Impedance Transmission Conditions for the Electric Potential across a Highly Conductive Casing

Hélène Barucq, Aralar Erdozain, Victor Péron

Impedance Transmission Conditions for the Electric Potential across a Highly Conductive Casing

Hélène Barucq, Aralar Erdozain, Victor Péron

RESEARCH REPORT
N° 8998
December 8, 2016
Project-Team Magique 3D

ISSN 0249-6399 ISRN INRIA/RR--8998--FR+ENG

Impedance Transmission Conditions for the Electric Potential across a Highly Conductive Casing

Hélène Barucq*, Aralar Erdozain*, Victor Péron * †

Project-Team Magique 3D

Research Report n° 8998 — December 8, 2016 — 88 pages

Abstract: Borehole resistivity measurements are a common procedure when trying to obtain a better characterization of the Earth's subsurface. The use of a casing surrounding the borehole highly complicates the numerical simulations due to a large contrast between the conductivities of the casing and the rock formations. In this work, we consider the casing to be a thin layer of uniform thickness and motivated by realistic scenarios, we assume that the conductivity of such casing is proportional to the thickness of the casing to the power of -3. We derive Impedance Transmission Conditions (ITCs) for the static (zero frequency) electric potential for a 2D configuration. Then, we analyse these models by proving stability and convergence results. Next, we asses the numerical performance of these models by employing a Finite Element Method. Finally we present present asymptotic models for similar configurations including the time-harmonic configuration and a 3D axisymmetric scenario.

Key-words: Asymptotic Models, Impedance Conditions, Electric Potential, Borehole, Casing, Resistivity, Conductivity.

* INRIA Bordeaux Sud-Ouest, Team magique 3D, Pau, France
† Université de Pau et des Pays de l'Adour, Pau, France

RESEARCH CENTRE
BORDEAUX – SUD-OUEST

200 avenue de la Vieille Tour
33405 Talence Cedex

Conditions d'Impédance pour le Potentiel Électrique à travers un tube métallique fortement conducteur

Résumé : Les mesures de résistivité en forage sont communément utilisées pour obtenir une meilleure caractérisation du sous-sol de la Terre. Pour obtenir de telles mesures, on utilise typiquement un tube métallique qui protège le forage, mais cela complique énormément la simulation numérique à cause du fort contraste entre les conductivités du tube et des formations rocheuses. Dans ce travail , motivé par des configurations réalistes, on considère que la conductivité du tube est proportionnelle à l'épaisseur du tube à la puissance -3. On développe des conditions de transmission d'impédance (ITCs en Anglais) pour le potentiel électrique dans le cas statique, dans un domaine bidimensionnel. On présente la construction des modèles asymptotiques, validés par des résultats de convergence. On illustre les résultats théoriques avec des simulations numériques obtenues en utilisant une discrétisation par éléments finis. On présente aussi des modèles asymptotiques pour d'autres problèmes et configurations, à fréquence non-nulle et en 3D.

Mots-clés : Modèles Asymptotiques, Conditions d'Impedance, Potentiel Électrique, Forage, tube, Résistivité, Conductivité.

Contents

Introduction

Resistivity measurements are commonly used when trying to obtain a better characterization of the earth's subsurface. The standard procedure for acquiring these resistivity measurements consists in employing one or several transmitters and receiver antennas. These transmitters and receivers, which we will refer to as the instrument, are located inside a well, then they are employed both to transmit electromagnetic waves into the layer formations and record the received waves. According to the results shown in [26, 27], the second derivative of electric potential in the vertical direction, measured at the receiving antennas, can be employed to determine the conductivity of the different layer formations composing the Earth's subsurface. This technique has been widely employed in the literature for acquiring borehole resistivity measurements, we refer the reader to [32, 33, 34, 29, 35, 36, 9, 47] for more information regarding this matter.

Electrical logging through casing is of special interest because the well is often surrounded by a steel-made casing. The use of such casings allows to protect the well and avoid possible collapses on one hand, but it also highly complicates the numerical simulations for the electric potential on the other hand due to the high conductivity and thinness of the casing compared to that of the layer formations. Thus, when performing these kind of studies, the results are often inaccurate or simply too costly to be performed in real time.

These kind of problems have been faced by two different approaches, the use of analytical methods and the use of numerical methods. The use of analytical methods [28, 23, 37] limits the types of geometries that one can consider, so it is not very suitable for modelling realistic physical configurations. The use of numerical methods seems the best answer for dealing with complex geometries. A wide range of techniques can be found regarding the numerical methods. The Discontinuous Petrov-Galerking method [15, 58], the Isogeometric analysis [24, 48] and the hp-adaptive Finite Element Method [31, 30, 32, 55] are examples of techniques which are worth mentioning. However, this option becomes challenging too due to the high electrical conductivity contrast between the metallic casing and the layer formations, as well as the small thickness of the casing. In particular, when dealing with this kind of thin layers, the computational cost greatly increases when trying to mesh it. Moreover, the numerical methods employed to solve these problems do not perform well when high contrasted media is considered. These facts lead to an unavoidable increase of the computational cost, so it is relevant to avoid the thin layer by employing mathematical techniques which allow to construct reduced problems involving appropriate boundary or transmission conditions

To overcome these difficulties we adopt an asymptotic method which is motivated by realistic configurations [36], where the conductivity in the casing takes much higher values than that in the layer formations. We intend to work in the context of this application for which we assume that the conductivity in the casing has the following form

$$\sigma_{\text{lay}} \approx \varepsilon^{-3},$$

where ε denotes the thickness of the casing, which is presented as a thin layer of uniform thickness. We can motivate this choice according to the paper [36], where we can observe the following values for the conductivity and the thickness of the casing

$$\begin{cases} \quad \varepsilon = 1.27 \cdot 10^{-2} \text{ m}, \\ \sigma_{\text{lay}} = 4.34 \cdot 10^{6} \ \Omega^{-1} \text{ m}^{-1}. \end{cases}$$

From these values we infer the following relation between these physical parameters

$$\sigma_{\text{lay}} = 8.89 \cdot \varepsilon^{-3}.$$

In this framework our aim is to derive Impedance Transmission Conditions (ITCs) for the electric potential across such a casing. The naturally small thickness of the casing compared to the rest of the domain makes it ideal for applying this kind of method. The concept of Impedance Conditions (ICs) and ITCs is classical in the modelling of wave propagation phenomena, such a condition is derived by performing an asymptotic expansion and is designed to replace one part of the computational domain. Asymptotic techniques are widely employed in the field of wave propagation problems, for instance we cite the works [8, 7, 21, 22, 5, 25] related to boundary layer phenomena in Electromagnetism (skin effect and eddy current problem).

Similar studies regarding the derivation ICs for electromagnetism include [11, 45, 46, 20, 1, 54] where ICs are derived to substitute a thin layer present in one border of the domain. The question of ITCs is more related to the present work, but it is also more delicate than that of ICs. Even so, we can also find a wide variety of works related to this topic, [17, 16, 13, 10, 46, 41, 18, 50, 53, 52, 44, 38, 49].

This study is performed in the framework of high contrasted media where the physical parameters have a dependence on the thickness of the thin layer. Several works can be found with similarities in this matter, for instance, in [51], the authors derive ITCs for eddy current models with a dependence on the conductivity parameter of the thin layer of the form ε^{-1} and ε^{-2}. In the same way, in [42], we find a thin layer problem for the time-harmonic Maxwell equations, whose conductivity depends on the thickness of the thin layer in the form of ε^{-2}. In [43] and [17], where a problem for the static potential and an electromagnetic problem is considered respectively, and in both works a resistive thin layer is present.

There exist also similar studies regarding the derivation of ITCs for other physical models, for example we can mention [4] regarding the study of Elastodynamics, [39, 40] regarding the study of a problem with elastic and acoustic media and [14] in the field of Acoustics. There exist also models where the physical parameters depend on the thickness of the thin layer, [2] perfoms a study about the problem of an elastic shell-like inclusion with a rigidity of the form ε^{-1} and ε^{-3}.

In this work, we consider non-smooth computational domains, which include vertices and edges. In general, this framework greatly complicates the analysis compared to the smooth case (see for example [12]) and the presence of geometrical singularities (such as corners) reduces the performance of standard impedance conditions, see for example [5, 6, 56, 57]. In this work, we consider mainly a transmission problem for the electric potential

$$\operatorname{div}\left[(\sigma - i\epsilon_0\omega)\nabla u\right] = f \quad \text{in} \quad \Omega,$$

with an homogeneous Dirichlet boundary condition. We first consider the domain Ω to be a rectangular shaped domain in $\mathbb{R}^2$, and then we consider the domain Ω to be an axisymmetric borehole shaped domain in $\mathbb{R}^3$. This domain is composed of three subdomains $\Omega^\varepsilon_{\text{int}}$, $\Omega^\varepsilon_{\text{ext}}$ and $\Omega^\varepsilon_{\text{lay}}$, where the last one corresponds to the casing and is a thin layer of uniform thickness. Here ω represents the frequency and ϵ_0 represents the permittivity. The parameter σ corresponds to the conductivity and it is a piecewise constant function that takes different values in each subdomain. The function f corresponds to the right-hand side and it is a function that vanishes in the casing.

In this framework, we address the issue of ITCs for u as ε tends to zero. We derive two different classes of ITCs employing different approaches. The first one consists in deriving ITCs across the casing itself, whereas the second approach tackles the problem by deriving ITCs on an artificial interface located in the middle of the casing. Both classes have their advantages and drawbacks, matter which we shall discuss in this work. We shall present the mathematical justification for these ITCs and we shall also concentrate on studying the numerical performance of the models derived. First class of ITCs appear as second and fourth order approximations whereas second class of ITCs are derived from order 1, up to fourth order 4.

The asymptotic method we follow can be summarized in the following steps. First of all we perform a scaling in the subdomain corresponding to the casing, $\Omega^{\varepsilon}_{\text{lay}}$, along the direction normal to the thin layer. Then we perform an *Ansatz* in form of power expansion of ε and we obtain a collection of problems. They can be alternately solved to determine the elementary problems satisfied by each term of the asymptotic expansion. Then we truncate the series and collect the first terms of the expansion to infer equivalent conditions by neglecting residual terms depending on ε. Finally we prove convergence results for the derived asymptotic models. We follow this methodology for both the 2D and the 3D configurations.

This document is divided in several sections. We begin by presenting the model problem we are interested in, we study the well posedness of this problem and we present the first step towards the derivation of asymptotic models. Sections 2 and 3 are devoted to the formal calculi of the derivation of the asymptotic models for the two different classes. Section 4 corresponds to the validation of the asymptotic models derived for both the first and second classes. Then, Section 5 presents some numerical results to illustrate the theoretical results of the previous sections, and finally, the last section corresponds to an appendix summarizing similar results for other configurations of interest, including a time-harmonic problem and a 3D axisymmetric scenario.

1 Model problem and scaling

This section acts as an introduction to the following two sections. Here, we present a transmission problem for the electric potential in a configuration where a highly conductive thin layer of uniform thickness ε is present. Later on, in the following two sections, we will employ this model problem for deriving two different classes of ITCs employing two different approaches which correspond to these two sections.

In this section we first present the model problem we are interested in studying. Then, we begin the process of deriving asymptotic models by performing the first step of the procedure, which corresponds to a scaling in the part of the domain corresponding to the casing. This step is common to both approaches we have considered.

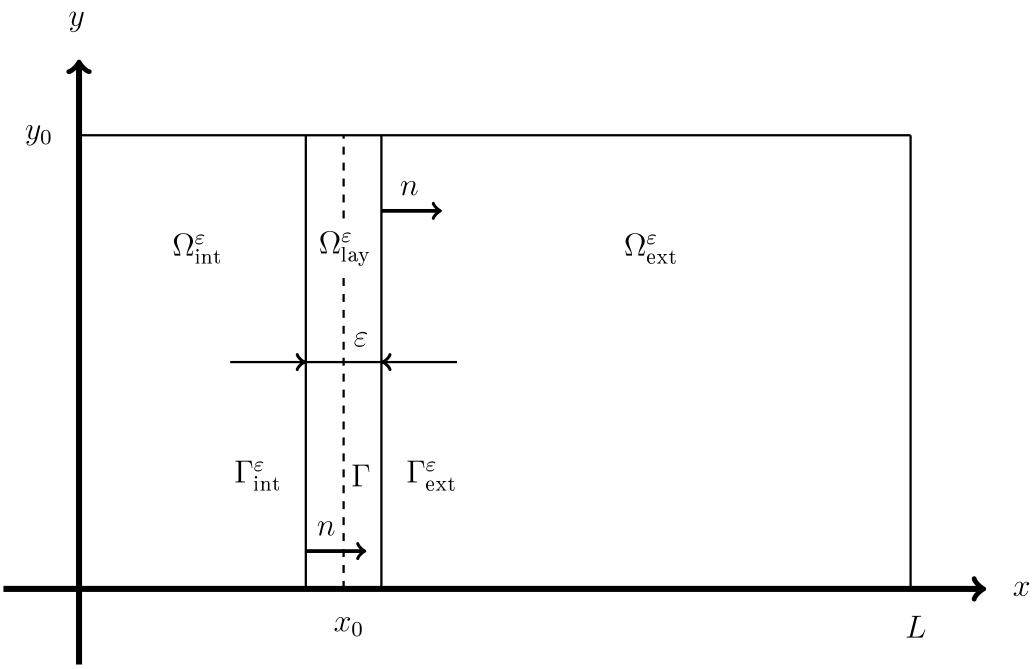


Figure 1: Domain of interest, composed of a thin layer, an interior domain and an exterior domain.

Let $\Omega \subset \mathbb{R}^2$ be the domain of interest described in Figure 1. The domain Ω is rectangular shaped and is composed of three rectangular shaped subdomains $\Omega^\varepsilon_{\text{int}}, \Omega^\varepsilon_{\text{ext}}$ and $\Omega^\varepsilon_{\text{lay}}$. The subdomain $\Omega^\varepsilon_{\text{lay}}$ is a thin layer of uniform thickness $\varepsilon > 0$. We denote the interfaces the interface between $\Omega^\varepsilon_{\text{int}}$ and $\Omega^\varepsilon_{\text{lay}}$ by $\Gamma^\varepsilon_{\text{int}}$, and the interface between $\Omega^\varepsilon_{\text{lay}}$ and $\Omega^\varepsilon_{\text{ext}}$ by $\Gamma^\varepsilon_{\text{ext}}$. In this domain, we study the equations of the static electric potential, which read as follows

$$\operatorname{div}(\sigma \nabla u) = f, \tag{1}$$

along with an homogeneous Dirichlet boundary condition. Here, u represents the electric potential, σ is the conductivity and f stands for a current source. The conductivity is a piecewise constant function, with a different value in each subdomain. Specifically, the value of the conductivity inside the thin layer $\Omega^\varepsilon_{\text{lay}}$ is much larger than the one in the other subdomains and we assume that it depends on the parameter ε. We consider a conductivity of the following form

$$\sigma = \begin{cases} \sigma_{\text{int}} & \text{in} \quad \Omega^{\varepsilon}_{\text{int}}, \\ \sigma_{\text{lay}} = \widehat{\sigma}_0 \varepsilon^{-3} & \text{in} \quad \Omega^{\varepsilon}_{\text{lay}}, \\ \sigma_{\text{ext}} & \text{in} \quad \Omega^{\varepsilon}_{\text{ext}}, \end{cases}$$

where $\widehat{\sigma}_0 > 0$ is a given constant. We assume that the right-hand side f, is a piecewise smooth function independent of ε, and it vanishes inside the layer.

$$f = \begin{cases} f_{\text{int}} & \text{in} \quad \Omega^{\varepsilon}_{\text{int}}, \\ f_{\text{lay}} = 0 & \text{in} \quad \Omega^{\varepsilon}_{\text{lay}}, \\ f_{\text{ext}} & \text{in} \quad \Omega^{\varepsilon}_{\text{ext}}. \end{cases}$$

It is possible to prove that Problem (1) has a unique solution $u \in H^1_0(\Omega)$. Then, representing the solution u in each subdomain as follows

$$u = \begin{cases} u_{\text{int}} & \text{in} \quad \Omega^{\varepsilon}_{\text{int}}, \\ u_{\text{lay}} & \text{in} \quad \Omega^{\varepsilon}_{\text{lay}}, \\ u_{\text{ext}} & \text{in} \quad \Omega^{\varepsilon}_{\text{ext}}, \end{cases}$$

the Problem (1) reads as

$$\begin{cases} \sigma_{\text{int}} \Delta u_{\text{int}} = f_{\text{int}} & \text{in} \quad \Omega^{\varepsilon}_{\text{int}}, \\ \sigma_{\text{ext}} \Delta u_{\text{ext}} = f_{\text{ext}} & \text{in} \quad \Omega^{\varepsilon}_{\text{ext}}, \\ \Delta u_{\text{lay}} = 0 & \text{in} \quad \Omega^{\varepsilon}_{\text{lay}}, \\ u_{\text{int}} = u_{\text{lay}} & \text{on} \quad \Gamma^{\varepsilon}_{\text{int}}, \\ u_{\text{lay}} = u_{\text{ext}} & \text{on} \quad \Gamma^{\varepsilon}_{\text{ext}}, \\ \sigma_{\text{int}} \partial_n u_{\text{int}} = \widehat{\sigma}_0 \varepsilon^{-3} \partial_n u_{\text{lay}} & \text{on} \quad \Gamma^{\varepsilon}_{\text{int}}, \\ \widehat{\sigma}_0 \varepsilon^{-3} \partial_n u_{\text{lay}} = \sigma_{\text{ext}} \partial_n u_{\text{ext}} & \text{on} \quad \Gamma^{\varepsilon}_{\text{ext}}, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{2}$$

where ∂_n represents the normal derivative in the direction of the normal vector, inwardly directed to $\Omega^{\varepsilon}_{\text{ext}}$ on $\Gamma^{\varepsilon}_{\text{ext}}$, and outwardly directed to $\Omega^{\varepsilon}_{\text{int}}$ on $\Gamma^{\varepsilon}_{\text{int}}$, see Figure 1.

Remark 1. *In this document we present the results for a problem where we consider homogeneous Dirichlet boundary conditions. Similar results can be found in [19] for a problem where mixed (Dirichlet and Neumann) conditions are considered, configuration which is more relevant towards the applications.*

First of all, before starting with the derivation of a multiscale expansion, we will prove that there exists a solution to Problem (2) and that this solution is unique in $H^1_0(\Omega)$. Instead of considering directly Problem (2), we will consider a similar one. This problem is defined employing the same configuration we have defined in this section. We remember that the constants σ_{int}, σ_{ext} and $\widehat{\sigma}_0$ are strictly positive as this fact will play an important role in the following proofs. We recall Figure 1 shows the configuration of the domain we are working with. In this framework we consider the following problem

$$\left\{\begin{aligned} \sigma_{\text{int}}\Delta u_{\text{int}} &= f_{\text{int}} && \text{in} && \Omega^{\varepsilon}_{\text{int}},\\ \sigma_{\text{ext}}\Delta u_{\text{ext}} &= f_{\text{ext}} && \text{in} && \Omega^{\varepsilon}_{\text{ext}},\\ \widehat{\sigma}_0\varepsilon^{-3}\Delta u_{\text{lay}} &= f_{\text{lay}} && \text{in} && \Omega^{\varepsilon}_{\text{lay}},\\ u_{\text{int}} &= u_{\text{lay}} && \text{on} && \Gamma^{\varepsilon}_{\text{int}},\\ u_{\text{lay}} &= u_{\text{ext}} && \text{on} && \Gamma^{\varepsilon}_{\text{ext}},\\ \widehat{\sigma}_0\varepsilon^{-3}\partial_n u_{\text{lay}} - \sigma_{\text{int}}\partial_n u_{\text{int}} &= g_{\text{int}} && \text{on} && \Gamma^{\varepsilon}_{\text{int}},\\ \widehat{\sigma}_0\varepsilon^{-3}\partial_n u_{\text{lay}} - \sigma_{\text{ext}}\partial_n u_{\text{ext}} &= g_{\text{ext}} && \text{on} && \Gamma^{\varepsilon}_{\text{ext}},\\ u &= 0 && \text{on} && \partial\Omega. \end{aligned}\right. \tag{3}$$

This problem is similar to Problem (2) and it generalises it as the right-hand side f_{lay} does not vanish inside the layer any more and as it includes the right-hand side functions g_{int} and g_{ext}. The results obtained for this problem will be useful in later sections, where we prove the convergence of the asymptotic models.

First of all we can write the variational formulation of Problem (3). Assuming $f \in L^2(\Omega)$, $g_{\text{int}} \in L^2(\Gamma^{\varepsilon}_{\text{int}})$ and $g_{\text{ext}} \in L^2(\Gamma^{\varepsilon}_{\text{ext}})$, we look for $u \in H^1_0(\Omega)$, such that for all $w \in H^1_0(\Omega)$

$$a(u,w) = l(w) \tag{4}$$

where

$$a(u,w) = \sigma_{\text{int}}\int_{\Omega^{\varepsilon}_{\text{int}}} \nabla u\cdot\nabla w\,\mathrm{d}x + \sigma_{\text{ext}}\int_{\Omega^{\varepsilon}_{\text{ext}}} \nabla u\cdot\nabla w\,\mathrm{d}x + \widehat{\sigma}_0\varepsilon^{-3}\int_{\Omega^{\varepsilon}_{\text{lay}}} \nabla u\cdot\nabla w\,\mathrm{d}x,$$

$$l(w) = -\int_{\Omega^{\varepsilon}_{\text{int}}} f_{\text{int}} w\,\mathrm{d}x - \int_{\Omega^{\varepsilon}_{\text{ext}}} f_{\text{ext}} w\,\mathrm{d}x - \int_{\Omega^{\varepsilon}_{\text{lay}}} f_{\text{lay}} w\,\mathrm{d}x + \int_{\Gamma^{\varepsilon}_{\text{int}}} g_{\text{int}} w\,\mathrm{d}s + \int_{\Gamma^{\varepsilon}_{\text{ext}}} g_{\text{ext}} w\,\mathrm{d}s.$$

Now we give the theorem and proof that guarantees the existence and uniqueness of a solution to this problem and which presents some uniform estimates for the solution, but before we will present the following notation for the different norms we employ in this document.

Notation 1. *For any function $u \in L^2(\Omega)$, we denote the norm in L^2 by*

$$\|u\|_{0,\Omega} = \|u\|_{L^2(\Omega)}\,.$$

In the same way, for any function $u \in H^1(\Omega)$, we denote the norm in H^1 by

$$\|u\|_{1,\Omega} = \left(\|u\|^2_{0,\Omega} + \|\nabla u\|^2_{0,\Omega}\right)^{\frac{1}{2}}.$$

Theorem 1. *For all $\varepsilon > 0$ there exists a unique $u \in H_0^1(\Omega)$, solution to Problem (4) with data $f \in L^2(\Omega)$, $g_{int} \in L^2(\Gamma^\varepsilon_{int})$, $g_{ext} \in L^2(\Gamma^\varepsilon_{ext})$. Moreover, there exists $\varepsilon_0 > 0$ and a constant $C > 0$, such that for all $\varepsilon \in (0, \varepsilon_0)$,*

$$\|u\|_{1,\Omega} \leq C \left(\|f\|_{0,\Omega} + \|g_{int}\|_{0,\Gamma^\varepsilon_{int}} + \|g_{ext}\|_{0,\Gamma^\varepsilon_{ext}} \right).$$

Proof. The proof can be summarized in the following steps. As the bilinear form a is coercive and continuous in $H_0^1(\Omega)$, and the linear form l is continuous in $H_0^1(\Omega)$, the existence and uniqueness of a weak solution follows from the Lax-Milgram Lemma. Then, for proving the uniform estimates, we select ε_0 as

$$\varepsilon_0 = \sqrt[3]{\frac{\widehat{\sigma}_0}{\min(\sigma_{\text{int}}, \sigma_{\text{ext}})}}.$$

Then for $\varepsilon < \varepsilon_0$ and employing the Poincaré inequality, there exists a constant $k_1 > 0$ such that

$$a(w, w) \geq \frac{1}{k_1} \min\left(\sigma_{\text{int}}, \sigma_{\text{ext}}\right) \|w\|^2_{1,\Omega}. \tag{5}$$

Applying a trace theorem and Cauchy-Schwarz inequality to the definition of l we obtain that for a constant $k_2 > 0$, we have

$$|l(w)| \leq k_2 \|w\|_{1,\Omega} \left(\|f_{\text{int}}\|_{0,\Omega^\varepsilon_{\text{int}}} + \|f_{\text{ext}}\|_{0,\Omega^\varepsilon_{\text{ext}}} + \|f_{\text{lay}}\|_{0,\Omega^\varepsilon_{\text{lay}}} + \|g_{\text{int}}\|_{0,\Gamma^\varepsilon_{\text{int}}} + \|g_{\text{ext}}\|_{0,\Gamma^\varepsilon_{\text{ext}}} \right). \tag{6}$$

Finally, employing Equations (5) and (6) we obtain

$$\|u\|_{1,\Omega} \leq C \left(\|f_{\text{int}}\|_{0,\Omega^\varepsilon_{\text{int}}} + \|f_{\text{ext}}\|_{0,\Omega^\varepsilon_{\text{ext}}} + \|f_{\text{lay}}\|_{0,\Omega^\varepsilon_{\text{lay}}} + \|g_{\text{int}}\|_{0,\Gamma^\varepsilon_{\text{int}}} + \|g_{\text{ext}}\|_{0,\Gamma^\varepsilon_{\text{ext}}} \right),$$

where $C = \dfrac{k_2 k_1}{\min\left(\sigma_{\text{int}}, \sigma_{\text{ext}}\right)}$. □

Introduction of a scaling

A key point for the derivation of a multiscale expansion for the solution to Problem (2) consists in performing a scaling along the normal direction to the thin layer. We begin by describing the domain $\Omega^\varepsilon_{\text{lay}}$ in the following way

$$\Omega^\varepsilon_{\text{lay}} = \left\{ \gamma(y) + \varepsilon X n : \gamma(y) \in \Gamma, X \in \left(-\frac{1}{2}, \frac{1}{2} \right) \right\},$$

where γ is a parametrization of the curve Γ (see Figure 1), which is defined as

$$\gamma(y) = (x_0, y), \text{ for all } y \in (0, y_0)$$

and $n = (1, 0)$ is the normal vector to the curve Γ. This geometry of the domain induces the following scaling

$$x = x_0 + \varepsilon X \quad \Leftrightarrow \quad X = \varepsilon^{-1} (x - x_0).$$

As a consequence, we have

$$\partial_X^k = \varepsilon^k \partial_x^k, \quad k \in \mathbb{N}.$$

This scaling allows us to write the Laplace operator in the following way

$$\Delta = \partial_x^2 + \partial_y^2 = \varepsilon^{-2}\partial_X^2 + \partial_y^2.$$

Besides, we notice that on the interfaces $\Gamma_{\text{int}}^{\varepsilon}$ and $\Gamma_{\text{ext}}^{\varepsilon}$ we can rewrite the normal derivative in the following form $\partial_n = \partial_x = \varepsilon^{-1}\partial_X$. Finally we denote by U the function that meets

$$u_{\text{lay}}(x,y) = u_{\text{lay}}(x_0 + \varepsilon X, y) = U(X,y), \qquad (X,y) \in \left(-\frac{1}{2},\frac{1}{2}\right) \times (0,y_0)\,.$$

We rewrite Equations (2) with the newly defined variables and functions and they take the following form

$$\begin{cases} \sigma_{\text{int}}\Delta u_{\text{int}} = f_{\text{int}} & \text{in} \quad \Omega_{\text{int}}^{\varepsilon}, \\ \sigma_{\text{ext}}\Delta u_{\text{ext}} = f_{\text{ext}} & \text{in} \quad \Omega_{\text{ext}}^{\varepsilon}, \\ \varepsilon^{-2}\partial_X^2 U + \partial_y^2 U = 0 & \text{in} \left(\frac{-1}{2},\frac{1}{2}\right) \times (0,y_0)\,, \\ u_{\text{int}}\left(x_0 - \frac{\varepsilon}{2}, y\right) = U\left(-\frac{1}{2}, y\right) & y \in (0,y_0), \\ u_{\text{ext}}\left(x_0 + \frac{\varepsilon}{2}, y\right) = U\left(\frac{1}{2}, y\right) & y \in (0,y_0), \\ \sigma_{\text{int}}\partial_n u_{\text{int}}\left(x_0 - \frac{\varepsilon}{2}, y\right) = \widehat{\sigma}_0\varepsilon^{-4}\partial_X U\left(-\frac{1}{2}, y\right) & y \in (0,y_0), \\ \sigma_{\text{ext}}\partial_n u_{\text{ext}}\left(x_0 + \frac{\varepsilon}{2}, y\right) = \widehat{\sigma}_0\varepsilon^{-4}\partial_X U\left(\frac{1}{2}, y\right) & y \in (0,y_0), \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{7}$$

2 First class of ITCs

The objective of this section is the derivation of asymptotic models for the reference model (2). In Section 2.1 we make an asymptotic expansion of the solution in power series of ε. Then, by truncating this series and neglecting higher order terms in ε, we derive approximate models composed of equivalent transmission conditions across the thin layer in Section 2.2. The last section, Section 2.3, shows a comparison of the models derived here with more standard models.

2.1 Construction of a multiscale expansion

First of all, we define the jump and mean value of a function across the thin layer.

Definition 1. *Let u be a smooth function defined over Ω. We define its jump and mean value across the thin layer as*

$$\begin{cases} [u]_{\Gamma^\varepsilon} = u_{ext}|_{\Gamma^\varepsilon_{ext}} - u_{int}|_{\Gamma^\varepsilon_{int}}, \\ \{u\}_{\Gamma^\varepsilon} = \dfrac{1}{2}\left(u_{ext}|_{\Gamma^\varepsilon_{ext}} + u_{int}|_{\Gamma^\varepsilon_{int}}\right). \end{cases}$$

Now we proceed to derive the asymptotic expansion. To begin with, we perform an *Ansatz* in the form of power series of ε for the solution to Problem (7). We look for solutions

$$\begin{cases} u_{\text{int}}(x,y) \approx \displaystyle\sum_{k\geq 0} \varepsilon^k u^k_{\text{int}}(x,y) & \text{in} \quad \Omega^\varepsilon_{\text{int}}, \\ u_{\text{ext}}(x,y) \approx \displaystyle\sum_{k\geq 0} \varepsilon^k u^k_{\text{ext}}(x,y) & \text{in} \quad \Omega^\varepsilon_{\text{ext}}, \\ U(X,y) \approx \displaystyle\sum_{k\geq 0} \varepsilon^k U^k(X,y) & \text{in} \quad \left(-\dfrac{1}{2},\dfrac{1}{2}\right)\times(0,y_0). \end{cases} \tag{8}$$

2.1.1 Equations for the coefficients of the electric potential

Substituting the previous expressions (8) into the Equations (7) and collecting the terms with the same powers in ε, for every $k \in \mathbb{N}$ we obtain the following set of equations

$$\begin{cases} \sigma_{\text{int}}\Delta u^k_{\text{int}}(x,y) = f_{\text{int}}(x,y)\delta^k_0 & \text{in} \quad \Omega^\varepsilon_{\text{ext}}, \\ \sigma_{\text{ext}}\Delta u^k_{\text{ext}}(x,y) = f_{\text{ext}}(x,y)\delta^k_0 & \text{in} \quad \Omega^\varepsilon_{\text{ext}}, \\ \partial^2_X U^k(X,y) = -\partial^2_y U^{k-2}(X,y) & \text{in} \quad \left(-\dfrac{1}{2},\dfrac{1}{2}\right)\times(0,y_0), \end{cases} \tag{9}$$

along with the following transmission conditions

$$\begin{cases} U^k\left(-\dfrac{1}{2},y\right) = u^k_{\text{int}}\left(x_0-\dfrac{\varepsilon}{2},y\right) & y\in(0,y_0), \\ U^k\left(\dfrac{1}{2},y\right) = u^k_{\text{ext}}\left(x_0+\dfrac{\varepsilon}{2},y\right) & y\in(0,y_0), \\ \widehat{\sigma}_0\partial_X U^k\left(-\dfrac{1}{2},y\right) = \sigma_{\text{int}}\partial_n u^{k-4}_{\text{int}}\left(x_0-\dfrac{\varepsilon}{2},y\right) & y\in(0,y_0), \\ \widehat{\sigma}_0\partial_X U^k\left(\dfrac{1}{2},y\right) = \sigma_{\text{ext}}\partial_n u^{k-4}_{\text{ext}}\left(x_0+\dfrac{\varepsilon}{2},y\right) & y\in(0,y_0), \end{cases} \tag{10}$$

and the following boundary conditions

$$\begin{cases} u^k(0,y) = u^k(L,y) = 0 & y\in(0,y_0), \\ u^k(x,0) = u^k(x,y_0) = 0 & x\in\left(0,x_0-\dfrac{\varepsilon}{2}\right)\cup\left(x_0+\dfrac{\varepsilon}{2},L\right), \\ U^k(X,0) = U^k(X,y_0) = 0 & X\in\left(-\dfrac{1}{2},\dfrac{1}{2}\right), \end{cases} \tag{11}$$

where δ_0^k represents the Kronecker symbol. Besides, we will also need the following equation obtained by applying the fundamental theorem of calculus for a smooth function U^k,

$$\int_{\frac{-1}{2}}^{\frac{1}{2}} \partial_X^2 U^k(X,y)\,\mathrm{d}X = \partial_X U^k\left(\frac{1}{2},y\right) - \partial_X U^k\left(-\frac{1}{2},y\right). \tag{12}$$

If we apply the third equation of (9) on the left hand side and the third and fourth equations of (10) on the right-hand side we obtain the following compatibility condition

$$-\int_{\frac{-1}{2}}^{\frac{1}{2}} \partial_y^2 U^{k-2}(X,y)\,\mathrm{d}X = \frac{1}{\widehat{\sigma}_0}\left[\sigma\partial_n u^{k-4}\right]_{\Gamma^\varepsilon}(y). \tag{13}$$

In these equations, we adopt the convention that the terms with negative indices are equal to 0. Employing these equations ((9) - (13)) we can deduce the elementary problems satisfied outside and inside the layer for any $k\in\mathbb{N}$. For that purpose, we employ the following algorithm composed of three steps.

2.1.2 Algorithm for the determination of the coefficients

We assume that the first terms of the expansion (8) up to the order ε^{k-1} have already been calculated, and we derive the equations for the k-th term. The first two steps consist in determining U^k and the third step consist in determining u^k_{int} and u^k_{ext}.

First step:

We begin by selecting the third equation from (9), along with third and fourth equations from (10), and we build the following differential problem in the variable X for U^k (the variable y plays the role of a parameter)

$$
\begin{cases}
\partial_X^2 U^k(X,y) = -\partial_y^2 U^{k-2}(X,y) & X \in \left(-\frac{1}{2}, \frac{1}{2}\right), \\
\widehat{\sigma}_0 \partial_X U^k \left(-\frac{1}{2}, y\right) = \sigma_{\text{int}} \partial_n u_{\text{int}}^{k-4} \left(x_0 - \frac{\varepsilon}{2}, y\right), \\
\widehat{\sigma}_0 \partial_X U^k \left(\frac{1}{2}, y\right) = \sigma_{\text{ext}} \partial_n u_{\text{ext}}^{k-4} \left(x_0 + \frac{\varepsilon}{2}, y\right).
\end{cases} \tag{14}
$$

There exists a solution U^k of (14) provided the compatibility condition (13) is satisfied. We deduce the expression of U^k up to a function in the variable y, $\varphi_0^k(y)$. The function U^k has the following form

$$
U^k(X,y) = V^k(X,y) + \varphi_0^k(y),
$$

where V^k represents the part of U^k that can be determined at this step and has the following form (see Proposition 1)

$$
V^k(X,y) = \begin{cases}
0 & \text{if} \quad k = 0,1,2,3, \\
\varphi_{k-2}^k(y) X^{k-2} + \varphi_{k-3}^k(y) X^{k-3} + \ldots + \varphi_1^k(y) X & \text{if} \quad k > 3.
\end{cases}
$$

The function φ_0^k represents the part of U^k that is determined at the second step.

