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Resolution strategy for the Hybridizable Discontinuous Galerkin system for solving Helmholtz elastic wave equations

M. Bonnasse-Gahot^{1,2}, H. Calandra³, J. Diaz¹ and S. Lanteri²

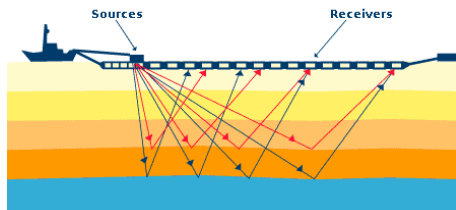
¹ INRIA Bordeaux-Sud-Ouest, team-project Magique 3D

² INRIA Sophia-Antipolis-Méditerranée, team-project Nachos

³ TOTAL Exploration-Production

Motivations

Principles of seismic imaging



Motivations

Examples of seismic imaging campaigns



Motivations

Imaging methods

- ▶ Reverse Time Migration (RTM) : based on the **reversibility of wave equation**
- ▶ Full Wave Inversion (FWI) : **inversion process** requiring to solve **many forward problems**

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Seismic imaging : time-domain or harmonic-domain ?

- ▶ **Time-domain** : **imaging condition complicated** but **quite low computational cost**
- ▶ **Harmonic-domain** : **imaging condition simple** but **huge computational cost**

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Memory usage



Motivations

Resolution of the forward problem of the inversion process

- ▶ Elastic wave propagation in the frequency domain : **Helmholtz equation**

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First order formulation of Helmholtz wave equations

$$\mathbf{x} = (x, y, z) \in \Omega \subset \mathbb{R}^3,$$

$$\begin{cases} i\omega\rho(\mathbf{x})\mathbf{v}(\mathbf{x}) = \nabla \cdot \underline{\underline{\sigma}}(\mathbf{x}) + \mathbf{f}_s(\mathbf{x}) \\ i\omega\underline{\underline{\sigma}}(\mathbf{x}) = \underline{\underline{C}}(\mathbf{x}) \underline{\underline{\varepsilon}}(\mathbf{v}(\mathbf{x})) \end{cases}$$

- ▶ \mathbf{v} : velocity vector
- ▶ $\underline{\underline{\sigma}}$: stress tensor
- ▶ $\underline{\underline{\varepsilon}}$: strain tensor

Motivations

Resolution of the forward problem of the inversion process

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First order formulation of Helmholtz wave equations

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- ▶ ρ : mass density
- ▶ $\underline{\underline{C}}$: elasticity tensor
- ▶ \mathbf{f}_s : source term, $\mathbf{f}_s \in L^2(\Omega)$

Approximation methods

Discontinuous Galerkin Methods

- ✓ unstructured tetrahedral meshes
- ✓ combination between FEM and finite volume method (FVM)
- ✓ *hp*-adaptivity
- ✓ easily parallelizable method

Approximation methods

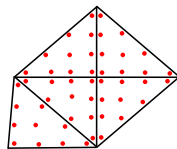
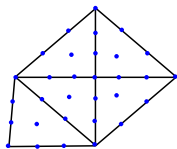
Discontinuous Galerkin Methods

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- ✗ ✗ large number of DOF as compared to classical FEM

Approximation methods

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Approximation methods

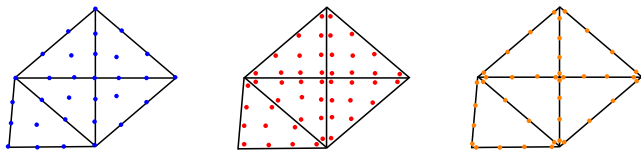
Hybridizable Discontinuous Galerkin Methods

- ✓ same advantages as DG methods : unstructured tetrahedral meshes, *hp*-adaptivity, easily parallelizable method, discontinuous basis functions
- ✓ introduction of a new variable defined only on the interfaces
- ✓ lower number of coupled DOF than classical DG methods

