Resolution strategy for the Hybridizable Discontinuous Galerkin system for solving Helmholtz elastic wave equations

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Motivations

Principles of seismic imaging
Motivations

Examples of seismic imaging campaigns
Motivations

Imaging methods

- Reverse Time Migration (RTM) : based on the **reversibility of wave equation**
- Full Wave Inversion (FWI) : **inversion process** requiring to solve many forward problems
Motivations

Imaging methods

- Reverse Time Migration (RTM) : based on the reversibility of wave equation
- Full Wave Inversion (FWI) : inversion process requiring to solve many forward problems

Seismic imaging : time-domain or harmonic-domain?

- Time-domain : imaging condition complicated but quite low computational cost
- Harmonic-domain : imaging condition simple but huge computational cost
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- Full Wave Inversion (FWI): inversion process requiring to solve many forward problems

Seismic imaging: time-domain or harmonic-domain?

- **Time-domain**: imaging condition complicated but quite low computational cost
- **Harmonic-domain**: imaging condition simple but huge computational cost

⚠️ Memory usage ⚠️
Motivations

Resolution of the forward problem of the inversion process

- Elastic wave propagation in the frequency domain: Helmholtz equation
Motivations

Resolution of the forward problem of the inversion process

- Elastic wave propagation in the frequency domain: Helmholz equation

First order formulation of Helmholz wave equations

\[ x = (x, y, z) \in \Omega \subset \mathbb{R}^3, \]

\[
\begin{aligned}
    i\omega \rho(x)v(x) &= \nabla \cdot \sigma(x) + f_s(x) \\
    i\omega \sigma(x) &= C(x) \varepsilon(v(x))
\end{aligned}
\]

- \( v \): velocity vector
- \( \sigma \): stress tensor
- \( \varepsilon \): strain tensor
Motivations

Resolution of the forward problem of the inversion process

▶ Elastic wave propagation in the frequency domain: Helmholtz equation

First order formulation of Helmholtz wave equations

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\begin{align*}
  i\omega \rho(\mathbf{x}) \mathbf{v}(\mathbf{x}) &= \nabla \cdot \mathbf{\sigma}(\mathbf{x}) + f_s(\mathbf{x}) \\
  i\omega \mathbf{\sigma}(\mathbf{x}) &= \mathbf{C}(\mathbf{x}) \varepsilon(\mathbf{v}(\mathbf{x}))
\end{align*}
\]

▶ \( \rho \): mass density

▶ \( \mathbf{C} \): elasticity tensor

▶ \( f_s \): source term, \( f_s \in L^2(\Omega) \)
Approximation methods

Discontinuous Galerkin Methods

✓ unstructured tetrahedral meshes
✓ combination between FEM and finite volume method (FVM)
✓ $hp$-adaptivity
✓ easily parallelizable method
Approximation methods

Discontinuous Galerkin Methods

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- **large number of DOF as compared to classical FEM**
Approximation methods

Discontinuous Galerkin Methods

- unstructured tetrahedral meshes
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- \( \times \times \) large number of DOF as compared to classical FEM
Approximation methods

Hybridizable Discontinuous Galerkin Methods

✅ same advantages as DG methods: unstructured tetrahedral meshes, $hp$-adaptivity, easily parallelizable method, discontinuous basis functions

✅ introduction of a new variable defined only on the interfaces

✅ lower number of coupled DOF than classical DG methods
Approximation methods

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Hybridizable Discontinuous Galerkin Methods

- Same advantages as DG methods: unstructured tetrahedral meshes, $hp$-adaptivity, easily parallelizable method, discontinuous basis functions
- Introduction of a new variable defined only on the interfaces
- Lower number of coupled DOF than classical DG methods
- Time-domain increases computational costs
Hybridizable Discontinuous Galerkin method


Contents

Hybridizable Discontinuous Galerkin method
  Formulation
  Algorithm

Numerical results
HDG formulation of the equations

Local HDG formulation

\[
\begin{align*}
    i\omega \rho \mathbf{v} - \nabla \cdot \sigma &= 0 \\
    i\omega \sigma - C\varepsilon(\mathbf{v}) &= 0
\end{align*}
\]
HDG formulation of the equations

Local HDG formulation

\[
\begin{align*}
\int_K i\omega \rho^K \mathbf{v}^K \cdot \mathbf{w} + \int_K \sigma^K : \nabla \mathbf{w} - \int_{\partial K} \hat{\sigma} \partial K \cdot \mathbf{n} \cdot \mathbf{w} &= 0 \\
\int_K i\omega \sigma^K : \xi + \int_K \mathbf{v}^K \cdot \nabla \left( C^K \xi \right) - \int_{\partial K} \hat{\mathbf{v}} \partial K \cdot C^K \xi \cdot \mathbf{n} &= 0
\end{align*}
\]

\(\hat{\sigma}^K\) and \(\hat{\mathbf{v}}^K\) are numerical traces of \(\sigma^K\) and \(\mathbf{v}^K\) respectively on \(\partial K\).
We define:

\[ \tilde{v} \partial K = \lambda^F, \quad \forall F \in \mathcal{F}_h, \]
HDG formulation of the equations