Second step:

We employ the compatibility condition (13) (at rank $k+2$), along with third equation of (11) to write the following differential problem in the variable y for the function φ_0^k, involved into the expression of U^k.

$$
\begin{cases}
\frac{\mathrm{d}^2}{\mathrm{d}y^2} \varphi_0^k(y) = -\frac{1}{\widehat{\sigma}_0} \left[\sigma \partial_n u^{k-2}\right]_{\Gamma^\varepsilon}(y) - \int_{\frac{-1}{2}}^{\frac{1}{2}} \partial_y^2 V^k(X,y) \,\mathrm{d}X & y \in (0, y_0), \\
\varphi_0^k(0) = 0, \\
\varphi_0^k(y_0) = 0.
\end{cases} \tag{15}
$$

Solving this differential equation we obtain the function φ_0^k and thus the complete expression of U^k.

Third step:

Now we can finally derive the equations outside the layer by employing first and second equations of (9), first and second equations of (10), and first and second equations of (11). We infer that u_{int}^k and u_{ext}^k are defined independently in the two subdomains $\Omega_{\text{int}}^\varepsilon$ and $\Omega_{\text{ext}}^\varepsilon$ by the following differential problems

$$
\begin{cases}
\sigma_{\text{int}} \Delta u^k_{\text{int}} = f_{\text{int}} \delta^k_0 & \text{in} \quad \Omega^\varepsilon_{\text{int}}, \\[2ex]
u^k_{\text{int}} \left(x_0 - \dfrac{\varepsilon}{2}, y \right) = U^k \left(-\dfrac{1}{2}, y \right), & \\[2ex]
u^k_{\text{int}} = 0 & \text{on} \quad \partial\Omega \cap \partial\Omega^\varepsilon_{\text{int}}.
\end{cases}
$$

$$
\begin{cases}
\sigma_{\text{ext}} \Delta u^k_{\text{ext}} = f_{\text{ext}} \delta^k_0 & \text{in} \quad \Omega^\varepsilon_{\text{ext}}, \\[2ex]
u^k_{\text{ext}} \left(x_0 + \dfrac{\varepsilon}{2}, y \right) = U^k \left(\dfrac{1}{2}, y \right), & \\[2ex]
u^k_{\text{ext}} = 0 & \text{on} \quad \partial\Omega \cap \partial\Omega^\varepsilon_{\text{ext}}.
\end{cases}
\tag{16}
$$

Subsequently, it is convenient to define u^k, for $k \in \mathbb{N}$, by

$$
u^k = \begin{cases}
u^k_{\text{int}} & \text{in} \quad \Omega^\varepsilon_{\text{int}}, \\[1ex]
u^k_{\text{ext}} & \text{in} \quad \Omega^\varepsilon_{\text{ext}}.
\end{cases}
$$

We will now employ this algorithm to determine the first terms of the expansion.

2.1.3 First terms of the asymptotics

<u>Case k=0</u>

We consider Problem (14) for U^0

$$
\begin{cases}
\partial^2_X U^0(X, y) = 0 \quad X \in \left(\dfrac{-1}{2}, \dfrac{1}{2} \right), \\[2ex]
\partial_X U^0 \left(-\dfrac{1}{2}, y \right) = 0, \\[2ex]
\partial_X U^0 \left(\dfrac{1}{2}, y \right) = 0.
\end{cases}
$$

We deduce that the solution to this equation has the form $U^0(X, y) = \varphi^0_0(y)$. Then we employ (15) and we build the following problem for φ^0_0

$$
\begin{cases}
\dfrac{\mathrm{d}^2}{\mathrm{d}y^2} \varphi^0_0(y) = 0 \qquad y \in (0, y_0), \\[2ex]
\varphi^0_0(0) = 0, \\[1ex]
\varphi^0_0(y_0) = 0.
\end{cases}
$$

We deduce that $\varphi^0_0(y) = 0$ and thus $U^0(X, y) = 0$. Finally, employing (16), we obtain that the limit solution u^0 satisfies homogeneous Dirichlet boundary conditions on $\Gamma^\varepsilon_{\text{int}}$ and $\Gamma^\varepsilon_{\text{ext}}$. Thus, we write the problem satisfied by u^0 as

$$\begin{cases} \sigma_{\text{int}} \Delta u^0_{\text{int}} = f_{\text{int}} & \text{in} \quad \Omega^\varepsilon_{\text{int}}, \\ u^0_{\text{int}} = 0 & \text{on} \quad \partial\Omega^\varepsilon_{\text{int}}. \end{cases}$$
$$\begin{cases} \sigma_{\text{ext}} \Delta u^0_{\text{ext}} = f_{\text{ext}} & \text{in} \quad \Omega^\varepsilon_{\text{ext}}, \\ u^0_{\text{ext}} = 0 & \text{on} \quad \partial\Omega^\varepsilon_{\text{ext}}. \end{cases} \tag{17}$$

<u>Case k=1</u>

We consider Problem (14) for U^1

$$\begin{cases} \partial^2_X U^1(X,y) = 0 \quad X \in \left(\frac{-1}{2}, \frac{1}{2}\right), \\ \partial_X U^1\left(-\frac{1}{2}, y\right) = 0, \\ \partial_X U^1\left(\frac{1}{2}, y\right) = 0. \end{cases}$$

We deduce that the solution to this equation has the form $U^1(X,y) = \varphi^1_0(y)$. Then we employ (15) and we build the following problem for φ^1_0

$$\begin{cases} \frac{\mathrm{d}^2}{\mathrm{d}y^2}\varphi^1_0(y) = 0 \qquad y \in (0, y_0), \\ \varphi^1_0(0) = 0, \\ \varphi^1_0(y_0) = 0. \end{cases}$$

We deduce that $\varphi^1_0(y) = 0$ and thus $U^1(X,y) = 0$. Finally, employing (16) we can write the problem satisfied by u^1 outside the layer as two uncoupled problems

$$\begin{cases} \Delta u^1_{\text{int}} = 0 & \text{in} \quad \Omega^\varepsilon_{\text{int}}, \\ u^1_{\text{int}} = 0 & \text{on} \quad \partial\Omega^\varepsilon_{\text{int}}. \end{cases}$$
$$\begin{cases} \Delta u^1_{\text{ext}} = 0 & \text{in} \quad \Omega^\varepsilon_{\text{ext}}, \\ u^1_{\text{ext}} = 0 & \text{on} \quad \partial\Omega^\varepsilon_{\text{ext}}. \end{cases} \tag{18}$$

We deduce that $u^1 \equiv 0$.
<u>Case k=2</u>

We consider Problem (14) for U^2

$$\begin{cases} \partial_X^2 U^2(X,y) = 0 & X \in \left(\frac{-1}{2}, \frac{1}{2}\right), \\ \partial_X U^2\left(-\frac{1}{2}, y\right) = 0, \\ \partial_X U^2\left(\frac{1}{2}, y\right) = 0. \end{cases}$$

We deduce that the solution to this equation has the form $U^2(X,y) = \varphi_0^2(y)$. Then we employ (15) and we build the following problem for φ_0^2

$$\begin{cases} \frac{\mathrm{d}^2}{\mathrm{d}y^2}\varphi_0^2(y) = -\frac{1}{\widehat{\sigma}_0}\left[\sigma \partial_n u^0\right](y) & y \in (0, y_0), \\ \varphi_0^2(0) = 0, \\ \varphi_0^2(y_0) = 0. \end{cases}$$

We deduce that $\varphi_0^2(y)$ and thus $U^2(X,y)$ have the following form.

$$\begin{aligned} U^2(X,y) &= \varphi_0^2(y) \\ &= -\frac{1}{\widehat{\sigma}_0}\int_0^y (y-t)\left[\sigma\partial_n u^0\right]_{\Gamma^\varepsilon}(t)\,\mathrm{d}t + \frac{y}{\sigma_0 y_0}\int_0^{y_0}(y_0-t)\left[\sigma\partial_n u^0\right]_{\Gamma^\varepsilon}(t)\,\mathrm{d}t. \end{aligned} \tag{19}$$

We assume the integrals in the expression of U^2 make sense and we make the same assumption for the rest of integrals that appear in this section. Finally, employing (16) we can write the problem satisfied outside the layer by u^2 as two uncoupled problems

$$\begin{cases} \Delta u_{\text{int}}^2 = 0 & \text{in} \quad \Omega_{\text{int}}^\varepsilon, \\ u_{\text{int}}^2\left(x_0 - \frac{\varepsilon}{2}, y\right) = \varphi_0^2(y), \\ u_{\text{int}}^2 = 0 & \text{on} \quad \partial\Omega \cap \partial\Omega_{\text{int}}^\varepsilon. \end{cases}$$
$$\begin{cases} \Delta u_{\text{ext}}^2 = 0 & \text{in} \quad \Omega_{\text{ext}}^\varepsilon, \\ u_{\text{ext}}^2\left(x_0 + \frac{\varepsilon}{2}, y\right) = \varphi_0^2(y), \\ u_{\text{ext}}^2 = 0 & \text{on} \quad \partial\Omega \cap \partial\Omega_{\text{ext}}^\varepsilon. \end{cases} \tag{20}$$

Case k=3

We consider Problem (14) for U^3

$$\begin{cases} \partial_X^2 U^3(X,y) = 0 & X \in \left(\dfrac{-1}{2}, \dfrac{1}{2}\right), \\ \partial_X U^3\left(-\dfrac{1}{2}, y\right) = 0, & \\ \partial_X U^3\left(\dfrac{1}{2}, y\right) = 0. & \end{cases}$$

We deduce that the solution to this equation has the form $U^3(X,y) = \varphi_0^3(y)$. Then we employ (15) and we build the following problem for φ_0^3

$$\begin{cases} \dfrac{\mathrm{d}^2}{\mathrm{d}y^2}\varphi_0^3(y) = 0 & y \in (0, y_0), \\ \varphi_0^3(0) = 0, & \\ \varphi_0^3(y_0) = 0. & \end{cases}$$

We deduce that $\varphi_0^3(y) = 0$ and thus $U^3(X,y) = 0$. Finally, employing (16) we can write the problem satisfied outside the layer by u^3 as two uncoupled problems

$$\begin{cases} \Delta u^3_{\text{int}} = 0 & \text{in} & \Omega^\varepsilon_{\text{int}}, \\ u^3_{\text{int}} = 0 & \text{on} & \partial\Omega^\varepsilon_{\text{int}}. \end{cases}$$
$$\begin{cases} \Delta u^3_{\text{ext}} = 0 & \text{in} & \Omega^\varepsilon_{\text{ext}}, \\ u^3_{\text{ext}} = 0 & \text{on} & \partial\Omega^\varepsilon_{\text{ext}}. \end{cases} \tag{21}$$

We deduce that $u^3 \equiv 0$.

2.1.4 Recapitulation of the asymptotic expansion

Proposition 1. *The asymptotic expansion (8), has the following form*

$$\begin{cases} u_{int}(x,y) = u^0_{int}(x,y) + \varepsilon^2 u^2_{int}(x,y) + O\left(\varepsilon^4\right) & in \quad \Omega^\varepsilon_{int}, \\ u_{ext}(x,y) = u^0_{ext}(x,y) + \varepsilon^2 u^2_{ext}(x,y) + O\left(\varepsilon^4\right) & in \quad \Omega^\varepsilon_{ext}, \\ U\left(X,y\right) = \varepsilon^2 \varphi_0^2\left(y\right) + O\left(\varepsilon^4\right) & in \quad \left(-\dfrac{1}{2}, \dfrac{1}{2}\right) \times (0, y_0), \end{cases}$$

where the functions φ_0^2, u^0 and u^2 are defined by Equations (19), (17) and (20) respectively. Besides, for $k \in \mathbb{N}$, the solution U^k to Equation (14) has the following form

$$U^k\left(X,y\right) = \begin{cases} 0 & if \quad k = 0 \text{ or } k \text{ odd}, \\ \displaystyle\sum_{j=0}^{k-2} \varphi_j^k(y) X^j & if \quad k \text{ even}, \end{cases}$$

and the solution $u^k = (u^k_{int}, u^k_{ext})$ *to Problem* (16) *satisfies*

$$u^k_{int} \equiv u^k_{ext} \equiv 0, \qquad \textit{if} \quad k \textit{ odd.}$$

Proof. We perform the proof by induction on k. For $k = 0, 1, 2, 3$, we have already calculated the expressions of u^k and U^k in the previous section. Now let us assume that for all even number $i \in \mathbb{N}$, such that $i < k$, the function U^i has the form

$$U^i\,(X, y) = \varphi^i_{i-2}(y)X^{i-2} + \varphi^i_{i-3}(y)X^{i-3} + \ldots + \varphi^i_1(y)X + \varphi^i_0(y),$$

We begin by considering Problem (14) for any even number $k \geq 4$. Solving this problem we obtain a solution of the form

$$U^k\,(X, y) = \varphi^k_{k-2}(y)X^{k-2} + \varphi^k_{k-3}(y)X^{k-3} + \ldots + \varphi^k_1(y)X + \varphi^k_0(y),$$

where $k \geq 4$ and

$$\begin{cases} \varphi^k_1(y) = \dfrac{1}{\widehat{\sigma}_0}\left\{\sigma\partial_n u^{k-4}\right\}(y), \\[2ex] \varphi^k_2(y) = \dfrac{1}{2\widehat{\sigma}_0}\left[\sigma\partial_n u^{k-4}\right](y), \\[2ex] \varphi^k_{k-j}(y) = \dfrac{-\frac{\mathrm{d}^2}{\mathrm{d}y^2}\varphi^{k-2}_{k-j-2}(y)}{(k-j)(k-j-1)} \qquad j = 2, \ldots, k-2. \end{cases}$$

We denote by V^k the following function

$$V^k\,(X, y) = \varphi^k_{k-2}(y)X^{k-2} + \varphi^k_{k-3}(y)X^{k-3} + \ldots + \varphi^k_1(y)X.$$

This expression corresponds to the function V^k we have defined at the first step of the algorithm. The only thing left to prove is that if k is an odd number, $U^k \equiv 0$ and $u^k_{\mathrm{int}} \equiv u^k_{\mathrm{ext}} \equiv 0$. We assume that for all odd number $j \in \mathbb{N}$, such that $j < k$, $U^j \equiv 0$ and $u^j_{\mathrm{int}} \equiv u^j_{\mathrm{ext}} \equiv 0$. Employing Equation (14), we have the following problem for U^k

$$\begin{cases} \partial^2_X U^k(X, y) = -\partial^2_y U^{k-2}(X, y) & X \in \left(-\dfrac{1}{2}, \dfrac{1}{2}\right), \\[2ex] \widehat{\sigma}_0 \partial_X U^k\left(-\dfrac{1}{2}, y\right) = \sigma_{\mathrm{int}}\partial_n u^{k-4}_{\mathrm{int}}\left(x_0 - \dfrac{\varepsilon}{2}, y\right), \\[2ex] \widehat{\sigma}_0 \partial_X U^k\left(\dfrac{1}{2}, y\right) = \sigma_{\mathrm{ext}}\partial_n u^{k-4}_{\mathrm{ext}}\left(x_0 + \dfrac{\varepsilon}{2}, y\right). \end{cases}$$

Thanks to the inductive assumptions we know that $U^{k-2} \equiv 0$ and $u^{k-4}_{\mathrm{int}} \equiv u^{k-4}_{\mathrm{ext}} \equiv 0$. Thus, we deduce that U^k has the following form

$$U^k(X, y) = \varphi^k_0(y).$$

Now we employ Equation (15) to build the following problem

$$\begin{cases} \dfrac{\mathrm{d}^2}{\mathrm{d}y^2}\varphi_0^k(y) = -\dfrac{1}{\widehat{\sigma}_0}\left[\sigma\partial_n u^{k-2}\right]_{\Gamma^\varepsilon}(y) & y \in (0, y_0), \\ \varphi_0^k(0) = 0, \\ \varphi_0^k(y_0) = 0. \end{cases}$$

Again, thanks to the inductive assumptions we know that $u^{k-2} \equiv 0$. Thus, we deduce that $\varphi_0^k \equiv 0$ and consequently $U^k \equiv 0$. Now employing (16), we can write the problems satisfied by u_{int}^k and u_{ext}^k.

$$\begin{cases} \Delta u_{\mathrm{int}}^k = 0 & \text{in} \quad \Omega_{\mathrm{int}}^\varepsilon, \\ u_{\mathrm{int}}^k = 0 & \text{on} \quad \Gamma_{\mathrm{int}}^\varepsilon, \\ u_{\mathrm{int}}^k = 0 & \text{on} \quad \partial\Omega \cap \partial\Omega_{\mathrm{int}}^\varepsilon. \end{cases}$$

$$\begin{cases} \Delta u_{\mathrm{ext}}^k = 0 & \text{in} \quad \Omega_{\mathrm{ext}}^\varepsilon, \\ u_{\mathrm{ext}}^k = 0 & \text{on} \quad \Gamma_{\mathrm{ext}}^\varepsilon, \\ u_{\mathrm{ext}}^k = 0 & \text{on} \quad \partial\Omega \cap \partial\Omega_{\mathrm{ext}}^\varepsilon. \end{cases}$$

We deduce that $u^k \equiv 0$. □

2.2 Equivalent models

Now that we know the expressions for the first terms of the expansion, we truncate the series for a given $k \in \mathbb{N}$ and we identify a simpler problem satisfied by

$$u^{(k)} = u^0 + \varepsilon u^1 + \ldots + \varepsilon^k u^k \quad \text{in} \quad \Omega_{\mathrm{int}}^\varepsilon \cup \Omega_{\mathrm{ext}}^\varepsilon$$

up to a residual term of order ε^{k+1}. We neglect the residual term of order ε^{k+1} to obtain an approximate model satisfied by the function $u^{[k]}$. For the sake of simplicity of the notation we will employ the following notation for the domain.

Notation 2. *We denote by Ω^ε the domain*

$$\Omega^\varepsilon = \Omega_{int}^\varepsilon \cup \Omega_{ext}^\varepsilon,$$

where Ω_{int}^ε and Ω_{ext}^ε are the domains defined in Section 1.

In the following we define what the order of convergence for an asymptotic model is. Here, we formally derive two approximate models of order 2 and order 4 respectively. These orders of convergence have still to be proven, we refer the reader to Section 4 for more details concerning these convergence proofs.

Definition 2. *Let $u^{[k]}$ be the solution to an asymptotic model, and let u be the solution to the reference problem. We say that the asymptotic model is of order $k+1$, if there exists a constant C independent of ε, such that the following relation is satisfied*

$$\left\| u - u^{[k]} \right\|_{1,\Omega^\varepsilon} \leq C\varepsilon^{k+1},$$

for ε sufficiently small.

2.2.1 Second-order model

For deriving the second-order model, we truncate the series from the second term and we define $u^{(1)}$ as

$$u^{(1)} = u^0 + \varepsilon u^1 = u^0 \quad \text{in} \quad \Omega^\varepsilon_{\text{int}} \cup \Omega^\varepsilon_{\text{ext}} \qquad \text{(see Proposition 1).}$$

From (17), we can deduce that $u^{(1)}$ solves the following uncoupled problem

$$\begin{cases} \sigma_{\text{int}} \Delta u^{(1)}_{\text{int}} = f_{\text{int}} & \text{in} \quad \Omega^\varepsilon_{\text{int}}, \\ u^{(1)}_{\text{int}} = 0 & \text{on} \quad \partial\Omega^\varepsilon_{\text{int}}. \end{cases}$$
$$\begin{cases} \sigma_{\text{ext}} \Delta u^{(1)}_{\text{ext}} = f_{\text{ext}} & \text{in} \quad \Omega^\varepsilon_{\text{ext}}, \\ u^{(1)}_{\text{ext}} = 0 & \text{on} \quad \partial\Omega^\varepsilon_{\text{ext}}. \end{cases} \tag{22}$$

In this case, we have $u^{[1]} = u^{(1)}$ as $u^{(1)}$ does not depend on ε. We infer a second order model satisfied by $u^{[1]}$ solution to Problem (22).

2.2.2 Fourth-order model

For deriving the fourth-order model, we truncate the series from the fourth term and we define $u^{(3)}$ as

$$u^{(3)} = u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \varepsilon^3 u^3 = u^0 + \varepsilon^2 u^2 \quad \text{in} \quad \Omega^\varepsilon_{\text{int}} \cup \Omega^\varepsilon_{\text{ext}} \quad \text{(see Proposition 1).}$$

From (17), (18), (20) and (21), we can deduce that $u^{(3)}$ satisfies the following equations

$$\begin{cases} \sigma_{\text{int}} \Delta u^{(3)}_{\text{int}} = f_{\text{int}} & \text{in} \quad \Omega^\varepsilon_{\text{int}}, \\ \sigma_{\text{ext}} \Delta u^{(3)}_{\text{ext}} = f_{\text{ext}} & \text{in} \quad \Omega^\varepsilon_{\text{ext}}, \\ \left[u^{(3)}\right]_{\Gamma^\varepsilon} = 0, & \\ \dfrac{\mathrm{d}^2}{\mathrm{d}y^2} \left\{u^{(3)}\right\}_{\Gamma^\varepsilon} = -\varepsilon^2 \dfrac{1}{\widehat{\sigma}_0} \left[\sigma \partial_n u^0\right]_{\Gamma^\varepsilon}, & \\ u^{(3)} = 0 & \text{on} \quad \partial\Omega \cap \partial\Omega^\varepsilon. \end{cases}$$

Then, we employ the expression $u^0 = u^{(3)} - \varepsilon^2 u^2$ to rewrite the right-hand side of the second transmission condition

$$-\varepsilon^2 \frac{1}{\widehat{\sigma}_0} \left[\sigma \partial_n u^0\right]_{\Gamma^\varepsilon} = -\varepsilon^2 \frac{1}{\widehat{\sigma}_0} \left[\sigma \partial_n u^{(3)}\right]_{\Gamma^\varepsilon} + \varepsilon^4 \frac{1}{\widehat{\sigma}_0} \left[\sigma \partial_n u^2\right]_{\Gamma^\varepsilon} = -\varepsilon^2 \frac{1}{\widehat{\sigma}_0} \left[\sigma \partial_n u^{(3)}\right]_{\Gamma^\varepsilon} + O(\varepsilon^4).$$

Now we deduce that $u^{(3)}$ satisfies the following equations

$$\begin{cases} \sigma_{\text{int}}\Delta u_{\text{int}}^{(3)} = f_{\text{int}} & \text{in} \quad \Omega_{\text{int}}^{\varepsilon}, \\ \sigma_{\text{ext}}\Delta u_{\text{ext}}^{(3)} = f_{\text{ext}} & \text{in} \quad \Omega_{\text{ext}}^{\varepsilon}, \\ \left[u^{(3)}\right]_{\Gamma^{\varepsilon}} = 0, & \\ \dfrac{\mathrm{d}^2}{\mathrm{d}y^2}\left\{u^{(3)}\right\}_{\Gamma^{\varepsilon}} = -\varepsilon^2 \dfrac{1}{\widehat{\sigma}_0}\left[\sigma\partial_n u^{(3)}\right]_{\Gamma^{\varepsilon}} + O(\varepsilon^4), & \\ u^{(3)} = 0 & \text{on} \quad \partial\Omega\cap\partial\Omega^{\varepsilon}. \end{cases}$$

We define as $u^{[3]}$ the function we obtain when truncating the solution at the fourth element of the expansion and neglecting the terms of order 4 or higher in ε. Then, $u^{[3]}$ satisfies the following equations

$$\begin{cases} \sigma_{\text{int}}\Delta u_{\text{int}}^{[3]} = f_{\text{int}} & \text{in} \quad \Omega_{\text{int}}^{\varepsilon}, \\ \sigma_{\text{ext}}\Delta u_{\text{ext}}^{[3]} = f_{\text{ext}} & \text{in} \quad \Omega_{\text{ext}}^{\varepsilon}, \\ \left[u^{[3]}\right]_{\Gamma^{\varepsilon}} = 0, & \\ \left[\sigma\partial_n u^{[3]}\right]_{\Gamma^{\varepsilon}} = -\dfrac{\widehat{\sigma}_0}{\varepsilon^2}\dfrac{\mathrm{d}^2}{\mathrm{d}y^2}\left\{u^{[3]}\right\}_{\Gamma^{\varepsilon}}, & \\ u^{[3]} = 0 & \text{on} \quad \partial\Omega\cap\partial\Omega^{\varepsilon}. \end{cases} \tag{23}$$

2.3 Classical conditions and comparison with equivalent conditions

In this section we show the results we obtain when we no longer consider the conductivity in the thin layer to be dependent on its thickness to remark the different results we obtain with our approach compared to this one. The model problem remains the same, we consider Equations (2) set in the domain showed in Figure 1, but know we consider a conductivity of the following form

$$\sigma = \begin{cases} \sigma_{\text{int}} & \text{in} \quad \Omega_{\text{int}}^{\varepsilon}, \\ \sigma_{\text{lay}} & \text{in} \quad \Omega_{\text{lay}}^{\varepsilon}, \\ \sigma_{\text{ext}} & \text{in} \quad \Omega_{\text{ext}}^{\varepsilon}, \end{cases}$$

where σ_{lay} is just a constant and not dependent on ε any more. With this configuration, the model problem we consider writes as follows

$$
\left\{
\begin{aligned}
\sigma_{\text{int}}\Delta u_{\text{int}} &= f_{\text{int}} && \text{in} \quad \Omega^{\varepsilon}_{\text{int}},\\
\sigma_{\text{ext}}\Delta u_{\text{ext}} &= f_{\text{ext}} && \text{in} \quad \Omega^{\varepsilon}_{\text{ext}},\\
\Delta u_{\text{lay}} &= 0 && \text{in} \quad \Omega^{\varepsilon}_{\text{lay}},\\
u_{\text{int}} &= u_{\text{lay}} && \text{on} \quad \Gamma^{\varepsilon}_{\text{int}},\\
u_{\text{lay}} &= u_{\text{ext}} && \text{on} \quad \Gamma^{\varepsilon}_{\text{ext}},\\
\sigma_{\text{int}}\partial_n u_{\text{int}} &= \sigma_{\text{lay}}\partial_n u_{\text{lay}} && \text{on} \quad \Gamma^{\varepsilon}_{\text{int}},\\
\sigma_{\text{lay}}\partial_n u_{\text{lay}} &= \sigma_{\text{ext}}\partial_n u_{\text{ext}} && \text{on} \quad \Gamma^{\varepsilon}_{\text{ext}},\\
u &= 0 && \text{on} \quad \partial\Omega.
\end{aligned}
\right.
\tag{24}
$$

Considering this model, and applying the same asymptotic method developed in the previous section to derive approximate models, we obtain a first-order model and a third-order model. The expression for these models are the following.

First-order model

$$
\left\{
\begin{aligned}
\sigma_{\text{int}}\Delta u^{[0]}_{\text{int}} &= f_{\text{int}} && \text{in} \quad \Omega^{\varepsilon}_{\text{int}},\\
\sigma_{\text{ext}}\Delta u^{[0]}_{\text{ext}} &= f_{\text{ext}} && \text{in} \quad \Omega^{\varepsilon}_{\text{ext}},\\
\left[u^{[0]}\right]_{\Gamma^{\varepsilon}} &= 0, &&\\
\left[\sigma\partial_n u^{[0]}\right]_{\Gamma^{\varepsilon}} &= 0, &&\\
u^{[0]} &= 0 && \text{on} \quad \partial\Omega\cap\partial\Omega^{\varepsilon}.
\end{aligned}
\right.
\tag{25}
$$

Third-order model

$$
\left\{
\begin{aligned}
\sigma_{\text{int}}\Delta u^{[2]}_{\text{int}} &= f_{\text{int}} && \text{in} \quad \Omega^{\varepsilon}_{\text{int}},\\
\sigma_{\text{ext}}\Delta u^{[2]}_{\text{ext}} &= f_{\text{ext}} && \text{in} \quad \Omega^{\varepsilon}_{\text{ext}},\\
\left[u^{[2]}\right]_{\Gamma^{\varepsilon}} &= \frac{\varepsilon}{\sigma_{\text{lay}}}\left\{\sigma\partial_n u^{[2]}\right\}_{\Gamma^{\varepsilon}}, &&\\
\left[\sigma\partial_n u^{[2]}\right]_{\Gamma^{\varepsilon}} &= -\varepsilon\sigma_{\text{lay}}\frac{\mathrm{d}^2}{\mathrm{d}y^2}\left\{u^{[2]}\right\}_{\Gamma^{\varepsilon}}, &&\\
u^{[2]} &= 0 && \text{on} \quad \partial\Omega\cap\partial\Omega^{\varepsilon}.
\end{aligned}
\right.
\tag{26}
$$

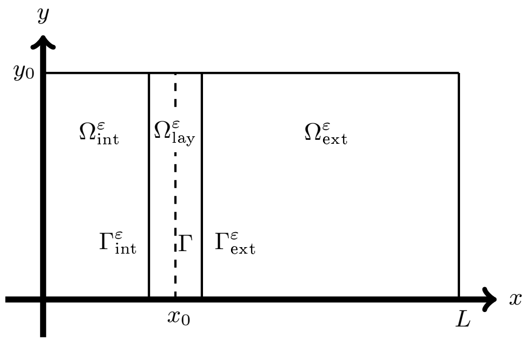
We notice that these models are different from (22) and (23). A main difference comparing with our approach (i.e. when $\sigma_{\text{lay}} = \widehat{\sigma}_0\varepsilon^{-3}$) comes from the fact that now the lower order model (25) is coupled, whereas model (22) is governed by two independent problems. Moreover, the lower order model (25) has order 1, whereas model (22) has order 2. In the same way, the higher order model (26) is of order 3, whereas model (23) is of order 4.

3 Second class of ITCs

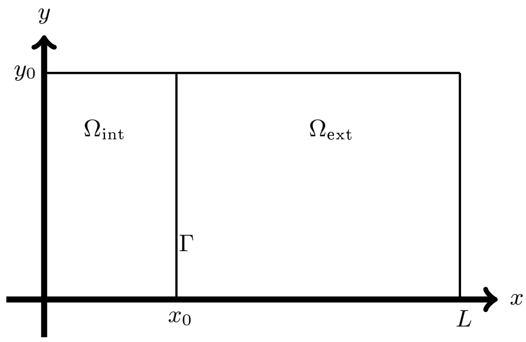
The objective of this section is the derivation of asymptotic models for the reference Problem (2). The main difference with the first class of ITCs is that now we employ some formal Taylor expansions to write the terms of the expansion across an artificial interface Γ situated in the middle of the thin layer. The resulting asymptotic models will be defined in the domain depicted in Figure 2b. In this section we perform an expansion of the solution in power series of ε, following the same way as in Section 2.1. Then, by truncating this series and neglecting higher order terms in ε, we derive approximate models composed by equivalent transmission conditions in Section 3.2. Finally in Section 3.3 we present a technique for solving a stability problem related with one of the derived asymptotic models. The last section, Section 3.4, shows a comparison of the models derived here with more standard models.

3.1 Construction of a multiscale expansion

First of all, we define the jump and mean value of a function across the interface Γ, in the same way we have done with the jump and the mean value across the thin layer in Definition 1.



(a) Reference problem domain.



(b) Domain for the second class of ITCs.

Figure 2: Domains for the reference model and the second class of asymptotic models.

