



HAL
open science

Resolution strategy for the Hybridizable Discontinuous Galerkin system for solving Helmholtz elastic wave equations

Marie Bonnasse-Gahot, Henri Calandra, Julien Diaz, Stephane Lanteri

► **To cite this version:**

Marie Bonnasse-Gahot, Henri Calandra, Julien Diaz, Stephane Lanteri. Resolution strategy for the Hybridizable Discontinuous Galerkin system for solving Helmholtz elastic wave equations. Face to face meeting HPC4E Brazilian-European project, Sep 2016, Gramado, Brazil. hal-01400643

HAL Id: hal-01400643

<https://inria.hal.science/hal-01400643v1>

Submitted on 24 Nov 2016

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Resolution strategy for the Hybridizable Discontinuous Galerkin system for solving Helmholtz elastic wave equations

M. Bonnasse-Gahot^{1,2}, H. Calandra³, J. Diaz¹ and S. Lanteri²

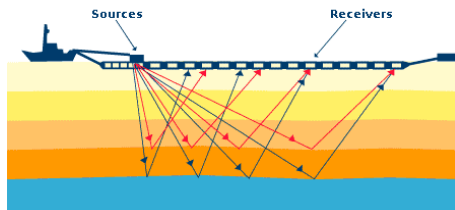
¹ INRIA Bordeaux-Sud-Ouest, team-project Magique 3D

² INRIA Sophia-Antipolis-Méditerranée, team-project Nachos

³ TOTAL Exploration-Production

Motivations

Principles of seismic imaging



Motivations

Examples of seismic imaging campaigns



Motivations

Imaging methods

- ▶ Reverse Time Migration (RTM) : based on the **reversibility of wave equation**
- ▶ Full Wave Inversion (FWI) : **inversion process** requiring to solve **many forward problems**

Motivations

Imaging methods

- ▶ Reverse Time Migration (RTM) : based on the **reversibility of wave equation**
- ▶ Full Wave Inversion (FWI) : **inversion process** requiring to solve **many forward problems**

Seismic imaging : time-domain or harmonic-domain ?

- ▶ **Time-domain** : **imaging condition complicated** but **quite low computational cost**
- ▶ **Harmonic-domain** : **imaging condition simple** but **huge computational cost**

Motivations

Imaging methods

- ▶ Reverse Time Migration (RTM) : based on the **reversibility of wave equation**
- ▶ Full Wave Inversion (FWI) : **inversion process** requiring to solve **many forward problems**

Seismic imaging : time-domain or harmonic-domain ?

- ▶ **Time-domain** : **imaging condition complicated** but **quite low computational cost**
- ▶ **Harmonic-domain** : **imaging condition simple** but **huge computational cost**



Memory usage



Motivations

Resolution of the forward problem of the inversion process

- ▶ Elastic wave propagation in the frequency domain : **Helmholtz equation**

Motivations

Resolution of the forward problem of the inversion process

- ▶ Elastic wave propagation in the frequency domain : **Helmholtz equation**

First order formulation of Helmholtz wave equations

$$\mathbf{x} = (x, y, z) \in \Omega \subset \mathbb{R}^3,$$

$$\begin{cases} i\omega\rho(\mathbf{x})\mathbf{v}(\mathbf{x}) = \nabla \cdot \underline{\underline{\sigma}}(\mathbf{x}) + \mathbf{f}_s(\mathbf{x}) \\ i\omega\underline{\underline{\sigma}}(\mathbf{x}) = \underline{\underline{C}}(\mathbf{x}) \underline{\underline{\varepsilon}}(\mathbf{v}(\mathbf{x})) \end{cases}$$

- ▶ \mathbf{v} : velocity vector
- ▶ $\underline{\underline{\sigma}}$: stress tensor
- ▶ $\underline{\underline{\varepsilon}}$: strain tensor

Motivations

Resolution of the forward problem of the inversion process

- ▶ Elastic wave propagation in the frequency domain : **Helmholtz equation**

First order formulation of Helmholtz wave equations

$$\mathbf{x} = (x, y, z) \in \Omega \subset \mathbb{R}^3,$$

$$\begin{cases} i\omega\rho(\mathbf{x})\mathbf{v}(\mathbf{x}) = \nabla \cdot \underline{\underline{\sigma}}(\mathbf{x}) + \mathbf{f}_s(\mathbf{x}) \\ i\omega\underline{\underline{\sigma}}(\mathbf{x}) = \underline{\underline{C}}(\mathbf{x}) \underline{\underline{\varepsilon}}(\mathbf{v}(\mathbf{x})) \end{cases}$$