Approximation methods

Hybridizable Discontinuous Galerkin Methods

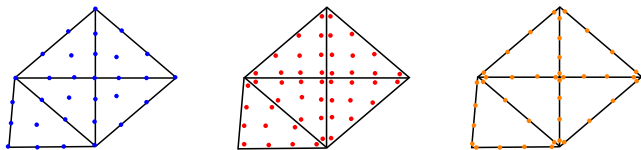
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- ✓ same advantages as DG methods : unstructured tetrahedral meshes, hp -adaptivity, easily parallelizable method, discontinuous basis functions
- ✓ introduction of a new variable defined only on the interfaces
- ✓ lower number of coupled DOF than classical DG methods
- ✗ time-domain increases computational costs



Hybridizable Discontinuous Galerkin method

-  B. Cockburn, J. Gopalakrishnan and R. Lazarov. Unified hybridization of discontinuous Galerkin, mixed and continuous Galerkin methods for second order elliptic problems. *SIAM Journal on Numerical Analysis*, Vol. 47 :1319-1365, 2009.
-  S. Lanteri, L. Li and R. Perrussel. Numerical investigation of a high order hybridizable discontinuous Galerkin method for 2d time-harmonic Maxwell's equations. *COMPEL*, 32(3)1112-1138, 2013.
-  N.C. Nguyen, J. Peraire and B. Cockburn. High-order implicit hybridizable discontinuous Galerkin methods for acoustics and elastodynamics. *Journal of Computational Physics*, 230 :7151-7175, 2011
-  N.C. Nguyen and B. Cockburn. Hybridizable discontinuous Galerkin methods for partial differential equations in continuum mechanics. *Journal of Computational Physics* 231 :5955-5988, 2012

Contents

Hybridizable Discontinuous Galerkin method

Formulation

Algorithm

Numerical results

HDG formulation of the equations

Local HDG formulation

$$\begin{cases} i\omega\rho\mathbf{v} - \nabla \cdot \underline{\underline{\sigma}} = 0 \\ i\omega\underline{\underline{\sigma}} - \underline{\underline{C}}\varepsilon(\mathbf{v}) = 0 \end{cases}$$

HDG formulation of the equations

Local HDG formulation

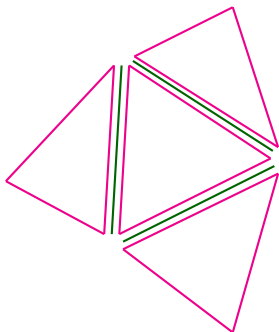
$$\begin{cases} \int_K i\omega \rho^K \mathbf{v}^K \cdot \mathbf{w} + \int_K \underline{\underline{\sigma}}^K : \nabla \mathbf{w} - \int_{\partial K} \widehat{\underline{\underline{\sigma}}}^{\partial K} \cdot \mathbf{n} \cdot \mathbf{w} = 0 \\ \int_K i\omega \underline{\underline{\sigma}}^K : \underline{\underline{\xi}} + \int_K \mathbf{v}^K \cdot \nabla \cdot (\underline{\underline{C}}^K \underline{\underline{\xi}}) - \int_{\partial K} \widehat{\mathbf{v}}^{\partial K} \cdot \underline{\underline{C}}^K \underline{\underline{\xi}} \cdot \mathbf{n} = 0 \end{cases}$$

$\widehat{\underline{\underline{\sigma}}}^K$ and $\widehat{\mathbf{v}}^K$ are numerical traces of $\underline{\underline{\sigma}}^K$ and \mathbf{v}^K respectively on ∂K

HDG formulation of the equations

We define :

$$\hat{v}^{\partial K} = \lambda^F, \quad \forall F \in \mathcal{F}_h,$$

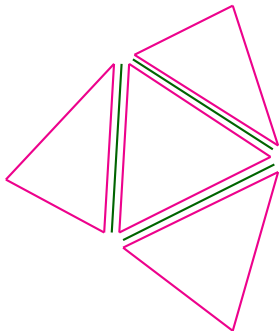


HDG formulation of the equations

We define :

$$\begin{aligned} \hat{\mathbf{v}}^{\partial K} &= \lambda^F, & \forall F \in \mathcal{F}_h, \\ \underline{\hat{\sigma}}^{\partial K} \cdot \mathbf{n} &= \underline{\sigma}^K \cdot \mathbf{n} - \tau \mathbf{l} (\mathbf{v}^K - \lambda^F), & \text{on } \partial K \end{aligned}$$

where τ is the stabilization parameter ($\tau > 0$)