Definition 3. *Let u be a function defined over Ω. We define its jump and mean value across the interface Γ as*

$$\begin{cases} [u]_\Gamma = u_{ext}|_\Gamma - u_{int}|_\Gamma, \\ \{u\}_\Gamma = \dfrac{1}{2}\left(u_{ext}|_\Gamma + u_{int}|_\Gamma\right). \end{cases}$$

Now we derive the asymptotic expansion. To begin with, we perform an *Ansatz* in the form of power series of ε for the solution to Problem (7). We look for solutions

$$
\begin{cases}
u_{\text{int}}(x,y) \approx \displaystyle\sum_{k\geq 0} \varepsilon^k u_{\text{int}}^k(x,y) & \text{in} \quad \Omega_{\text{int}}^\varepsilon, \\
u_{\text{ext}}(x,y) \approx \displaystyle\sum_{k\geq 0} \varepsilon^k u_{\text{ext}}^k(x,y) & \text{in} \quad \Omega_{\text{ext}}^\varepsilon, \\
U(X,y) \approx \displaystyle\sum_{k\geq 0} \varepsilon^k U^k(X,y) & \text{in} \quad \left(-\frac{1}{2},\frac{1}{2}\right) \times (0,y_0)\,,
\end{cases}
\tag{27}
$$

where the functions $\left(u_{\text{int}}^k\right)_{k\in\mathbb{N}}$ and $\left(u_{\text{ext}}^k\right)_{k\in\mathbb{N}}$ are now defined in ε-independent domains, contrary to the first approach. We emphasize that the sequence $\left(u_{\text{int}}^k\right)_{k\in\mathbb{N}}$ (respectively $\left(u_{\text{ext}}^k\right)_{k\in\mathbb{N}}$) is defined in Ω_{int} (respectively Ω_{ext}) even if its associated series does not approach u in the thin layer. We assume that for $k \in \mathbb{N}$, the terms u_{int}^k and u_{ext}^k are as regular as necessary, see [16]. Then, we perform a formal Taylor expansion of the terms $u_{\text{int}}^k\big|_{\Gamma_{\text{int}}^\varepsilon}$ and $u_{\text{ext}}^k\big|_{\Gamma_{\text{ext}}^\varepsilon}$ of the series along the direction normal to the thin layer, the variable x in this case, in order to write the transmission conditions across the interface Γ. The formal Taylor expansion writes as follows

$$
\begin{cases}
u_{\text{int}}^k\left(x_0 - \dfrac{\varepsilon}{2}, y\right) = \displaystyle\sum_{i\geq 0} \varepsilon^i \frac{(-1)^i}{2^i i!} \partial_n^i u_{\text{int}}^k\left(x_0,y\right), \\
u_{\text{ext}}^k\left(x_0 + \dfrac{\varepsilon}{2}, y\right) = \displaystyle\sum_{i\geq 0} \varepsilon^i \frac{1}{2^i i!} \partial_n^i u_{\text{ext}}^k\left(x_0,y\right).
\end{cases}
$$

We also perform a formal Taylor expansion of the same form for the derivatives $\partial_n u_{\text{int}}^k\big|_{\Gamma_{\text{int}}^\varepsilon}$ and $\partial_n u_{\text{ext}}^k\big|_{\Gamma_{\text{ext}}^\varepsilon}$

$$
\begin{cases}
\partial_n u_{\text{int}}^k\left(x_0 - \dfrac{\varepsilon}{2}, y\right) = \displaystyle\sum_{i\geq 0} \frac{(-1)^i}{2^i i!} \partial_n^{i+1} u_{\text{int}}^k\left(x_0,y\right), \\
\partial_n u_{\text{ext}}^k\left(x_0 + \dfrac{\varepsilon}{2}, y\right) = \displaystyle\sum_{i\geq 0} \varepsilon^i \frac{1}{2^i i!} \partial_n^{i+1} u_{\text{ext}}^k\left(x_0,y\right).
\end{cases}
$$

Employing these formal Taylor expansions and the *Ansatz* (27) we develop the terms $u_{\text{int}}\big|_{\Gamma_{\text{int}}^\varepsilon}$ and $u_{\text{ext}}\big|_{\Gamma_{\text{ext}}^\varepsilon}$ in the following way

$$
\begin{aligned}
u_{\text{int}}\big|_{\Gamma_{\text{int}}^\varepsilon} &= u_{\text{int}}\left(x_0 - \frac{\varepsilon}{2}, y\right) = \sum_{k\geq 0} \varepsilon^k u_{\text{int}}^k\left(x_0 - \frac{\varepsilon}{2}, y\right) \\
&= \sum_{k\geq 0} \varepsilon^k \sum_{i\geq 0} \varepsilon^i \frac{(-1)^i}{2^i i!} \partial_n^i u_{\text{int}}^k(x_0,y) = \sum_{k\geq 0} \varepsilon^k \sum_{i=0}^{k} \frac{(-1)^i}{2^i i!} \partial_n^i u_{\text{int}}^{k-i}(x_0,y), \\
u_{\text{ext}}\big|_{\Gamma_{\text{ext}}^\varepsilon} &= u_{\text{ext}}\left(x_0 + \frac{\varepsilon}{2}, y\right) = \sum_{k\geq 0} \varepsilon^k u_{\text{ext}}^k\left(x_0 + \frac{\varepsilon}{2}, y\right) \\
&= \sum_{k\geq 0} \varepsilon^k \sum_{i\geq 0} \varepsilon^i \frac{1}{2^i i!} \partial_n^i u_{\text{ext}}^k(x_0,y) = \sum_{k\geq 0} \varepsilon^k \sum_{i=0}^{k} \frac{1}{2^i i!} \partial_n^i u_{\text{ext}}^{k-i}(x_0,y),
\end{aligned}
\tag{28}
$$

and the terms $\partial_n u^k_{\text{int}}\big|_{\Gamma^\varepsilon_{\text{int}}}$, $\partial_n u^k_{\text{ext}}\big|_{\Gamma^\varepsilon_{\text{ext}}}$ in the following way

$$\begin{aligned}
\partial_n u_{\text{int}}\big|_{\Gamma^\varepsilon_{\text{int}}} &= \partial_n u_{\text{int}}\left(x_0-\frac{\varepsilon}{2},y\right) = \sum_{k\geq 0}\varepsilon^k\partial_n u^k_{\text{int}}\left(x_0-\frac{\varepsilon}{2},y\right)\\
&= \sum_{k\geq 0}\varepsilon^k\sum_{i\geq 0}\varepsilon^i\frac{(-1)^i}{2^i i!}\partial_n^{i+1}u^k_{\text{int}}(x_0,y) = \sum_{k\geq 0}\varepsilon^k\sum_{i=0}^{k}\frac{(-1)^i}{2^i i!}\partial_n^{i+1}u^{k-i}_{\text{int}}(x_0,y),\\
\partial_n u_{\text{ext}}\big|_{\Gamma^\varepsilon_{\text{ext}}} &= \partial_n u_{\text{ext}}\left(x_0+\frac{\varepsilon}{2},y\right) = \sum_{k\geq 0}\varepsilon^k\partial_n u^k_{\text{ext}}\left(x_0+\frac{\varepsilon}{2},y\right)\\
&= \sum_{k\geq 0}\varepsilon^k\sum_{i\geq 0}\varepsilon^i\frac{1}{2^i i!}\partial_n^{i+1}u^k_{\text{ext}}(x_0,y) = \sum_{k\geq 0}\varepsilon^k\sum_{i=0}^{k}\frac{1}{2^i i!}\partial_n^{i+1}u^{k-i}_{\text{ext}}(x_0,y).
\end{aligned}\tag{29}$$

3.1.1 Equations for the coefficients of the electric potential

Substituting the *Ansatz* (27) and the identities (28), (29) in the Equations (7) and collecting the terms with the same powers in ε, for every $k\in\mathbb{N}$ we obtain the following set of equations

$$\left\{\begin{aligned}
&\sigma_{\text{int}}\Delta u^k_{\text{int}}(x,y) = f_{\text{int}}(x,y)\delta^k_0 && \text{in} \quad \Omega_{\text{int}},\\
&\sigma_{\text{ext}}\Delta u^k_{\text{ext}}(x,y) = f_{\text{ext}}(x,y)\delta^k_0 && \text{in} \quad \Omega_{\text{ext}},\\
&\partial_X^2 U^k(X,y)+\partial_y^2 U^{k-2}(X,y) = 0 && \text{in} \quad \left(-\frac{1}{2},\frac{1}{2}\right)\times(0,y_0),
\end{aligned}\right.\tag{30}$$

along with the following transmission conditions

$$\left\{\begin{aligned}
&\sum_{i=0}^{k}\frac{(-1)^i}{2^i i!}\partial_n^i u^{k-i}_{\text{int}}(x_0,y) = U^k\left(\frac{-1}{2},y\right) && y\in(0,y_0),\\
&\sum_{i=0}^{k}\frac{1}{2^i i!}\partial_n^i u^{k-i}_{\text{ext}}(x_0,y) = U^k\left(\frac{1}{2},y\right) && y\in(0,y_0),\\
&\sigma_{\text{int}}\sum_{i=0}^{k-4}\frac{(-1)^i}{2^i i!}\partial_n^{i+1}u^{k-4-i}_{\text{int}}(x_0,y) = \widehat{\sigma}_0\partial_X U^k\left(\frac{-1}{2},y\right) && y\in(0,y_0),\\
&\sigma_{\text{ext}}\sum_{i=0}^{k-4}\frac{1}{2^i i!}\partial_n^{i+1}u^{k-4-i}_{\text{ext}}(x_0,y) = \widehat{\sigma}_0\partial_X U^k\left(\frac{1}{2},y\right) && y\in(0,y_0),
\end{aligned}\right.\tag{31}$$

and the following boundary conditions

$$
\begin{cases}
u^k(0,y) = u^k(L,y) = 0 & y \in (0,y_0)\,, \\
u^k(x,0) = u^k(x,y_0) = 0 & x \in (0,x_0)\cup(x_0,L)\,, \\
U^k(X,0) = U^k(X,y_0) = 0 & X \in \left(-\frac{1}{2},\frac{1}{2}\right),
\end{cases}
\tag{32}
$$

where δ_0^k represents the Kronecker symbol. Besides, we will also employ the following compatibility condition obtained by applying the fundamental theorem of calculus (12) for a smooth function U^{k+2}, along with the third equation of (30) and the third and fourth equations of (32)

$$
\begin{aligned}
&-\int_{\frac{-1}{2}}^{\frac{1}{2}} \partial_y^2 U^k\,(X,y)\,\mathrm{d}X \\
&= \frac{1}{\widehat{\sigma}_0}\sum_{i=0}^{k-2}\Big(\frac{\sigma_{\text{ext}}}{2^i i!}\partial_n^{i+1}u_{\text{ext}}^{k-2-i}\,(x_0,y) + (-1)^{i+1}\frac{\sigma_{\text{int}}}{2^i i!}\partial_n^{i+1}u_{\text{int}}^{k-2-i}\,(x_0,y)\Big)
\end{aligned}
\tag{33}
$$

For these equations, we adopt the convention that the terms with negative indices are equal to 0. Employing these equations ((30) - (33)) we can determine the elementary problems satisfied outside and inside the layer for any $k \in \mathbb{N}$. For that purpose, we use the following algorithm composed of three steps.

3.1.2 Algorithm for the determination of the coefficients

We assume that the first terms of the expansion (27) up to order ε^{k-1} have already been calculated and we calculate the equations for the k-th term. The first two steps consist in fixing U^k and the third step consist in determining u_{int}^k and u_{ext}^k.

First step:

We begin by selecting the third equation from (30), along with third and fourth equations from (31), and we build the following differential problem in the variable X for U^k (the variable y plays the role of a parameter)

$$
\begin{cases}
\partial_X^2 U^k(X,y) = -\partial_y^2 U^{k-2}(X,y) & X \in \left(-\frac{1}{2},\frac{1}{2}\right), \\
\widehat{\sigma}_0 \partial_X U^k\left(\frac{-1}{2},y\right) = \sigma_{\text{int}}\sum_{i=0}^{k-4}\frac{(-1)^i}{2^i i!}\partial_n^{i+1}u_{\text{int}}^{k-4-i}\,(x_0,y)\,, & \\
\widehat{\sigma}_0 \partial_X U^k\left(\frac{1}{2},y\right) = \sigma_{\text{ext}}\sum_{i=0}^{k-4}\frac{1}{2^i i!}\partial_n^{i+1}u_{\text{ext}}^{k-4-i}\,(x_0,y)\,. &
\end{cases}
\tag{34}
$$

There exists a solution U^k to (34) provided the compatibility condition (33) is satisfied. We deduce the expression of U^k up to a function in the variable y, $\psi_0^k(y)$. The function U^k has the following form

$$
U^k(X,y) = V^k(X,y) + \psi_0^k(y),
$$

where V^k represents the part of U^k that can be determined at this step and has the form (see Proposition 2)

$$V^k(X,y) = \begin{cases} 0 & \text{if} \quad k = 0,1,2,3, \\ \psi^k_{k-2}(y)X^{k-2} + \psi^k_{k-3}(y)X^{k-3} + \ldots + \psi^k_1(y)X & \text{if} \quad k > 3. \end{cases}$$

The function ψ_0^k represents the part of U^k that is determined at the second step.

Second step:

We involve the compatibility condition (33), along with third equation of (32) to write the following differential problem in the variable y for the function ψ_0^k, present in the expression of U^k.

$$\begin{cases} \dfrac{\mathrm{d}^2}{\mathrm{d}y^2}\psi_0^k(y) = h^k(y) \quad y \in (0, y_0), \\ \quad \psi_0^k(0) = 0, \\ \quad \psi_0^k(y_0) = 0, \end{cases} \tag{35}$$

where

$$h^k(y) = -\frac{1}{\widehat{\sigma}_0}\sum_{i=0}^{k-2}\Big(\frac{\sigma_{\text{ext}}}{2^i i!}\partial_n^{i+1}u_{\text{ext}}^{k-2-i}(x_0,y) + (-1)^{i+1}\frac{\sigma_{\text{int}}}{2^i i!}\partial_n^{i+1}u_{\text{int}}^{k-2-i}(x_0,y)\Big)$$
$$-\int_{\frac{-1}{2}}^{\frac{1}{2}} \partial_y^2 V^k(X,y)\,\mathrm{d}X.$$

Solving this differential equation we obtain the function ψ_0^k and thus the complete expression of U^k.

Third step:

Now we can finally derive the equations outside the layer by employing first and second equations of (30), first and second equations of (31), and first and second equations of (32). We infer that u^k_{int} and u^k_{ext} are defined independently in the two subdomains Ω_{int} and Ω_{ext}

$$
\left\{
\begin{aligned}
\sigma_{\mathrm{int}} \Delta u^k_{\mathrm{int}} &= f_{\mathrm{int}} \delta^k_0 && \text{in} \quad \Omega_{\mathrm{int}},\\
u^k_{\mathrm{int}}\left(x_0, y\right) &= U^k\left(-\frac{1}{2}, y\right) - \sum_{i=1}^{k} \frac{(-1)^i}{2^i i!} \partial^i_n u^{k-i}_{\mathrm{int}}\left(x_0, y\right), &&\\
u^k_{\mathrm{int}} &= 0 && \text{on} \quad \partial\Omega \cap \partial\Omega_{\mathrm{int}}.
\end{aligned}
\right.
$$

$$
\left\{
\begin{aligned}
\sigma_{\mathrm{ext}} \Delta u^k_{\mathrm{ext}} &= f_{\mathrm{ext}} \delta^k_0 && \text{in} \quad \Omega_{\mathrm{ext}},\\
u^k_{\mathrm{ext}}\left(x_0, y\right) &= U^k\left(\frac{1}{2}, y\right) - \sum_{i=1}^{k} \frac{1}{2^i i!} \partial^i_n u^{k-i}_{\mathrm{ext}}\left(x_0, y\right), &&\\
u^k_{\mathrm{ext}} &= 0 && \text{on} \quad \partial\Omega \cap \partial\Omega_{\mathrm{ext}}.
\end{aligned}
\right.
\tag{36}
$$

Now this algorithm is used to determine the first terms of the expansion.

3.1.3 First terms of the asymptotics

Case k=0

We consider Problem (34) for U^0

$$
\left\{
\begin{aligned}
\partial^2_X U^0(X, y) &= 0 \quad X \in \left(\frac{-1}{2}, \frac{1}{2}\right),\\
\partial_X U^0\left(-\frac{1}{2}, y\right) &= 0,\\
\partial_X U^0\left(\frac{1}{2}, y\right) &= 0.
\end{aligned}
\right.
$$

We deduce that the solution to this equation has the form $U^0(X, y) = \psi^0_0(y)$. Then using (35) we build the following problem for ψ^0_0

$$
\left\{
\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}y^2} \psi^0_0(y) &= 0 \qquad y \in (0, y_0),\\
\psi^0_0(0) &= 0,\\
\psi^0_0(y_0) &= 0.
\end{aligned}
\right.
$$

We deduce that $\psi^0_0(y) = 0$ and thus $U^0(X, y) = 0$. Finally, employing (36) we obtain that the limit solution u^0 satisfies homogeneous Dirichlet boundary conditions on Γ. Thus, the problem satisfied by u^0 reads as

$$
\begin{cases} \Delta u^0_{\text{int}} = f_{\text{int}} & \text{in} \quad \Omega_{\text{int}}, \\ u^0_{\text{int}} = 0 & \text{on} \quad \partial\Omega_{\text{int}}. \end{cases}
\qquad
\begin{cases} \Delta u^0_{\text{ext}} = f_{\text{ext}} & \text{in} \quad \Omega_{\text{ext}}, \\ u^0_{\text{ext}} = 0 & \text{on} \quad \partial\Omega_{\text{ext}}. \end{cases} \tag{37}
$$

<u>Case k=1</u>

We consider Problem (34) for U^1

$$
\begin{cases} \partial^2_X U^1(X,y) = 0 \quad X \in \left(\frac{-1}{2}, \frac{1}{2}\right), \\ \partial_X U^1\left(-\frac{1}{2}, y\right) = 0, \\ \partial_X U^1\left(\frac{1}{2}, y\right) = 0. \end{cases}
$$

We deduce that the solution to this equation has the form $U^1(X,y) = \psi^1_0(y)$. Then from (35) we deduce the following problem for ψ^1_0

$$
\begin{cases} \frac{\mathrm{d}^2}{\mathrm{d}y^2}\psi^1_0(y) = 0 \qquad y \in (0, y_0), \\ \psi^1_0(0) = 0, \\ \psi^1_0(y_0) = 0. \end{cases}
$$

We thus have that $\psi^1_0(y) = 0$ and $U^1(X,y) = 0$. Finally, employing (36) we can write the problem satisfied outside the layer by u^1 as two uncoupled problems

$$
\begin{cases} \Delta u^1_{\text{int}} = 0 & \text{in} \quad \Omega_{\text{int}}, \\ u^1_{\text{int}} = \frac{1}{2}\partial_n u^0_{\text{int}} & \text{on} \quad \Gamma, \\ u^1_{\text{int}} = 0 & \text{on} \quad \partial\Omega \cap \partial\Omega_{\text{int}}. \end{cases}
\qquad
\begin{cases} \Delta u^1_{\text{ext}} = 0 & \text{in} \quad \Omega_{\text{ext}}, \\ u^1_{\text{ext}} = -\frac{1}{2}\partial_n u^0_{\text{ext}} & \text{on} \quad \Gamma, \\ u^1_{\text{ext}} = 0 & \text{on} \quad \partial\Omega \cap \partial\Omega_{\text{ext}}. \end{cases} \tag{38}
$$

<u>Case k=2</u>

We consider Problem (34) for U^2

$$\begin{cases} \partial_X^2 U^2(X,y) = 0 \quad X \in \left(\frac{-1}{2}, \frac{1}{2}\right), \\ \partial_X U^2\left(-\frac{1}{2}, y\right) = 0, \\ \partial_X U^2\left(\frac{1}{2}, y\right) = 0. \end{cases}$$

We deduce that the solution to this equation has the form $U^2(X,y) = \psi_0^2(y)$. Then according to (35) we can build the following problem for ψ_0^2

$$\begin{cases} \frac{\mathrm{d}^2}{\mathrm{d}y^2}\psi_0^2(y) = -\frac{1}{\widehat{\sigma}_0}\left[\sigma_{\mathrm{ext}}\partial_n u^0\right](y) \qquad y \in (0, y_0), \\ \psi_0^2(0) = 0, \\ \psi_0^2(y_0) = 0. \end{cases}$$

We deduce that ψ_0^2 has the following form

$$\psi_0^2(y) = \frac{-1}{\widehat{\sigma}_0}\int_0^y (y-t)\left[\sigma\partial_n u^0\right](t)\,\mathrm{d}t + \frac{y}{\widehat{\sigma}_0 y_0}\int_0^{y_0}(y_0 - t)\left[\sigma\partial_n u^0\right](t)\,\mathrm{d}t. \tag{39}$$

Finally, (36) implies that the problem satisfied outside the layer by u^2 is composed of two uncoupled problems

$$\begin{cases} \Delta u_{\mathrm{int}}^2 = 0 & \text{in} \quad \Omega_{\mathrm{int}}, \\ u_{\mathrm{int}}^2(x_0, y) = \psi_0^2(y) & \qquad y \in (0, y_0), \\ u_{\mathrm{int}}^2 = 0 & \text{on} \quad \partial\Omega \cap \partial\Omega_{\mathrm{int}}. \end{cases}$$
$$\begin{cases} \Delta u_{\mathrm{ext}}^2 = 0 & \text{in} \quad \Omega_{\mathrm{ext}}, \\ u_{\mathrm{ext}}^2(x_0, y) = \psi_0^2(y) & \qquad y \in (0, y_0), \\ u_{\mathrm{ext}}^2 = 0 & \text{on} \quad \partial\Omega \cap \partial\Omega_{\mathrm{ext}}. \end{cases} \tag{40}$$

<u>Case k=3</u>

We consider Problem (34) for U^3

$$\begin{cases} \partial_X^2 U^3(X,y) = 0 \quad X \in \left(\frac{-1}{2}, \frac{1}{2}\right), \\ \partial_X U^3\left(-\frac{1}{2}, y\right) = 0, \\ \partial_X U^3\left(\frac{1}{2}, y\right) = 0. \end{cases}$$

We deduce that the solution to this equation has the form $U^3(X,y) = \psi_0^3(y)$. Then we employ (35) and we build the following problem for ψ_0^3

$$\begin{cases} \dfrac{\mathrm{d}^2}{\mathrm{d}y^2}\psi_0^3(y) = -\dfrac{1}{\widehat{\sigma}_0}\left[\sigma_{\text{ext}}\partial_n u^1\right](y) - \dfrac{1}{\widehat{\sigma}_0}\left\{\sigma_{\text{ext}}\partial_n u^0\right\}(y) & y \in (0, y_0), \\ \psi_0^3(0) = 0, \\ \psi_0^3(y_0) = 0. \end{cases}$$

We deduce that ψ_0^3 has the following form

$$\begin{aligned} \psi_0^3(y) = & \frac{-1}{\widehat{\sigma}_0}\int_0^y (y-t)\Big(\left[\sigma\partial_n u^1\right](t) + \left\{\sigma\partial_n u^0\right\}(t)\Big)\,\mathrm{d}t \\ & + \frac{y}{\widehat{\sigma}_0 y_0}\int_0^{y_0}(y_0-t)\Big(\left[\sigma\partial_n u^1\right](t) + \left\{\sigma\partial_n u^0\right\}(t)\Big)\,\mathrm{d}t. \end{aligned} \tag{41}$$

Finally, employing (36) we can write the problem satisfied outside the layer by u^3 as two uncoupled problems

$$\begin{aligned} &\begin{cases} \Delta u_{\text{int}}^3 = 0 & \text{in} \quad \Omega_{\text{int}}, \\ u_{\text{int}}^3(x_0, y) = \psi_0^3(y) & \qquad y \in (0, y_0), \\ u_{\text{int}}^3 = 0 & \text{on} \quad \partial\Omega \cap \partial\Omega_{\text{int}}. \end{cases} \\ &\begin{cases} \Delta u_{\text{ext}}^3 = 0 & \text{in} \quad \Omega_{\text{ext}}, \\ u_{\text{ext}}^3(x_0, y) = \psi_0^3(y) & \qquad y \in (0, y_0), \\ u_{\text{ext}}^3 = 0 & \text{on} \quad \partial\Omega \cap \partial\Omega_{\text{ext}}. \end{cases} \end{aligned} \tag{42}$$

3.1.4 Recapitulation of the asymptotic expansion

Proposition 2. *The asymptotic expansion* (27)*, has the following form,*

$$\begin{cases} u_{int}(x,y) = u_{int}^0(x,y) + \varepsilon u_{int}^1(x,y) + \varepsilon^2 u_{int}^2(x,y) + \varepsilon^3 u_{int}^3(x,y) + O\left(\varepsilon^4\right), \\ u_{ext}(x,y) = u_{ext}^0(x,y) + \varepsilon u_{ext}^1(x,y) + \varepsilon^2 u_{int}^2(x,y) + \varepsilon^3 u_{int}^3(x,y) + O\left(\varepsilon^4\right), \end{cases}$$

and the following form inside the layer, for $(X, y) \in \left(-\dfrac{1}{2}, \dfrac{1}{2}\right) \times (0, y_0)$

$$U(X, y) = \varepsilon^2\psi_0^2(y) + \varepsilon^3\psi_0^3(y) + O\left(\varepsilon^4\right),$$

where the functions u^0, u^1, u^2 and u^3 are defined by Equations (37)*,* (38)*,* (40) *and* (42) *respectively, and the functions ψ_0^2 and ψ_0^3 are defined by Equations* (39) *and* (41) *respectively. Besides, for $k \in \mathbb{N}$, the solution U^k to Equation* (34) *has the following form*

$$U^k\left(X,y\right)=\begin{cases}0 & k=0,1,\\ \displaystyle\sum_{j=0}^{k-2}\psi_j^k(y)X^j & k=2l+2,\quad l\in\mathbb{N},\\ \displaystyle\sum_{j=0}^{k-3}\psi_j^k(y)X^j & k=2l+3,\quad l\in\mathbb{N}.\end{cases}$$

Proof. We perform the proof by induction on k. For $k=0,1,2,3$ we have already calculated the expressions of u^k and U^k in the previous section. Now let us assume that for any even number $i\in\mathbb{N}$, such that $i<k$, the function U^i has the form

$$U^i\left(X,y\right)=\psi_{i-2}^i(y)X^{i-2}+\psi_{i-3}^i(y)X^{i-3}+\ldots+\psi_1^i(y)X+\psi_0^i(y),$$

We begin by considering Problem (34) for any even number k. Solving this problem we obtain a solution of the form

$$U^k\left(X,y\right)=\psi_{k-2}^k(y)X^{k-2}+\psi_{k-3}^k(y)X^{k-3}+\ldots+\psi_1^k(y)X+\psi_0^k(y),$$

where $k\geq 2$ and

$$\begin{cases}\psi_1^k(y)=\dfrac{1}{\widehat{\sigma}_0}\displaystyle\sum_{i=0}^{k-4}\left(\frac{\sigma_{\text{ext}}}{2^i i!}\partial_n^{i+1}u_{\text{ext}}^{k-4-i}\left(x_0,y\right)+(-1)^i\frac{\sigma_{\text{int}}}{2^i i!}\partial_n^{i+1}u_{\text{int}}^{k-4-i}\left(x_0,y\right)\right),\\[2ex] \psi_2^k(y)=\dfrac{1}{2\widehat{\sigma}_0}\left(\displaystyle\sum_{i=0}^{k-4}\left(\frac{\sigma_{\text{ext}}}{2^i i!}\partial_n^{i+1}u_{\text{ext}}^{k-4-i}\left(x_0,y\right)+(-1)^{i+1}\frac{\sigma_{\text{int}}}{2^i i!}\partial_n^{i+1}u_{\text{int}}^{k-4-i}\left(x_0,y\right)\right)\right),\\[2ex] \psi_{k-j}^k(y)=\dfrac{-\frac{\mathrm{d}^2}{\mathrm{d}y^2}\psi_{k-j-2}^{k-2}(y)}{(k-j)(k-j-1)}\qquad j=2,\ldots,k-2.\end{cases}$$

We denote by V^k the following function

$$V^k\left(X,y\right)=\psi_{k-2}^k(y)X^{k-2}+\psi_{k-3}^k(y)X^{k-3}+\ldots+\psi_1^k(y)X.$$

This expression corresponds to the function V^k we have defined in the first step of the algorithm. A similar argument can be involved when k is an odd number.

□

3.2 Equivalent models

Now that we know the expressions for the first terms of the expansion, we truncate the series and we identify a simpler problem satisfied by

$$u^{(k)}=u^0+\varepsilon u^1+\ldots+\varepsilon^k u^k\quad\text{in}\quad\Omega_{\text{int}}\cup\Omega_{\text{ext}}$$

up to a residual term of order ε^{k+1}. We neglect the residual term of order ε^{k+1} to obtain an approximate model satisfied by the function $u^{[k]}$. Here, we formally derive several approximate models of order 1, order 2, order 3 and order 4 respectively.

3.2.1 First-order model

For deriving the first-order model, we truncate the series from the first term and we define $u^{(0)}$ as

$$u^{(0)} = u^0 \quad \text{in} \quad \Omega_{\text{int}} \cup \Omega_{\text{ext}} \qquad \text{(see Proposition 2).}$$

From (37), we can deduce that $u^{(0)}$ solves the problem

$$\begin{cases} \sigma_{\text{int}} \Delta u^{(0)}_{\text{int}} = f_{\text{int}} & \text{in} \quad \Omega_{\text{int}}, \\ u^{(0)}_{\text{int}} = 0 & \text{on} \quad \partial\Omega_{\text{int}}. \end{cases}$$
$$\begin{cases} \sigma_{\text{ext}} \Delta u^{(0)}_{\text{ext}} = f_{\text{ext}} & \text{in} \quad \Omega_{\text{ext}}, \\ u^{(0)}_{\text{ext}} = 0 & \text{on} \quad \partial\Omega_{\text{ext}}. \end{cases} \tag{43}$$

In this case, we have $u^{[0]} = u^{(0)}$ as $u^{(0)}$ does not depend on ε. We thus infer a first-order model satisfied by $u^{[0]}$ solution to Problem (43).

3.2.2 Second-order model

For deriving the second-order model, we truncate the series from the second term and we define $u^{(1)}$ as

$$u^{(1)} = u^0 + \varepsilon u^1 \quad \text{in} \quad \Omega_{\text{int}} \cup \Omega_{\text{ext}} \qquad \text{(see Proposition 2).}$$

From (37) and (38) we can deduce that $u^{(1)}$ satisfies the following equations

$$\begin{cases} \sigma_{\text{int}} \Delta u^{(1)}_{\text{int}} = f_{\text{int}} & \text{in} \quad \Omega_{\text{int}}, \\ u^{(1)}_{\text{int}} = \dfrac{\varepsilon}{2} \partial_n u^0_{\text{int}} & \text{on} \quad \Gamma, \\ u^{(1)}_{\text{int}} = 0 & \text{on} \quad \partial\Omega \cap \partial\Omega_{\text{int}}. \end{cases}$$
$$\begin{cases} \sigma_{\text{ext}} \Delta u^{(1)}_{\text{ext}} = f_{\text{ext}} & \text{in} \quad \Omega_{\text{ext}}, \\ u^{(1)}_{\text{ext}} = -\dfrac{\varepsilon}{2} \partial_n u^0_{\text{ext}} & \text{on} \quad \Gamma, \\ u^{(1)}_{\text{ext}} = 0 & \text{on} \quad \partial\Omega \cap \partial\Omega_{\text{ext}}. \end{cases} \tag{44}$$

Following the same procedure as in Section 2.2 we obtain the following second-order asymptotic model for $u^{[1]}$

$$\left\{\begin{aligned} \sigma_{\text{int}}\Delta u_{\text{int}}^{[1]} &= f_{\text{int}} && \text{in} \quad \Omega_{\text{int}}, \\ u_{\text{int}}^{[1]} &= \frac{\varepsilon}{2}\partial_n u_{\text{int}}^{[1]} && \text{on} \quad \Gamma, \\ u_{\text{int}}^{[1]} &= 0 && \text{on} \quad \partial\Omega\cap\partial\Omega_{\text{int}}. \end{aligned}\right.$$
$$\left\{\begin{aligned} \sigma_{\text{ext}}\Delta u_{\text{ext}}^{[1]} &= f_{\text{ext}} && \text{in} \quad \Omega_{\text{ext}}, \\ u_{\text{ext}}^{[1]} &= -\frac{\varepsilon}{2}\partial_n u_{\text{ext}}^{[1]} && \text{on} \quad \Gamma, \\ u_{\text{ext}}^{[1]} &= 0 && \text{on} \quad \partial\Omega\cap\partial\Omega_{\text{ext}}. \end{aligned}\right. \tag{45}$$

3.2.3 Third-order model

For deriving the third-order model, we truncate the series from the third term and we define $u^{(2)}$ as

$$u^{(2)} = u^0 + \varepsilon u^1 + \varepsilon^2 u^2 \quad \text{in} \quad \Omega_{\text{int}}\cup\Omega_{\text{ext}} \qquad \text{(see Proposition 2).}$$

From (37), (38) and (40) we can deduce that $u^{(2)}$ satisfies the following equations

$$\left\{\begin{aligned} \sigma_{\text{int}}\Delta u_{\text{int}}^{(2)} &= f_{\text{int}} && \text{in} \quad \Omega_{\text{int}}, \\ u_{\text{int}}^{(2)}(x_0,y) &= g_1^2(y) && \qquad y\in(0,y_0), \\ u_{\text{int}}^{(2)} &= 0 && \text{on} \quad \partial\Omega\cap\partial\Omega_{\text{int}}. \end{aligned}\right.$$
$$\left\{\begin{aligned} \sigma_{\text{ext}}\Delta u_{\text{ext}}^{(2)} &= f_{\text{ext}} && \text{in} \quad \Omega_{\text{ext}}, \\ u_{\text{ext}}^{(2)}(x_0,y) &= g_2^2(y) && \qquad y\in(0,y_0), \\ u_{\text{ext}}^{(2)} &= 0 && \text{on} \quad \partial\Omega\cap\partial\Omega_{\text{ext}}. \end{aligned}\right. \tag{46}$$

where g_1^2 and g_2^2 are defined as follows

$$\left\{\begin{aligned} g_1^2(y) =& \frac{\varepsilon}{2}\partial_n u_{\text{int}}^0(x_0,y) - \frac{\varepsilon^2}{\widehat{\sigma}_0}\int_0^y (y-t)\left[\sigma\partial_n u^0\right](t)\,\mathrm{d}t \\ &+ \frac{\varepsilon^2 y}{\widehat{\sigma}_0 y_0}\int_0^{y_0}(y_0-t)\left[\sigma\partial_n u^0\right](t)\,\mathrm{d}t, \\ g_2^2(y) =& -\frac{\varepsilon}{2}\partial_n u_{\text{ext}}^0(x_0,y) - \frac{\varepsilon^2}{\widehat{\sigma}_0}\int_0^y (y-t)\left[\sigma\partial_n u^0\right](t)\,\mathrm{d}t \\ &+ \frac{\varepsilon^2 y}{\widehat{\sigma}_0 y_0}\int_0^{y_0}(y_0-t)\left[\sigma\partial_n u^0\right](t)\,\mathrm{d}t. \end{aligned}\right.$$

Following the same procedure as in Section 2.2 we obtain the following third-order asymptotic model for $u^{[2]}$

$$\left\{\begin{array}{lll}\sigma_{\text{int}}\Delta u^{[2]}_{\text{int}}=f_{\text{int}} & \text{in} & \Omega_{\text{int}},\\ \sigma_{\text{ext}}\Delta u^{[2]}_{\text{ext}}=f_{\text{ext}} & \text{in} & \Omega_{\text{ext}},\\ T^2_1\left(u^{[2]}\right)=0 & \text{on} & \Gamma,\\ T^2_2\left(u^{[2]}\right)=0 & \text{on} & \Gamma,\\ u^{[2]}_{\text{ext}}=0 & \text{on} & \partial\Omega.\end{array}\right. \tag{47}$$

where T^2_1 and T^2_2 are defined as follows

$$\left\{\begin{array}{l}T^2_1\left(u^{[2]}\right)=\left[u^{[2]}\right]_\Gamma+\varepsilon\left\{\partial_n u^{[2]}\right\}_\Gamma+\dfrac{\varepsilon^2}{8}\left[\partial_n u^{[2]}\right]_\Gamma,\\ T^2_2\left(u^{[2]}\right)=\dfrac{\varepsilon^2}{\widehat{\sigma}_0}\left[\sigma\partial_n u^{[2]}\right]_\Gamma+\dfrac{\varepsilon^2}{8}\dfrac{\mathrm{d}^2}{\mathrm{d}y^2}\left\{\partial^2_n u^{[2]}\right\}_\Gamma+\dfrac{\varepsilon}{4}\dfrac{\mathrm{d}^2}{\mathrm{d}y^2}\left[\partial_n u^{[2]}\right]_\Gamma+\dfrac{\mathrm{d}^2}{\mathrm{d}y^2}\left\{u^{[2]}\right\}_\Gamma.\end{array}\right.$$

3.2.4 Fourth-order model

For deriving the fourth-order model, we truncate the series from the fourth term and we define $u^{(3)}$ as

$$u^{(3)}=u^0+\varepsilon u^1+\varepsilon^2u^2+\varepsilon^3u^3 \quad \text{in} \quad \Omega_{\text{int}}\cup\Omega_{\text{ext}} \qquad \text{(see Proposition 2).}$$

From (37), (38), (40) and (42) we can deduce that $u^{(3)}$ satisfies the following equations

$$\left\{\begin{array}{lll}\sigma_{\text{int}}\Delta u^{(3)}_{\text{int}}=f_{\text{int}} & \text{in} & \Omega_{\text{int}},\\ u^{(3)}_{\text{int}}(x_0,y)=g^3_1(y) & & y\in(0,y_0),\\ u^{(3)}_{\text{int}}=0 & \text{on} & \partial\Omega\cap\partial\Omega_{\text{int}}.\end{array}\right.$$
$$\left\{\begin{array}{lll}\sigma_{\text{ext}}\Delta u^{(3)}_{\text{ext}}=f_{\text{ext}} & \text{in} & \Omega_{\text{ext}},\\ u^{(3)}_{\text{ext}}(x_0,y)=g^3_2(y) & & y\in(0,y_0),\\ u^{(3)}_{\text{ext}}=0 & \text{on} & \partial\Omega\cap\partial\Omega_{\text{ext}}.\end{array}\right. \tag{48}$$

where g^3_1 and g^3_2 are defined as follows

$$\left\{\begin{aligned} g_1^3(y) =& \frac{\varepsilon}{2}\partial_n u_{\text{int}}^0(x_0,y) - \frac{\varepsilon^2}{\widehat{\sigma}_0}\int_0^y (y-t)\left[\sigma\partial_n u^0\right](t)\,\mathrm{d}t \\ &+ \frac{\varepsilon^3 y}{\widehat{\sigma}_0 y_0}\int_0^{y_0}(y_0-t)\left[\sigma\partial_n u^0\right](t)\,\mathrm{d}t \\ &- \frac{\varepsilon^3}{\widehat{\sigma}_0}\int_0^y (y-t)\left(\left[\sigma\partial_n u^1\right](t) + \left\{\sigma\partial_n u^0\right\}(t)\right)\mathrm{d}t \\ &+ \frac{\varepsilon^3 y}{\widehat{\sigma}_0 y_0}\int_0^{y_0}(y_0-t)\left(\left[\sigma\partial_n u^1\right](t) + \left\{\sigma\partial_n u^0\right\}(t)\right)\mathrm{d}t, \\ g_2^3(y) =& -\frac{\varepsilon}{2}\partial_n u_{\text{ext}}^0(x_0,y) - \frac{\varepsilon^2}{\widehat{\sigma}_0}\int_0^y (y-t)\left[\sigma\partial_n u^0\right](t)\,\mathrm{d}t \\ &+ \frac{\varepsilon^2 y}{\widehat{\sigma}_0 y_0}\int_0^{y_0}(y_0-t)\left[\sigma\partial_n u^0\right](t)\,\mathrm{d}t \\ &- \frac{\varepsilon^3}{\widehat{\sigma}_0}\int_0^y (y-t)\left(\left[\sigma\partial_n u^1\right](t) + \left\{\sigma\partial_n u^0\right\}(t)\right)\mathrm{d}t \\ &+ \frac{\varepsilon^3 y}{\widehat{\sigma}_0 y_0}\int_0^{y_0}(y_0-t)\left(\left[\sigma\partial_n u^1\right](t) + \left\{\sigma\partial_n u^0\right\}(t)\right)\mathrm{d}t. \end{aligned}\right.$$

