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Scheduling Series-Parallel Task Graphs to Minimize Peak Memory

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Abstract: We consider a variant of the well-known, NP-complete problem of minimum cut linear arrangement for directed acyclic graphs. In this variant, we are given a directed acyclic graph and asked to find a topological ordering such that the maximum number of cut edges at any point in this ordering is minimum. In our main variant the vertices and edges have weights, and the aim is to minimize the maximum weight of cut edges in addition to the weight of the last vertex before the cut. There is a known, polynomial time algorithm [Liu, SIAM J. Algebra. Discr., 1987] for the cases where the input graph is a rooted tree. We focus on the variant where the input graph is a directed series-parallel graph, and propose a polynomial time algorithm. Directed acyclic graphs are used to model scientific applications where the vertices correspond to the tasks of a given application and the edges represent the dependencies between the tasks. In such models, the problem we address reads as minimizing the peak memory requirement in an execution of the application. Our work, combined with Liu's work on rooted trees addresses this practical problem in two important classes of applications.

Key-words: series-parallel graphs; scheduling; peak memory minimization.

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Ordonnancement de graphes séries-parallèles pour minimiser le pic mémoire

Résumé : Nous considérons dans ce rapport une variante d'un problème classique et NP-complet: la coupe minimale de l'arrangement linéaire pour les graphes acycliques dirigés. Dans cette variante, étant donné un tel graphe, nous cherchons un ordre topologique qui minimise le nombre maximum d'arêtes coupées à tout instant. Plus précisément, nous considérons que les sommets et les arêtes sont munis de poids, et que le but est de minimiser le poids maximum des arêtes coupées en plus du poids du dernier sommet avant la coupe. Il existe un algorithme polynomial dans le cas où le graphe considéré est un arbre enraciné et que les arêtes sont dirigées vers la racine [Liu, SIAM J. Algebra. Discr., 1987]. Nous nous concentrons ici sur les graphes séries-parallèles et proposons un algorithme polynomial pour résoudre ce problème. Les graphes acycliques dirigés sont souvent utilisés pour modéliser les applications scientifiques: les sommets correspondent aux tâches d'une application et les arêtes représentent les dépendances entre ces tâches. Dans ce modèle, le problème que nous étudions correspond à la minimisation du pic mémoire requis pour l'exécution d'une telle application. Notre travail, combiné à celui de Liu, permet de résoudre ce problème pour deux classes importantes d'applications.

Mots-clés : graphes séries-parallèles; ordonnancement; minimisation du pic mémoire.

1 Introduction

Scientific applications are often modeled as task graphs, where vertices represent tasks and edges represent the dependencies between tasks. In these models, each task consumes one or more *input file/data*, and produces one or more *output file/data*. To be executed, a task also requires an *execution file/data*. As the size of the data to be processed increases, minimizing the peak memory of an application arises as an important objective. This is so for the following reason. Modern computing platforms exhibit a complex memory hierarchy ranging from caches to RAM and disks and even sometimes tape storage, with the classical property that the smaller the memory, the faster. Thus, to avoid large makespan one must keep the application's peak memory low, so as to avoid using low bandwidth memory as much as possible. We consider the execution of a task graph on a single compute resource, i.e., a single processor and a single memory hierarchy. An execution of the application is defined by a traversal the graph, that is, a schedule of the vertices which respects the dependencies (or topological order). The peak memory of a traversal is the maximum amount of memory needed to store application files/data at any given time throughout the application execution. The question we study in this work is: How should the task graph be traversed so as to minimize the peak memory? This can be seen as a variant of the minimum cut linear arrangement (MCLA) problem on directed acyclic graphs and is NP-complete [27]. Our focus is on a certain class of applications whose graphs are series-parallel; these applications have already receive some attention in the scheduling literature [7, 10, 21]. We propose a polynomial time algorithm for minimizing the peak memory of this class of task graphs on serial execution environments.