- ▶ ρ : mass density
- ▶ $\underline{\underline{C}}$: elasticity tensor
- ▶ \mathbf{f}_s : source term, $\mathbf{f}_s \in L^2(\Omega)$

Approximation methods

Discontinuous Galerkin Methods

- ✓ unstructured tetrahedral meshes
- ✓ combination between FEM and finite volume method (FVM)
- ✓ *hp*-adaptivity
- ✓ easily parallelizable method

Approximation methods

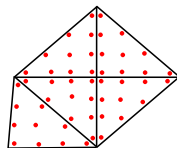
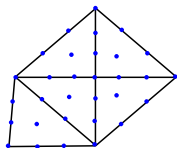
Discontinuous Galerkin Methods

- ✓ unstructured tetrahedral meshes
- ✓ combination between FEM and finite volume method (FVM)
- ✓ *hp*-adaptivity
- ✓ easily parallelizable method
- ✗ ✗ large number of DOF as compared to classical FEM

Approximation methods

Discontinuous Galerkin Methods

- ✓ unstructured tetrahedral meshes
- ✓ combination between FEM and finite volume method (FVM)
- ✓ *hp*-adaptivity
- ✓ easily parallelizable method
- ✗ ✗ large number of DOF as compared to classical FEM



Approximation methods

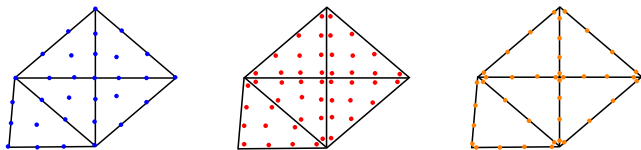
Hybridizable Discontinuous Galerkin Methods

- ✓ same advantages as DG methods : unstructured tetrahedral meshes, *hp*-adaptivity, easily parallelizable method, discontinuous basis functions
- ✓ introduction of a new variable defined only on the interfaces
- ✓ lower number of coupled DOF than classical DG methods

Approximation methods

Hybridizable Discontinuous Galerkin Methods

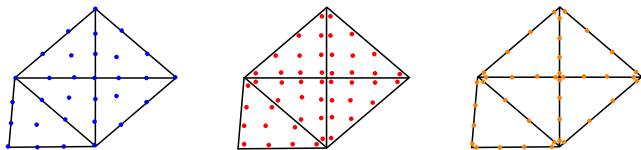
- ✓ same advantages as DG methods : unstructured tetrahedral meshes, *hp*-adaptivity, easily parallelizable method, discontinuous basis functions
- ✓ introduction of a new variable defined only on the interfaces
- ✓ lower number of coupled DOF than classical DG methods







Approximation methods

Hybridizable Discontinuous Galerkin Methods

- ✓ same advantages as DG methods : unstructured tetrahedral meshes, hp -adaptivity, easily parallelizable method, discontinuous basis functions
- ✓ introduction of a new variable defined only on the interfaces
- ✓ lower number of coupled DOF than classical DG methods
- ✗ time-domain increases computational costs



Hybridizable Discontinuous Galerkin method

-  B. Cockburn, J. Gopalakrishnan and R. Lazarov. Unified hybridization of discontinuous Galerkin, mixed and continuous Galerkin methods for second order elliptic problems. *SIAM Journal on Numerical Analysis*, Vol. 47 :1319-1365, 2009.
-  S. Lanteri, L. Li and R. Perrussel. Numerical investigation of a high order hybridizable discontinuous Galerkin method for 2d time-harmonic Maxwell's equations. *COMPEL*, 32(3)1112-1138, 2013.
-  N.C. Nguyen, J. Peraire and B. Cockburn. High-order implicit hybridizable discontinuous Galerkin methods for acoustics and elastodynamics. *Journal of Computational Physics*, 230 :7151-7175, 2011
-  N.C. Nguyen and B. Cockburn. Hybridizable discontinuous Galerkin methods for partial differential equations in continuum mechanics. *Journal of Computational Physics* 231 :5955-5988, 2012