Following the same procedure as in Section 2.2 we obtain the following fourth-order asymptotic model for $u^{[3]}$

$$\left\{\begin{aligned} \sigma_{\text{int}}\Delta u_{\text{int}}^{[3]} &= f_{\text{int}} && \text{in} && \Omega_{\text{int}}, \\ \sigma_{\text{ext}}\Delta u_{\text{ext}}^{[3]} &= f_{\text{ext}} && \text{in} && \Omega_{\text{ext}}, \\ T_1^3\left(u^{[3]}\right) &= 0 && \text{on} && \Gamma, \\ T_2^3\left(u^{[3]}\right) &= 0 && \text{on} && \Gamma, \\ u_{\text{ext}}^{[3]} &= 0 && \text{on} && \partial\Omega. \end{aligned}\right. \tag{49}$$

where T_1^3 and T_2^3 are defined as follows

$$\left\{\begin{aligned} T_1^3\left(u^{[3]}\right) =& \left[u^{[3]}\right]_\Gamma + \varepsilon\left\{\partial_n u^{[3]}\right\}_\Gamma + \frac{\varepsilon^2}{8}\left[\partial_n u^{[3]}\right]_\Gamma + \frac{\varepsilon^3}{24}\left\{\partial_n^3 u^{[3]}\right\}_\Gamma, \\ T_2^3\left(u^{[3]}\right) =& \frac{\varepsilon^2}{\widehat{\sigma}_0}\left[\sigma\partial_n u^{[3]}\right]_\Gamma + \frac{\varepsilon^3}{\widehat{\sigma}_0}\left\{\sigma\partial_n u^{[3]}\right\}_\Gamma + \frac{\varepsilon^3}{96}\frac{\mathrm{d}^2}{\mathrm{d}y^2}\left[\partial_n^3 u^{[3]}\right]_\Gamma \\ &+ \frac{\varepsilon^2}{8}\frac{\mathrm{d}^2}{\mathrm{d}y^2}\left\{\partial_n^2 u^{[3]}\right\}_\Gamma + \frac{\varepsilon}{4}\frac{\mathrm{d}^2}{\mathrm{d}y^2}\left[\partial_n u^{[3]}\right]_\Gamma + \frac{\mathrm{d}^2}{\mathrm{d}y^2}\left\{u^{[3]}\right\}_\Gamma. \end{aligned}\right.$$

3.3 Artificial boundaries

If we derive the variational formulation of the second-order model (45), as we do later in the Section 4.3.2, we notice that we cannot prove the coerciveness of the bilinear form, defined in

Section 4.3.2, due to a negative term. This negative term could cause instabilities when solving the problem with the finite element method as we observe in Section 5.2. However, to overcome this problem and restore stability we are going to use a technique based on introducing some new artificial boundaries, across of which we are going to rewrite the transmission conditions, see for example [13, 14]. We define these new artificial boundaries as follows

Definition 4. *We define the artificial boundaries Γ^δ_{int} and Γ^δ_{ext} as*

$$\Gamma^\delta_{int} = \{(x_0 - \delta\varepsilon, y) : \delta > 0, y \in (0, y_0)\},$$

$$\Gamma^\delta_{ext} = \{(x_0 + \delta\varepsilon, y) : \delta > 0, y \in (0, y_0)\}.$$

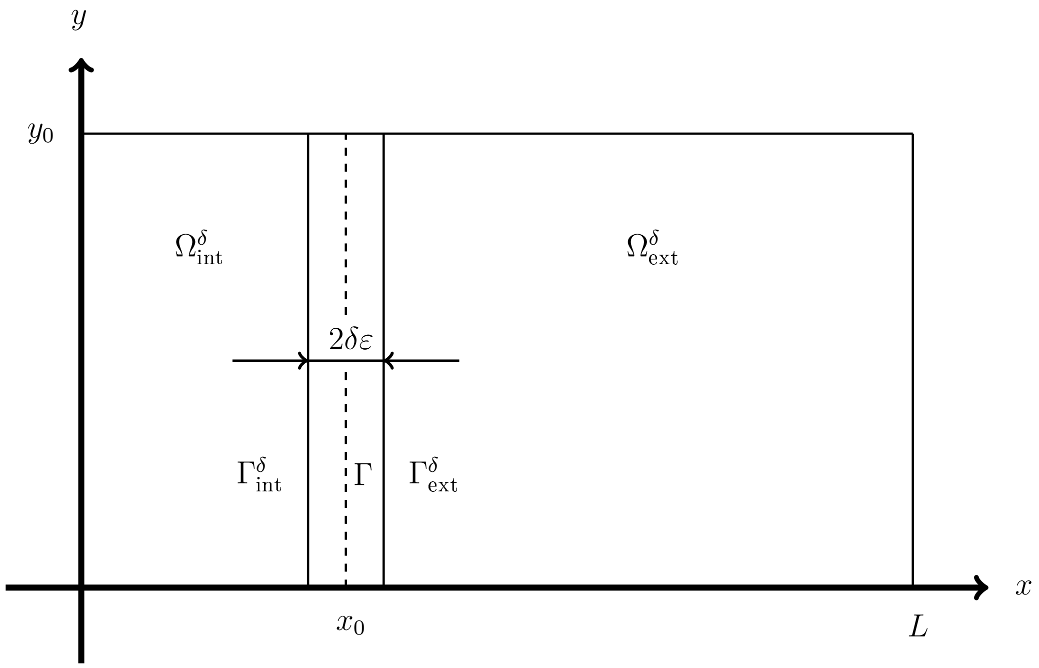


Figure 3: New configuration for the domain composed of two artificial boundaries.

We can observe the new configuration defined by the artificial boundaries in Figure 3.

Remark 2. *The domains Ω^δ_{int} and Ω^δ_{ext}, and the boundaries Γ^δ_{int} and Γ^δ_{ext}, all depend on ε, but we do not include it in the notation for the sake of simplicity.*

We apply a formal Taylor expansion on the variable normal to the thin layer, x in this case, in order to write the transmission conditions across the artificial boundaries.

$$\begin{cases} u_{\text{ext}}(x_0, y) = u_{\text{ext}}(x_0 + \delta\varepsilon, y) - \delta\varepsilon\partial_n u_{\text{ext}}(x_0 + \delta\varepsilon, y) + O(\varepsilon^2), \\ u_{\text{int}}(x_0, y) = u_{\text{int}}(x_0 - \delta\varepsilon, y) + \delta\varepsilon\partial_n u_{\text{int}}(x_0 - \delta\varepsilon, y) + O(\varepsilon^2), \\ \partial_n u_{\text{ext}}(x_0, y) = \partial_n u_{\text{ext}}(x_0 + \delta\varepsilon, y) - \delta\varepsilon\partial^2_n u_{\text{ext}}(x_0 + \delta\varepsilon, y) + O(\varepsilon^2), \\ \partial_n u_{\text{int}}(x_0, y) = \partial_n u_{\text{int}}(x_0 - \delta\varepsilon, y) + \delta\varepsilon\partial^2_n u_{\text{int}}(x_0 - \delta\varepsilon, y) + O(\varepsilon^2). \end{cases}$$

Now we substitute these expressions in the boundary conditions over Γ of Equation (45) and neglecting the terms of order 2 or higher in ε, we obtain the new boundary conditions written over the new artificial boundaries. The resulting asymptotic model writes as follows.

$$
\begin{cases}
\sigma_{\text{int}} \Delta u^{[1]}_{\delta,\text{int}} = f_{\text{int}} & \text{in} \quad \Omega^{\delta}_{\text{int}}, \\
u^{[1]}_{\delta,\text{int}} = \dfrac{\varepsilon(1-2\delta)}{2} \partial_n u^{[1]}_{\delta,\text{int}} & \text{on} \quad \Gamma^{\delta}_{\text{int}}, \\
u^{[1]}_{\delta,\text{int}} = 0 & \text{on} \quad \partial\Omega \cap \partial\Omega^{\delta}_{\text{int}}.
\end{cases}
\qquad
\begin{cases}
\sigma_{\text{ext}} \Delta u^{[1]}_{\delta,\text{ext}} = f_{\text{ext}} & \text{in} \quad \Omega^{\delta}_{\text{ext}}, \\
u^{[1]}_{\delta,\text{ext}} = -\dfrac{\varepsilon(1-2\delta)}{2} \partial_n u^{[1]}_{\delta,\text{ext}} & \text{on} \quad \Gamma^{\delta}_{\text{ext}}, \\
u^{[1]}_{\delta,\text{ext}} = 0 & \text{on} \quad \partial\Omega \cap \partial\Omega^{\delta}_{\text{ext}}.
\end{cases}
\tag{50}
$$

As we will see later in Section 4.3, with this new formulation, if we select $\delta > \frac{1}{2}$, the negative term of the bilinear form in the variational formulation becomes positive and stability is restored. Henceforth, we will refer to this new stable model as stabilized δ-order 2 model.

Notation 3. *We denote by Ω^{δ} the domain*

$$\Omega^{\delta} = \Omega^{\delta}_{int} \cup \Omega^{\delta}_{ext},$$

where Ω^{δ}_{int} and Ω^{δ}_{ext} are the domains defined in Figure 3.

3.4 Classical conditions and comparison with equivalent conditions

In this section we show the results we obtain when we no longer consider the conductivity in the thin layer to be dependent on its thickness to remark the different results we obtain with our approach compared to this one. The model problem remains the same, we consider Equations (2) set in the domain showed in Figure 1, but know we consider a conductivity of the following form

$$
\sigma = \begin{cases}
\sigma_{\text{int}} & \text{in} \quad \Omega^{\varepsilon}_{\text{int}}, \\
\sigma_{\text{lay}} & \text{in} \quad \Omega^{\varepsilon}_{\text{lay}}, \\
\sigma_{\text{ext}} & \text{in} \quad \Omega^{\varepsilon}_{\text{ext}},
\end{cases}
$$

where σ_{lay} is just a constant and not dependent on ε any more. With this configuration, the model problem we are considering is the one described by Equations (24). Considering this model, and applying the same asymptotic method developed in the previous section to derive approximate models, we obtain an first-order model and a second-order model. The expression for these models are the following.

First-order model

$$
\begin{cases}
\sigma_{\mathrm{int}}\Delta u_{\mathrm{int}}^{[0]} = f_{\mathrm{int}} & \text{in} \quad \Omega_{\mathrm{int}}, \\
\sigma_{\mathrm{ext}}\Delta u_{\mathrm{ext}}^{[0]} = f_{\mathrm{ext}} & \text{in} \quad \Omega_{\mathrm{ext}}, \\
\left[u^{[0]}\right]_{\Gamma} = 0, & \\
\left[\sigma\partial_n u^{[0]}\right]_{\Gamma} = 0, & \\
u^{[0]} = 0 & \text{on} \quad \partial\Omega.
\end{cases}
$$

Second-order model

$$
\begin{cases}
\sigma_{\mathrm{int}}\Delta u_{\mathrm{int}}^{[1]} = f_{\mathrm{int}} & \text{in} \quad \Omega_{\mathrm{int}}, \\
\sigma_{\mathrm{ext}}\Delta u_{\mathrm{ext}}^{[1]} = f_{\mathrm{ext}} & \text{in} \quad \Omega_{\mathrm{ext}}, \\
\left[u^{[1]}\right]_{\Gamma} = \dfrac{\varepsilon}{\sigma_{\mathrm{lay}}}\left\{\sigma\partial_n u^{[1]}\right\}_{\Gamma} - \varepsilon\left\{\partial_n u^{[1]}\right\}_{\Gamma}, & \\
\left[\sigma\partial_n u^{[1]}\right]_{\Gamma} = -\varepsilon\sigma_{\mathrm{lay}}\dfrac{\mathrm{d}^2}{\mathrm{d}y^2}\left\{u^{[1]}\right\}_{\Gamma} - \varepsilon\left\{\sigma\partial_n^2 u^{[1]}\right\}_{\Gamma}, & \\
u^{[1]} = 0 & \text{on} \quad \partial\Omega.
\end{cases}
$$

We notice that these models are different from (43) and (45). A main difference comparing with our approach (i.e. when $\sigma_{\mathrm{lay}} = \widehat{\sigma}_0\varepsilon^{-3}$) comes from the fact that now both models are coupled, whereas employing our approach, the models are uncoupled, given thus by two independent problems. In this case, contrary to the first class of ITCs, the order of these models, (3.4) and (3.4), coincide with the one of the models (43) and (45).

4 Stability and convergence results

This section is devoted to the validation of the multiscale expansions we have derived in the previous sections. We perform the proofs of existence, uniqueness and uniform estimates for the derived asymptotic models and we derive the convergence results. This section is structured as follows. In Section 4.1 we study the convergence of the asymptotic expansion for the reference model and give estimates for the residue. Then, Sections 4.2 and 4.3 are devoted to the study of the asymptotic models of the first class and second class respectively. In these sections we derive the variational formulations for such asymptotic models and we prove stability and convergence results.

4.1 Convergence of the asymptotic expansion for the reference model

We begin by defining the residue for the reference Problem (2). We remind the domain Ω is defined in Section 1 and we can observe it in Figure 1. The residue of the asymptotic expansion (8) is defined by removing the first terms to the solution u of the reference Problem (2).

Definition 5. *Let u be the solution to Problem (2), given the expansion in power series (8) and a specific order $N \in \mathbb{N}$, we define the residue r^N as*

$$\begin{cases} r^N_{int}(x,y) = u_{int}(x,y) - \displaystyle\sum_{k=0}^{N} \varepsilon^k u^k_{int}(x,y) & \text{in} \quad \Omega^\varepsilon_{int}, \\ r^N_{ext}(x,y) = u_{ext}(x,y) - \displaystyle\sum_{k=0}^{N} \varepsilon^k u^k_{ext}(x,y) & \text{in} \quad \Omega^\varepsilon_{ext}, \\ r^N_{lay}(x,y) = u_{lay}(x,y) - \displaystyle\sum_{k=0}^{N} \varepsilon^k U^k \left(\frac{x - x_0}{\varepsilon}, y \right) & \text{in} \quad \Omega^\varepsilon_{lay}. \end{cases}$$

Proposition 3. *Let $N \in \mathbb{N}$, the residue r^N defined in Definition 5 satisfies the following equations*

$$\begin{cases} \sigma_{int} \Delta r^N_{int} = 0 & \text{in} \quad \Omega^\varepsilon_{int}, \\ \sigma_{ext} \Delta r^N_{ext} = 0 & \text{in} \quad \Omega^\varepsilon_{ext}, \\ \widehat{\sigma}_0 \varepsilon^{-3} \Delta r^N_{lay} = f^N_{lay} & \text{in} \quad \Omega^\varepsilon_{lay}, \\ r^N_{int} = r^N_{lay} & \text{on} \quad \Gamma^\varepsilon_{int}, \\ r^N_{lay} = r^N_{ext} & \text{on} \quad \Gamma^\varepsilon_{ext}, \\ \widehat{\sigma}_0 \varepsilon^{-3} \partial_n r^N_{lay} - \sigma_{int} \partial_n r^N_{int} = g^N_{int} & \text{on} \quad \Gamma^\varepsilon_{int}, \\ \widehat{\sigma}_0 \varepsilon^{-3} \partial_n r^N_{lay} - \sigma_{ext} \partial_n r^N_{ext} = g^N_{ext} & \text{on} \quad \Gamma^\varepsilon_{ext}, \\ r^N = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where

$$
\begin{aligned}
f_{lay}^N &= \varepsilon^{N-4}\left(-\widehat{\sigma}_0\partial_y^2 u_{lay}^{N-1} - \varepsilon\widehat{\sigma}_0\partial_y^2 u_{lay}^N\right), \\
g_{int}^N &= \varepsilon^{N-3}\left(\sigma_{int}\partial_n u_{int}^{N-3} + \ldots + \varepsilon^3\sigma_{int}\partial_n u_{int}^N\right), \\
g_{ext}^N &= \varepsilon^{N-4}\left(\sigma_{ext}\partial_n u_{ext}^{N-3} + \ldots + \varepsilon^3\sigma_{ext}\partial_n u_{ext}^N\right).
\end{aligned}
\tag{51}
$$

Proof. We can deduce this result by applying Equations (9), (10), (11) and (3) to the definition of the residue. □

Theorem 2. *Let $N \in \mathbb{N}$. For $\varepsilon \in (0, \varepsilon_0)$ and under the assumptions $f_{lay}^N \in L^2\left(\Omega_{lay}^\varepsilon\right)$, $g_{int}^N \in L^2\left(\Gamma_{int}^\varepsilon\right)$, $g_{ext}^N \in L^2\left(\Gamma_{ext}^\varepsilon\right)$, $f_{lay}^{N+5} \in L^2\left(\Omega_{lay}^\varepsilon\right)$, $g_{int}^{N+5} \in L^2\left(\Gamma_{int}^\varepsilon\right)$, $g_{ext}^{N+5} \in L^2\left(\Gamma_{ext}^\varepsilon\right)$, and $u^k \in H^1\left(\Omega^\varepsilon\right)$ for $k \leq N+5$, the following estimate holds for the residue defined in Definition 5,*

$$
\left\|r_{ext}^N\right\|_{1,\Omega_{ext}^\varepsilon} + \left\|r_{int}^N\right\|_{1,\Omega_{int}^\varepsilon} + \sqrt{\varepsilon}\left\|r_{lay}^N\right\|_{1,\Omega_{lay}^\varepsilon} \leq C\varepsilon^{N+1},
$$

for a positive constant $C > 0$ independent of ε.

Proof. Applying Theorem 1 and Proposition 3, for $\varepsilon \in (0, \varepsilon_0)$, we deduce that there exists a unique $r^N \in H_0^1(\Omega)$ and $\left\|r^N\right\|_{1,\Omega} = O(\varepsilon^{N-4})$. Rewriting r^N in the following way

$$
r^N = r^{N+5} + \varepsilon^{N+1}u^{N+1} + \varepsilon^{N+2}u^{N+2} + \varepsilon^{N+3}u^{N+3} + \varepsilon^{N+4}u^{N+4} + \varepsilon^{N+5}u^{N+5},
$$

we deduce that $\left\|r_{\mathrm{int}}^N\right\|_{1,\Omega_{\mathrm{int}}^\varepsilon} + \left\|r_{\mathrm{ext}}^N\right\|_{1,\Omega_{\mathrm{ext}}^\varepsilon} = O(\varepsilon^{N+1})$. Finally taking into account that $\left\|u_{\mathrm{lay}}^k\right\|_{1,\Omega_{\mathrm{lay}}^\varepsilon} = O\left(\varepsilon^{-\frac{1}{2}}\right)$, we deduce that $\left\|r_{\mathrm{lay}}^N\right\|_{1,\Omega_{\mathrm{lay}}^\varepsilon} = O\left(\varepsilon^{N+\frac{1}{2}}\right)$ and the desired result. □

Remark 3. *We remark that the assumptions of Theorem 2 are rather strong assumptions because in general the expansion consumes regularity at each order and the first term of the expansion u^0, solution to (17), belongs only to $H^3\left(\Omega^\varepsilon\right)$.*

4.2 Validation of the first class of ITCs

This section is devoted to the derivation of convergence results for the order 2 and order 4 asymptotic models we have derived in Section 2.2. We remark that the domain and configuration for these models have been presented in Section 1 and Figure 1.

4.2.1 Second order model: variational formulation

Problem (22) is uncoupled into two independent problems, therefore we write two variational formulations, one for each problem. First of all we introduce the functional spaces $H_0^1\left(\Omega_{\mathrm{int}}^\varepsilon\right)$ and $H_0^1\left(\Omega_{\mathrm{ext}}^\varepsilon\right)$ as the functional framework. Assuming $f_{\mathrm{int}} \in L^2\left(\Omega_{\mathrm{int}}^\varepsilon\right)$ and $f_{\mathrm{ext}} \in L^2\left(\Omega_{\mathrm{ext}}^\varepsilon\right)$, the variational formulations reduce to finding $u_{\mathrm{int}} \in H_0^1\left(\Omega_{\mathrm{int}}^\varepsilon\right)$ such that for all $w_{\mathrm{int}} \in H_0^1\left(\Omega_{\mathrm{int}}^\varepsilon\right)$

$$
-\int_{\Omega_{\mathrm{int}}^\varepsilon} f_{\mathrm{int}} w_{\mathrm{int}}\,\mathrm{d}x = \int_{\Omega_{\mathrm{int}}^\varepsilon} \sigma_{\mathrm{int}}\nabla u_{\mathrm{int}}\cdot\nabla w_{\mathrm{int}}\,\mathrm{d}x, \tag{52}
$$

and finding $u_{\mathrm{ext}} \in H_0^1\left(\Omega_{\mathrm{ext}}^\varepsilon\right)$ such that for alld $w_{\mathrm{ext}} \in H_0^1\left(\Omega_{\mathrm{ext}}^\varepsilon\right)$

$$
-\int_{\Omega_{\mathrm{ext}}^\varepsilon} f_{\mathrm{ext}} w_{\mathrm{ext}}\,\mathrm{d}x = \int_{\Omega_{\mathrm{ext}}^\varepsilon} \sigma_{\mathrm{ext}}\nabla u_{\mathrm{ext}}\cdot\nabla w_{\mathrm{ext}}\,\mathrm{d}x. \tag{53}
$$

4.2.2 Fourth order model: variational formulation

Instead of considering Problem (23) directly, we will consider the following problem

$$\left\{\begin{aligned} \sigma_{\text{int}}\Delta u_{\text{int}} &= f_{\text{int}} \quad \text{in} \quad \Omega^{\varepsilon}_{\text{int}},\\ \sigma_{\text{ext}}\Delta u_{\text{ext}} &= f_{\text{ext}} \quad \text{in} \quad \Omega^{\varepsilon}_{\text{ext}},\\ [u]_{\Gamma^{\varepsilon}} &= 0,\\ \varepsilon^{-2}\widehat{\sigma}_0\frac{\mathrm{d}^2}{\mathrm{d}y^2}\{u\}_{\Gamma^{\varepsilon}} + [\sigma\partial_n u]_{\Gamma^{\varepsilon}} &= g,\\ u &= 0 \quad \text{on} \quad \partial\Omega\cap\partial\Omega^{\varepsilon}. \end{aligned}\right. \tag{54}$$

This problem is similar to Problem (23) and it generalizes it by including the right-hand side function g. We begin by selecting the functional space denoted by V_4 and defined as follows

Definition 6. *The functional space V_4 is given by*

$$V_4 = \Big\{ w : w_{int} \in H^1\left(\Omega^{\varepsilon}_{int}\right), w_{ext} \in H^1\left(\Omega^{\varepsilon}_{ext}\right), \nabla_{\Gamma^{\varepsilon}}\{w\} \in L^2\left(\Gamma^{\varepsilon}\right),$$
$$w|_{\Gamma^{\varepsilon}_{int}} = w|_{\Gamma^{\varepsilon}_{ext}}, w|_{\partial\Omega\cap\partial\Omega^{\varepsilon}_{int}} = 0, w|_{\partial\Omega\cap\partial\Omega^{\varepsilon}_{ext}} = 0 \Big\}$$

As $H^1\left(\Omega^{\varepsilon}\right)$ is a Hilbert space and V_4 is a closed subspace of $H^1\left(\Omega^{\varepsilon}\right)$, we can deduce that the functional space V_4, characterized in Definition 6 equipped with the norm

$$\|w\|_{V_4} = \left(\|w\|^2_{1,\Omega^{\varepsilon}} + \|\nabla_{\Gamma^{\varepsilon}}\{w\}\|^2_{0,\Gamma^{\varepsilon}}\right)^{\frac{1}{2}},$$

is a Hilbert space.

Remark 4. *Mean values and jumps are defined over the interfaces $\Gamma^{\varepsilon}_{int}$ and $\Gamma^{\varepsilon}_{ext}$. As jump and mean values only depend on the variable y, when we write Γ^{ε} we are referring to the interval $y \in (0, y_0)$.*

Assuming $f_{\text{int}} \in L^2\left(\Omega^{\varepsilon}_{\text{int}}\right)$ and $f_{\text{ext}} \in L^2\left(\Omega^{\varepsilon}_{\text{ext}}\right)$, the variational formulation for problem 6 reduces to finding $u \in V_4$, such that for all $w \in V_4$,

$$a(u, w) = l(w), \tag{55}$$

where

$$a(u, w) = \sigma_{\text{int}}\int_{\Omega^{\varepsilon}_{\text{int}}} \nabla u\cdot\nabla w\,\mathrm{d}x + \sigma_{\text{ext}}\int_{\Omega^{\varepsilon}_{\text{ext}}} \nabla u\cdot\nabla w\,\mathrm{d}x + \widehat{\sigma}_0\varepsilon^{-2}\int_{\Gamma^{\varepsilon}} \nabla_{\Gamma^{\varepsilon}}\{u\}_{\Gamma^{\varepsilon}}\nabla_{\Gamma^{\varepsilon}}\{w\}_{\Gamma^{\varepsilon}}\,\mathrm{d}s,$$

$$l(w) = -\int_{\Omega^{\varepsilon}_{\text{int}}} f_{\text{int}} w\,\mathrm{d}x - \int_{\Omega^{\varepsilon}_{\text{ext}}} f_{\text{ext}} w\,\mathrm{d}x - \int_{\Gamma^{\varepsilon}} g\{w\}_{\Gamma^{\varepsilon}}\,\mathrm{d}s.$$

4.2.3 Stability results

First of all we will develop an expansion in power series of ε for the Problem (23) in the form

$$\begin{cases} u_{\text{ext}}^{[3]} \approx \displaystyle\sum_{k\geq 0} \varepsilon^k \widehat{u}_{\text{ext}}^k & \text{in} \quad \Omega_{\text{int}}^{\varepsilon}, \\ u_{\text{int}}^{[3]} \approx \displaystyle\sum_{k\geq 0} \varepsilon^k \widehat{u}_{\text{int}}^k & \text{in} \quad \Omega_{\text{ext}}^{\varepsilon}. \end{cases} \tag{56}$$

We substitute these series in Equations (23) and we collect the terms with the same power in ε. For every $k \in \mathbb{N}$ we obtain the following set of equations

$$\begin{cases} \sigma_{\text{int}} \Delta \widehat{u}_{\text{int}}^k = f_{\text{int}} \delta_k^0 & \text{in} \quad \Omega_{\text{int}}^{\varepsilon}, \\ \sigma_{\text{ext}} \Delta \widehat{u}_{\text{ext}}^k = f_{\text{ext}} \delta_k^0 & \text{in} \quad \Omega_{\text{ext}}^{\varepsilon}, \\ \left[\widehat{u}^k\right]_{\Gamma^{\varepsilon}} = 0, & \\ -\widehat{\sigma}_0 \dfrac{\mathrm{d}^2}{\mathrm{d}y^2} \left\{\widehat{u}^k\right\}_{\Gamma^{\varepsilon}} = \left[\sigma \partial_n \widehat{u}^{k-2}\right]_{\Gamma^{\varepsilon}}, & \\ \widehat{u}^k = 0 & \text{on} \quad \partial\Omega \cap \partial\Omega^{\varepsilon}. \end{cases} \tag{57}$$

Definition 7. *Given the expansion in power series (56) and $N \in \mathbb{N}$, we define the residue $\widehat{r}^N$ as*

$$\begin{cases} \widehat{r}_{int}^N(x,y) = u_{int}^{[3]}(x,y) - \displaystyle\sum_{k=0}^{N} \varepsilon^k \widehat{u}_{int}^k(x,y) & \mathit{in} \quad \Omega_{int}^{\varepsilon}, \\ \widehat{r}_{ext}^N(x,y) = u_{ext}^{[3]}(x,y) - \displaystyle\sum_{k=0}^{N} \varepsilon^k \widehat{u}_{ext}^k(x,y) & \mathit{in} \quad \Omega_{ext}^{\varepsilon}. \end{cases}$$

We now prove the existence, uniqueness and uniform estimates of the solution to Problem (55), but before starting with the proofs of existence, uniqueness and estimates, we will write a Poincaré inequality for the configuration we are working on.

Theorem 3. *We can write a Poincaré inequality of the following form: there exists a constant $C > 0$ such that for all $u \in V_4$,*

$$\int_{\Omega^{\varepsilon}} |u|^2 \,\mathrm{d}x \leq C \int_{\Omega^{\varepsilon}} |\nabla u|^2 \,\mathrm{d}x.$$

Proof. We follow a similar reasoning to the one presented in [3] (proposition 8.13 and corollary 9.19). For all $u \in V_4$, u is continuous across the interfaces $\Gamma_{\text{int}}^{\varepsilon}$ and $\Gamma_{\text{ext}}^{\varepsilon}$, and it vanishes in the rest of the boundary. We can deduce that for all $(x,y) \in \Omega^{\varepsilon}$

$$|u(x,y)| \leq \int_I |\partial_t u(t,y)| \,dt,$$

where we have defined $I = \left(0, x_0 - \frac{\varepsilon}{2}\right) \cup \left(x_0 + \frac{\varepsilon}{2}, L\right)$. We apply Cauchy-Schwartz inequality and by integrating first in the x variable and then in the y variable, we obtain the desired result

$$\int_{\Omega^\varepsilon} |u|^2 \,\mathrm{d}x \leq L^2 \int_{\Omega^\varepsilon} |\partial_x u|^2 \,\mathrm{d}x \leq L^2 \int_{\Omega^\varepsilon} |\nabla u|^2 \,\mathrm{d}x.$$

□

Theorem 4. *For all $\varepsilon > 0$ there exists a unique $u \in V_4$ solution to Problem (55) with data $f_{int} \in L^2(\Omega^\varepsilon_{int})$, $f_{ext} \in L^2(\Omega^\varepsilon_{ext})$ and $g \in L^2(\Gamma^\varepsilon)$. Moreover, there exists $\varepsilon_0 > 0$ and a constant $C > 0$, such that for all $\varepsilon \in (0, \varepsilon_0)$,*

$$\|u\|_{V_4} \leq C \left(\|f_{int}\|_{0,\Omega^\varepsilon_{int}} + \|f_{ext}\|_{0,\Omega^\varepsilon_{ext}} + \|g\|_{0,\Gamma^\varepsilon} \right).$$

Proof. First of all, as the bilinear form a is coercive and continuous in V_4, and the linear form l is continuous in V_4, the existence and uniqueness of a weak solution follows from the Lax-Milgram Lemma. Then, for proving the uniform estimates, we select ε_0 as

$$\varepsilon_0 = \sqrt{\frac{\widehat{\sigma}_0}{\min(\sigma_{\rm int}, \sigma_{\rm ext})}}.$$

Then for $\varepsilon < \varepsilon_0$ and employing theorem 3, there exists a constant $k_1 > 0$ such that for all $w \in V_4$

$$a(w, w) \geq \frac{1}{k_1} \min\left(\sigma_{\rm int}, \sigma_{\rm ext}\right) \|w\|^2_{V_4}. \tag{58}$$

Applying a trace theorem and Cauchy-Schwarz inequality to the definition of l we obtain that l is continuous in V_4, more specifically, there exists a constant $k_2 > 0$, such that

$$|l(w)| \leq k_2 \|w\|_{V_4} \left(\|f_{\rm int}\|_{0,\Omega^\varepsilon_{\rm int}} + \|f_{\rm ext}\|_{0,\Omega^\varepsilon_{\rm ext}} + \|g\|_{0,\Gamma^\varepsilon} \right). \tag{59}$$

Finally, employing Equations (58) and (59) we obtain

$$\|u\|_{V_4} \leq C \left(\|f_{\rm int}\|_{0,\Omega^\varepsilon_{\rm int}} + \|f_{\rm ext}\|_{0,\Omega^\varepsilon_{\rm ext}} + \|g\|_{0,\Gamma^\varepsilon} \right),$$

where $C = \dfrac{k_2 k_1}{\min\left(\sigma_{\rm int}, \sigma_{\rm ext}\right)}$. □

Proposition 4. *Let $N \in \mathbb{N}$, the residue $\widehat{r}^N$ defined in Definition 7 satisfies the following equations*

$$\begin{cases} \sigma_{int} \Delta \widehat{r}^N_{int} = 0 & in \quad \Omega^\varepsilon_{int}, \\ \sigma_{ext} \Delta \widehat{r}^N_{ext} = 0 & in \quad \Omega^\varepsilon_{ext}, \\ \left[\widehat{r}^N\right]_{\Gamma^\varepsilon} = 0, & \\ \varepsilon^{-2} \widehat{\sigma}_0 \dfrac{\mathrm{d}^2}{\mathrm{d}y^2} \left\{\widehat{r}^N\right\}_{\Gamma^\varepsilon} + \left[\sigma \partial_n \widehat{r}^N\right]_{\Gamma^\varepsilon} = g^N, & \\ \widehat{r}^N = 0 & on \quad \partial\Omega \cap \partial\Omega^\varepsilon, \end{cases}$$

where

$$g^N = \varepsilon^{N+1} \left(- \left[\sigma \partial_n \widehat{u}^{N+1}\right]_{\Gamma^\varepsilon} - \varepsilon \left[\sigma \partial_n \widehat{u}^{N+2}\right]_{\Gamma^\varepsilon} \right).$$

Proof. We can deduce this result by applying Equations (57) and (23) to the definition of the residue. □

Theorem 5. *Let $N \in \mathbb{N}$. For $\varepsilon \in (0, \varepsilon_0)$ and under the assumption $g^N \in L^2(\Gamma^\varepsilon)$, the following estimate holds for the residue $\widehat{r}^N$ defined in Definition 7, for a constant $C > 0$ independent of ε*

$$\left\|\widehat{r}^N_{ext}\right\|_{1,\Omega^\varepsilon_{ext}} + \left\|\widehat{r}^N_{int}\right\|_{1,\Omega^\varepsilon_{int}} \leq C\varepsilon^{N+1}.$$

Proof. We can deduce this result directly from Theorem 4 and Proposition 4. □

4.2.4 Convergence results

Theorem 6. *Under the assumptions of Theorem 2 for $N = 1$, $\varepsilon \in (0, \varepsilon_0)$ and with a data $f_{int} \in L^2(\Omega^\varepsilon_{int})$ and $f_{ext} \in L^2(\Omega^\varepsilon_{ext})$, the following estimate holds for the function $u^{[1]}$, solution to the second order asymptotic model (22), which writes as*

$$\begin{cases} \sigma_{int}\Delta u^{[1]}_{int} = f_{int} & \text{in} \quad \Omega^\varepsilon_{int}, \\ u^{[1]}_{int} = 0 & \text{on} \quad \partial\Omega^\varepsilon_{int}. \end{cases}$$

$$\begin{cases} \sigma_{ext}\Delta u^{[1]}_{ext} = f_{ext} & \text{in} \quad \Omega^\varepsilon_{ext}, \\ u^{[1]}_{ext} = 0 & \text{on} \quad \partial\Omega^\varepsilon_{ext}. \end{cases}$$

and u, solution to the reference Problem (2): there exists a constant $C > 0$ independent of ε, such that

$$\left\|u_{int} - u^{[1]}_{int}\right\|_{1,\Omega^\varepsilon_{int}} + \left\|u_{ext} - u^{[1]}_{ext}\right\|_{1,\Omega^\varepsilon_{ext}} \leq C\varepsilon^2.$$

Proof. We can deduce this result directly from Theorem 2. □

Theorem 7. *Under the assumptions of Theorem 2 and Theorem 5 for $N = 3$, $\varepsilon \in (0, \varepsilon_0)$ and with a data $f_{int} \in L^2(\Omega^\varepsilon_{int})$ and $f_{ext} \in L^2(\Omega^\varepsilon_{ext})$, the following estimate holds for the function $u^{[3]}$, solution to the fourth order asymptotic model (23), which writes as*

$$\begin{cases} \sigma_{int}\Delta u^{[3]}_{int} = f_{int} & \text{in} \quad \Omega^\varepsilon_{int}, \\ \sigma_{ext}\Delta u^{[3]}_{ext} = f_{ext} & \text{in} \quad \Omega^\varepsilon_{ext}, \\ \left[u^{[3]}\right]_{\Gamma^\varepsilon} = 0, & \\ \left[\sigma\partial_n u^{[3]}\right]_{\Gamma^\varepsilon} = -\dfrac{\widehat{\sigma}_0}{\varepsilon^2}\dfrac{\mathrm{d}^2}{\mathrm{d}y^2}\left\{u^{[3]}\right\}_{\Gamma^\varepsilon}, & \\ u^{[3]} = 0 & \text{on} \quad \partial\Omega \cap \partial\Omega^\varepsilon. \end{cases}$$

and u, solution to the reference Problem (2): there exists a constant $C > 0$ independent of ε, such that

$$\left\|u_{int} - u^{[3]}_{int}\right\|_{1,\Omega^\varepsilon_{int}} + \left\|u_{ext} - u^{[3]}_{ext}\right\|_{1,\Omega^\varepsilon_{ext}} \leq C\varepsilon^4.$$

Proof. To begin with, we consider the expansion (56) of $u^{[3]}$ we have done above. More specifically, we consider Equations (57). We deduce that for $k = 0, 1, 2, 3$, we obtain

$$\widehat{u}^k \equiv u^k \qquad k = 0, 1, 2, 3.$$

Thus, using Theorem 2 and Theorem 5 we deduce the desired result. $\square$

4.3 Validation of the second class of ITCs

This section is devoted to the derivation of convergence results for the first-order and second-order asymptotic models we have derived in Section 3.2. We remind that the domain and configuration for these models have been presented in Section 3.1 and Figure 2.

