4×4	1 600	9	$4.11 \cdot 10^5$	726	50 640	14.9
	5×5	14 641	11	$4.89 \cdot 10^7$	—	—	$4.02 \cdot 10^4$
		(Sym.) 121	12	$3.93 \cdot 10^4$	31	6 460	0.14
	6×6	(Sym.) 364	15	$2.37 \cdot 10^8$	4	1 024	$3.79 \cdot 10^3$
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	5 × 5	961	13	$9.65 \cdot 10^9$	27	27	$2.28 \cdot 10^5$
	6 × 6	3 969	14	$4.37 \cdot 10^9$	—	—	$6.03 \cdot 10^5$
		(Sym.) 63	17	$8.08 \cdot 10^6$	6	54	17.7
	7 × 7	(Sym.) 127	22	$3.38 \cdot 10^{12}$	2 618	19 550	$1.59 \cdot 10^7$
$F_3[X]$	2 × 2	16	3	1	1	4	0.00
	3 × 3	169	6	24	22	1 493	0.00
	4 × 4	1 600	9	$4.11 \cdot 10^5$	726	50 640	14.9
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		(Sym.) 121	12	$3.93 \cdot 10^4$	31	6 460	0.14
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n × m-term polynomial multiplication

Ring	n × m	#$\mathcal{G}$	k	# of tests	# of solutions	# of formulae	Calculation time [s]
$\mathbf{F}_2[X]$	2 × 2	9	3	1	1	1	0.00
	3 × 3	49	6	9	3	9	0.00
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	3×3	49	6	9	3	9	0.00
	4×4	225	9	$6.60 \cdot 10^3$	4	4	0.03
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Multiplication in small finite-field extensions

Finite field	$\#\mathcal{G}$	k	# of tests	# of solutions	# of formulae	Calculation time [s]
$\mathbf{F}_{2^2}$	9	3	3	3	3	0.00
$\mathbf{F}_{2^3}$	49	6	$7.03 \cdot 10^3$	105	147	0.01
$\mathbf{F}_{2^4}$	225	9	$2.57 \cdot 10^9$	2 025	2 025	$1.13 \cdot 10^4$
$\mathbf{F}_{2^5}$	961	9	$3.10 \cdot 10^{10}$	—	—	$8.11 \cdot 10^5$
	(Sym.) 31	13	$3.49 \cdot 10^6$	2 015	2 015	6.24
$\mathbf{F}_{2^6}$	(Sym.) 63	15	$2.21 \cdot 10^{10}$	21	21	$6.63 \cdot 10^4$
$\mathbf{F}_{2^7}$	(Sym.) 127	15	$1.34 \cdot 10^{12}$	—	—	$6.17 \cdot 10^6$
$\mathbf{F}_{3^2}$	16	3	3	4	16	0.00
$\mathbf{F}_{3^3}$	169	6	$2.42 \cdot 10^5$	11 843	105 963	1.08
$\mathbf{F}_{3^4}$	1 600	8	$2.27 \cdot 10^{11}$	—	—	$1.08 \cdot 10^7$
	(Sym.) 40	9	$1.10 \cdot 10^5$	234	615 240	0.45
$\mathbf{F}_{3^5}$	(Sym.) 121	11	$2.66 \cdot 10^9$	121	121	$1.45 \cdot 10^4$
$\mathbf{F}_{3^6}$	(Sym.) 364	12	$3.01 \cdot 10^{12}$	—	—	$4.50 \cdot 10^7$

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  - We can cherry-pick the one with minimum number of additions and scalar multiplications
- Work in progress and perspectives
  - Lifting formulae for higher-characteristic or characteristic-0 fields
  - Find formulae for your bilinear application!

Thank you for your attention

Questions?

Representing elements

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  - machine-specific bound on N: ≤ 64, ≤ 128 (SSE2), or ≤ 256 (AVX)

Representing subspaces

- Subspaces $W \subset V$ represented using matrices in row echelon form

$$W = \text{RowSpace}\left(\overbrace{\begin{pmatrix} 0 & w_{1,j_1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & w_{2,j_2} & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & 0 & 0 & \ddots & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & w_{r,j_r} & \cdots \end{pmatrix}}^{\dim V = N \text{ columns}}\right\} \dim W = r \text{ rows}$$

where each pivot $w_{i,j_i} \neq 0$, and with $j_1 < j_2 < \cdots < j_r$

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  - ☹ waste of memory
  - ☺ easier to insert extra rows when expanding W

Reduction by a subspace

- Given a vector $v \in V$ and a subspace W:
  - sequentially cancel the contribution of pivot rows of W to v
  - stop when no pivot corresponds to the leading coefficient of v

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2: **if** $v = 0$ **then return** 0
3: **if** $v_i \neq 0$ **then**
4: **if** $w_{i,i} \neq 0$ **then** $v \leftarrow v - (v_i / w_{i,i}) w_i$
5: **else return** v
6: **end for**

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- Test if $v \in W$:
  - $v \in W$ if and only if $\texttt{reduce}(v, W) = 0$
- Construct $W \oplus \mathrm{Span}(g)$, for $g \in \mathcal{G} \setminus W$:
  - compute $\tilde{g} \leftarrow \texttt{reduce}(g, W)$
  - let i be the iteration at which the algorithm returns $\tilde{g}$ $(\neq 0)$
  - then $w_i \leftarrow \tilde{g}$

Reduction by a subspace

- Given a vector $v \in V$ and a subspace W:
  - sequentially cancel the contribution of pivot rows of W to v
  - stop when no pivot corresponds to the leading coefficient of v

```
1: for i ← 1 to N do
2:     if v = 0 then return 0
3:     if v_i ≠ 0 then
4:         if w_{i,i} ≠ 0 then v ← v − (v_i / w_{i,i}) w_i
5:                        else return v
6: end for
```

- Test if $v \in W$:
  - $v \in W$ if and only if $\texttt{reduce}(v, W) = 0$
- Construct $W \oplus \mathrm{Span}(g)$, for $g \in \mathcal{G} \setminus W$:
  - compute $\tilde{g} \leftarrow \texttt{reduce}(g, W)$
  - let i be the iteration at which the algorithm returns $\tilde{g}$ $(\neq 0)$
  - then $w_i \leftarrow \tilde{g}$
- Improvement: since W is constructed incrementally
  - keep the vectors of $\mathcal{G}$ reduced by W at all times
  - when expanding W by $\mathrm{Span}(g)$, reduce $\mathcal{G}$ by g only

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$$W = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & w_{j_1,j_1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & w_{j_2,j_2} & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & w_{j_r,j_r} & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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  - batch 64, 128, or 256 products $g^{\mathsf{T}} \cdot (W' - \mathrm{Id})$