Contents

Hybridizable Discontinuous Galerkin method

Formulation

Algorithm

Numerical results

HDG formulation of the equations

Local HDG formulation

$$\begin{cases} i\omega\rho\mathbf{v} - \nabla \cdot \underline{\underline{\sigma}} = 0 \\ i\omega\underline{\underline{\sigma}} - \underline{\underline{C}}\varepsilon(\mathbf{v}) = 0 \end{cases}$$

HDG formulation of the equations

Local HDG formulation

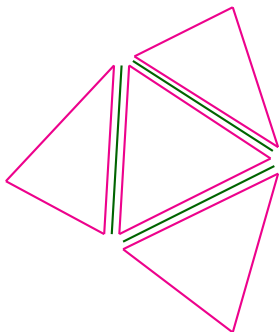
$$\begin{cases} \int_K i\omega \rho^K \mathbf{v}^K \cdot \mathbf{w} + \int_K \underline{\underline{\sigma}}^K : \nabla \mathbf{w} - \int_{\partial K} \widehat{\underline{\underline{\sigma}}}^{\partial K} \cdot \mathbf{n} \cdot \mathbf{w} = 0 \\ \int_K i\omega \underline{\underline{\sigma}}^K : \underline{\underline{\xi}} + \int_K \mathbf{v}^K \cdot \nabla \cdot (\underline{\underline{C}}^K \underline{\underline{\xi}}) - \int_{\partial K} \widehat{\mathbf{v}}^{\partial K} \cdot \underline{\underline{C}}^K \underline{\underline{\xi}} \cdot \mathbf{n} = 0 \end{cases}$$

$\widehat{\underline{\underline{\sigma}}}^K$ and $\widehat{\mathbf{v}}^K$ are numerical traces of $\underline{\underline{\sigma}}^K$ and \mathbf{v}^K respectively on ∂K

HDG formulation of the equations

We define :

$$\hat{\mathbf{v}}^{\partial K} = \lambda^F, \quad \forall F \in \mathcal{F}_h,$$

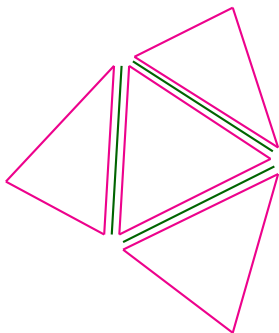


HDG formulation of the equations

We define :

$$\begin{aligned} \hat{\mathbf{v}}^{\partial K} &= \lambda^F, & \forall F \in \mathcal{F}_h, \\ \underline{\hat{\sigma}}^{\partial K} \cdot \mathbf{n} &= \underline{\sigma}^K \cdot \mathbf{n} - \tau \mathbf{l} (\mathbf{v}^K - \lambda^F), & \text{on } \partial K \end{aligned}$$

where τ is the stabilization parameter ($\tau > 0$)



HDG formulation of the equations

Local HDG formulation

$$\left\{ \begin{array}{l} \int_K i\omega \rho^K \mathbf{v}^K \cdot \mathbf{w} - \int_K (\nabla \cdot \underline{\underline{\sigma}}^K) \cdot \mathbf{w} + \int_{\partial K} \tau \mathbf{l} (\mathbf{v}^K - \lambda^F) \cdot \mathbf{w} = 0 \\ \int_K i\omega \underline{\underline{\sigma}}^K : \underline{\underline{\xi}} + \int_K \mathbf{v}^K \cdot \nabla \cdot (\underline{\underline{C}}^K \underline{\underline{\xi}}) - \int_{\partial K} \lambda^F \cdot \underline{\underline{C}}^K \underline{\underline{\xi}} \cdot \mathbf{n} = 0 \end{array} \right.$$

HDG formulation of the equations

Local HDG formulation

$$\begin{cases} \int_K i\omega \rho^K \mathbf{v}^K \cdot \mathbf{w} - \int_K (\nabla \cdot \underline{\underline{\sigma}}^K) \cdot \mathbf{w} + \int_{\partial K} \tau \mathbf{l} (\mathbf{v}^K - \lambda^F) \cdot \mathbf{w} = 0 \\ \int_K i\omega \underline{\underline{\sigma}}^K : \underline{\underline{\xi}} + \int_K \mathbf{v}^K \cdot \nabla \cdot (\underline{\underline{C}}^K \underline{\underline{\xi}}) - \int_{\partial K} \lambda^F \cdot \underline{\underline{C}}^K \underline{\underline{\xi}} \cdot \mathbf{n} = 0 \end{cases}$$