4.3.1 First order model: variational formulation

Problem (43) is uncoupled into two independent problems, therefore we write two variational formulations, one for each problem. First of all we introduce the functional spaces $H_0^1\left(\Omega_{\mathrm{int}}\right)$ and $H_0^1\left(\Omega_{\mathrm{ext}}\right)$ as the functional framework. Assuming $f_{\mathrm{int}} \in L^2\left(\Omega_{\mathrm{int}}\right)$ and $f_{\mathrm{ext}} \in L^2\left(\Omega_{\mathrm{ext}}\right)$, the variational formulations reduce to finding $u_{\mathrm{int}} \in H_0^1\left(\Omega_{\mathrm{int}}\right)$ such that for all $w_{\mathrm{int}} \in H_0^1\left(\Omega_{\mathrm{int}}\right)$

$$-\int_{\Omega_{\mathrm{int}}} f_{\mathrm{int}} w_{\mathrm{int}} \,\mathrm{d}x = \int_{\Omega_{\mathrm{int}}} \sigma_{\mathrm{int}} \nabla u_{\mathrm{int}} \cdot \nabla w_{\mathrm{int}} \,\mathrm{d}x, \tag{60}$$

and finding $u_{\mathrm{ext}} \in H_0^1\left(\Omega_{\mathrm{ext}}\right)$ such that for all $w_{\mathrm{ext}} \in H_0^1\left(\Omega_{\mathrm{ext}}\right)$

$$-\int_{\Omega_{\mathrm{ext}}} f_{\mathrm{ext}} w_{\mathrm{ext}} \,\mathrm{d}x = \int_{\Omega_{\mathrm{ext}}} \sigma_{\mathrm{ext}} \nabla u_{\mathrm{ext}} \cdot \nabla w_{\mathrm{ext}} \,\mathrm{d}x. \tag{61}$$

4.3.2 Second order model: variational formulation

In this section we derive a variational formulation for the second order asymptotic model (45) we have derived in Section 3.2. We begin by selecting the functional spaces V_{int} and V_{ext} as the functional framework, which are defined as follows

$$\begin{aligned} V_{\mathrm{int}} &= \left\{ w \in H^1\left(\Omega_{\mathrm{int}}\right) : w|_{\partial\Omega \cap \partial\Omega_{\mathrm{int}}} = 0 \right\}, \\ V_{\mathrm{ext}} &= \left\{ w \in H^1\left(\Omega_{\mathrm{ext}}\right) : w|_{\partial\Omega \cap \partial\Omega_{\mathrm{ext}}} = 0 \right\}. \end{aligned} \tag{62}$$

Assuming $f_{\mathrm{int}} \in L^2\left(\Omega_{\mathrm{int}}\right)$ and $f_{\mathrm{ext}} \in L^2\left(\Omega_{\mathrm{ext}}\right)$, the variational formulations consist in finding $u_{\mathrm{int}} \in V_{\mathrm{int}}$, such that for all $w_{\mathrm{int}} \in V_{\mathrm{int}}$

$$-\int_{\Omega_{\mathrm{int}}} f_{\mathrm{int}} w_{\mathrm{int}} \,\mathrm{d}x = \int_{\Omega_{\mathrm{int}}} \sigma_{\mathrm{int}} \nabla u_{\mathrm{int}} \cdot \nabla w_{\mathrm{int}} \,\mathrm{d}x - \int_{\Gamma} \frac{2\sigma_{\mathrm{int}}}{\varepsilon} u_{\mathrm{int}} w_{\mathrm{int}} \,\mathrm{d}s,$$

and finding $u_{\mathrm{ext}} \in V_{\mathrm{ext}}$, such that for all $w_{\mathrm{ext}} \in V_{\mathrm{ext}}$

$$-\int_{\Omega_{\mathrm{ext}}} f_{\mathrm{ext}} w_{\mathrm{ext}} \,\mathrm{d}x = \int_{\Omega_{\mathrm{ext}}} \sigma_{\mathrm{ext}} \nabla u_{\mathrm{ext}} \cdot \nabla w_{\mathrm{ext}} \,\mathrm{d}x - \int_{\Gamma} \frac{2\sigma_{\mathrm{ext}}}{\varepsilon} u_{\mathrm{ext}} w_{\mathrm{ext}} \,\mathrm{d}s.$$

Observing these variational formulations we notice that we cannot prove the coerciveness of the bilinear forms due to the terms

$$-\int_{\Gamma} \frac{2\sigma_{\mathrm{int}}}{\varepsilon} u_{\mathrm{int}} w_{\mathrm{int}} \,\mathrm{d}s \quad \text{and} \quad -\int_{\Gamma} \frac{2\sigma_{\mathrm{ext}}}{\varepsilon} u_{\mathrm{ext}} w_{\mathrm{ext}} \,\mathrm{d}s$$

being negative. These negative terms could cause instabilities when numerically solving the problem with the finite element method. However, to overcome this problem and recover stability, we have derived new models across some artificial boundaries in Section 3.3. With these models we will no longer have instability problems, as we will prove in the following section.

4.3.3 Stabilized δ-order 2 model: variational formulation

In this section we derive a variational formulation for the stabilized δ-order 2 asymptotic model (50) we have derived in Section 3.3. Instead of directly considering Problem (50), we will consider the following problem

$$\begin{cases} \sigma_{\text{int}} \Delta u_{\text{int}} = f_{\text{int}} & \text{in} \quad \Omega^\delta_{\text{int}}, \\ u_{\text{int}} - \dfrac{\varepsilon(1-2\delta)}{2} \partial_n u_{\text{int}} = g_{\text{int}} & \text{on} \quad \Gamma^\delta_{\text{int}}, \\ u_{\text{int}} = 0 & \text{on} \quad \partial\Omega \cap \partial\Omega^\delta_{\text{int}}. \end{cases}$$

$$\begin{cases} \sigma_{\text{ext}} \Delta u_{\text{ext}} = f_{\text{ext}} & \text{in} \quad \Omega^\delta_{\text{ext}}, \\ u_{\text{ext}} + \dfrac{\varepsilon(1-2\delta)}{2} \partial_n u_{\text{ext}} = g_{\text{ext}} & \text{on} \quad \Gamma^\delta_{\text{ext}}, \\ u_{\text{ext}} = 0 & \text{on} \quad \partial\Omega \cap \partial\Omega^\delta_{\text{ext}}. \end{cases}$$

This problem is similar to Problem (50) and it generalizes it by including the right-hand side functions g_1 and g_2. We begin by selecting the functional framework. We introduce the functional spaces V^δ_{int} and V^δ_{ext}, which are defined as follows

$$V^\delta_{\text{int}} = \left\{ w \in H^1\left(\Omega^\delta_{\text{int}}\right) : w|_{\partial\Omega\cap\partial\Omega^\delta_{\text{int}}} = 0 \right\},$$

$$V^\delta_{\text{ext}} = \left\{ w \in H^1\left(\Omega^\delta_{\text{ext}}\right) : w|_{\partial\Omega\cap\partial\Omega^\delta_{\text{ext}}} = 0 \right\}.$$

Assuming $f_{\text{int}} \in L^2\left(\Omega^\delta_{\text{int}}\right)$ and $f_{\text{ext}} \in L^2\left(\Omega^\delta_{\text{ext}}\right)$, the variational formulations reduce to finding $u_{\text{int}} \in V^\delta_{\text{int}}$, such that for all $w_{\text{int}} \in V^\delta_{\text{int}}$

$$a_{\text{int}}(u_{\text{int}}, w_{\text{int}}) = l_{\text{int}}(w_{\text{int}}), \tag{63}$$

and finding $u_{\text{ext}} \in V^\delta_{\text{ext}}$, such that for all $w_{\text{ext}} \in V^\delta_{\text{ext}}$

$$a_{\text{ext}}(u_{\text{ext}}, w_{\text{ext}}) = l_{\text{ext}}(w_{\text{ext}}), \tag{64}$$

where

$$a_{\text{int}}(u_{\text{int}}, w_{\text{int}}) = \int_{\Omega^\delta_{\text{int}}} \sigma_{\text{int}} \nabla u_{\text{int}} \cdot \nabla w_{\text{int}} \, \mathrm{d}x - \int_{\Gamma^\delta_{\text{int}}} \frac{2\sigma_{\text{int}}}{\varepsilon(1-2\delta)} u_{\text{int}} w_{\text{int}} \, \mathrm{d}2,$$

$$a_{\text{ext}}(u_{\text{ext}}, w_{\text{ext}}) = \int_{\Omega^\delta_{\text{ext}}} \sigma_{\text{ext}} \nabla u_{\text{ext}} \cdot \nabla w_{\text{ext}} \, \mathrm{d}x - \int_{\Gamma^\delta_{\text{ext}}} \frac{2\sigma_{\text{ext}}}{\varepsilon(1-2\delta)} u_{\text{ext}} w_{\text{ext}} \, \mathrm{d}s,$$

and

$$l_{\text{int}}(w_{\text{int}}) = -\int_{\Omega^\delta_{\text{int}}} f_{\text{int}} w_{\text{int}}\,\mathrm{d}x - \int_{\Gamma^\delta_{\text{int}}} \frac{2\sigma_{\text{int}}}{\varepsilon(1-2\delta)} g_{\text{int}} w_{\text{int}}\,\mathrm{d}s,$$

$$l_{\text{ext}}(w_{\text{ext}}) = -\int_{\Omega^\delta_{\text{ext}}} f_{\text{ext}} w_{\text{ext}}\,\mathrm{d}x - \int_{\Gamma^\delta_{\text{ext}}} \frac{2\sigma_{\text{ext}}}{\varepsilon(1-2\delta)} g_{\text{ext}} w_{\text{ext}}\,\mathrm{d}s.$$

Now, with these variational formulations, we can observe that if we select $\delta > \frac{1}{2}$, the last terms of the bilinear forms will be positive and thus enforce the coerciveness of the corresponding bilinear forms.

4.3.4 Stability results

First of all we will develop an expansion in power series of ε for the Problem (50) in the form

$$\begin{cases} u^{[1]}_{\delta,\text{ext}} \approx \sum_{k\geq 0} \varepsilon^k \widehat{u}^k_{\delta,\text{ext}} & \text{in} \quad \Omega^\delta_{\text{int}}, \\ u^{[1]}_{\delta,\text{int}} \approx \sum_{k\geq 0} \varepsilon^k \widehat{u}^k_{\delta,\text{int}} & \text{in} \quad \Omega^\delta_{\text{ext}}. \end{cases} \tag{65}$$

We substitute these series into the Equations (50) and we collect the terms with the same powers in ε. For every $k \in \mathbb{N}$ we obtain the following set of equations

$$\begin{cases} \sigma_{\text{int}} \Delta \widehat{u}^k_{\delta,\text{int}} = f_{\text{int}} \delta^0_k & \text{in} \quad \Omega^\delta_{\text{int}}, \\ \widehat{u}^k_{\delta,\text{int}} = \dfrac{\varepsilon(1-2\delta)}{2} \partial_n \widehat{u}^{k-1}_{\delta,\text{int}} & \text{in} \quad \Gamma^\delta_{\text{int}}, \\ \widehat{u}^k_{\delta,\text{int}} = 0 & \text{on} \quad \partial\Omega \cap \partial\Omega^\delta_{\text{int}}. \end{cases}$$
$$\begin{cases} \sigma_{\text{ext}} \Delta \widehat{u}^k_{\delta,\text{ext}} = f_{\text{ext}} \delta^0_k & \text{in} \quad \Omega^\delta_{\text{ext}}, \\ \widehat{u}^k_{\delta,\text{ext}} = -\dfrac{\varepsilon(1-2\delta)}{2} \partial_n \widehat{u}^{k-1}_{\delta,\text{ext}} & \text{in} \quad \Gamma^\delta_{\text{ext}}, \\ \widehat{u}^k_{\delta,\text{ext}} = 0 & \text{on} \quad \partial\Omega \cap \partial\Omega^\delta_{\text{ext}}. \end{cases} \tag{66}$$

Definition 8. *Given the expansion in power series (65) and $N \in \mathbb{N}$, we define the residue $\widehat{r}^N_\delta$ as*

$$\begin{cases} \widehat{r}^N_{\delta,int}(x,y) = u^{[3]}_{\delta,int}(x,y) - \sum_{k=0}^{N} \varepsilon^k \widehat{u}^k_{\delta,int}(x,y), \\ \widehat{r}^N_{\delta,ext}(x,y) = u^{[3]}_{\delta,ext}(x,y) - \sum_{k=0}^{N} \varepsilon^k \widehat{u}^k_{\delta,ext}(x,y). \end{cases}$$

We now prove the existence, uniqueness and uniform estimates of the solution to Problems (63) and (64).

Theorem 8. *For all $\varepsilon > 0$ and $\delta > \frac{1}{2}$ there exists a unique $u = (u_{int}, u_{ext})$ where $u_{int} \in V^\delta_{int}$ and $u_{ext} \in V^\delta_{ext}$ are solutions to (63) and (64) respectively, with data $f_{int} \in L^2(\Omega^\delta_{int})$, $f_{ext} \in L^2(\Omega^\delta_{ext})$, $g_{int} \in L^2(\Gamma^\delta_{int})$, $g_{ext} \in L^2(\Gamma^\delta_{ext})$. Moreover there exists ε_0 and a constant $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$*

$$\|u\|_{1,\Omega^\delta} \leq \varepsilon^{-1}C\left(\|f_{int}\|_{0,\Omega^\delta_{int}} + \|f_{ext}\|_{0,\Omega^\delta_{ext}} + \|g_{int}\|_{0,\Gamma^\delta_{int}} + \|g_{ext}\|_{0,\Gamma^\delta_{ext}}\right). \tag{67}$$

Proof. First of all as the bilinear form a_{int} is coercive and continuous in V^δ_{int} and the linear form l_{int} is also continuous in V^δ_{int}, the existence and uniqueness of a weak solution follows from the Lax-Milgram Lemma. Then, for proving the uniform estimates, employing Poincaré inequality, there exists a constant k_1 such that for all $w \in V^\delta_{\text{int}}$

$$a_{\text{int}}(w,w) \geq \frac{\sigma_{\text{int}}}{k_1}\|w\|^2_{1,\Omega^\delta_{\text{int}}}. \tag{68}$$

Applying a trace theorem for the Dirichlet trace operator $\gamma_{\Gamma^\delta_{\text{int}}}$ we have that there exists a constant k_2 such that

$$\left\|\gamma_{\Gamma^\delta_{\text{int}}}(w)\right\|_{\frac{1}{2},\Gamma^\delta_{\text{int}}} \leq k_2\|w\|_{1,\Omega^\delta_{\text{int}}}.$$

We select ε_0 as

$$\varepsilon_0 = \frac{2k_2\sigma_{\text{int}}}{2\delta - 1}.$$

Then, for $\varepsilon \in (0,\varepsilon_0)$, and applying Cauchy-Schwarz inequality to the definition of l_{int} we obtain that there exists a constant k_3, such that

$$|l_{\text{int}}(w)| \leq \varepsilon^{-1}k_3\|w\|_{1,\Omega^\delta_{\text{int}}}\left(\|f_{\text{int}}\|_{0,\Omega^\delta_{\text{int}}} + \|g_{\text{int}}\|_{0,\Gamma^\delta_{\text{int}}}\right). \tag{69}$$

Finally, employing Equations (68) and (69) we obtain

$$\|u_{\text{int}}\|_{1,\Omega^\delta_{\text{int}}} \leq \varepsilon^{-1}C\left(\|f_{\text{int}}\|_{0,\Omega^\delta_{\text{int}}} + \|g_{\text{int}}\|_{0,\Gamma^\delta_{\text{int}}}\right),$$

where $C = \dfrac{k_1k_2}{\sigma_{\text{int}}}$. The same proof holds for the equations in $\Omega^\delta_{\text{ext}}$, and employing these two results we obtain the desired result. $\square$

Proposition 5. *Let $N \in \mathbb{N}$, the residue $\widehat{r}^N_\delta$ defined in Definition 8 satisfies the following equations*

$$\begin{cases} \sigma_{int}\Delta\widehat{r}^N_{\delta,int} = 0 & in \quad \Omega^\delta_{int}, \\ \widehat{r}^N_{\delta,int} - \dfrac{\varepsilon(1-2\delta)}{2}\widehat{r}^N_{\delta,int} = g^N_{int} & on \quad \Gamma^\delta_{int}, \\ \widehat{r}^N_{\delta,int} = 0 & on \quad \partial\Omega \cap \partial\Omega^\delta_{int}, \end{cases}$$

$$\begin{cases} \sigma_{int}\Delta\widehat{r}^N_{\delta,ext} = 0 & in \quad \Omega^\delta_{ext}, \\ \widehat{r}^N_{\delta,ext} + \dfrac{\varepsilon(1-2\delta)}{2}\widehat{r}^N_{\delta,ext} = g^N_{ext} & on \quad \Gamma^\delta_{ext}, \\ \widehat{r}^N_{\delta,ext} = 0 & on \quad \partial\Omega \cap \partial\Omega^\delta_{ext}, \end{cases}$$

where

$$g_{int}^{N}=\varepsilon^{N+1}\frac{1-2\delta}{2}\partial_{n}u_{int}^{N},$$

$$g_{ext}^{N}=\varepsilon^{N+1}\frac{1-2\delta}{2}\partial_{n}u_{ext}^{N}.$$

Proof. We can deduce this result by applying Equations (66) and (50) to the definition of the residue. □

Theorem 9. *Let $N\in\mathbb{N}$. For $\varepsilon\in(0,\varepsilon_0)$ and under the assumptions $g_{int}^{N}\in L^2(\Gamma_{int}^{\delta})$, $g_{ext}^{N}\in L^2(\Gamma_{ext}^{\delta})$, $g_{int}^{N+1}\in L^2(\Gamma_{int}^{\delta})$, $g_{ext}^{N+1}\in L^2(\Gamma_{ext}^{\delta})$, functions defined in Proposition 5, and $u^k\in H^1(\Omega^{\varepsilon})$ for $k\leq N+1$, there exists a constant $C>0$, independent of ε, for which the following estimate holds for the residue $\widehat{r}_{\delta}^{N}$ defined in Definition 8,*

$$\left\|\widehat{r}_{\delta,ext}^{N}\right\|_{1,\Omega_{ext}^{\delta}}+\left\|\widehat{r}_{\delta,int}^{N}\right\|_{1,\Omega_{int}^{\delta}}\leq C\varepsilon^{N+1}.$$

Proof. From Theorem 8 and Proposition 5 we deduce that

$$\left\|\widehat{r}_{\delta,\mathrm{ext}}^{N}\right\|_{1,\Omega_{\mathrm{ext}}^{\delta}}+\left\|\widehat{r}_{\delta,\mathrm{int}}^{N}\right\|_{1,\Omega_{\mathrm{int}}^{\delta}}=O(\varepsilon^{N}).$$

Finally, writing

$$\begin{cases}\widehat{r}_{\delta,\mathrm{int}}^{N}=\widehat{r}_{\delta,\mathrm{int}}^{N+1}+\varepsilon^{N+1}\widehat{u}_{\delta,\mathrm{int}}^{N+1},\\[2ex] \widehat{r}_{\delta,\mathrm{ext}}^{N}=\widehat{r}_{\delta,\mathrm{ext}}^{N+1}+\varepsilon^{N+1}\widehat{u}_{\delta,\mathrm{ext}}^{N+1},\end{cases}$$

we deduce the desired result. □

4.3.5 Convergence results

Theorem 10. *Under the assumptions of Theorem 2 for $N=0$ and $\varepsilon\in(0,\varepsilon_0)$, with the data $f_{int}\in L^2(\Omega_{int})$ and $f_{ext}\in L^2(\Omega_{ext})$, the following estimate holds for the function $u^{[0]}$ solution to the first order asymptotic model (43), which writes as*

$$\begin{cases}\sigma_{int}\Delta u_{int}^{[0]}=f_{int} & \textit{in}\quad \Omega_{int},\\[2ex] u_{int}^{[0]}=0 & \textit{on}\quad \partial\Omega_{int},\end{cases}$$

$$\begin{cases}\sigma_{ext}\Delta u_{ext}^{[0]}=f_{ext} & \textit{in}\quad \Omega_{ext},\\[2ex] u_{ext}^{[0]}=0 & \textit{on}\quad \partial\Omega_{ext}.\end{cases}$$

and u, solution to the reference Problem (2): there exists a constant $C>0$ independent of ε, such that

$$\left\|u_{int}-u_{int}^{[0]}\right\|_{1,\Omega_{int}^{\varepsilon}}+\left\|u_{ext}-u_{ext}^{[0]}\right\|_{1,\Omega_{ext}^{\varepsilon}}\leq C\varepsilon.$$

Proof. We can deduce this result directly from Theorem 2. □

Theorem 11. *Under the assumptions of Theorem 2 and Theorem 9 for $N = 1$ and $\varepsilon \in (0, \varepsilon_0)$, with the data $f_{int} \in L^2(\Omega_{int})$ and $f_{ext} \in L^2(\Omega_{ext})$, the following estimate holds for $u_\delta^{[1]}$, solution to the stabilized δ-order 2 asymptotic model (50), which writes as*

$$\begin{cases} \sigma_{int}\Delta u_{\delta,int}^{[1]} = f_{int} & \text{in} \quad \Omega_{int}^\delta, \\ u_{\delta,int}^{[1]} = \dfrac{\varepsilon(1-2\delta)}{2}\partial_n u_{\delta,int}^{[1]} & \text{on} \quad \Gamma_{int}^\delta, \\ u_{\delta,int}^{[1]} = 0 & \text{on} \quad \partial\Omega \cap \partial\Omega_{int}^\delta. \end{cases}$$

$$\begin{cases} \sigma_{ext}\Delta u_{\delta,ext}^{[1]} = f_{ext} & \text{in} \quad \Omega_{ext}^\delta, \\ u_{\delta,ext}^{[1]} = -\dfrac{\varepsilon(1-2\delta)}{2}\partial_n u_{\delta,ext}^{[1]} & \text{on} \quad \Gamma_{ext}^\delta, \\ u_{\delta,ext}^{[1]} = 0 & \text{on} \quad \partial\Omega \cap \partial\Omega_{ext}^\delta. \end{cases}$$

and u, solution to the reference Problem (2): there exists a constant $C > 0$ independent form ε, such that

$$\left\| u_{int} - u_{\delta,int}^{[1]} \right\|_{1,\Omega_{int}^\delta} + \left\| u_{ext} - u_{\delta,ext}^{[1]} \right\|_{1,\Omega_{ext}^\delta} \leq C\varepsilon^2.$$

Proof. To begin with, we consider the expansion (65) of $u_\delta^{[1]}$ we have obtained above. More specifically, we consider Equations (66). We truncate the series from the second term and we consider the truncated series $\widehat{u}_\delta^{(1)} = \widehat{u}_\delta^0 + \varepsilon \widehat{u}_\delta^1$. We rewrite the equations of Problem (44) to derive the conditions on the artificial boundaries $\Gamma_{\mathrm{int}}^\delta$ and $\Gamma_{\mathrm{ext}}^\delta$ for $u_\delta^{(1)}$ following the same procedure as in Section 3.3. If we apply Theorem 8, we deduce

$$\left\| \widehat{u}_\delta^{(1)} - u_\delta^{(1)} \right\|_{1,\Omega^\delta} \leq C\varepsilon^2.$$

Finally applying Theorem 2 and Theorem 9 we deduce the desired result. □

5 Numerical results

The previous sections were devoted to the derivation and analysis of several asymptotic models. In this section we numerically asses the performance of these models and we check if the obtained numerical results match the theoretical results. For obtaining such numerical results, a Finite Element Method has been implemented employing Matlab and C programming language. The code could be divided into two main parts. The first one consists in the assembling of a linear system and the second one consists in solving such linear system. The part corresponding to the assembling is mainly coded in C and the part corresponding to the resolution of the linear system is mainly coded in Matlab. Moreover, Matlab is employed for the post processing of the solution, including tasks like the visualization of the solution or the calculus of the error in different norms.

This implementation corresponds to the classical Finite Element Method. It is based on straight triangular elements for discretizing the domain of the problem and piecewise polynomials of any given degree for representing the solution. Such polynomials correspond to the Lagrange interpolating Polynomials. We mainly employ structured meshes like the one showed in Figure 4 due to the considered domains being mainly rectangles, but the code is adapted to work with unstructured meshes generated with other mesh generators.

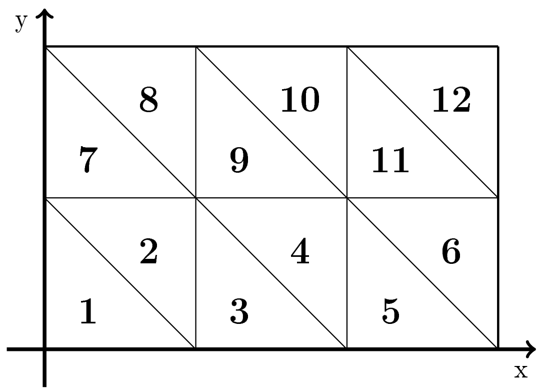


Figure 4: Structured mesh for a rectangular domain.

The problem we are interested in solving is the Poisson's equation

$$\sigma \Delta u = f,$$

where the conductivity σ and the right-hand side function f are known data. In general this equation is set in a domain $\Omega \subset \mathbb{R}^2$, composed of several subdomains and the conductivity is considered to be a piecewise constant function which takes a different value inside each subdomain. The code is capable of dealing with this kind of complex configuration, as well as with the different transmission conditions required to solve some of the derived asymptotic models.

5.1 First class of ITCs

In this section we show some numerical tests regarding the approximate problems we have derived. For this purpose, we use the Finite Element Method along with the variational formulations derived in Sections 1, 4.2 and 4.3 to obtain approximate solutions of these models.

The objective of these numerical tests is to illustrate the theory by checking if the theoretical orders of convergence coincide with the ones obtained numerically. For these experiments, the considered domain is a 2 m $\times$ 1 m rectangle domain. In addition, we consider a conductivity of the following form

$$\sigma = \begin{cases} \sigma_{\text{int}} = 5 \text{ S/m} & \text{in} \quad \Omega^{\varepsilon}_{\text{int}}, \\ \sigma_{\text{lay}} = 8.89\varepsilon^{-3} \text{ S/m} & \text{in} \quad \Omega^{\varepsilon}_{\text{lay}}, \\ \sigma_{\text{ext}} = 3 \text{ S/m} & \text{in} \quad \Omega^{\varepsilon}_{\text{ext}}, \end{cases}$$

where ε represents the thickness of the thin layer, and the following right-hand side

$$f = \begin{cases} f_{\text{int}} = 1 \text{ C} & \text{in} \quad \Omega^{\varepsilon}_{\text{int}}, \\ f_{\text{lay}} = 0 \text{ C} & \text{in} \quad \Omega^{\varepsilon}_{\text{lay}}, \\ f_{\text{ext}} = 1 \text{ C} & \text{in} \quad \Omega^{\varepsilon}_{\text{ext}}. \end{cases}$$

where the units S/m and C correspond to Siemens per meter and Coulomb respectively. With these configurations and parameters, we solve the reference Problem (2) and asymptotic models (22) and (23) by employing the Finite Element Method. For discretizing the domain, we have used 384 triangular shaped elements and Lagrange interpolating Polynomials of second degree.

We begin by performing a qualitative comparison between the solution to the reference model (2) and the approximate models (22) and (23). Employing the stated parameters and configuration, we have solved these models and their solutions are shown in Figure 5. We notice that the fourth-order model is more accurate and approximates better the effect of the high conductive thin layer than the second-order model.

We calculate the H^1 errors between the solution to the reference model (2) and the approximate models (22) and (23) for different thicknesses of the thin layer. These results can be observed in Figure 6, where we show the obtained convergence rates for the H^1 relative error. From these results we observe that the numerical convergence rates we obtain coincide with the theoretical convergence rates proved in Section 4.2.

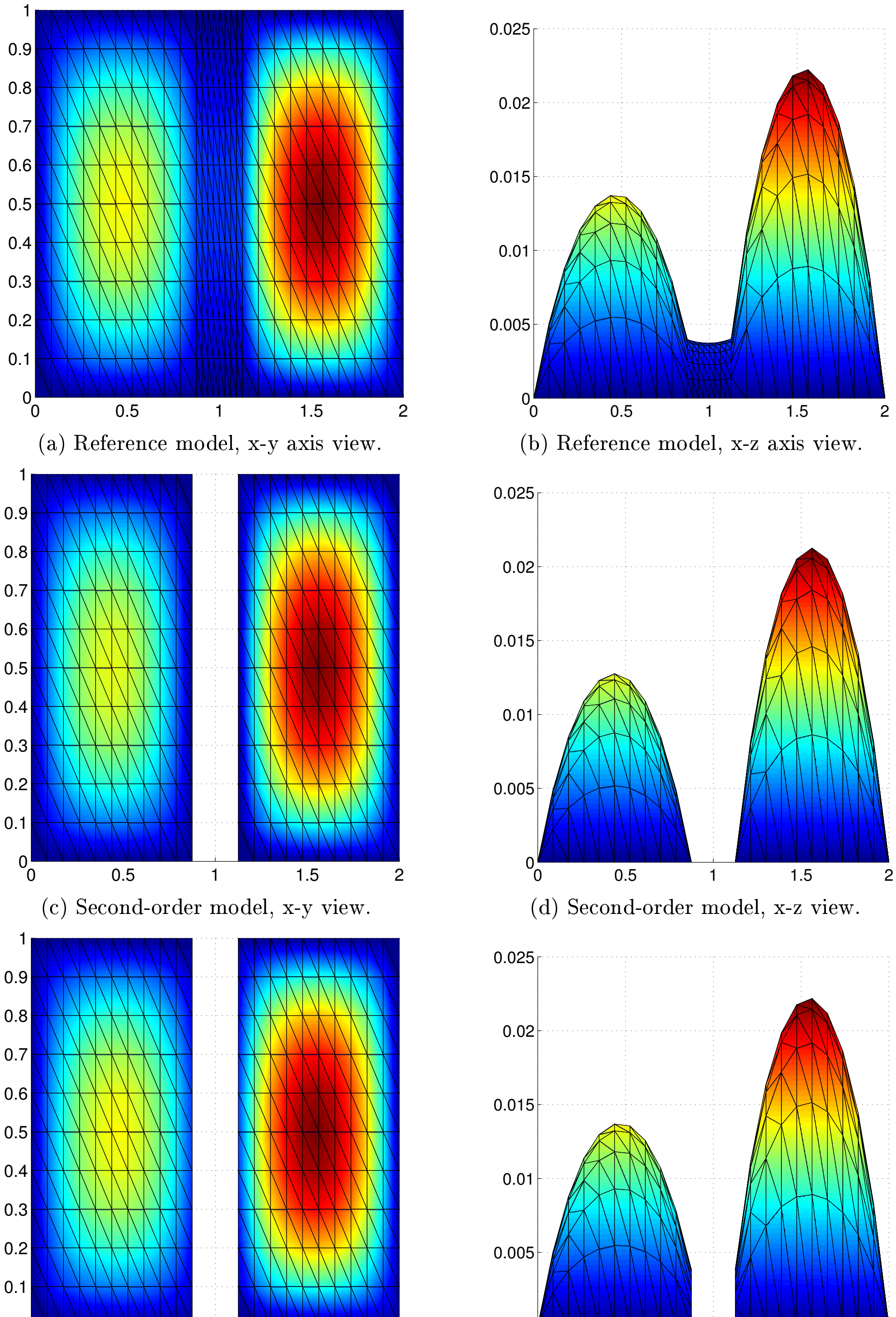
5.2 Second class of ITCs

In the same way we have done in the previous section, we would like to begin by performing a qualitative comparison of the solution to the reference model and the asymptotic models of the second class. We have employed the same physical parameters as in the previous section for these new tests. In Figure 7, we observe the solution we obtain for the reference Problem (2) and the asymptotic models of order 1 (43) and order 2 (45) of the second class.

As we stated in the previous sections, the stability of the second-order model cannot be guaranteed, this fact induces big changes in the solution when we slightly change the parameters of the problem. We illustrate this fact in Figure 8, where we show the solution for the second-order model for different values of ε. We observe that the solution drastically changes around the transmission conditions for every small change in the value of ε.

To deliver a more quantitative comparison of the models we have derived. We have calculated the H^1 error between the reference solution (2) and the approximate models (43) and (45) for different thicknesses of the thin layer. Before showing the H^1 norm results, we show similar results regarding the L^2 norm to remark how the second-order model behaves differently depending on the norm we choose. In Figure 9 we observe the obtained convergence rates for the L^2 absolute and relative errors and in Figure 10 the convergence rates for the H^1 absolute and relative errors.

We observe that the numerical convergence rates we have obtained coincide with the theoretical convergence rates proved in Section 4.3 for the first-order model. On the other hand, for the



(a) Reference model, x-y axis view.

(b) Reference model, x-z axis view.

(c) Second-order model, x-y view.

(d) Second-order model, x-z view.

(e) Fourth-order model, x-y view.

(f) Order4 model, x-z view.

Figure 5: Solution to the reference Problem (2), the second-order model (22) and the fourth-order model (23) of the first class.