We define:

\[ \hat{v} \partial K = \lambda^F, \quad \forall F \in \mathcal{F}_h, \]
\[ \hat{\sigma} \partial K \cdot n = \sigma^K \cdot n - \tau I (v^K - \lambda^F), \quad \text{on } \partial K \]

where \( \tau \) is the stabilization parameter \((\tau > 0)\)
HDG formulation of the equations

Local HDG formulation

\[
\begin{align*}
\forall K & \int_K i \omega \rho^K v^K \cdot w - \int_K (\nabla \cdot \sigma^K) \cdot w + \int_{\partial K} \tau I (v^K - \lambda F) \cdot w = 0 \\
\forall K & \int_K i \omega \sigma^K : \xi + \int_K v^K \cdot \nabla \left( C^K \xi \right) - \int_{\partial K} \lambda F \cdot C^K \xi \cdot n = 0
\end{align*}
\]
HDG formulation of the equations

Local HDG formulation

\[
\begin{cases}
\int_K i\omega \rho^K \mathbf{v}^K \cdot \mathbf{w} - \int_K (\nabla \cdot \sigma^K) \cdot \mathbf{w} + \int_{\partial K} \tau I (\mathbf{v}^K - \lambda^F) \cdot \mathbf{w} = 0 \\
\int_K i\omega \sigma^K : \xi + \int_K \mathbf{v}^K \cdot \nabla \cdot (C^K \xi) - \int_{\partial K} \lambda^F \cdot C^K \xi \cdot n = 0
\end{cases}
\]

We define:

\[ W^K = \left( V^K_x, V^K_y, V^K_z, \sigma^K_{xx}, \sigma^K_{yy}, \sigma^K_{zz}, \sigma^K_{xy}, \sigma^K_{xz}, \sigma^K_{yz} \right)^T \]

\[ \Lambda = \left( \Lambda^F_1, \Lambda^F_2, \ldots, \Lambda^F_{n_f} \right)^T \]

where \( n_f = \text{card}(\mathcal{F}_h) \)

Discretization of the local HDG formulation

\[ \Lambda^K W^K + \sum_{F \in \partial K} C^K,F \Lambda = 0 \]
HDG formulation of the equations

Local HDG formulation

\[
\begin{cases}
\int_K i \omega \rho^K \mathbf{v}^K \cdot \mathbf{w} - \int_K (\nabla \cdot \sigma^K) \cdot \mathbf{w} + \int_{\partial K} \tau I (\mathbf{v}^K - \lambda^F) \cdot \mathbf{w} = 0 \\
\int_K i \omega \sigma^K : \xi + \int_K \mathbf{v}^K \cdot \nabla \cdot (C^K \xi) - \int_{\partial K} \lambda^F \cdot C^K \xi \cdot \mathbf{n} = 0
\end{cases}
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\[
W^K = \left( V^K_x, V^K_y, V^K_z, \sigma^K_{xx}, \sigma^K_{yy}, \sigma^K_{zz}, \sigma^K_{xy}, \sigma^K_{xz}, \sigma^K_{yz} \right)^T
\]
\[
\Lambda = (\Lambda^{F_1}, \Lambda^{F_2}, ..., \Lambda^{F_{nf}})^T, \text{ where } n_f = \text{card}(\mathcal{F}_h)
\]

Discretization of the local HDG formulation

\[
\Lambda^K W^K + C^K \Lambda = 0
\]
HDG formulation of the equations

Transmission condition

In order to determine $\lambda^F$, the continuity of the normal component of $\hat{\sigma} \partial K$ is weakly enforced, rendering this numerical trace conservative:

$$\int_F \left[ \hat{\sigma} \partial K \cdot n \right] \cdot \eta = 0$$
Transmission condition

In order to determine $\lambda^F$, the continuity of the normal component of $\hat{\sigma} \partial K$ is weakly enforced, rendering this numerical trace conservative:

$$\int_F \left[ \hat{\sigma} \partial K \cdot \mathbf{n} \right] \cdot \eta = 0$$

Discretization of the transmission condition

$$\sum_{K \in \mathcal{T}_h} \left[ B^K W^K + L^K \Lambda \right] = 0$$
HDG formulation of the equations

Global HDG discretization

\[
\begin{align*}
A^K \mathbf{w}^K + C^K \Lambda &= 0 \\
\sum_{K \in \mathcal{T}_h} \left( B^K \mathbf{w}^K + L^K \Lambda \right) &= 0
\end{align*}
\]
HDG formulation of the equations

Global HDG discretization

\[
\begin{aligned}
\mathbf{W}^K &= - (\mathbf{A}^K)^{-1} \mathbf{C}^K \Lambda \\
\sum_{K \in \mathcal{T}_h} \left[ \mathbf{B}^K \mathbf{W}^K + \mathbf{L}^K \Lambda \right] &= 0
\end{aligned}
\]
HDG formulation of the equations