We define :

$$\underline{\underline{W}}^K = \left(\underline{V}_x^K, \underline{V}_y^K, \underline{V}_z^K, \underline{\sigma}_{xx}^K, \underline{\sigma}_{yy}^K, \underline{\sigma}_{zz}^K, \underline{\sigma}_{xy}^K, \underline{\sigma}_{xz}^K, \underline{\sigma}_{yz}^K \right)^T$$

$$\underline{\underline{\Lambda}} = \left(\underline{\Lambda}^{F_1}, \underline{\Lambda}^{F_2}, \dots, \underline{\Lambda}^{F_{n_f}} \right)^T, \text{ where } n_f = \text{card}(\mathcal{F}_h)$$

Discretization of the local HDG formulation

$$\underline{\underline{A}}^K \underline{\underline{W}}^K + \sum_{F \in \partial K} \underline{\underline{C}}^{K,F} \underline{\underline{\Lambda}} = 0$$

HDG formulation of the equations

Local HDG formulation

$$\begin{cases} \int_K i\omega\rho^K \mathbf{v}^K \cdot \mathbf{w} - \int_K (\nabla \cdot \underline{\underline{\sigma}}^K) \cdot \mathbf{w} + \int_{\partial K} \tau \mathbf{l} (\mathbf{v}^K - \lambda^F) \cdot \mathbf{w} = 0 \\ \int_K i\omega \underline{\underline{\sigma}}^K : \underline{\underline{\xi}} + \int_K \mathbf{v}^K \cdot \nabla \cdot (\underline{\underline{C}}^K \underline{\underline{\xi}}) - \int_{\partial K} \lambda^F \cdot \underline{\underline{C}}^K \underline{\underline{\xi}} \cdot \mathbf{n} = 0 \end{cases}$$

We define :

$$\underline{\underline{W}}^K = \left(\underline{\underline{V}}_x^K, \underline{\underline{V}}_y^K, \underline{\underline{V}}_z^K, \underline{\underline{\sigma}}_{xx}^K, \underline{\underline{\sigma}}_{yy}^K, \underline{\underline{\sigma}}_{zz}^K, \underline{\underline{\sigma}}_{xy}^K, \underline{\underline{\sigma}}_{xz}^K, \underline{\underline{\sigma}}_{yz}^K \right)^T$$

$$\underline{\underline{\Lambda}} = \left(\underline{\underline{\Lambda}}^{F_1}, \underline{\underline{\Lambda}}^{F_2}, \dots, \underline{\underline{\Lambda}}^{F_{n_f}} \right)^T, \text{ where } n_f = \text{card}(\mathcal{F}_h)$$

Discretization of the local HDG formulation

$$\mathbf{A}^K \underline{\underline{W}}^K + \mathbf{C}^K \underline{\underline{\Lambda}} = 0$$

HDG formulation of the equations

Transmission condition

In order to determine λ^F , the continuity of the normal component of $\underline{\hat{\underline{\sigma}}}^{\partial K}$ is weakly enforced, rendering this numerical trace conservative :

$$\int_F \llbracket \underline{\hat{\underline{\sigma}}}^{\partial K} \cdot \mathbf{n} \rrbracket \cdot \eta = 0$$

HDG formulation of the equations

Transmission condition

In order to determine λ^F , the continuity of the normal component of $\underline{\underline{\hat{\sigma}}}^{\partial K}$ is weakly enforced, rendering this numerical trace conservative :

$$\int_F [[\underline{\underline{\hat{\sigma}}}^{\partial K} \cdot \mathbf{n}]] \cdot \eta = 0$$

Discretization of the transmission condition

$$\sum_{K \in \mathcal{T}_h} [\mathbb{B}^K \underline{W}^K + \mathbb{L}^K \underline{\Lambda}] = 0$$

HDG formulation of the equations

Global HDG discretization

$$\left\{ \begin{array}{l} \mathbf{A}^K \underline{\mathbf{W}}^K + \mathbf{C}^K \underline{\Lambda} = 0 \\ \sum_{K \in \mathcal{T}_h} [\mathbf{B}^K \underline{\mathbf{W}}^K + \mathbf{L}^K \underline{\Lambda}] = 0 \end{array} \right.$$