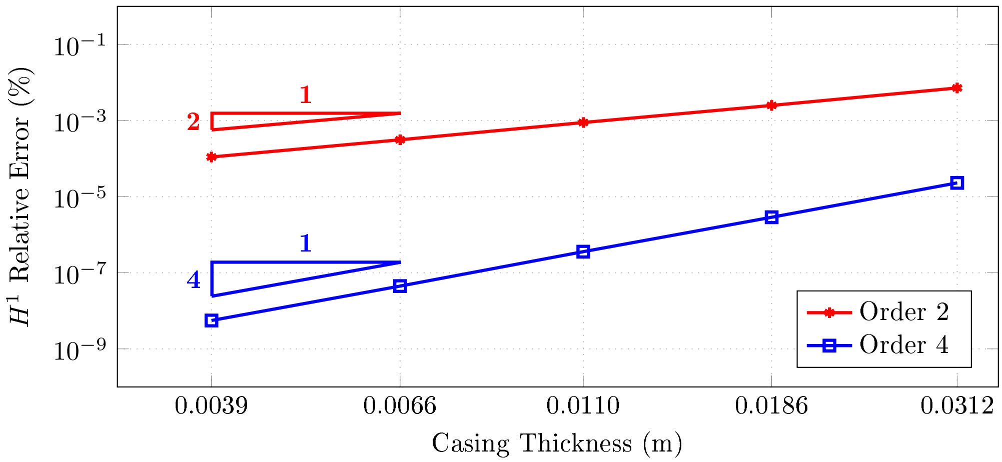


Figure 6: H^1 relative error of the second-order model (22) and fourth-order model (23) of the first class for different values of ε.

second-order model, even though we still recover the expected theoretical order of convergence for the L^2 absolute and relative errors, due to the instabilities it does not perform as well for the H^1 absolute and relative errors. Even so, it still outperforms the first-order model.

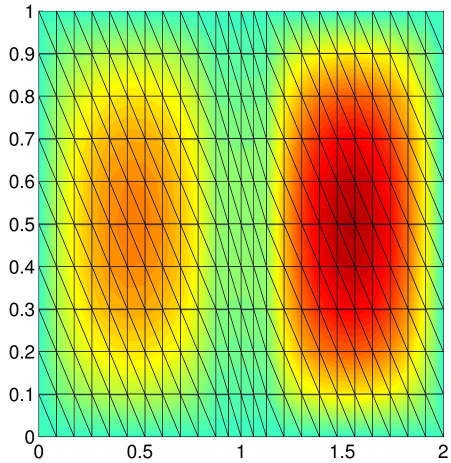
To face these instabilities, in Section 3.3, we have derived a new second-order model by defining some artificial conditions and moving the boundary conditions to these new boundaries. To derive this new model (50) we employ a parameter δ that controls the distance between the artificial boundaries. In Section 4.3.3 we prove that for $\delta > 0.5$ this approach solves the problem of instabilities. Figure 11 shows a problem with instabilities and how they can be eliminated when applying a δ parameter which is greater than 0.5. However, the instabilities are not completely removed if it is not greater than 0.5. To illustrate this fact, the example of Figure 11 shows that for δ=0.1 the instabilities are still present, whereas when $\delta = 0.51$ is applied, we do not have instabilities any more.

In Figure 9, we compare the obtained convergence rates for the L^2 relative error and in Figure 10 the convergence rates for the H^1 relative error for the unstable order 2 model and for the stabilized δ-order 2 model. We observe that for the L^2 error both models behave similarly but for the H^1 error, the second-order model does not converge properly, whereas the stabilized δ-order 2 model delivers the correct convergence rates.

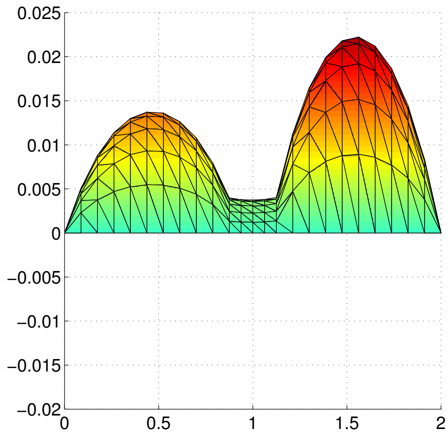
From these results we observe that if we apply the artificial boundary approach with a δ greater than 0.5, we have no longer instability problems and the numerical convergence rates coincide with the theoretical convergence rates proved in Section 4.3, for both the L^2 and the H^1 errors.

5.3 Comparison between the first and the second class

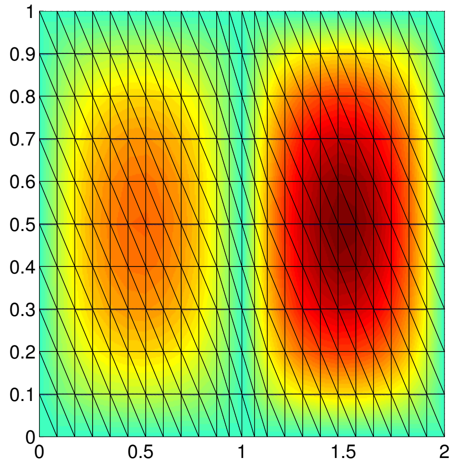
In this section we will do a brief comparison between the asymptotic models we have derived, mentioning the strong and weak points of each class. Regarding the convergence, considering the first class of ITCs, the model with highest order reaches a convergence of order 4, whereas for the second class, the model with highest order only reaches a convergence of order 2. We can observe these convergence rates in L^2 norm for the five models we have derived in Figure 12 and in H^1 norm in Figure 13. We see that all the models converge with the expected order of



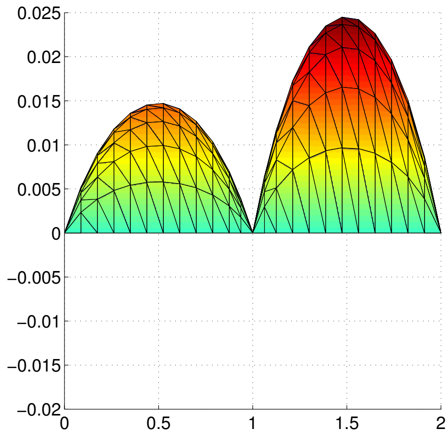
(a) x-y axis view.



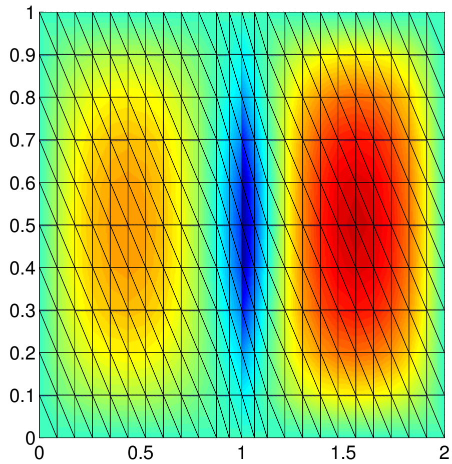
(b) x-z axis view.



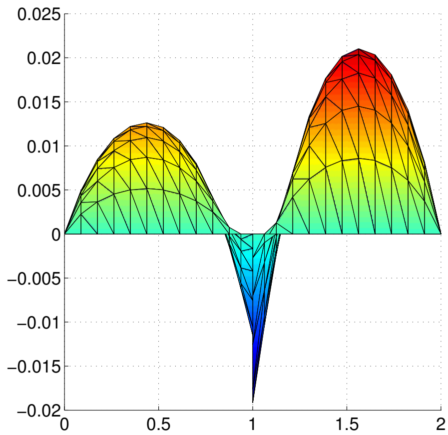
(c) First-order model, x-y view.



(d) First-order model, x-z view.

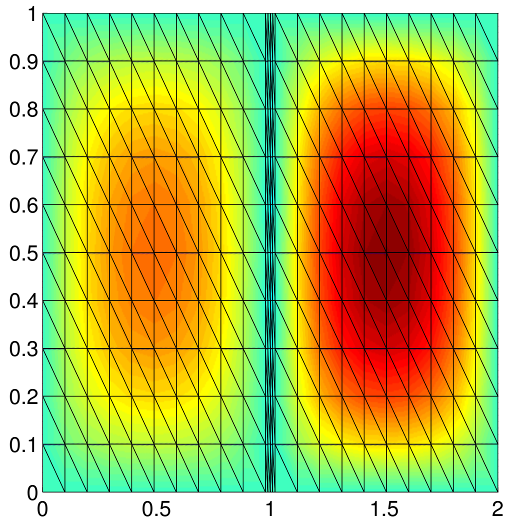


(e) Second-order model, x-y view.

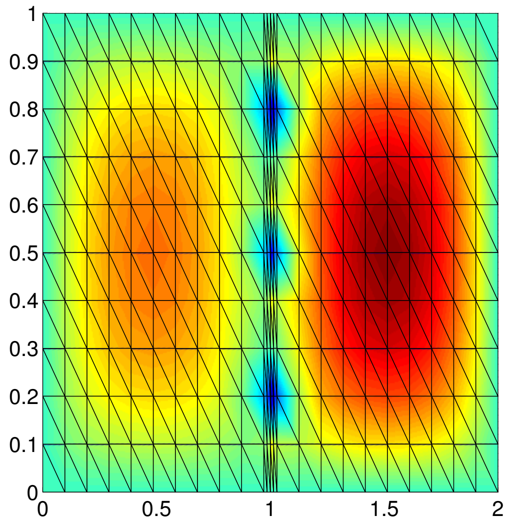


(f) Second-order model, x-z view.

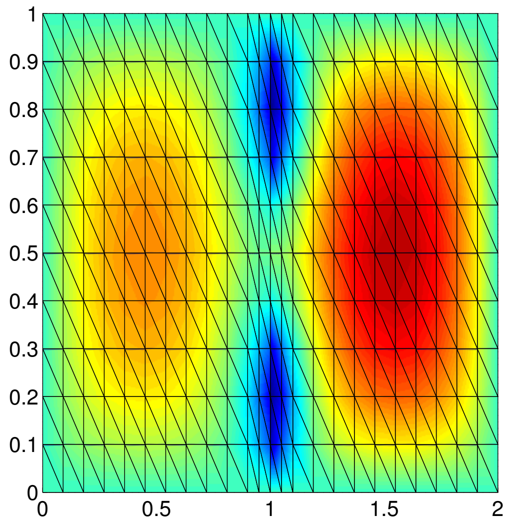
Figure 7: Solution to the reference Problem (2), the first-order model (43) and the second-order model (45) of the second class.



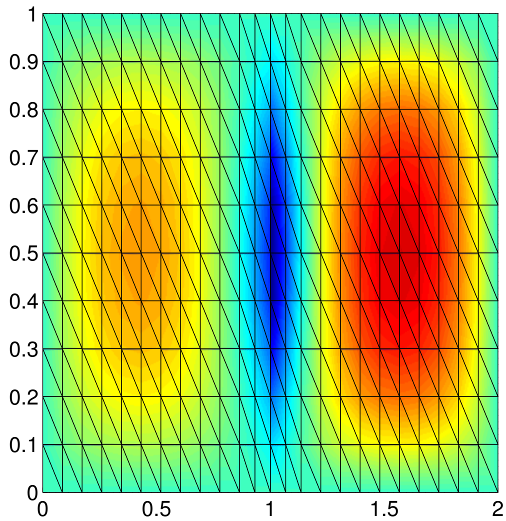
(a) Second-order model, $\varepsilon = 0.0426$.



(b) Second-order model, $\varepsilon = 0.0578$.



(c) Second-order model, $\varepsilon = 0.1984$.



(d) Second-order model, $\varepsilon = 0.27$.

Figure 8: Instabilities of the solutions to the second-order model (45) for different values of ε.

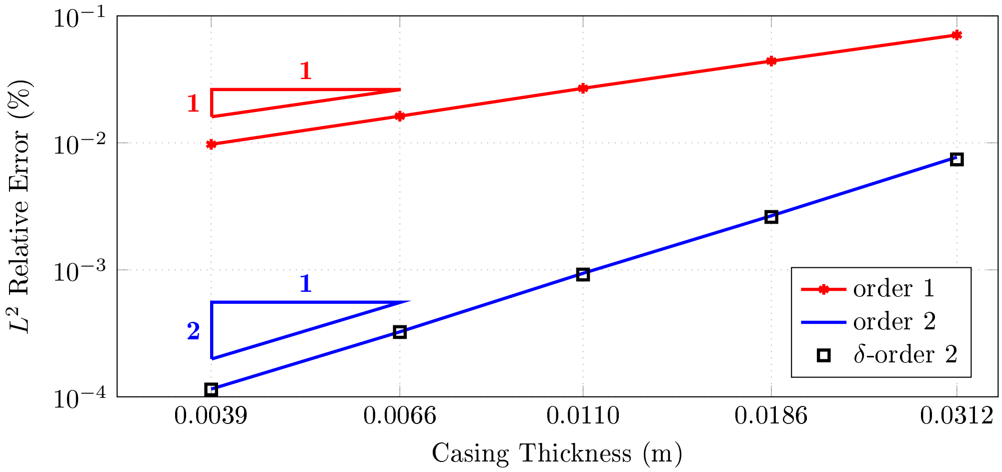


Figure 9: L^2 relative error of the first-order model (43), second-order model (45) and the stabilized δ-order 2 model (50) of the second class for different values of ε.

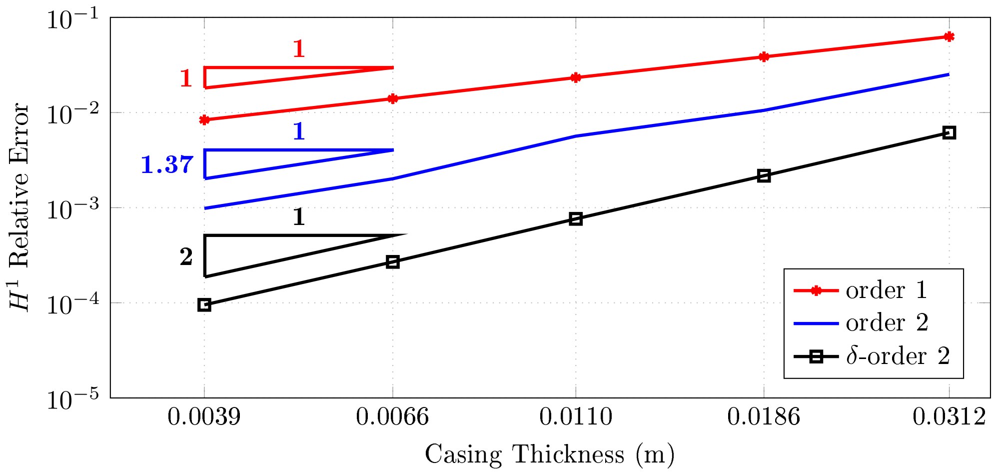
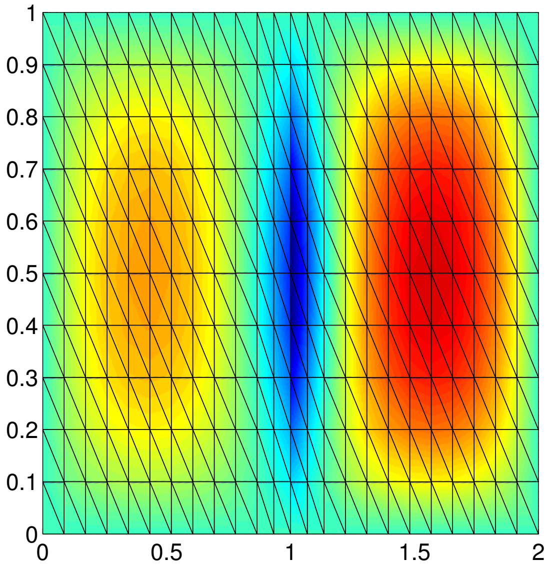
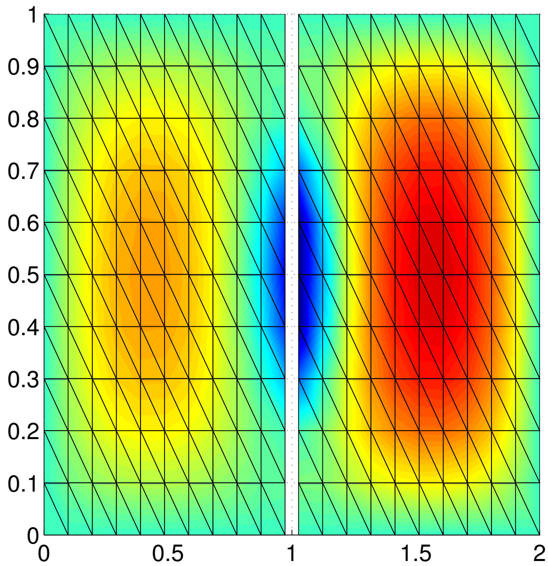


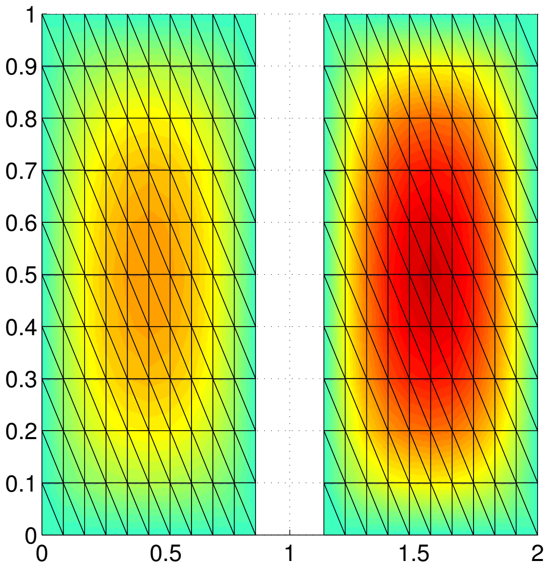
Figure 10: H^1 relative error of the first-order model (43), second-order model (45) and the stabilized δ-order 2 model (50) of the second class for different values of ε.



(a) Second-order model with instabilities.



(b) Stabilized δ-order 2 model for $\delta = 0.1$.



(c) Stabilized δ-order 2 model for $\delta = 0.51$.

Figure 11: Removing the instabilities of the order 2 model (45) with the artificial boundaries.

accuracy in L^2 norm. On the other hand, in H^1 norm, all models converge with the expected order of accuracy except the second-order model of the second class due to the instabilities.

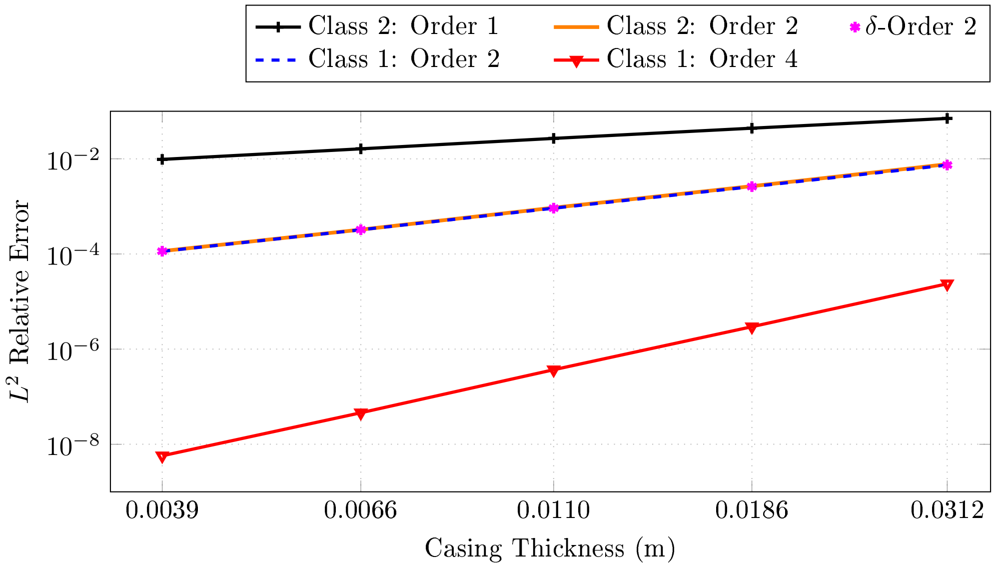


Figure 12: L^2 relative error of the different asymptotic models for different values of ε

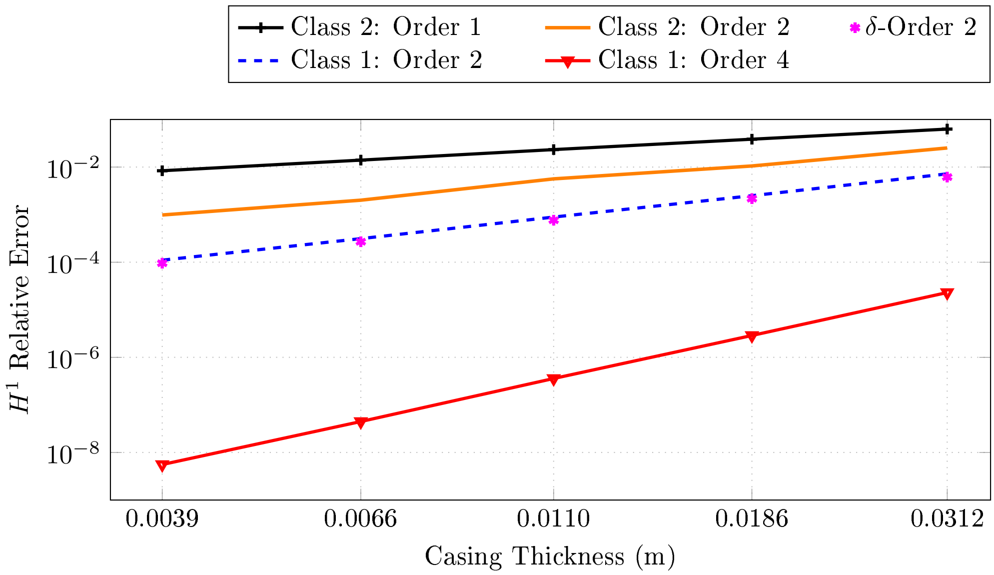


Figure 13: H^1 relative error of the different asymptotic models for different values of ε

Another drawback of the second class of ITCs is that the model of order 2 presents instabilities, whereas the models derived for the first class are both stable.

Regarding the domain, a strong point of the second class is that the domain does not depend on ε, while the domain for the first class depends on ε. Even though this point is not very relevant for our configuration, due to the thin layer having a straight shape, it could be very interesting when considering more complex configurations, in which the shape of the thin layer is curved. In such a case, the fact of having a single interface between the two subdomains instead of having a gap greatly reduces the numerical complexity of the model. All these features are summarized in table 1.

Model	Numerical order	Stability	ε-independent domain
Class 1: Order 2	**2**	✓	✗
Class 1: Order 4	**4**	✓	✗
Class 2: Order 1	**1**	✓	✓
Class 2: Order 2	**1-2**	✗	✓
δ-Order 2	**2**	✓	✗

Table 1: Comparison of the different derived models.

Appendix A Additional results

In this section we present asymptotic models for similar configurations to the one presented in the previous sections. The configurations considered in this section include a 2D configuration for a time-harmonic problem, and a 3D axisymmetric borehole shaped configuration. Due to similarities in the procedure, in this section we will concentrate in presenting the resulting asymptotic models.

A.1 Time-harmonic problem

A.1.1 Model problem

Here we consider again the equation for the electric potential, but now we consider the frequency to be non-zero. The problem writes as follows

$$\mathrm{div}\ \left[\left(\sigma - i\epsilon_0\omega\right)\nabla u\right] = f, \tag{70}$$

where u represents the electric potential, σ stands for the conductivity, f denotes a current source, ω is the frequency and ϵ_0 is the permittivity. We consider the same domain we had in Section 1, which is depicted at Figure 1. We consider the conductivity to be piecewise constant and to have a different value in each subdomain, being of the form $\sigma_{\mathrm{lay}} = \widehat{\sigma}_0\varepsilon^{-3}$ inside the thin layer. Both the right-hand side f and the conductivity σ have the same form as the ones considered in Section 1. In this framework, the Problem (70) writes as follows

$$\left\{\begin{array}{rll}
\left(\sigma_{\mathrm{int}} - i\epsilon_0\omega\right)\Delta u_{\mathrm{int}} = f_{\mathrm{int}} & \text{in} & \Omega^{\varepsilon}_{\mathrm{int}}, \\
\left(\sigma_{\mathrm{ext}} - i\epsilon_0\omega\right)\Delta u_{\mathrm{ext}} = f_{\mathrm{ext}} & \text{in} & \Omega^{\varepsilon}_{\mathrm{ext}}, \\
\left(\widehat{\sigma}_0\varepsilon^{-3} - i\epsilon_0\omega\right)\Delta u_{\mathrm{lay}} = 0 & \text{in} & \Omega^{\varepsilon}_{\mathrm{lay}}, \\
u_{\mathrm{int}} = u_{\mathrm{lay}} & \text{on} & \Gamma^{\varepsilon}_{\mathrm{int}}, \\
u_{\mathrm{lay}} = u_{\mathrm{ext}} & \text{on} & \Gamma^{\varepsilon}_{\mathrm{ext}}, \\
\left(\sigma_{\mathrm{int}} - i\epsilon_0\omega\right)\partial_n u_{\mathrm{int}} = \left(\widehat{\sigma}_0\varepsilon^{-3} - i\epsilon_0\omega\right)\partial_n u_{\mathrm{lay}} & \text{on} & \Gamma^{\varepsilon}_{\mathrm{int}}, \\
\left(\widehat{\sigma}_0\varepsilon^{-3} - i\epsilon_0\omega\right)\partial_n u_{\mathrm{lay}} = \left(\sigma_{\mathrm{ext}} - i\epsilon_0\omega\right)\partial_n u_{\mathrm{ext}} & \text{on} & \Gamma^{\varepsilon}_{\mathrm{ext}}, \\
u = 0 & \text{on} & \partial\Omega,
\end{array}\right. \tag{71}$$

where ∂_n represents the normal derivative in the direction of the normal vector, inwardly directed to $\Omega^{\varepsilon}_{\mathrm{ext}}$ on $\Gamma^{\varepsilon}_{\mathrm{ext}}$, and outwardly directed to $\Omega^{\varepsilon}_{\mathrm{int}}$ on $\Gamma^{\varepsilon}_{\mathrm{int}}$, see Figure 1. Assuming $f \in L^2(\Omega)$, the variational formulation for this problem writes as follows: we look for $u \in H^1_0(\Omega)$, such that for all $w \in H^1_0(\Omega)$

$$\left(\sigma_{\mathrm{int}} - i\epsilon_0\omega\right)\int_{\Omega^{\varepsilon}_{\mathrm{int}}} \nabla u \cdot \nabla \overline{w}\,\mathrm{d}x + \left(\sigma_{\mathrm{ext}} - i\epsilon_0\omega\right)\int_{\Omega^{\varepsilon}_{\mathrm{ext}}} \nabla u \cdot \nabla \overline{w}\,\mathrm{d}x$$

$$+\left(\widehat{\sigma}_0\varepsilon^{-3} - i\epsilon_0\omega\right)\int_{\Omega^{\varepsilon}_{\mathrm{lay}}} \nabla u \cdot \nabla \overline{w}\,\mathrm{d}x = -\int_{\Omega^{\varepsilon}_{\mathrm{int}}} f_{\mathrm{int}}\overline{w}\,\mathrm{d}x - \int_{\Omega^{\varepsilon}_{\mathrm{ext}}} f_{\mathrm{ext}}\overline{w}\,\mathrm{d}x.$$

We apply the first approach developed in Section 2 to derive approximate models of the first class (Sections A.1.2 and A.1.3) and the second approach developed in Section 3 to derive approximate models of the second class (Sections A.1.4 and A.1.5) for this configuration.

A.1.2 First class: second-order model

$$\left\{\begin{aligned}(\sigma_{\text{int}} - i\epsilon_0\omega)\,\Delta u_{\text{int}}^{[1]} &= f_{\text{int}} && \text{in} \quad \Omega_{\text{int}}^{\varepsilon},\\ u_{\text{int}}^{[1]} &= 0 && \text{on} \quad \partial\Omega_{\text{int}}^{\varepsilon}.\end{aligned}\right.$$
$$\left\{\begin{aligned}(\sigma_{\text{ext}} - i\epsilon_0\omega)\,\Delta u_{\text{ext}}^{[1]} &= f_{\text{ext}} && \text{in} \quad \Omega_{\text{ext}}^{\varepsilon},\\ u_{\text{ext}}^{[1]} &= 0 && \text{on} \quad \partial\Omega_{\text{ext}}^{\varepsilon}.\end{aligned}\right. \tag{72}$$

Assuming $f_{\text{int}} \in L^2(\Omega_{\text{int}}^{\varepsilon})$ and $f_{\text{ext}} \in L^2(\Omega_{\text{ext}}^{\varepsilon})$, the variational formulations consist in finding $u_{\text{int}} \in H_0^1(\Omega_{\text{int}}^{\varepsilon})$, such that for all $w_{\text{int}} \in H_0^1(\Omega_{\text{int}}^{\varepsilon})$

$$-\int_{\Omega_{\text{int}}^{\varepsilon}} f_{\text{int}}\overline{w}_{\text{int}}\,\mathrm{d}x = \int_{\Omega_{\text{int}}^{\varepsilon}} (\sigma_{\text{int}} - i\epsilon_0\omega)\,\nabla u_{\text{int}}\cdot\nabla\overline{w}_{\text{int}}\,\mathrm{d}x,$$

and finding $u_{\text{ext}} \in H_0^1(\Omega_{\text{ext}}^{\varepsilon})$ such that for all $w_{\text{ext}} \in H_0^1(\Omega_{\text{ext}}^{\varepsilon})$

$$-\int_{\Omega_{\text{ext}}^{\varepsilon}} f_{\text{ext}}\overline{w}_{\text{ext}}\,\mathrm{d}x = \int_{\Omega_{\text{ext}}^{\varepsilon}} (\sigma_{\text{ext}} - i\epsilon_0\omega)\,\nabla u_{\text{ext}}\cdot\nabla\overline{w}_{\text{ext}}\,\mathrm{d}x.$$

A.1.3 First class: fourth-order model

$$\left\{\begin{aligned}(\sigma_{\text{int}} - i\epsilon_0\omega)\,\Delta u_{\text{int}}^{[3]} &= f_{\text{int}} && \text{in} \quad \Omega_{\text{int}}^{\varepsilon},\\ (\sigma_{\text{ext}} - i\epsilon_0\omega)\,\Delta u_{\text{ext}}^{[3]} &= f_{\text{ext}} && \text{in} \quad \Omega_{\text{ext}}^{\varepsilon},\\ \left[u^{[3]}\right]_{\Gamma^\varepsilon} &= 0, &&\\ \left[(\sigma - i\epsilon_0\omega)\,\partial_n u^{[3]}\right]_{\Gamma^\varepsilon} &= -\frac{\widehat{\sigma}_0}{\varepsilon^2}\frac{\mathrm{d}^2}{\mathrm{d}y^2}\left\{u^{[3]}\right\}_{\Gamma^\varepsilon}, &&\\ u^{[3]} &= 0 && \text{on} \quad \partial\Omega\cap\partial\Omega^{\varepsilon}.\end{aligned}\right. \tag{73}$$

Assuming $f_{\text{int}} \in L^2(\Omega_{\text{int}}^{\varepsilon})$ and $f_{\text{ext}} \in L^2(\Omega_{\text{ext}}^{\varepsilon})$, the variational problem reduces to finding $u \in V_4$, such that for all $w \in V_4$,

$$(\sigma_{\text{int}} - i\epsilon_0\omega)\int_{\Omega_{\text{int}}^{\varepsilon}} \nabla u\cdot\nabla\overline{w}\,\mathrm{d}x + (\sigma_{\text{int}} - i\epsilon_0\omega)\int_{\Omega_{\text{ext}}^{\varepsilon}} \nabla u\cdot\nabla\overline{w}\,\mathrm{d}x$$
$$+\widehat{\sigma}_0\varepsilon^{-2}\int_{\Gamma^\varepsilon} \nabla_{\Gamma^\varepsilon}\{u\}_{\Gamma^\varepsilon}\,\nabla_{\Gamma^\varepsilon}\{\overline{w}\}_{\Gamma^\varepsilon}\,\mathrm{d}s = -\int_{\Omega_{\text{int}}^{\varepsilon}} f_{\text{int}}\overline{w}\,\mathrm{d}x - \int_{\Omega_{\text{ext}}^{\varepsilon}} f_{\text{ext}}\overline{w}\,\mathrm{d}x.$$

A.1.4 Second class: first-order model

$$\left\{\begin{aligned}(\sigma_{\text{int}} - i\epsilon_0\omega)\,\Delta u_{\text{int}}^{[0]} &= f_{\text{int}} && \text{in} \quad \Omega_{\text{int}},\\ u_{\text{int}}^{[0]} &= 0 && \text{on} \quad \partial\Omega_{\text{int}}.\end{aligned}\right.$$
$$\left\{\begin{aligned}(\sigma_{\text{ext}} - i\epsilon_0\omega)\,\Delta u_{\text{ext}}^{[0]} &= f_{\text{ext}} && \text{in} \quad \Omega_{\text{ext}},\\ u_{\text{ext}}^{[0]} &= 0 && \text{on} \quad \partial\Omega_{\text{ext}}.\end{aligned}\right. \tag{74}$$

Assuming $f_{\text{int}} \in L^2(\Omega_{\text{int}})$ and $f_{\text{ext}} \in L^2(\Omega_{\text{ext}})$, the variational formulations of these problems consist in finding $u_{\text{int}} \in H_0^1(\Omega_{\text{int}})$ such that for all $w_{\text{int}} \in H_0^1(\Omega_{\text{int}})$

$$-\int_{\Omega_{\text{int}}} f_{\text{int}} \overline{w}_{\text{int}} \, \mathrm{d}x = \int_{\Omega_{\text{int}}} (\sigma_{\text{int}} - i\epsilon_0\omega) \nabla u_{\text{int}} \cdot \nabla \overline{w}_{\text{int}} \, \mathrm{d}x,$$

and finding $u_{\text{ext}} \in H_0^1(\Omega_{\text{ext}})$ such that for all $w_{\text{ext}} \in H_0^1(\Omega_{\text{ext}})$

$$-\int_{\Omega_{\text{ext}}} f_{\text{ext}} \overline{w}_{\text{ext}} \, \mathrm{d}x = \int_{\Omega_{\text{ext}}} (\sigma_{\text{ext}} - i\epsilon_0\omega) \nabla u_{\text{ext}} \cdot \nabla \overline{w}_{\text{ext}} \, \mathrm{d}x.$$

A.1.5 Second class: second-order model

$$\begin{cases} (\sigma_{\text{int}} - i\epsilon_0\omega) \Delta u_{\text{int}}^{[1]} = f_{\text{int}} & \text{in} \quad \Omega_{\text{int}}, \\ u_{\text{int}}^{[1]} = \dfrac{\varepsilon}{2} \partial_n u_{\text{int}}^{[1]} & \text{on} \quad \Gamma, \\ u_{\text{int}}^{[1]} = 0 & \text{on} \quad \partial\Omega \cap \partial\Omega_{\text{int}}. \end{cases}$$
$$\begin{cases} (\sigma_{\text{ext}} - i\epsilon_0\omega) \Delta u_{\text{ext}}^{[1]} = f_{\text{ext}} & \text{in} \quad \Omega_{\text{ext}}, \\ u_{\text{ext}}^{[1]} = -\dfrac{\varepsilon}{2} \partial_n u_{\text{ext}}^{[1]} & \text{on} \quad \Gamma, \\ u_{\text{ext}}^{[1]} = 0 & \text{on} \quad \partial\Omega \cap \partial\Omega_{\text{ext}}. \end{cases} \tag{75}$$

Assuming $f_{\text{int}} \in L^2(\Omega_{\text{int}})$ and $f_{\text{ext}} \in L^2(\Omega_{\text{ext}})$, the variational formulations reduce to looking for $u_{\text{int}} \in V_{\text{int}}$, such that for all $w_{\text{int}} \in V_{\text{int}}$

$$-\int_{\Omega_{\text{int}}} f_{\text{int}} \overline{w}_{\text{int}} \, \mathrm{d}x = \int_{\Omega_{\text{int}}} (\sigma_{\text{int}} - i\epsilon_0\omega) \nabla u_{\text{int}} \cdot \nabla \overline{w}_{\text{int}} \, \mathrm{d}x - \int_{\Gamma} \frac{2(\sigma_{\text{int}} - i\epsilon_0\omega)}{\varepsilon} u_{\text{int}} \overline{w}_{\text{int}} \, \mathrm{d}s,$$

and looking for $u_{\text{ext}} \in V_{\text{ext}}$, such that for all $w_{\text{ext}} \in V_{\text{ext}}$

$$-\int_{\Omega_{\text{ext}}} f_{\text{ext}} \overline{w}_{\text{ext}} \, \mathrm{d}x = \int_{\Omega_{\text{ext}}} (\sigma_{\text{ext}} - i\epsilon_0\omega) \nabla u_{\text{ext}} \cdot \nabla \overline{w}_{\text{ext}} \, \mathrm{d}x - \int_{\Gamma} \frac{2(\sigma_{\text{ext}} - i\epsilon_0\omega)}{\varepsilon} u_{\text{ext}} \overline{w}_{\text{ext}} \, \mathrm{d}s,$$

where the spaces V_{int} and V_{ext} have been defined in (62).