Global HDG discretization

$$\sum_{K \in T_h} \left[ -B^K (A^K)^{-1} C^K + L^K \right] \Lambda = 0$$
Main steps of the HDG algorithm

1. Construction of the global matrix $\mathbf{M}$
   with $\mathbf{M} = \sum_{K \in T_h} \left[ -\mathbf{B}^K (\mathbf{A}^K)^{-1} \mathbf{C}^K + \mathbf{L}^K \right]$

   for $K = 1$ to $Nb_{tri}$ do
     Computation of matrices $\mathbf{B}^K, (\mathbf{A}^K)^{-1}, \mathbf{C}^K$ and $\mathbf{L}^K$
     Construction of the corresponding section of $\mathbf{M}$
   end for
Main steps of the HDG algorithm

1. Construction of the global matrix $M$
2. Construction of the right hand side $S$
Main steps of the HDG algorithm

1. Construction of the global matrix $\mathbf{M}$
2. Construction of the right hand side $\mathbf{S}$
3. Resolution $\mathbf{M} \Lambda = \mathbf{S}$, with a direct solver (MUMPS) or hybrid solver (MaPhys)
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4. Computation of the solutions of the initial problem
Main steps of the HDG algorithm

1. Construction of the global matrix $\mathbf{M}$
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3. Resolution $\mathbf{M}\Lambda = \mathbf{S}$, with a direct solver (MUMPS) or hybrid solver (MaPhys)
4. Computation of the solutions of the initial problem

\begin{verbatim}
for $K = 1$ to $Nb_{tri}$ do
    Compute $\mathbf{W}^K = - (\mathbf{A}^K)^{-1} \mathbf{C}^K \Lambda$
end for
\end{verbatim}
MaPhys Vs MUMPS

Pattern of the HDG global matrix for $\mathbb{P}_1$ interpolation and for a 3D mesh composed of 21,000 elements
MaPhys Vs MUMPS

Software packages for solving systems of linear equations $Ax = b$, where $A$ is a sparse matrix

- **MUMPS** (MUltifrontal Massively Parallel sparse direct Solver):
  - Direct factorization $A = LU$ or $A = LDL^T$
  - Multifrontal approach

- **MaPhys** (Massively Parallel Hybrid Solver):
  - Direct and iterative methods
  - non-overlapping algebraic domain decomposition method (Schur complement method)
  - resolution of each local problem thanks to direct solver such as MUMPS or PaStiX.
3D plane wave in an homogeneous medium

Physical parameters:
- $\rho = 1 \text{ kg.m}^{-3}$
- $\lambda = 16 \text{ GPa}$
- $\mu = 8 \text{ GPa}$

Plane wave:
$$u = \nabla e^{i(k_x x + k_y y + k_z z)}$$

where
$$k_x = \frac{\omega}{v_p} \cos \theta_0 \cos \theta_1,$$
$$k_y = \frac{\omega}{v_p} \sin \theta_0 \cos \theta_1,$$
$$k_z = \frac{\omega}{v_p} \sin \theta_1,$$

$\omega = 2\pi f, f = 8 \text{ Hz}$
$\theta_0 = 30^\circ, \theta_1 = 0^\circ$

Mesh composed of 21,000 elements
Cluster configuration

Features of the nodes :

- 2 Dodeca-core Haswell Intel Xeon E5-2680
- Frequency : 2,5 GHz
- RAM : 128 Go
- Storage : 500 Go
- Infiniband QDR TrueScale : 40Gb/s
- Ethernet : 1Gb/s
3D Plane wave : Memory consumption

Maximum local memory for HDG-$\mathbb{P}_2$ method

(matrix order = 772 416, $\#$ nz=107 495 424)
3D Plane wave : Memory consumption

Maximum local memory for HDG-${\mathbb{P}}_3$ method

(matrix order = 1 287 360, $\#$ nz=298 598 400 )
3D Plane wave : Execution time

Execution time for the resolution of the HDG-$\mathbb{P}_2$ system

(matrix order = 772 416, $\#$ nz=107 495 424 )
3D Plane wave : Execution time

Execution time for the resolution of the HDG-$P_3$ system

(matrix order = 1 287 360, # nz=298 598 400)

MaPhys
- 8 MPI, 3 threads
- 4 MPI, 6 threads
- 2 MPI, 12 threads

MUMPS
- 8 MPI, 3 threads
- 4 MPI, 6 threads
- 2 MPI, 12 threads
Conclusion-Perspectives

- HDG method implemented in Total program (WP6)
- more detailed analysis of the comparison between MUMPS and MaPhys (WP3)
- comparison between to PaStiX solver
- extension to elasto-acoustic case
- call for projects PRACE to test bigger test-cases
Thank you!
# Factorization time (s) for the HDG-$P_2$ system

(Matrix order = 772 416, $\#$ nz = 107 495 424)

<table>
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<th>4 nodes</th>
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<th>8 nodes</th>
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<tr>
<td>8 MPI/n., 3 t./MPI</td>
<td>21.77</td>
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