HDG formulation of the equations

Global HDG discretization

$$\left\{ \begin{array}{l} \underline{W}^K = -(\underline{A}^K)^{-1} \underline{C}^K \underline{\Lambda} \\ \sum_{K \in \mathcal{T}_h} [\underline{B}^K \underline{W}^K + \underline{L}^K \underline{\Lambda}] = 0 \end{array} \right.$$

HDG formulation of the equations

Global HDG discretization

$$\sum_{K \in \mathcal{T}_h} [-\mathbf{B}^K (\mathbf{A}^K)^{-1} \mathbf{C}^K + \mathbf{L}^K] \underline{\mathbf{\Lambda}} = 0$$

Main steps of the HDG algorithm

1. Construction of the global matrix \mathbf{M}

with
$$\mathbf{M} = \sum_{K \in \mathcal{T}_h} \left[-\mathbf{B}^K (\mathbf{A}^K)^{-1} \mathbf{C}^K + \mathbf{L}^K \right]$$

for $K = 1$ to Nb_{tri} **do**

 Computation of matrices \mathbf{B}^K , $(\mathbf{A}^K)^{-1}$, \mathbf{C}^K and \mathbf{L}^K

 Construction of the corresponding section of \mathbf{M}

end for

Main steps of the HDG algorithm

-
-
1. Construction of the global matrix \mathbf{M}
 2. Construction of the right hand side \mathbf{S}
-

Main steps of the HDG algorithm

-
1. Construction of the global matrix \mathbf{M}
 2. Construction of the right hand side \mathbf{S}
 3. Resolution $\mathbf{M}\underline{\Lambda} = \mathbf{S}$, with a direct solver (MUMPS) or hybrid solver (MaPhys)
-

Main steps of the HDG algorithm

-
-
1. Construction of the global matrix \mathbf{M}
 2. Construction of the right hand side \mathbf{S}
 3. Resolution $\mathbf{M}\underline{\Lambda} = \mathbf{S}$, with a direct solver (MUMPS) or hybrid solver (MaPhys)
 4. Computation of the solutions of the initial problem
-

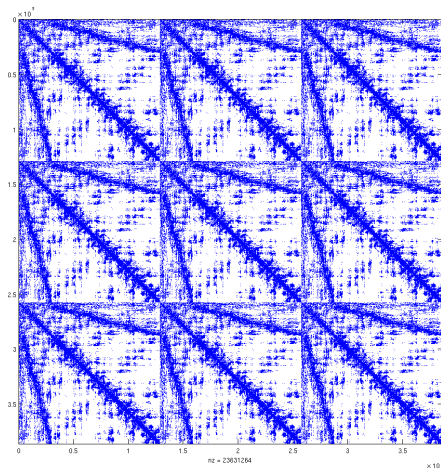
Main steps of the HDG algorithm

-
1. Construction of the global matrix $\underline{\mathbf{M}}$
 2. Construction of the right hand side $\underline{\mathbf{S}}$
 3. Resolution $\underline{\mathbf{M}}\underline{\mathbf{\Lambda}} = \underline{\mathbf{S}}$, with a direct solver (MUMPS) or hybrid solver (MaPhys)
 4. Computation of the solutions of the initial problem

for $K = 1$ to Nb_{tri} **do**
 Compute $\underline{\mathbf{W}}^K = -(\underline{\mathbf{A}}^K)^{-1}\underline{\mathbf{C}}^K\underline{\mathbf{\Lambda}}$
end for

MaPhys Vs MUMPS

Pattern of the HDG global matrix for \mathbb{P}_1 interpolation and for a 3D mesh composed of 21 000 elements

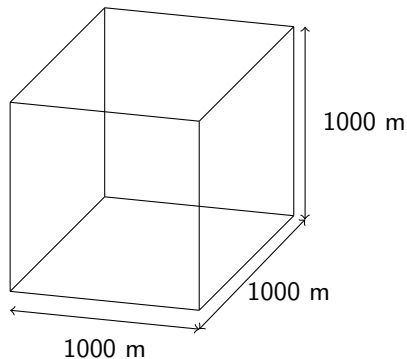


MaPhys Vs MUMPS

Software packages for solving systems of linear equations $Ax = b$, where A is a sparse matrix

- ▶ MUMPS (MULTifrontal Massively Parallel sparse direct Solver) :
 - ▶ Direct factorization $A = LU$ or $A = LDL^T$
 - ▶ Multifrontal approach
- ▶ MaPhys (Massively Parallel Hybrid Solver) :
 - ▶ Direct and iterative methods
 - ▶ non-overlapping algebraic domain decomposition method (Schur complement method)
 - ▶ resolution of each local problem thanks to direct solver such as MUMPS or PaStiX.