Remark 5. *The models obtained in this section are very similar to the ones obtained for the static case,* (22) *and* (23) *for the first class and* (43) *and* (45) *for the second class. In fact, it is possible to obtain the models we present in this section by simply substituting the conductivities* σ_{int} *and* σ_{ext} *by* $\sigma_{int} - i\epsilon_0\omega$ *and* $\sigma_{ext} - i\epsilon_0\omega$ *respectively.*

A.2 3D axisymmetric configuration

The main objective of this section is the derivation of approximate models in a 3D axisymmetric configuration. The plan of the section is the following. First we set the model problem we are interested in. Then, we develop a multiscale expansion in powers of ε for the solution to the model problem and we obtain the equations for the first terms of the expansion adopting the first approach. Finally, we derive the desired approximate models. We then address the second class of problems, and for avoiding repetition with the previous sections only the main results are presented.

A.2.1 Model problem and scaling

Let $\Omega \subset \mathbb{R}^3$ be the domain of interest described at Figure 14. The Domain Ω is a cylinder shaped domain and is decomposed into three subdomains: $\Omega^\varepsilon_{\text{int}}$, $\Omega^\varepsilon_{\text{ext}}$, and $\Omega^\varepsilon_{\text{lay}}$. Subdomain $\Omega^\varepsilon_{\text{lay}}$ is a thin layer of uniform thickness $\varepsilon > 0$. We denote by $\Gamma^\varepsilon_{\text{int}}$ the interface between $\Omega^\varepsilon_{\text{int}}$ and $\Omega^\varepsilon_{\text{lay}}$, and by $\Gamma^\varepsilon_{\text{ext}}$ the interface between $\Omega^\varepsilon_{\text{lay}}$ and $\Omega^\varepsilon_{\text{ext}}$. In this domain, we study the static electric potential equation, which read as follows

$$\operatorname{div}(\sigma \nabla u) = f. \tag{76}$$

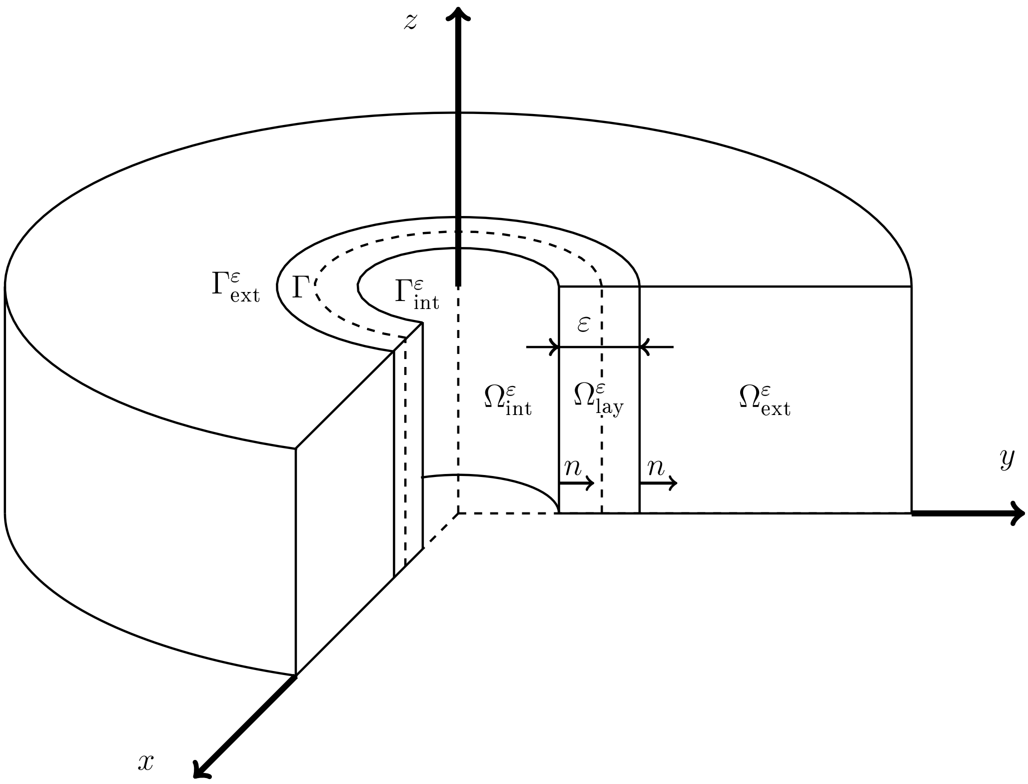


Figure 14: Sectioned three dimensional domain for the model problem and asymptotic models of the first class.

Here, u represents the electric potential, σ stands for the conductivity and f is the right-hand side, which corresponds to a current source. The conductivity is a piecewise constant function, with a different value in each subdomain. Specifically, the value of the conductivity inside the thin layer $\Omega^\varepsilon_{\text{lay}}$ is much larger than the one in the other subdomains and we assume that it depends on parameter ε. We consider a conductivity of the following form

$$\sigma = \begin{cases} \sigma_{\text{int}} & \text{in} \quad \Omega^\varepsilon_{\text{int}}, \\ \sigma_{\text{lay}} = \widehat{\sigma}_0 \varepsilon^{-3} & \text{in} \quad \Omega^\varepsilon_{\text{lay}}, \\ \sigma_{\text{ext}} & \text{in} \quad \Omega^\varepsilon_{\text{ext}}, \end{cases}$$

where $\widehat{\sigma}_0 > 0$ is a given constant. We assume the right-hand side f is a piecewise smooth function that is independent of ε and vanishes inside the layer.

$$f = \begin{cases} f_{\text{int}} & \text{in} \quad \Omega^\varepsilon_{\text{int}}, \\ f_{\text{lay}} = 0 & \text{in} \quad \Omega^\varepsilon_{\text{lay}}, \\ f_{\text{ext}} & \text{in} \quad \Omega^\varepsilon_{\text{ext}}. \end{cases}$$

We assume that we have a solution $u \in H^1(\Omega)$ to (76). Then, denoting the solution u by

$$u = \begin{cases} u_{\text{int}} & \text{in} \quad \Omega^\varepsilon_{\text{int}}, \\ u_{\text{lay}} & \text{in} \quad \Omega^\varepsilon_{\text{lay}}, \\ u_{\text{ext}} & \text{in} \quad \Omega^\varepsilon_{\text{ext}}, \end{cases}$$

Problem (76) becomes

$$\begin{cases} \sigma_{\text{int}} \Delta u_{\text{int}} = f_{\text{int}} & \text{in} \quad \Omega^\varepsilon_{\text{int}}, \\ \sigma_{\text{ext}} \Delta u_{\text{ext}} = f_{\text{ext}} & \text{in} \quad \Omega^\varepsilon_{\text{ext}}, \\ \Delta u_{\text{lay}} = 0 & \text{in} \quad \Omega^\varepsilon_{\text{lay}}, \\ u_{\text{int}} = u_{\text{lay}} & \text{on} \quad \Gamma^\varepsilon_{\text{int}}, \\ u_{\text{lay}} = u_{\text{ext}} & \text{on} \quad \Gamma^\varepsilon_{\text{ext}}, \\ \sigma_{\text{int}} \partial_n u_{\text{int}} = \widehat{\sigma}_0 \varepsilon^{-3} \partial_n u_{\text{lay}} & \text{on} \quad \Gamma^\varepsilon_{\text{int}}, \\ \widehat{\sigma}_0 \varepsilon^{-3} \partial_n u_{\text{lay}} = \sigma_{\text{ext}} \partial_n u_{\text{ext}} & \text{on} \quad \Gamma^\varepsilon_{\text{ext}}, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{77}$$

where ∂_n represents the normal derivative in the direction of the normal vector, which is interior to $\Omega^\varepsilon_{\text{ext}}$ on $\Gamma^\varepsilon_{\text{ext}}$, and exterior to $\Omega^\varepsilon_{\text{int}}$ on $\Gamma^\varepsilon_{\text{int}}$, as shown at Figure 14. Due to the cylindrical shape of the considered domain, we consider these equations to be written in cylindrical coordinates. Thus, the Laplacian operators of equation (77) have the following form

$$\Delta = \frac{1}{r} \partial_r (r \partial_r) + \frac{1}{r^2} \partial^2_\theta + \partial^2_z.$$

A key point for the derivation of a multiscale expansion for the solution to Problem (77) consists in performing a scaling along the direction normal to the thin layer. We begin by describing domain $\Omega^\varepsilon_{\text{lay}}$ in the following way

$$\Omega^\varepsilon_{\text{lay}} = \left\{ \gamma(\theta, z) + \varepsilon R n : \gamma(z) \in \Gamma, R \in \left(-\frac{1}{2}, \frac{1}{2} \right) \right\},$$

where γ is a parametrization of the interface Γ (see Figure 14), which in cylindrical coordinates is defined as

$$\gamma(\theta, z) = (r_0, \theta, z), \text{ for all } \theta \in [0, 2\pi), z \in (0, z_0),$$

and $n = (1, 0, 0)$ is the normal vector to the curve Γ. This domain geometry induces the following scaling

$$r = r_0 + \varepsilon R \quad \Leftrightarrow \quad R = \varepsilon^{-1} (r - r_0).$$

As a consequence, we have

$$\partial_R^k = \varepsilon^k \partial_r^k, \quad k \in \mathbb{N}.$$

This scaling allows us to write the scalar operator $\frac{1}{r}\partial_r\left(r\partial_r\right) + \frac{1}{r^2}\partial_\theta^2 + \partial_z^2$ in the following way

$$\varepsilon^{-2}\partial_R^2 + \varepsilon^{-1}\frac{1}{r_0+\varepsilon R}\partial_R + \frac{1}{(r_0+\varepsilon R)^2}\partial_\theta^2 + \partial_z^2.$$

Now we perform an expansion of the terms $\frac{1}{r_0+\varepsilon R}$ and $\frac{1}{(r_0+\varepsilon R)^2}$ in powers of ε so that we obtain the following expression

$$\varepsilon^{-2}\partial_R^2 + \sum_{k=0}^{\infty}\varepsilon^{k-1}\frac{(-R)^k}{r_0^{k+1}}\partial_R + \sum_{k=0}^{\infty}\varepsilon^{k}\frac{(-R)^k}{r_0^{k+2}}\partial_\theta^2 + \partial_z^2.$$

We also notice that on the interfaces $\Gamma_{\text{int}}^{\varepsilon}$ and $\Gamma_{\text{ext}}^{\varepsilon}$ we rewrite the normal derivative in the following form $\partial_n = \partial_r = \varepsilon^{-1}\partial_R$. Finally we denote by U the function that satisfies

$$u_{\text{lay}}(r,\theta,z) = u_{\text{lay}}(r_0+\varepsilon R,\theta,z) = U(R,\theta,z), \quad (R,\theta,z) \in \left(-\frac{1}{2},\frac{1}{2}\right) \times [0,2\pi) \times (0,z_0)\,.$$

We rewrite Equations (77) with the newly defined variables and functions and they satisfy the following equations outside the thin layer

$$\begin{cases} \sigma_{\text{int}}\Delta u_{\text{int}} = f_{\text{int}} & \text{in} \quad \Omega_{\text{int}}^{\varepsilon}, \\ \sigma_{\text{ext}}\Delta u_{\text{ext}} = f_{\text{ext}} & \text{in} \quad \Omega_{\text{ext}}^{\varepsilon}, \end{cases} \tag{78}$$

the following equation inside the thin layer

$$\varepsilon^{-2}\partial_R^2 U + \sum_{k=0}^{\infty}\varepsilon^{k-1}\frac{(-R)^k}{r_0^{k+1}}\partial_R U + \sum_{k=0}^{\infty}\varepsilon^{k}(k+1)\frac{(-R)^k}{r_0^{k+2}}\partial_\theta^2 U + \partial_z^2 U = 0 \quad \text{in} \quad \left(-\frac{1}{2},\frac{1}{2}\right)\times[0,2\pi)\times(0,z_0)\,, \tag{79}$$

and the following transmission and boundary conditions

$$\begin{cases} u_{\text{int}}\left(r_0 - \frac{\varepsilon}{2},\theta,z\right) = U\left(-\frac{1}{2},\theta,z\right) & (\theta,z) \in [0,2\pi)\times(0,z_0), \\ u_{\text{ext}}\left(r_0 + \frac{\varepsilon}{2},\theta,z\right) = U\left(\frac{1}{2},\theta,z\right) & (\theta,z) \in [0,2\pi)\times(0,z_0), \\ \sigma_{\text{int}}\partial_n u_{\text{int}}\left(r_0 - \frac{\varepsilon}{2},\theta,z\right) = \widehat{\sigma}_0\varepsilon^{-4}\partial_R U\left(-\frac{1}{2},\theta,z\right) & (\theta,z) \in [0,2\pi)\times(0,z_0), \\ \sigma_{\text{ext}}\partial_n u_{\text{ext}}\left(r_0 + \frac{\varepsilon}{2},\theta,z\right) = \widehat{\sigma}_0\varepsilon^{-4}\partial_R U\left(\frac{1}{2},\theta,z\right) & (\theta,z) \in [0,2\pi)\times(0,z_0), \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{80}$$

A.2.2 First class of ITCs: construction of a multiscale expansion

We now derive the asymptotic expansion. To begin with, we perform an *Ansatz* in the form of power series of ε for the solution to Problems (78), (79) and (80). We look for solutions

$$\begin{cases} u_{\text{int}}(r,\theta,z) \approx \sum_{k\geq 0} \varepsilon^k u_{\text{int}}^k(r,\theta,z) & \text{in} \quad \Omega_{\text{int}}^{\varepsilon}, \\ u_{\text{ext}}(r,\theta,z) \approx \sum_{k\geq 0} \varepsilon^k u_{\text{ext}}^k(r,\theta,z) & \text{in} \quad \Omega_{\text{ext}}^{\varepsilon}, \\ U(R,\theta,z) \approx \sum_{k\geq 0} \varepsilon^k U^k(R,\theta,z) & \text{in} \quad \left(-\frac{1}{2},\frac{1}{2}\right) \times [0,2\pi) \times (0,z_0)\,. \end{cases} \tag{81}$$

Equations for the coefficients of the electric potential

Substituting the previous expressions into Equations (78), (79), and (80), and collecting the terms with the same powers in ε, for every $k \in \mathbb{N}$, we obtain the following set of equations outside the layer

$$\begin{cases} \sigma_{\text{int}} \Delta u_{\text{int}}^k = f_{\text{int}} \delta_k^0 & \text{in} \quad \Omega_{\text{int}}^{\varepsilon}, & \text{(82a)} \\ \sigma_{\text{ext}} \Delta u_{\text{ext}}^k = f_{\text{ext}} \delta_k^0 & \text{in} \quad \Omega_{\text{ext}}^{\varepsilon}, & \text{(82b)} \end{cases}$$

and the following equations inside the layer

$$\partial_R^2 U^k + \sum_{l=0}^{k-1} \frac{(-R)^{k-l-1}}{r_0^{k-l}} \partial_R U^l + \sum_{l=0}^{k-2} (k-l-1) \frac{(-R)^{k-l-2}}{r_0^{k-l}} \partial_\theta^2 U^l + \partial_z^2 U^{k-2} = 0$$
$$\text{in} \quad \left(-\frac{1}{2},\frac{1}{2}\right) \times [0,2\pi) \times (0,z_0), \tag{83}$$

along with the following transmission conditions

$$\begin{cases} U^k\left(-\frac{1}{2},\theta,z\right) = u_{\text{int}}^k\left(r_0 - \frac{\varepsilon}{2},\theta,z\right) & (\theta,z) \in [0,2\pi) \times (0,z_0), & \text{(84a)} \\ U^k\left(\frac{1}{2},\theta,z\right) = u_{\text{ext}}^k\left(r_0 + \frac{\varepsilon}{2},\theta,z\right) & (\theta,z) \in [0,2\pi) \times (0,z_0), & \text{(84b)} \\ \widehat{\sigma}_0 \partial_R U^k\left(-\frac{1}{2},\theta,z\right) = \sigma_{\text{int}} \partial_n u_{\text{int}}^{k-4}\left(r_0 - \frac{\varepsilon}{2},\theta,z\right) & (\theta,z) \in [0,2\pi) \times (0,z_0), & \text{(84c)} \\ \widehat{\sigma}_0 \partial_R U^k\left(\frac{1}{2},\theta,z\right) = \sigma_{\text{ext}} \partial_n u_{\text{ext}}^{k-4}\left(r_0 + \frac{\varepsilon}{2},\theta,z\right) & (\theta,z) \in [0,2\pi) \times (0,z_0), & \text{(84d)} \end{cases}$$

and the following boundary conditions

$$\begin{cases} u^k(R_0, \theta, z) = 0 & (\theta, z) \in [0, 2\pi) \times (0, z_0), & \text{(85a)} \\ u^k(r, \theta, 0) = u^k(r, \theta, z_0) = 0 & (r, \theta) \in \left(0, r_0 - \frac{\varepsilon}{2}\right) \cup \left(r_0 + \frac{\varepsilon}{2}, R_0\right) \times [0, 2\pi), & \text{(85b)} \\ U^k(R, \theta, 0) = U^k(R, \theta, z_0) = 0 & (R, \theta) \in \left(-\frac{1}{2}, \frac{1}{2}\right) \times [0, 2\pi). & \text{(85c)} \end{cases}$$

For determining the elementary problem satisfied by each of the terms of the expansion, we will also need the following equation obtained by applying the fundamental theorem of calculus for a smooth function U^k,

$$\int_{\frac{-1}{2}}^{\frac{1}{2}} \partial_R^2 U^k(R, z)\,\mathrm{d}R = \partial_R U^k\left(\frac{1}{2}, z\right) - \partial_R U^k\left(-\frac{1}{2}, z\right).$$

If we substitute Equation (83) to the left-hand side and Equations (84c) and (84d) to the right-hand side, we obtain the following compatibility condition

$$\int_{\frac{-1}{2}}^{\frac{1}{2}} \left(\partial_z^2 U^{k-2}(R, \theta, z) + \sum_{l=0}^{k-1} \frac{(-R)^{k-1-l}}{r_0^{k-l}} \partial_R U^l(R, \theta, z) + \sum_{l=0}^{k-2} (k-l-1) \frac{(-R)^{k-2-l}}{r_0^{k-l}} \partial_\theta^2 U^l(R, \theta, z) \right) \mathrm{d}R = \frac{-1}{\widehat{\sigma}_0} \left[\sigma \partial_n u^{k-4}\right]_{\Gamma^\varepsilon}(z). \tag{86}$$

We adopt the convention that the terms with negative indices in Equations (82)- (86) are equal to 0. Employing Equations (82) - (86) we deduce the elementary problems satisfied outside and inside the layer for any $k \in \mathbb{N}$. For that purpose we employ the following algorithm composed of three steps.

Algorithm for the determination of the coefficients

<u>Initialization of the algorithm:</u>

Before showing the different steps to obtain function U^k and u^k for every k, we need to determine function U^0 up to a function in the variables θ and z, denoted by φ_0^0. For that purpose we consider Equations (83), (84c), and (84d), and we build the following differential problem in the variable R for U^0 (the variables θ and z play the role of a parameter)

$$\begin{cases} \partial_R^2 U^0(R, \theta, z) = 0 \quad R \in \left(-\frac{1}{2}, \frac{1}{2}\right), \\ \widehat{\sigma}_0 \partial_R U^0\left(-\frac{1}{2}, \theta, z\right) = 0, \\ \widehat{\sigma}_0 \partial_R U^0\left(\frac{1}{2}, \theta, z\right) = 0. \end{cases}$$

From these equations we deduce that U^0 has the following form $U^0(R, \theta, z) = \varphi_0^0(\theta, z)$, where function φ_0^0 has yet to be determined and this will be done during the first step of the algorithm. After these preliminary steps, we move onto determining U^k and u^k for any k.

We assume that the first terms of the expansion (81) up to the order ε^{k-1} have already been calculated and we calculate the equations for the k-th term. We also assume that at rank k we know the form of U^k up to a function in the variables θ and z, denoted by φ_0^k. We obtain the expression of U^k at rank $k-1$. The first step consists in determining the expression of function U^{k+1} up to function φ_0^{k+1}. Then, at the second step we determine function φ_0^k involved in the expression of function U^k. Finally, we determine u_{int}^k and u_{ext}^k at the third step. For every $k = 0, 1, 2, \ldots$, we perform the following steps:

First step:

We select Equations (83), (84c), and (84d), and we build the following differential problem in the variable R for U^{k+1} (the variables θ and z play the role of a parameter)

$$\begin{cases} \partial_R^2 U^{k+1}(R,\theta,z) = g^{k+1}(R,\theta,z) & R \in \left(-\dfrac{1}{2}, \dfrac{1}{2}\right), \\ \widehat{\sigma}_0 \partial_R U^{k+1}\left(-\dfrac{1}{2},\theta,z\right) = \sigma_{\text{int}} \partial_n u_{\text{int}}^{k-3}\left(r_0 - \dfrac{\varepsilon}{2},\theta,z\right), & \\ \widehat{\sigma}_0 \partial_R U^{k+1}\left(\dfrac{1}{2},\theta,z\right) = \sigma_{\text{ext}} \partial_n u_{\text{ext}}^{k-3}\left(r_0 + \dfrac{\varepsilon}{2},\theta,z\right), & \end{cases} \tag{87}$$

where

$$g^{k+1}(R,\theta,z) = -\sum_{l=0}^{k} \frac{(-R)^{k-l}}{r_0^{k-l+1}} \partial_R U^l(R,\theta,z) - \sum_{l=0}^{k-1} (k-l) \frac{(-R)^{k-l-1}}{r_0^{k-l+1}} \partial_\theta^2 U^l(R,\theta,z) - \partial_z^2 U^{k-1}(R,\theta,z).$$

There exists a solution U^{k+1} to (87) provided the compatibility condition (86) is satisfied. We deduce the expression of U^{k+1} up to a function in the variables θ and z, denoted by $\varphi_0^{k+1}(\theta,z)$. The function U^{k+1} has the following form

$$U^{k+1}(R,\theta,z) = V^{k+1}(R,\theta,z) + \varphi_0^{k+1}(\theta,z),$$

where V^{k+1} represents the part of U^{k+1} that is determined at this step and has the form (see Proposition 6)

$$V^k(R,\theta,z) = \begin{cases} 0 & \text{if} \quad k = 0,1,2,3 \\ \varphi_{k-2}^k(\theta,z)R^{k-2} + \varphi_{k-3}^k(\theta,z)R^{k-3} + \ldots + \varphi_1^k(\theta,z)R & \text{if} \quad k > 3. \end{cases}$$

Function φ_0^{k+1} represents the part of U^{k+1} that is determined at the following rank.

Second step:

We employ the compatibility condition (86) (at rank $k+2$), along with Equation (85c) to write the following differential problem in the variables θ and z for function φ_0^k, present in the expression of U^k.

$$\begin{cases} \partial_z^2 \varphi_0^k(\theta,z) + \dfrac{1}{r_0^2} \partial_\theta^2 \varphi_0^k(\theta,z) = h^k(\theta,z) & (\theta,z) \in [0,2\pi) \times (0,z_0), \\ \varphi_0^k(\theta,0) = 0 & \theta \in [0,2\pi), \\ \varphi_0^k(\theta,z_0) = 0 & \theta \in [0,2\pi), \end{cases} \tag{88}$$

where

$$h^k(\theta,z) = -\int_{\frac{-1}{2}}^{\frac{1}{2}} \left(\partial_z^2 V^k\left(R,\theta,z\right) + \sum_{l=0}^{k+1} \frac{(-R)^{k+1-l}}{r_0^{k+2-l}} \partial_R U^l\left(R,\theta,z\right) \right.$$
$$\left. + \sum_{l=0}^{k-1} (k-l+1) \frac{(-R)^{k-l}}{r_0^{k+2-l}} \partial_\theta^2 U^l\left(R,\theta,z\right) + \frac{1}{r_0^2}\partial_\theta^2 V^k(R,\theta,z) \right) \mathrm{d}R - \frac{1}{\widehat{\sigma}_0}\left[\sigma \partial_n u^{k-2}\right]_{\Gamma^\varepsilon}(\theta,z).$$

Solving this differential equation we obtain function φ_0^k and thus, the complete expression of U^k.

<u>Third step:</u>

We derive the equations outside the layer by employing Equations (82a), (82b), (84a), (84b), (85a), and (85b). We infer that u_{int}^k and u_{ext}^k are defined independently in the two subdomains $\Omega_{\text{int}}^\varepsilon$ and $\Omega_{\text{ext}}^\varepsilon$.

$$\begin{cases} \sigma_{\text{int}} \Delta u_{\text{int}}^k = 0 & \text{in} \quad \Omega_{\text{int}}^\varepsilon, \\ u_{\text{int}}^k\left(r_0 - \dfrac{\varepsilon}{2},\theta,z\right) = U^k\left(-\dfrac{1}{2},\theta,z\right), & \\ u_{\text{int}}^k = 0 & \text{on} \quad \partial\Omega \cap \partial\Omega_{\text{int}}^\varepsilon. \end{cases}$$

$$\begin{cases} \sigma_{\text{ext}} \Delta u_{\text{ext}}^k = 0 & \text{in} \quad \Omega_{\text{ext}}^\varepsilon, \\ u_{\text{ext}}^k\left(r_0 + \dfrac{\varepsilon}{2},\theta,z\right) = U^k\left(\dfrac{1}{2},\theta,z\right), & \\ u_{\text{ext}}^k = 0 & \text{on} \quad \partial\Omega \cap \partial\Omega_{\text{ext}}^\varepsilon. \end{cases} \tag{89}$$

We will now employ this algorithm to obtain the equations for the first terms of the expansion.

First terms of the asymptotics

<u>Terms of order zero</u>

Thanks to the preliminary steps formerly performed during the initialization of the algorithm we already know that U^0 has the form $U^0\left(R,\theta,z\right) = \varphi_0^0\left(\theta,z\right)$. In the same way we consider Problem (87) for U^1

$$\begin{cases} \partial_R^2 U^1\left(R,\theta,z\right) = 0 & R \in \left(\dfrac{-1}{2},\dfrac{1}{2}\right), \\ \partial_R U^1\left(-\dfrac{1}{2},\theta,z\right) = 0, & \\ \partial_R U^1\left(\dfrac{1}{2},\theta,z\right) = 0. & \end{cases}$$

We deduce that the solution to this equation has the form $U^1(R,\theta,z) = \varphi_0^1(\theta,z)$. Then, we employ (88) and we build the following problem for φ_0^0

$$\begin{cases} \partial_z^2\varphi_0^0(\theta,z) + \dfrac{1}{r_0^2}\partial_\theta^2\varphi_0^0(\theta,z) = 0 & (\theta,z)\in[0,2\pi)\times(0,z_0), \\ \varphi_0^0(\theta,0) = 0, & \\ \varphi_0^0(\theta,z_0) = 0. & \end{cases}$$

We conclude that $\varphi_0^0(\theta,z) = 0$ and thus, $U^0(R,\theta,z) = 0$. Finally, employing (89), we obtain that the limit solution u^0 satisfies homogeneous Dirichlet boundary conditions on $\Gamma_{\text{int}}^\varepsilon$ and $\Gamma_{\text{ext}}^\varepsilon$. Thus, the problem satisfied by u^0 reads as

$$\begin{cases} \sigma_{\text{int}}\Delta u_{\text{int}}^0 = f_{\text{int}} & \text{in} \quad \Omega_{\text{int}}^\varepsilon, \\ u_{\text{int}}^0 = 0 & \text{on} \quad \partial\Omega_{\text{int}}^\varepsilon. \end{cases}$$
$$\begin{cases} \sigma_{\text{ext}}\Delta u_{\text{ext}}^0 = f_{\text{ext}} & \text{in} \quad \Omega_{\text{ext}}^\varepsilon, \\ u_{\text{ext}}^0 = 0 & \text{on} \quad \partial\Omega_{\text{ext}}^\varepsilon. \end{cases} \tag{90}$$

Terms of order one

We consider Problem (87) for U^2

$$\begin{cases} \partial_R^2 U^2(R,\theta,z) = 0 & R\in\left(\dfrac{-1}{2},\dfrac{1}{2}\right), \\ \partial_R U^2\left(-\dfrac{1}{2},\theta,z\right) = 0, & \\ \partial_R U^2\left(\dfrac{1}{2},\theta,z\right) = 0. & \end{cases}$$

We deduce that the solution to this equation has the form $U^2(R,\theta,z) = \varphi_0^2(\theta,z)$. Then, we employ (88) and we obtain the following problem for φ_0^1

$$\begin{cases} \partial_z^2\varphi_0^1(\theta,z) + \dfrac{1}{r_0^2}\partial_\theta^2\varphi_0^1(\theta,z) = 0 & (\theta,z)\in[0,2\pi)\times(0,z_0), \\ \varphi_0^1(0,\theta) = 0, & \\ \varphi_0^1(z_0,\theta) = 0. & \end{cases}$$

We conclude that $\varphi_0^1(\theta,z) = 0$ and thus, $U^1(R,\theta,z) = 0$. Finally, employing (89) we write the problem satisfied outside the layer by u^1 as two uncoupled problems

$$\begin{cases} \Delta u_{\text{int}}^1 = 0 & \text{in} \quad \Omega_{\text{int}}^\varepsilon, \\ u_{\text{int}}^1 = 0 & \text{on} \quad \partial\Omega_{\text{int}}^\varepsilon. \end{cases}$$
$$\begin{cases} \Delta u_{\text{ext}}^1 = 0 & \text{in} \quad \Omega_{\text{ext}}^\varepsilon, \\ u_{\text{ext}}^1 = 0 & \text{on} \quad \partial\Omega_{\text{ext}}^\varepsilon. \end{cases} \tag{91}$$

We deduce that $u^1 \equiv 0$.

Terms of order two

We consider Problem (87) for U^3

$$\begin{cases} \partial_R^2 U^3\left(R,\theta,z\right) = 0 & R \in \left(\dfrac{-1}{2}, \dfrac{1}{2}\right), \\ \partial_R U^3\left(-\dfrac{1}{2},\theta,z\right) = 0, & \\ \partial_R U^3\left(\dfrac{1}{2},\theta,z\right) = 0. & \end{cases}$$

We deduce that the solution to this equation has the form $U^3\left(R,\theta,z\right) = \varphi_0^3\left(\theta,z\right)$. Then, we employ (88) and φ_0^2 satisfies

$$\begin{cases} \partial_z^2\varphi_0^2(\theta,z) + \dfrac{1}{r_0^2}\partial_\theta^2\varphi_0^2(\theta,z) = -\dfrac{1}{\widehat{\sigma}_0}\left[\sigma\partial_r u^0\right]_{\Gamma^\varepsilon}(\theta,z) & (\theta,z)\in[0,2\pi)\times(0,z_0), \\ \varphi_0^2(\theta,0) = 0, & \\ \varphi_0^2(\theta,z_0) = 0. & \end{cases} \tag{92}$$

Solving this problem we obtain the function $\varphi_0^2(\theta,z)$ and thus, the complete expression of $U^2(R,\theta,z)$. Finally, employing (89) we write the problem satisfied by u^2 outside the layer as two uncoupled problems

$$\begin{cases} \Delta u_{\text{int}}^2 = 0 & \text{in} \quad \Omega_{\text{int}}^\varepsilon, \\ u_{\text{int}}^2\left(r_0 - \dfrac{\varepsilon}{2},\theta,z\right) = \varphi_0^2(\theta,z), & \\ u_{\text{int}}^2 = 0 & \text{on} \quad \partial\Omega\cap\partial\Omega_{\text{int}}^\varepsilon. \end{cases}$$

$$\begin{cases} \Delta u_{\text{ext}}^2 = 0 & \text{in} \quad \Omega_{\text{ext}}^\varepsilon, \\ u_{\text{ext}}^2\left(r_0 + \dfrac{\varepsilon}{2},\theta,z\right) = \varphi_0^2(\theta,z), & \\ u_{\text{ext}}^2 = 0 & \text{on} \quad \partial\Omega\cap\partial\Omega_{\text{ext}}^\varepsilon. \end{cases} \tag{93}$$

Terms of order three

We consider Problem (87) for U^4

$$\begin{cases} \partial_R^2 U^4\left(R,\theta,z\right) = -\partial_z^2 U^2(\theta,z) - \dfrac{1}{r_0^2}\partial_\theta^2\varphi_0^2(\theta,z) & R \in \left(\dfrac{-1}{2}, \dfrac{1}{2}\right), \\ \widehat{\sigma}_0\partial_R U^4\left(-\dfrac{1}{2},\theta,z\right) = \sigma_{\text{int}}\partial_n u_{\text{int}}^0\left(r_0 - \dfrac{\varepsilon}{2},\theta,z\right), & \\ \widehat{\sigma}_0\partial_R U^4\left(\dfrac{1}{2},\theta,z\right) = \sigma_{\text{ext}}\partial_n u_{\text{ext}}^0\left(r_0 + \dfrac{\varepsilon}{2},\theta,z\right). & \end{cases}$$

We deduce that the solution to this equation has the form

$$U^4(R,\theta,z) = \frac{1}{\widehat{\sigma}_0}\left[\sigma\partial_n u^0\right]_{\Gamma^\varepsilon}(\theta,z)\frac{R^2}{2} + \frac{1}{\widehat{\sigma}_0}\left\{\sigma\partial_n u^0\right\}_{\Gamma^\varepsilon}(\theta,z)R + \varphi_0^4(\theta,z).$$

Then, we employ (88) and we build the following problem for φ_0^3

$$\begin{cases} \partial_z^2\varphi_0^3(\theta,z) + \dfrac{1}{r_0^2}\partial_\theta^2\varphi_0^3(\theta,z) = -\dfrac{1}{\widehat{\sigma}_0 r_0}\left\{\sigma\partial_n u^0\right\}_{\Gamma^\varepsilon}(\theta,z) \quad (\theta,z)\in[0,2\pi)\times(0,z_0), \\ \varphi_0^3(\theta,0) = 0, \\ \varphi_0^3(\theta,z_0) = 0. \end{cases} \tag{94}$$

Solving this problem we obtain the function $\varphi_0^2(\theta,z)$, and thus, the complete expression of $U^2(R,\theta,z)$. Finally, employing (89) we write the problem satisfied outside the layer by u^3 as two uncoupled problems

$$\begin{cases} \Delta u^3_{\text{int}} = 0 & \text{in} \quad \Omega^\varepsilon_{\text{int}}, \\ u^3_{\text{int}}\left(r_0 - \dfrac{\varepsilon}{2},\theta,z\right) = U^3\left(-\dfrac{1}{2},\theta,z\right), \\ u^3_{\text{int}} = 0 & \text{on} \quad \partial\Omega\cap\partial\Omega^\varepsilon_{\text{int}}. \end{cases}$$

$$\begin{cases} \Delta u^3_{\text{ext}} = 0 & \text{in} \quad \Omega^\varepsilon_{\text{ext}}, \\ u^3_{\text{ext}}\left(r_0 + \dfrac{\varepsilon}{2},\theta,z\right) = U^3\left(\dfrac{1}{2},\theta,z\right), \\ u^3_{\text{ext}} = 0 & \text{on} \quad \partial\Omega\cap\partial\Omega^\varepsilon_{\text{ext}}. \end{cases} \tag{95}$$

Recapitulation of the asymptotic expansion

Proposition 6. *The asymptotic expansion (81), has the following form*

$$\begin{cases} u_{int}(r,\theta,z) = u^0_{int}(r,\theta,z) + \varepsilon^2 u^2_{int}(r,\theta,z) + \varepsilon^3 u^3_{int}(r,\theta,z) + O\left(\varepsilon^4\right) & in \quad \Omega^\varepsilon_{int}, \\ u_{ext}(r,\theta,z) = u^0_{ext}(r,\theta,z) + \varepsilon^2 u^2_{ext}(r,\theta,z) + \varepsilon^3 u^3_{ext}(r,\theta,z) + O\left(\varepsilon^4\right) & in \quad \Omega^\varepsilon_{ext}, \\ U\left(R,\theta,z\right) = \varepsilon^2\varphi_0^2(\theta,z) + \varepsilon^3\varphi_0^3(\theta,z) + O\left(\varepsilon^4\right) & in \left(-\dfrac{1}{2},\dfrac{1}{2}\right) \\ & \times[0,2\pi)\times(0,z_0), \end{cases}$$

where functions u^0, u^2, u^3, φ_0^2, and φ_0^3 are defined by Equations (90), (93), (95), (92), and (94) respectively. For $k\in\mathbb{N}$, the solution U^k to Equation (87) has the following form

$$U^k\left(R,z\right) = \begin{cases} 0 & if \quad k=0,1, \\ \varphi_0^k(\theta,z) & if \quad k=2,3, \\ \displaystyle\sum_{j=0}^{k-2}\varphi_j^k(\theta,z)R^j & if \quad k\geq 4, \end{cases}$$

Proof. We conduct the proof by induction on k. For $k = 0, 1, 2, 3$, we have already calculated the expressions of u^k and U^k in the previous section. Now let us assume that for any number $i \in \mathbb{N}$, such that $i < k$, function U^i has the form

$$U^i\,(R,\theta,z) = \varphi^i_{i-2}(\theta,z)R^{i-2} + \varphi^i_{i-3}(\theta,z)R^{i-3} + \ldots + \varphi^i_1(\theta,z)R + \varphi^i_0(\theta,wz),$$

We begin by considering Problem (87) for U^k. Solving this problem we obtain a solution of the form

$$U^k\,(R,\theta,z) = \varphi^k_{k-2}(\theta,z)R^{k-2} + \varphi^k_{k-3}(\theta,z)R^{k-3} + \ldots + \varphi^k_1(\theta,z)R + \varphi^k_0(\theta,z),$$

In the above expression of U^k we find function V^k, defined as

$$V^k\,(R,\theta,z) = \varphi^k_{k-2}(\theta,z)R^{k-2} + \varphi^k_{k-3}(\theta,z)R^{k-3} + \ldots + \varphi^k_1(\theta,z)R$$

at the first step of the algorithm.