HDG formulation of the equations

Local HDG formulation

$$\left\{ \begin{array}{l} \int_K i\omega \rho^K \mathbf{v}^K \cdot \mathbf{w} - \int_K (\nabla \cdot \underline{\underline{\sigma}}^K) \cdot \mathbf{w} + \int_{\partial K} \tau \mathbf{l} (\mathbf{v}^K - \lambda^F) \cdot \mathbf{w} = 0 \\ \int_K i\omega \underline{\underline{\sigma}}^K : \underline{\underline{\xi}} + \int_K \mathbf{v}^K \cdot \nabla \cdot (\underline{\underline{C}}^K \underline{\underline{\xi}}) - \int_{\partial K} \lambda^F \cdot \underline{\underline{C}}^K \underline{\underline{\xi}} \cdot \mathbf{n} = 0 \end{array} \right.$$

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We define :

$$\underline{\underline{W}}^K = \left(\underline{V}_x^K, \underline{V}_y^K, \underline{V}_z^K, \underline{\sigma}_{xx}^K, \underline{\sigma}_{yy}^K, \underline{\sigma}_{zz}^K, \underline{\sigma}_{xy}^K, \underline{\sigma}_{xz}^K, \underline{\sigma}_{yz}^K \right)^T$$

$$\underline{\underline{\Lambda}} = \left(\underline{\Lambda}^{F_1}, \underline{\Lambda}^{F_2}, \dots, \underline{\Lambda}^{F_{n_f}} \right)^T, \text{ where } n_f = \text{card}(\mathcal{F}_h)$$

Discretization of the local HDG formulation

$$\underline{\underline{A}}^K \underline{\underline{W}}^K + \sum_{F \in \partial K} \underline{\underline{C}}^{K,F} \underline{\underline{\Lambda}} = 0$$

HDG formulation of the equations

Local HDG formulation

$$\begin{cases} \int_K i\omega\rho^K \mathbf{v}^K \cdot \mathbf{w} - \int_K (\nabla \cdot \underline{\underline{\sigma}}^K) \cdot \mathbf{w} + \int_{\partial K} \tau \mathbf{l} (\mathbf{v}^K - \lambda^F) \cdot \mathbf{w} = 0 \\ \int_K i\omega \underline{\underline{\sigma}}^K : \underline{\underline{\xi}} + \int_K \mathbf{v}^K \cdot \nabla \cdot (\underline{\underline{C}}^K \underline{\underline{\xi}}) - \int_{\partial K} \lambda^F \cdot \underline{\underline{C}}^K \underline{\underline{\xi}} \cdot \mathbf{n} = 0 \end{cases}$$

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Discretization of the local HDG formulation

$$\mathbf{A}^K \underline{\underline{W}}^K + \mathbf{C}^K \underline{\underline{\Lambda}} = 0$$

HDG formulation of the equations

Transmission condition

In order to determine λ^F , the continuity of the normal component of $\underline{\hat{\sigma}}^{\partial K}$ is weakly enforced, rendering this numerical trace conservative :

$$\int_F \llbracket \underline{\hat{\sigma}}^{\partial K} \cdot \mathbf{n} \rrbracket \cdot \eta = 0$$

HDG formulation of the equations

Transmission condition

In order to determine λ^F , the continuity of the normal component of $\underline{\underline{\hat{\sigma}}}^{\partial K}$ is weakly enforced, rendering this numerical trace conservative :

$$\int_F [[\underline{\underline{\hat{\sigma}}}^{\partial K} \cdot \mathbf{n}]] \cdot \eta = 0$$

Discretization of the transmission condition

$$\sum_{K \in \mathcal{T}_h} [\mathbb{B}^K \underline{W}^K + \mathbb{L}^K \underline{\Lambda}] = 0$$