3D plane wave in an homogeneous medium



Configuration of the computational domain Ω .

► Physical parameters :

- $\rho = 1 \text{ kg.m}^{-3}$
- $\lambda = 16 \text{ GPa}$
- $\mu = 8 \text{ GPa}$

► Plane wave :

$$u = \nabla e^{i(k_x x + k_y y + k_z z)}$$

where $k_x = \frac{\omega}{v_p} \cos \theta_0 \cos \theta_1$,

$k_y = \frac{\omega}{v_p} \sin \theta_0 \cos \theta_1$, and

$k_z = \frac{\omega}{v_p} \sin \theta_1$

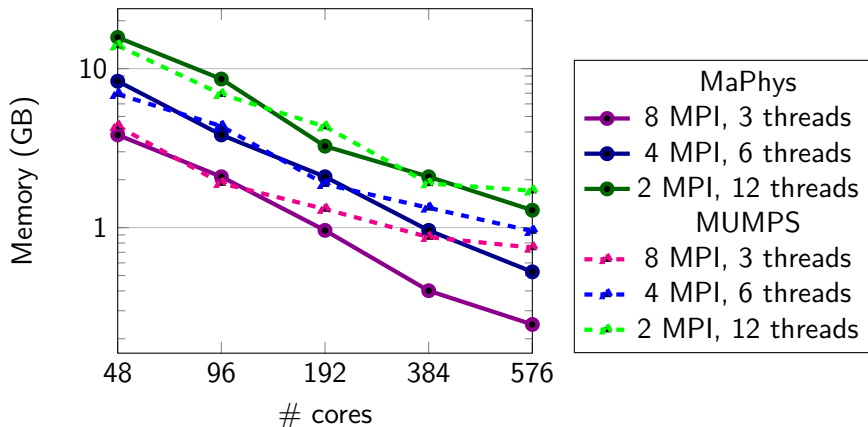
- $\omega = 2\pi f$, $f = 8 \text{ Hz}$
- $\theta_0 = 30^\circ$, $\theta_1 = 0^\circ$
- Mesh composed of 21 000 elements

Cluster configuration

Features of the nodes :

- ▶ 2 Dodeca-core Haswell Intel Xeon E5-2680
- ▶ Frequency : 2,5 GHz
- ▶ RAM : 128 Go
- ▶ Storage : 500 Go
- ▶ Infiniband QDR TrueScale : 40Gb/s
- ▶ Ethernet : 1Gb/s

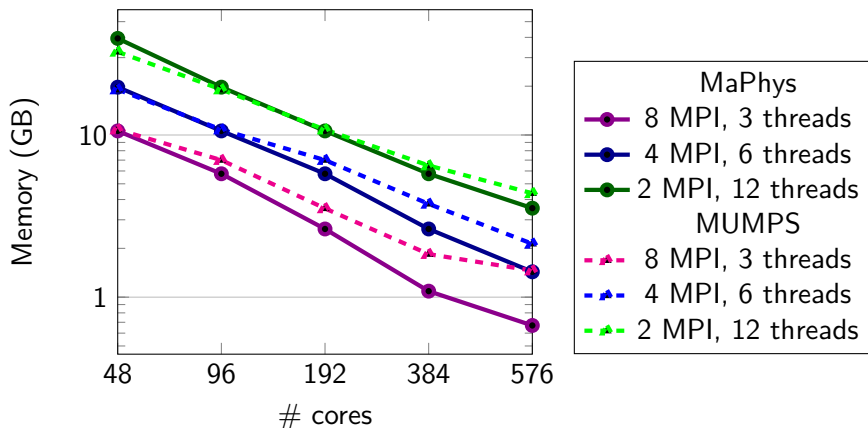
3D Plane wave : Memory consumption

Maximum local memory for HDG- \mathbb{P}_2 method

(matrix order = 772 416, # nz=107 495 424)