□

A.2.3 First class of ITCs: equivalent models

Now that we know the expressions for the first terms of the expansion, we truncate the series and we identify a simpler problem satisfied by

$$u^{(k)} = u^0 + \varepsilon u^1 + \ldots + \varepsilon^k u^k \quad \text{in} \quad \Omega^\varepsilon_{\text{int}} \cup \Omega^\varepsilon_{\text{ext}}$$

up to a residual term of order ε^{k+1}. We neglect the residual term of order ε^{k+1} to obtain an approximate model satisfied by function $u^{[k]}$. We formally derive three approximate models of second, third, and fourth order respectively.

Second-order model

For deriving the model of order two, we truncate the series from the second term and we define $u^{(1)}$ as

$$u^{(1)} = u^0 + \varepsilon u^1 = u^0 \quad \text{in} \quad \Omega^\varepsilon_{\text{int}} \cup \Omega^\varepsilon_{\text{ext}} \qquad \text{(see Proposition 6).}$$

From (90), we deduce that $u^{(1)}$ solves the following uncoupled problem

$$\begin{cases} \sigma_{\text{int}}\Delta u^{(1)}_{\text{int}} = f_{\text{int}} & \text{in} \quad \Omega^\varepsilon_{\text{int}}, \\ u^{(1)}_{\text{int}} = 0 & \text{on} \quad \partial\Omega^\varepsilon_{\text{int}}. \end{cases}$$
$$\begin{cases} \sigma_{\text{ext}}\Delta u^{(1)}_{\text{ext}} = f_{\text{ext}} & \text{in} \quad \Omega^\varepsilon_{\text{ext}}, \\ u^{(1)}_{\text{ext}} = 0 & \text{on} \quad \partial\Omega^\varepsilon_{\text{ext}}. \end{cases} \tag{96}$$

In this case, we have $u^{[1]} = u^{(1)}$ as $u^{(1)}$ does not depend on ε. We infer a second-order model satisfied by $u^{[1]}$ solution to Problem (96).

Third-order model

For deriving the model of order three, we truncate the series from the third term and we define $u^{(2)}$ as

$$u^{(2)} = u^0 + \varepsilon u^1 + \varepsilon^2 u^2 = u^0 + \varepsilon^2 u^2 \quad \text{in} \quad \Omega^\varepsilon_{\text{int}} \cup \Omega^\varepsilon_{\text{ext}} \qquad \text{(see Proposition 6).}$$

From (90), (91), and (93) we deduce that $u^{(2)}$ satisfies the following equations

$$\begin{cases} \sigma_{\text{int}} \Delta u^{(2)}_{\text{int}} = f_{\text{int}} & \text{in} \quad \Omega^\varepsilon_{\text{int}}, \\ \sigma_{\text{ext}} \Delta u^{(2)}_{\text{ext}} = f_{\text{ext}} & \text{in} \quad \Omega^\varepsilon_{\text{ext}}, \\ \left[u^{(2)} \right]_{\Gamma^\varepsilon} = 0, & \\ \Delta_\Gamma \left\{ u^{(2)} \right\}_{\Gamma^\varepsilon} = -\varepsilon^2 \dfrac{1}{\widehat{\sigma}_0} \left[\sigma \partial_n u^0 \right]_{\Gamma^\varepsilon} & \\ u^{(2)} = 0 & \text{on} \quad \partial\Omega \cap \partial\Omega^\varepsilon. \end{cases}$$

where $\Delta_\Gamma = \partial^2_z + \frac{1}{r_0^2} \partial^2_\theta$. Following the same procedure as in Section 2.2 we obtain the following third-order asymptotic model for $u^{[2]}$

$$\begin{cases} \sigma_{\text{int}} \Delta u^{[2]}_{\text{int}} = f_{\text{int}} & \text{in} \quad \Omega^\varepsilon_{\text{int}}, \\ \sigma_{\text{ext}} \Delta u^{[2]}_{\text{ext}} = f_{\text{ext}} & \text{in} \quad \Omega^\varepsilon_{\text{ext}}, \\ \left[u^{[2]} \right]_{\Gamma^\varepsilon} = 0, & \\ \Delta_\Gamma \left\{ u^{[2]} \right\}_{\Gamma^\varepsilon} = -\varepsilon^2 \dfrac{1}{\widehat{\sigma}_0} \left[\sigma \partial_n u^{[2]} \right]_{\Gamma^\varepsilon} & \\ u^{[2]} = 0 & \text{on} \quad \partial\Omega \cap \partial\Omega^\varepsilon. \end{cases} \tag{97}$$

Fourth-order model

For deriving the model of order four, we truncate the series from the fourth term and we define $u^{(3)}$ as

$$u^{(3)} = u^0 + \varepsilon u^1 + \varepsilon^2 u^2 + \varepsilon^3 u^3 = u^0 + \varepsilon^2 u^2 + \varepsilon^3 u^3 \quad \text{in} \quad \Omega^\varepsilon_{\text{int}} \cup \Omega^\varepsilon_{\text{ext}} \quad \text{(see Proposition 6).}$$

From (90), (91), (93), and (95), we deduce that $u^{(3)}$ satisfies the following equations

$$\begin{cases} \sigma_{\text{int}} \Delta u^{(3)}_{\text{int}} = f_{\text{int}} & \text{in} \quad \Omega^\varepsilon_{\text{int}}, \\ \sigma_{\text{ext}} \Delta u^{(3)}_{\text{ext}} = f_{\text{ext}} & \text{in} \quad \Omega^\varepsilon_{\text{ext}}, \\ \left[u^{(3)} \right]_{\Gamma^\varepsilon} = 0, & \\ \Delta_\Gamma \left\{ u^{(3)} \right\}_{\Gamma^\varepsilon} = g, & \\ u^{(3)} = 0 & \text{on} \quad \partial\Omega \cap \partial\Omega^\varepsilon, \end{cases}$$

where

$$g = -\varepsilon^2 \frac{1}{\widehat{\sigma}_0} \left[\sigma \partial_n u^0\right]_{\Gamma^\varepsilon} - \varepsilon^3 \frac{1}{\widehat{\sigma}_0} \left[\sigma \partial_n u^1\right]_{\Gamma^\varepsilon} - \varepsilon^3 \frac{1}{\widehat{\sigma}_0 r_0} \left\{\sigma \partial_n u^0\right\}_{\Gamma^\varepsilon}.$$

Following the same procedure as in Section 2.2 we obtain the following fourth-order asymptotic model for $u^{[3]}$

$$\begin{cases} \sigma_{\text{int}} \Delta u^{[3]}_{\text{int}} = f_{\text{int}} & \text{in} \quad \Omega^\varepsilon_{\text{int}}, \\ \sigma_{\text{ext}} \Delta u^{[3]}_{\text{ext}} = f_{\text{ext}} & \text{in} \quad \Omega^\varepsilon_{\text{ext}}, \\ \left[u^{[3]}\right]_{\Gamma^\varepsilon} = 0, & \\ \Delta_\Gamma \left\{u^{[3]}\right\}_{\Gamma^\varepsilon} = -\frac{\varepsilon^2}{\widehat{\sigma}_0} \left[\sigma \partial_n u^{[3]}\right]_{\Gamma^\varepsilon} - \frac{\varepsilon^3}{\widehat{\sigma}_0 r_0} \left\{\sigma \partial_n u^{[3]}\right\}_{\Gamma^\varepsilon}, & \\ u^{[3]} = 0 & \text{on} \quad \partial\Omega \cap \partial\Omega^\varepsilon. \end{cases} \tag{98}$$

A.2.4 Second class of ITCs: construction of a multiscale expansion

In this section we show the asymptotic models we obtain when we write the asymptotic conditions across the interface Γ. We expand the solution in power series of ε. Then, by truncating this series and neglecting higher order terms in ε, we derive approximate models coupled with equivalent transmission conditions across interface Γ. Since we use the same procedure as in the previous sections, we will concentrate on presenting the obtained results, regarding the multiscale expansion and the derivation of the asymptotic models. The domain where the approximate models are defined is depicted at Figure 15. For deriving the equivalent models, we first use an *Ansatz* in the form of power series of ε for the solution to problems (78) and (80). We look for solutions

$$\begin{cases} u_{\text{int}}(r,\theta,z) \approx \sum_{k\geq 0} \varepsilon^k u^k_{\text{int}}(r,\theta,z) & \text{in} \quad \Omega^\varepsilon_{\text{int}}, \\ u_{\text{ext}}(r,\theta,z) \approx \sum_{k\geq 0} \varepsilon^k u^k_{\text{ext}}(r,\theta,z) & \text{in} \quad \Omega^\varepsilon_{\text{ext}}, \\ U(R,\theta,z) \approx \sum_{k\geq 0} \varepsilon^k U^k(R,\theta,z) & \text{in} \quad \left(-\frac{1}{2},\frac{1}{2}\right) \times [0,2\pi) \times (0,z_0). \end{cases} \tag{99}$$

where functions $\left(u^k_{\text{int}}\right)_{k\in\mathbb{N}}$ and $\left(u^k_{\text{ext}}\right)_{k\in\mathbb{N}}$ are now defined in ε-independent domains, contrary to the first approach. We emphasize that the sequence $\left(u^k_{\text{int}}\right)_{k\in\mathbb{N}}$ (respectively $\left(u^k_{\text{ext}}\right)_{k\in\mathbb{N}}$) is defined in Ω_{int} (respectively Ω_{ext}) even if its associated series does not approach u in the thin layer. We assume that for $k \in \mathbb{N}$, the terms u^k_{int} and u^k_{ext} are as regular as necessary, we refer to [16] which provides some regularity results. Then, we conduct a formal Taylor series expansion of the terms $u^k_{\text{int}}\big|_{\Gamma^\varepsilon_{\text{int}}}$ and $u^k_{\text{ext}}\big|_{\Gamma^\varepsilon_{\text{ext}}}$ of the series, in order to write the transmission conditions across interface Γ. We perform the formal Taylor series expansion in the same way as formerly done in Section 3.1. Substituting this *Ansatz* and the formal Taylor series expansions into Equations (78) and (80), and grouping the terms with the same powers in ε together, for every $k \in \mathbb{N}$, we obtain the

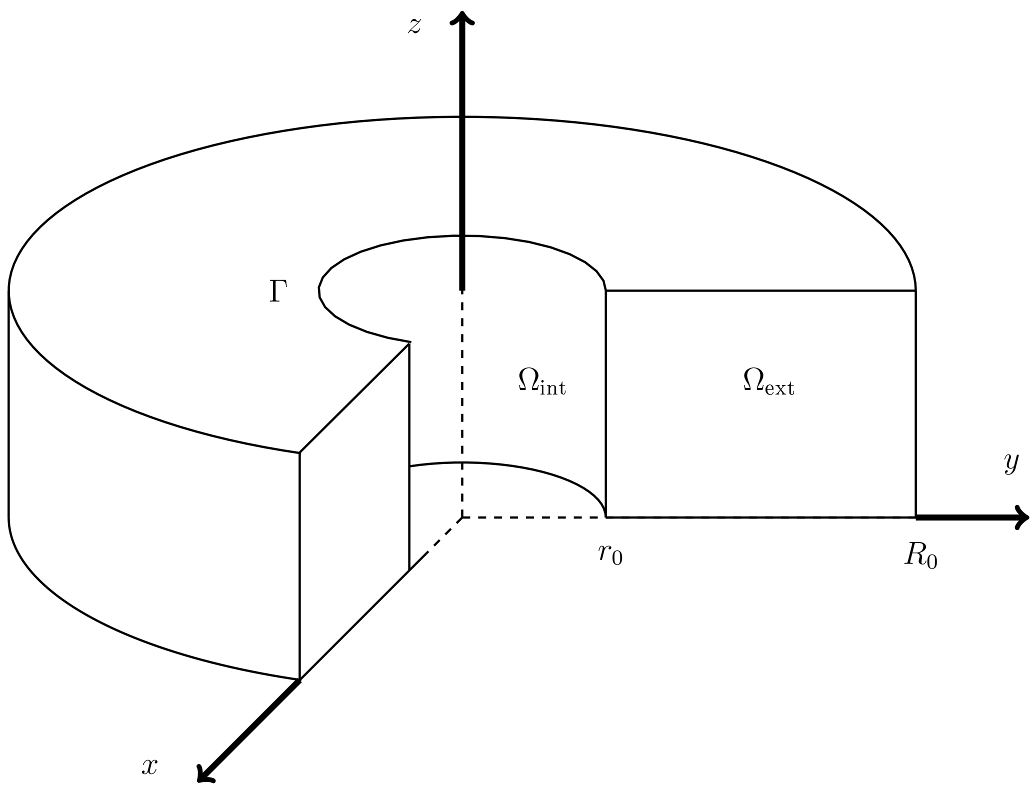


Figure 15: Sectioned domain for the asymptotic models of the second class.

following set of equations in Ω_{int} and Ω_{ext}

$$\begin{cases} \sigma_{\text{int}} \Delta u_{\text{int}}^k = f_{\text{int}} \delta_k^0 & \text{in} \quad \Omega_{\text{int}}, & (100a) \\ \sigma_{\text{ext}} \Delta u_{\text{ext}}^k = f_{\text{ext}} \delta_k^0 & \text{in} \quad \Omega_{\text{ext}}, & (100b) \end{cases}$$

and the following equations inside the layer

$$\partial_R^2 U^k + \sum_{l=0}^{k-1} \frac{(-R)^{k-l-1}}{r_0^{k-l}} \partial_R U^l + \sum_{l=0}^{k-2} (k-l-1) \frac{(-R)^{k-l-2}}{r_0^{k-l}} \partial_\theta^2 U^l + \partial_z^2 U^{k-2} = 0$$
$$\text{in} \quad \left(-\frac{1}{2}, \frac{1}{2}\right) \times [0, 2\pi) \times (0, z_0), \quad (101)$$

along with the following transmission conditions

$$\begin{cases} \displaystyle\sum_{i=0}^{k} \frac{(-1)^i}{2^i i!} \partial_n^i u_{\text{int}}^{k-i}(r_0, \theta, z) = U^k\left(-\frac{1}{2}, \theta, z\right) & (\theta, z) \in [0, 2\pi) \times (0, z_0), \\ \displaystyle\sum_{i=0}^{k} \frac{1}{2^i i!} \partial_n^i u_{\text{ext}}^{k-i}(r_0, \theta, z) = U^k\left(\frac{1}{2}, \theta, z\right) & (\theta, z) \in [0, 2\pi) \times (0, z_0), \\ \sigma_{\text{int}} \displaystyle\sum_{i=0}^{k-4} \frac{(-1)^i}{2^i i!} \partial_n^{i+1} u_{\text{int}}^{k-i-4}(r_0, \theta, z) = \widehat{\sigma}_0 \partial_R U^k\left(-\frac{1}{2}, \theta, z\right) & (\theta, z) \in [0, 2\pi) \times (0, z_0), \\ \sigma_{\text{ext}} \displaystyle\sum_{i=0}^{k-4} \frac{1}{2^i i!} \partial_n^{i+1} u_{\text{ext}}^{k-i-4}(r_0, \theta, z) = \widehat{\sigma}_0 \partial_R U^k\left(\frac{1}{2}, \theta, z\right) & (\theta, z) \in [0, 2\pi) \times (0, z_0), \end{cases} \quad (102)$$

and the following boundary conditions

$$\begin{cases} u^k(R_0, \theta, z) = 0 & (\theta, z) \in [0, 2\pi) \times (0, z_0), \\ u^k(r, \theta, 0) = u^k(r, \theta, z_0) = 0 & (r, \theta) \in \left(0, r_0 - \frac{\varepsilon}{2}\right) \cup \left(r_0 + \frac{\varepsilon}{2}, R_0\right) \times [0, 2\pi), \\ U^k(R, \theta, 0) = U^k(R, \theta, z_0) = 0 & (R, \theta) \in \left(-\frac{1}{2}, \frac{1}{2}\right) \times [0, 2\pi). \end{cases} \quad (103)$$

Employing these equations ((100) - (103)), we determine the elementary problems satisfied outside and inside the layer for any $k \in \mathbb{N}$.

Proposition 7. *Following a similar procedure as for the first class, we deduce that $U^0 \equiv U^1 \equiv 0$. Thus, the asymptotic expansion (99), has the following form*

$$\begin{cases} u_{int}(r, \theta, z) = u_{int}^0(r, \theta, z) + \varepsilon u_{int}^1(r, \theta, z) + O\left(\varepsilon^2\right) & in \quad \Omega_{int}^\varepsilon, \\ u_{ext}(r, \theta, z) = u_{int}^0(r, \theta, z) + \varepsilon u_{ext}^1(r, \theta, z) + O\left(\varepsilon^2\right) & in \quad \Omega_{ext}^\varepsilon, \\ U(R, \theta, z) = O\left(\varepsilon^2\right) & in \quad \left(-\frac{1}{2}, \frac{1}{2}\right) \times [0, 2\pi) \times (0, z_0), \end{cases}$$

where the functions u^0 and u^1 satisfy the following problems

$$\begin{cases} \sigma_{int}\Delta u^0_{int} = f_{int} & in \quad \Omega_{int}, \\ u^0_{int} = 0 & on \quad \partial\Omega_{int}, \end{cases}$$
$$\begin{cases} \sigma_{ext}\Delta u^0_{ext} = f_{ext} & in \quad \Omega_{ext}, \\ u^0_{ext} = 0 & on \quad \partial\Omega_{ext}, \end{cases} \tag{104}$$

and

$$\begin{cases} \sigma_{int}\Delta u^1_{int} = f_{int} & in \quad \Omega_{int}, \\ u^1_{int} = \dfrac{1}{2}\partial_n u^1_{int} & on \quad \Gamma \\ u^1_{int} = 0 & on \quad \partial\Omega_{int} \cap \partial\Omega, \end{cases}$$
$$\begin{cases} \sigma_{ext}\Delta u^1_{ext} = f_{ext} & in \quad \Omega_{ext}, \\ u^1_{ext} = -\dfrac{1}{2}\partial_n u^1_{ext} & on \quad \Gamma \\ u^1_{ext} = 0 & on \quad \partial\Omega_{ext} \cap \partial\Omega. \end{cases} \tag{105}$$

A.2.5 Second class of ITCs: equivalent models

Once we know the expressions for the first terms of the expansion, we truncate the series and we identify a simpler problem satisfied by

$$u^{(k)} = u^0 + \varepsilon u^1 + \ldots + \varepsilon^k u^k \quad \text{in} \quad \Omega_{\text{int}} \cup \Omega_{\text{ext}}$$

up to a residual term of order ε^{k+1}. We neglect the residual term of order ε^{k+1} to obtain an approximate model satisfied by function $u^{[k]}$. Here, we formally derive two approximate models of order one and order two respectively.

First-order model

$$\begin{cases} \sigma_{\text{int}}\Delta u^{[0]}_{\text{int}} = f_{\text{int}} & \text{in} \quad \Omega_{\text{int}}, \\ u^{[0]}_{\text{int}} = 0 & \text{on} \quad \partial\Omega_{\text{int}}. \end{cases}$$
$$\begin{cases} \sigma_{\text{ext}}\Delta u^{[0]}_{\text{ext}} = f_{\text{ext}} & \text{in} \quad \Omega_{\text{ext}}, \\ u^{[0]}_{\text{ext}} = 0 & \text{on} \quad \partial\Omega_{\text{ext}}. \end{cases} \tag{106}$$

Second-order model

$$
\left\{
\begin{aligned}
\sigma_{\text{int}} \Delta u^{[1]}_{\text{int}} &= f_{\text{int}} && \text{in} \quad \Omega_{\text{int}},\\
u^{[1]}_{\text{int}} &= \frac{\varepsilon}{2} \partial_n u^{[1]}_{\text{int}} && \text{on} \quad \Gamma,\\
u^{[1]}_{\text{int}} &= 0 && \text{on} \quad \partial\Omega \cap \partial\Omega_{\text{int}}.
\end{aligned}
\right.
\qquad
\left\{
\begin{aligned}
\sigma_{\text{ext}} \Delta u^{[1]}_{\text{ext}} &= f_{\text{ext}} && \text{in} \quad \Omega_{\text{ext}},\\
u^{[1]}_{\text{ext}} &= -\frac{\varepsilon}{2} \partial_n u^{[1]}_{\text{ext}} && \text{on} \quad \Gamma,\\
u^{[1]}_{\text{ext}} &= 0 && \text{on} \quad \partial\Omega \cap \partial\Omega_{\text{ext}}.
\end{aligned}
\right.
\tag{107}
$$

References

[1] A. Bendali and K. Lemrabet. The effect of a thin coating on the scattering of a time-harmonic wave for the Helmholtz equation. *SIAM Journal on Applied Mathematics*, 56(6):1664–1693, 1996.

[2] A.-L. Bessoud, F. Krasucki, and M. Serpilli. Asymptotic analysis of shell-like inclusions with high rigidity. *Journal of Elasticity*, 103(2):153–172, 2011.

[3] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Springer Science & Business Media, 2010.

[4] A. Burel. *Contributions à la simulation numérique en élastodynamique: découplage des ondes P et S, modèles asymptotiques pour la traversée de couches minces*. PhD thesis, Université Paris Sud-Paris XI, 2014.

[5] F. Buret, M. Dauge, P. Dular, L. Krähenbühl, V. Péron, R. Perrussel, C. Poignard, and D. Voyer. Eddy currents and corner singularities. *IEEE Transactions on Magnetics*, 48(2):679–682, Jan. 2012.

[6] G. Caloz, M. Costabel, M. Dauge, and G. Vial. Asymptotic expansion of the solution of an interface problem in a polygonal domain with thin layer. *Asymptotic Analysis*, 50(1, 2):121–173, 2006.

[7] G. Caloz, M. Dauge, E. Faou, and V. Péron. On the influence of the geometry on skin effect in electromagnetism. *Computer Methods in Applied Mechanics and Engineering*, 200(9):1053–1068, 2011.

[8] G. Caloz, M. Dauge, and V. Péron. Uniform estimates for transmission problems with high contrast in heat conduction and electromagnetism. *Journal of Mathematical Analysis and applications*, 370(2):555–572, 2010.

[9] Q. Chen, D. Pardo, H.-b. Li, and F.-r. Wang. New post-processing method for interpretation of through casing resistivity (TCR) measurements. *Journal of Applied Geophysics*, 74(1):19–25, 2011.

[10] S. Chun, H. Haddar, and J. S. Hesthaven. High-order accurate thin layer approximations for time-domain electromagnetics. Part II: transmission layers. *Journal of Computational and Applied Mathematics*, 234(8):2587–2608, 2010.

[11] S. Chun and J. S. Hesthaven. High-order accurate thin layer approximations for time-domain electromagnetics. Part I: General metal backed coatings. *Journal of computational and applied mathematics*, 231(2):598–611, 2009.

[12] M. Costabel, M. Dauge, and S. Nicaise. Singularities of Maxwell interface problems. *ESAIM: Mathematical Modelling and Numerical Analysis*, 33(03):627–649, 1999.

[13] B. Delourme. *Modèles et asymptotiques des interfaces fines et périodiques en électromagnétisme*. PhD thesis, Université Pierre et Marie Curie-Paris VI, 2010.

[14] B. Delourme, H. Haddar, and P. Joly. Approximate models for wave propagation across thin periodic interfaces. *Journal de mathématiques pures et appliquées*, 98(1):28–71, 2012.

[15] L. F. Demkowicz and J. Gopalakrishnan. A class of discontinuous Petrov–Galerkin methods. Part I: The transport equation. *Computer Methods in Applied Mechanics and Engineering*, 199(23):1558–1572, 2010.

[16] M. Duruflé, V. Péron, and C. Poignard. Time-harmonic Maxwell equations in biological cells. The differential form formalism to treat the thin layer. *Confluentes Mathematici*, 3(2):325–357, 2011. voir aussi http://hal.inria.fr/inria-00347971/fr/.

[17] M. Duruflé, V. Péron, and C. Poignard. Thin layer models for electromagnetism. *Communications in Computational Physics*, 16:213–238, 2014.

[18] B. Engquist and J.-C. Nédélec. Effective boundary conditions for acoustic and electromagnetic scattering in thin layers. Technical report, Technical Report of CMAP, 278, 1993.

[19] A. Erdozain. *Fast inversion of 3D Borehole Resistivity Measurements using Model Reduction Techniques based on 1D Semi-Analytical Solutions.* PhD thesis, INRIA, UPPA, UPV-EHU, 2016. unpublished.

[20] H. Haddar and P. Joly. Stability of thin layer approximation of electromagnetic waves scattering by linear and nonlinear coatings. *Journal of computational and applied mathematics*, 143(2):201–236, 2002.

[21] H. Haddar, P. Joly, and H.-M. Nguyen. Generalized impedance boundary conditions for scattering by strongly absorbing obstacles: the scalar case. *Mathematical Models and Methods in Applied Sciences*, 15(08):1273–1300, 2005.

[22] H. Haddar, P. Joly, and H. M. Nguyen. Construction and analysis of approximate models for electromagnetic scattering from imperfectly conducting scatterers. Research Report RR-6302, INRIA, 2007.

[23] J. Hernandes and A. K. T. Assis. Electric potential due to an infinite conducting cylinder with internal or external point charge. *Journal of electrostatics*, 63(12):1115–1131, 2005.

[24] T. J. Hughes, J. A. Cottrell, and Y. Bazilevs. Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement. *Computer methods in applied mechanics and engineering*, 194(39):4135–4195, 2005.

[25] N. Ida and S. Yuferev. Impedance boundary conditions for transient scattering problems. *IEEE Transactions on Magnetics*, 33(2):1444–1447, 1997.

[26] A. A. Kaufman. The electrical field in a borehole with a casing. *Geophysics*, 55(1):29–38, 1990.

[27] A. A. Kaufman and W. E. Wightman. A transmission-line model for electrical logging through casing. *Geophysics*, 58(12):1739–1747, 1993.

[28] I. Muga, D. Pardo, P. J. Matuszyk, and C. Torres-Verdín. Semi-analytical response of acoustic logging measurements in frequency domain. *Computers & Mathematics with Applications*, 70(4):314–329, 2015.

[29] D. Pardo, L. F. Demkowicz, C. Torres-Verdín, and C. Michler. PML enhanced with a self-adaptive goal-oriented hp-finite element method: Simulation of through-casing borehole resistivity measurements. *SIAM Journal on Scientific Computing*, 30(6):2948–2964, 2008.

[30] D. Pardo, L. F. Demkowicz, C. Torres-Verdín, and M. Paszynski. Two-Dimensional High-Accuracy Simulation of Resistivity Log ging-While-Drilling (LWD) Measurements Using a Self-Adaptive Goal-Oriented hp Finite Element Method. *SIAM Journal on Applied Mathematics*, 66(6):2085–2106, 2006.

[31] D. Pardo, L. F. Demkowicz, C. Torres-Verdín, and M. Paszynski. A self-adaptive goal-oriented hp-finite element method with electromagnetic applications. Part II: Electrodynamics. *Computer methods in applied mechanics and engineering*, 196(37):3585–3597, 2007.

[32] D. Pardo, L. F. Demkowicz, C. Torres-Verdín, and L. A. Tabarovsky. A goal-oriented hp-adaptive finite element method with electromagnetic applications. Part I: electrostatics. *International Journal for Numerical Methods in Engineering*, 65(8):1269–1309, 2006.

[33] D. Pardo, C. Torres-Verdín, and L. F. Demkowicz. Simulation of multifrequency borehole resistivity measurements through metal casing using a goal-oriented hp finite-element method. *Geoscience and Remote Sensing, IEEE Transactions on*, 44(8):2125–2134, 2006.

[34] D. Pardo, C. Torres-Verdín, and L. F. Demkowicz. Feasibility study for 2D frequency-dependent electromagnetic sensing through casing. *Geophysics*, 72(3):F111–F118, 2007.

[35] D. Pardo, C. Torres-Verdin, and M. Paszynski. Simulations of 3D DC borehole resistivity measurements with a goal-oriented hp finite-element method. Part II: through-casing resistivity instruments. *Computational Geosciences*, 12(1):83–89, 2008.

[36] D. Pardo, C. Torres-Verdín, and Z. Zhang. Sensitivity study of borehole-to-surface and crosswell electromagnetic measurements acquired with energized steel casing to water displacement in hydrocarbon-bearing layers. *Geophysics*, 73(6):F261–F268, 2008.

[37] C. Pérez-Arancibia and M. Durán. On the Green's function for the Helmholtz operator in an impedance circular cylindrical waveguide. *Journal of computational and applied mathematics*, 235(1):244–262, 2010.

[38] V. Péron. *Mathematical modeling of electromagnetic phenomena in high contrast media*. Theses, Université Rennes 1, Sept. 2009. date de rédaction: avril 2009.

[39] V. Péron. Equivalent Boundary Conditions for an Elasto-Acoustic Problem set in a Domain with a Thin Layer. Research Report RR-8163, INRIA, June 2013.

[40] V. Péron. Equivalent boundary conditions for an elasto-acoustic problem set in a domain with a thin layer. *ESAIM: Mathematical Modelling and Numerical Analysis*, 48(5):1431–1449, 2014.

[41] V. Péron and C. Poignard. Approximate transmission conditions for time-harmonic Maxwell equations in a domain with thin layer. Research Report RR-6775, INRIA, 2008.

[42] V. Péron, K. Schmidt, and M. Duruflé. Equivalent transmission conditions for the time-harmonic Maxwell equations in 3D for a medium with a highly conductive thin sheet. *SIAM Journal on Applied Mathematics*, 76(3):1031–1052, 2016.

[43] R. Perrussel and C. Poignard. Asymptotic expansion of steady-state potential in a high contrast medium with a thin resistive layer. *Applied Mathematics and Computation*, 221:48–65, 2013.

[44] C. Poignard. Rigorous asymptotics for the elecric field in TM mode at mid-frequency in a bidimensional medium with thin layer. *arXiv preprint math/0607403*, 2006.

[45] C. Poignard. Asymptotics for steady-state voltage potentials in a bidimensional highly contrasted medium with thin layer. *Mathematical Methods in the Applied Sciences*, 31(4):443–479, 2008.

[46] C. Poignard. Approximate transmission conditions through a weakly oscillating thin layer. *Mathematical Methods in the Applied Sciences*, 32(4):435–453, 2009.

[47] C. Qing, D. Pardo, L. Hong-Bin, and W. Fu-Rong. Compensation effect analysis in DIE method for through-casing measuring formation resistivity. *Journal of Applied Geophysics*, 74(4):287–293, 2011.

[48] D. Schillinger, S. J. Hossain, and T. J. Hughes. Reduced Bézier element quadrature rules for quadratic and cubic splines in isogeometric analysis. *Computer Methods in Applied Mechanics and Engineering*, 277:1–45, 2014.

[49] K. Schmidt. *High-order numerical modelling of highly conductive thin sheets*. PhD thesis, ETH Zurich, 2008.

[50] K. Schmidt and A. Chernov. A unified analysis of transmission conditions for thin conducting sheets in the time-harmonic eddy current model. *SIAM Journal on Applied Mathematics*, 73(6):1980–2003, 2013.

[51] K. Schmidt and A. Chernov. Robust transmission conditions of high order for thin conducting sheets in two dimensions. *Magnetics, IEEE Transactions on*, 50(2):41–44, 2014.

[52] K. Schmidt and S. Tordeux. Asymptotic modelling of conductive thin sheets. *Zeitschrift für angewandte Mathematik und Physik*, 61(4):603–626, 2010.

[53] K. Schmidt and S. Tordeux. High order transmission conditions for thin conductive sheets in magneto-quasistatics. *ESAIM: Mathematical Modelling and Numerical Analysis*, 45(06):1115–1140, 2011.

[54] T. Senior and J. L. Volakis. Generalized impedance boundary conditions in scattering. *Proceedings of the IEEE*, 79(10):1413–1420, 1991.

[55] P. Solin, L. Dubcova, J. Cerveny, and I. Dolezel. Adaptive hp-FEM with arbitrary-level hanging nodes for Maxwell's equations. *Adv. Appl. Math. Mech*, 2(4):518–532, 2010.

[56] G. Vial. *Analyse multi-échelle et conditions aux limites approchées pour un problème avec couche mince dans un domaine à coin*. PhD thesis, Rennes 1, 2003.

[57] G. Vial. Efficiency of approximate boundary conditions for corner domains coated with thin layers. *Comptes Rendus Mathematique*, 340(3):215–220, 2005.

[58] J. Zitelli, I. Muga, L. F. Demkowicz, J. Gopalakrishnan, D. Pardo, and V. M. Calo. A class of discontinuous Petrov–Galerkin methods. Part IV: The optimal test norm and time-harmonic wave propagation in 1D. *Journal of Computational Physics*, 230(7):2406–2432, 2011.

RESEARCH CENTRE
BORDEAUX – SUD-OUEST

200 avenue de la Vieille Tour
33405 Talence Cedex

Publisher
Inria
Domaine de Voluceau - Rocquencourt
BP 105 - 78153 Le Chesnay Cedex
inria.fr

ISSN 0249-6399