HDG formulation of the equations

Global HDG discretization

$$\left\{ \begin{array}{l} \mathbf{A}^K \underline{\mathbf{W}}^K + \mathbf{C}^K \underline{\Lambda} = 0 \\ \sum_{K \in \mathcal{T}_h} [\mathbf{B}^K \underline{\mathbf{W}}^K + \mathbf{L}^K \underline{\Lambda}] = 0 \end{array} \right.$$

HDG formulation of the equations

Global HDG discretization

$$\left\{ \begin{array}{l} \underline{W}^K = -(\underline{A}^K)^{-1} \underline{C}^K \underline{\Lambda} \\ \sum_{K \in \mathcal{T}_h} [\underline{B}^K \underline{W}^K + \underline{L}^K \underline{\Lambda}] = 0 \end{array} \right.$$

HDG formulation of the equations

Global HDG discretization

$$\sum_{K \in \mathcal{T}_h} [-\mathbf{B}^K (\mathbf{A}^K)^{-1} \mathbf{C}^K + \mathbf{L}^K] \underline{\mathbf{\Lambda}} = 0$$

Main steps of the HDG algorithm

1. Construction of the global matrix \mathbf{M}

with
$$\mathbf{M} = \sum_{K \in \mathcal{T}_h} \left[-\mathbf{B}^K (\mathbf{A}^K)^{-1} \mathbf{C}^K + \mathbf{L}^K \right]$$

for $K = 1$ to Nb_{tri} **do**

 Computation of matrices \mathbf{B}^K , $(\mathbf{A}^K)^{-1}$, \mathbf{C}^K and \mathbf{L}^K

 Construction of the corresponding section of \mathbf{M}

end for

Main steps of the HDG algorithm

-
-
1. Construction of the global matrix \mathbf{M}
 2. Construction of the right hand side \mathbf{S}
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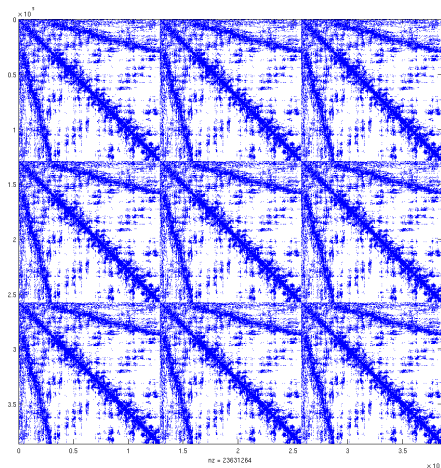
Main steps of the HDG algorithm

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for $K = 1$ to Nb_{tri} **do**
 Compute $\underline{\mathbf{W}}^K = -(\underline{\mathbf{A}}^K)^{-1}\underline{\mathbf{C}}^K\underline{\mathbf{\Lambda}}$
end for

MaPhys Vs MUMPS

Pattern of the HDG global matrix for \mathbb{P}_1 interpolation and for a 3D mesh composed of 21 000 elements

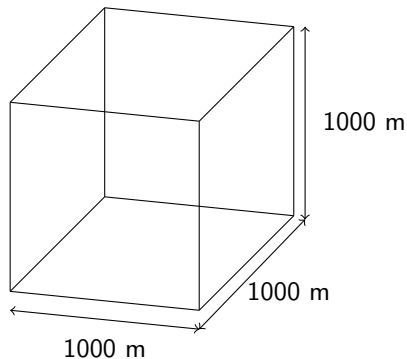


MaPhys Vs MUMPS

Software packages for solving systems of linear equations $Ax = b$, where A is a sparse matrix

- ▶ MUMPS (MULTifrontal Massively Parallel sparse direct Solver) :
 - ▶ Direct factorization $A = LU$ or $A = LDL^T$
 - ▶ Multifrontal approach
- ▶ MaPhys (Massively Parallel Hybrid Solver) :
 - ▶ Direct and iterative methods
 - ▶ non-overlapping algebraic domain decomposition method (Schur complement method)
 - ▶ resolution of each local problem thanks to direct solver such as MUMPS or PaStiX.