3D Plane wave : Memory consumption

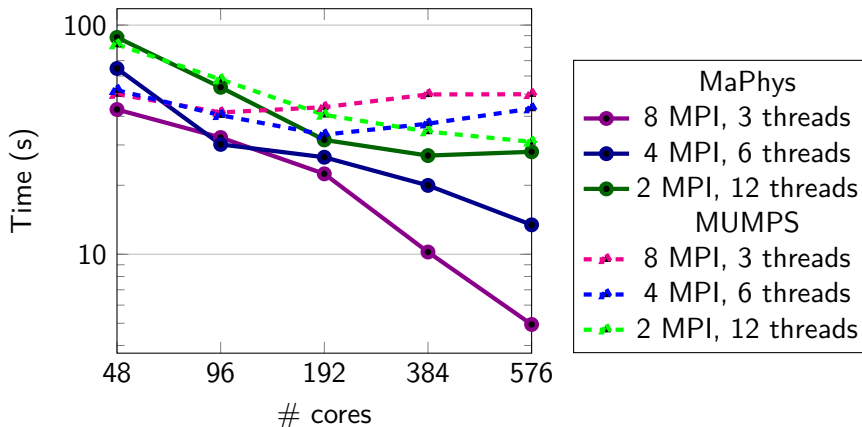
Maximum local memory for HDG- \mathbb{P}_3 method



(matrix order = 1 287 360, # nz=298 598 400)

3D Plane wave : Execution time

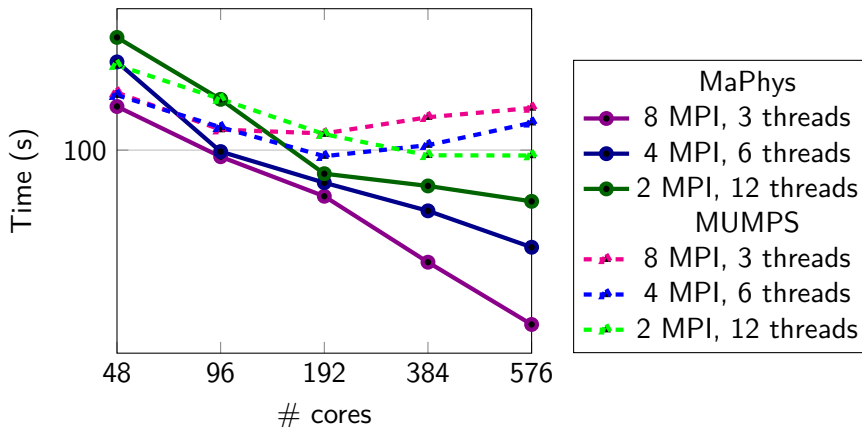
Execution time for the resolution of the HDG- \mathbb{P}_2 system



(matrix order = 772 416, # nz=107 495 424)

3D Plane wave : Execution time

Execution time for the resolution of the HDG- \mathbb{P}_3 system



(matrix order = 1 287 360, # nz=298 598 400)

Conclusion-Perspectives

- ▶ HDG method implemented in Total program (WP6)
- ▶ more detailed analysis of the comparison between MUMPS and MaPhys (WP3)
- ▶ comparison between to PaStiX solver
- ▶ extension to elasto-acoustic case
- ▶ call for projects PRACE to test bigger test-cases

Thank you !

The logo for Inria, featuring the word "inria" in a stylized, cursive font with a color gradient from red to orange. Above the "ria" part, the words "informatiques" and "mathématiques" are written in a smaller, black, sans-serif font, separated by a small red asterisk.

inria
informatiques mathématiques

Factorization time (s) for the HDG- \mathbb{P}_2 system
 (Matrix order = 772 416, # nz = 107 495 424)

	2 nodes		4 nodes		8 nodes		16 nodes
	Maphys	Mumps	Maphys	Mumps	Maphys	Mumps	Maphys
8 MPI/n., 3 t./MPI	21.77	42.55	7.18	35.06	2.62	37.54	1.32
4 MPI/n. 6 t./MPI	42.37	44.66	14.05	33.69	5.28	26.80	2.48
2 MPI/n. 12 t./MPI	70.20	69.48	29.11	49.69	10.79	33.44	4.22