3D plane wave in an homogeneous medium



Configuration of the computational domain Ω .

► Physical parameters :

- $\rho = 1 \text{ kg.m}^{-3}$
- $\lambda = 16 \text{ GPa}$
- $\mu = 8 \text{ GPa}$

► Plane wave :

$$u = \nabla e^{i(k_x x + k_y y + k_z z)}$$

where $k_x = \frac{\omega}{v_p} \cos \theta_0 \cos \theta_1$,

$k_y = \frac{\omega}{v_p} \sin \theta_0 \cos \theta_1$, and

$k_z = \frac{\omega}{v_p} \sin \theta_1$

- $\omega = 2\pi f$, $f = 8 \text{ Hz}$
- $\theta_0 = 30^\circ$, $\theta_1 = 0^\circ$
- Mesh composed of 21 000 elements

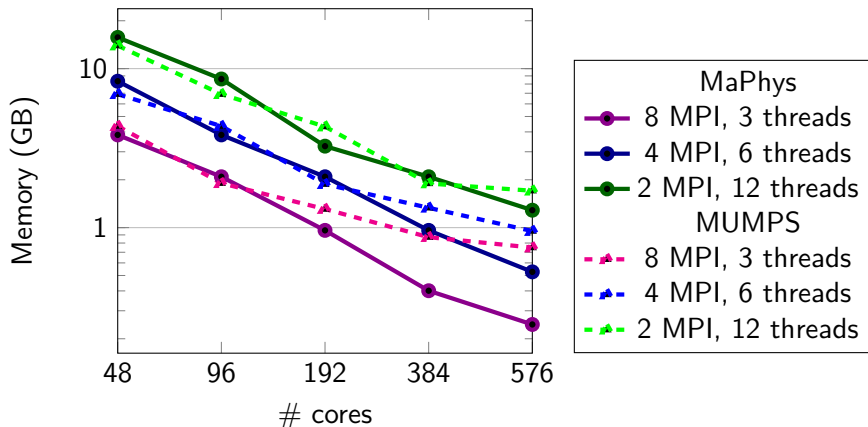
Cluster configuration

Features of the nodes :

- ▶ 2 Dodeca-core Haswell Intel Xeon E5-2680
- ▶ Frequency : 2,5 GHz
- ▶ RAM : 128 Go
- ▶ Storage : 500 Go
- ▶ Infiniband QDR TrueScale : 40Gb/s
- ▶ Ethernet : 1Gb/s

3D Plane wave : Memory consumption

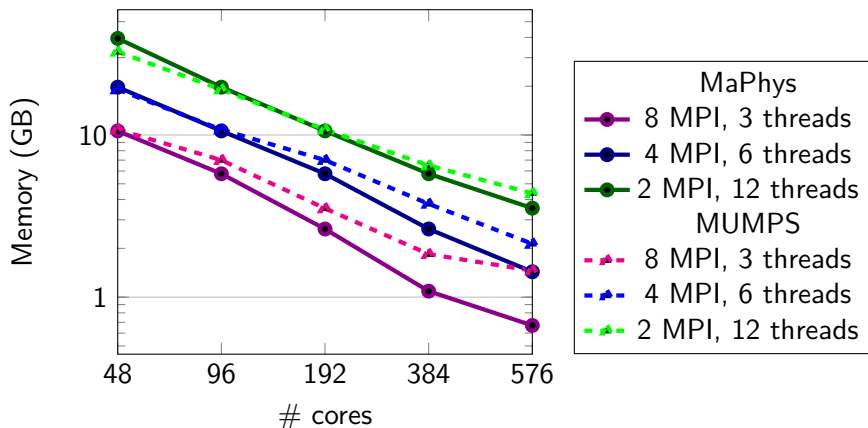
Maximum local memory for HDG- \mathbb{P}_2 method



(matrix order = 772 416, # nz=107 495 424)

3D Plane wave : Memory consumption

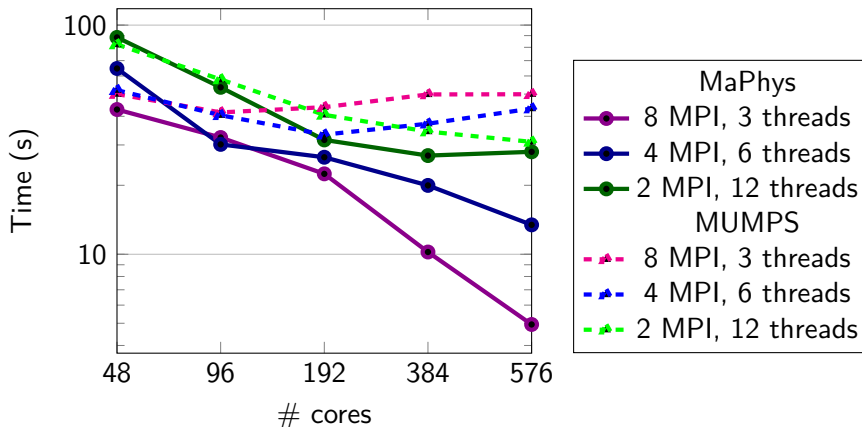
Maximum local memory for HDG- \mathbb{P}_3 method



(matrix order = 1 287 360, # nz=298 598 400)

3D Plane wave : Execution time

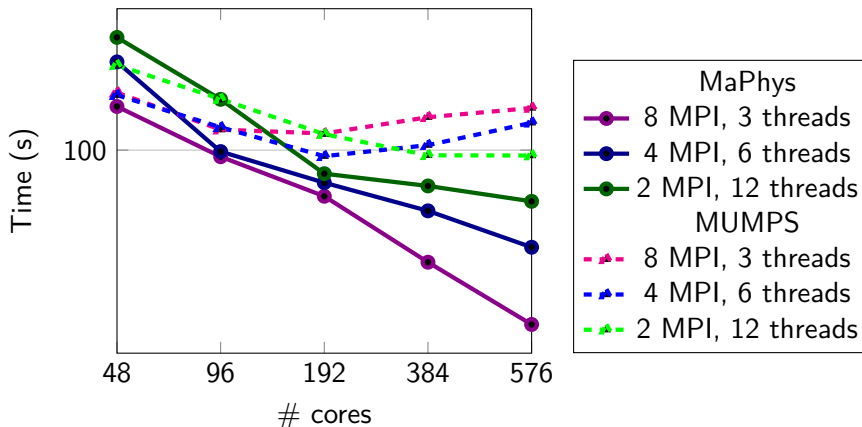
Execution time for the resolution of the HDG- \mathbb{P}_2 system



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3D Plane wave : Execution time

Execution time for the resolution of the HDG- \mathbb{P}_3 system



(matrix order = 1 287 360, # nz=298 598 400)

Conclusion-Perspectives

- ▶ HDG method implemented in Total program (WP6)
- ▶ more detailed analysis of the comparison between MUMPS and MaPhys (WP3)
- ▶ comparison between to PaStiX solver
- ▶ extension to elasto-acoustic case
- ▶ call for projects PRACE to test bigger test-cases

Thank you !

The logo for Inria, featuring the word "inria" in a stylized, cursive font with a color gradient from red to orange. Above the "ria" part, the words "informatiques" and "mathématiques" are written in a smaller, black, sans-serif font, separated by a small red asterisk.

inria
informatiques mathématiques

Factorization time (s) for the HDG- \mathbb{P}_2 system
 (Matrix order = 772 416, # nz = 107 495 424)

	2 nodes		4 nodes		8 nodes		16 nodes
	Maphys	Mumps	Maphys	Mumps	Maphys	Mumps	Maphys
8 MPI/n., 3 t./MPI	21.77	42.55	7.18	35.06	2.62	37.54	1.32
4 MPI/n. 6 t./MPI	42.37	44.66	14.05	33.69	5.28	26.80	2.48
2 MPI/n. 12 t./MPI	70.20	69.48	29.11	49.69	10.79	33.44	4.22