The problem of minimizing the peak memory is also important for parallel execution. While there is an abundant literature on parallel task graph scheduling (see [16] for a survey) where the objective is to minimize the total completion time, or *makespan*, there is little work in minimizing the peak memory. Although our focus in this paper is on the sequential execution case, these schedules can apply to parallel jobs, where each vertex of the task graph now is a parallel task. Also, our study establishes a theoretical ground needed to study parallel task graph scheduling mixing the makespan and the peak memory objectives in distributed memory systems (as we did for tree-shaped task graphs in previous work [9]).

Liu [17, 18] investigates the peak memory minimization problem for trees, where the traversal is constrained to be a post-order [17] or any topological order [18]. In the more general case, Liu makes use of a previous work by Yannakakis [28]. We, in return, are inspired by Liu's work [18] in designing the proposed algorithm for series-parallel graphs (SP-graphs). This way, we aim to contribute to the theoretical body of the known results for efficiently solvable instances of the peak memory problem.

This paper is organized as follows. We briefly discuss previous work in Section 2. We present the problem formally in Section 2.1. We then describe the existing optimal algorithm for trees in Section 3, since this algorithm forms the basis of our proposed algorithms. In the same section, we present the principle of our algorithms on a subclass of SP-graphs. Then, Section 4 presents the proposed algorithm for general SP-graphs as well as the new notion on min-cut optimality needed to prove its correctness. Some of the arguments in our proofs are very lengthy and involved. For the sake of readability, the most demanding proofs are detailed in Appendix.

2 Problem definition and related work

2.1 Peak memory minimization: Model and objective

We define the problem in general directed acyclic graphs (DAGs) with vertex and edge weights modeling applications. In this model, a DAG $G = (V, E, w_n, w_e)$ contains a vertex for each task of the application and a directed edge (p, q) between two vertices if the task corresponding to the vertex q needs a data produced by the task corresponding to the vertex p . Each task in the graph may have several input data, some execution data (or program), and several output data. We denote by $w_e(p, q) \geq 0$ the weight of edge (p, q) which represents the size of the data produced by task p for task q , and by $w_n(p) \geq 0$ the weight of a vertex p which is the size of the execution data of task p .

During the execution of a task (vertex) p , the memory must contain its input data, the execution data, and its output data. The memory needed for executing p is thus:

$$\left(\sum_{(r,p) \in E} w_e(r,p) \right) + w_n(p) + \left(\sum_{(p,q) \in E} w_e(p,q) \right)$$

After p has been processed, its input and execution data are discarded, while its output data are kept in memory until they are consumed at the end of the execution of the corresponding tasks. For example, in the graph depicted in Figure 1a, if task A is processed first, 8 units of memory are needed for its processing, but only 4 remains in the memory after its completion. If task C is processed right after task A , 7 units of memory are needed during its processing (4 to store the data produced by A for B , and 3 for the execution and output data of C).

While processing the task graph, memory consumption changes as data are created and deleted. Our objective is to minimize the *peak memory*, i.e., the maximum amount of memory used in the graph traversal.

More formally, we define a schedule of the task graph as a total order π on the vertices, denoted with \leq_π , such that $p \leq_\pi q$ means that vertex p is scheduled before vertex q . The precedence constraints of the task graphs impose that this is a topological order: we have $p \leq_\pi q$ for any edge $(p, q) \in E$. We use the notation \max^π (resp. \min^π) to express the maximum (resp. minimum) according to the order π : $\min^\pi G$ is for instance the first vertex scheduled by π . A schedule is also represented as a list of vertices: $\langle 1, 2, 3 \rangle$ denotes the schedule π such that $1 \leq_\pi 2 \leq_\pi 3$.

We define $\mu(p, \pi)$ as the memory required during the execution of a vertex $p \in V$ under the schedule π as

$$\mu(p, \pi) = w_n(p) + \sum_{q \leq_\pi p \leq_\pi r} \{w_e(q, r) : (q, r) \in E\}.$$

Note that the edges (q, r) such that $q \leq_\pi p \leq_\pi r$ correspond to the data that have been created but not yet consume while p is being processed. The objective is to find a schedule π for a graph G that minimizes the *peak memory*, defined as

$$\mu(\pi) = \max_{p \in V} \mu(p, \pi).$$

Given a directed graph $G = (V, E, w_n, w_e)$, we define the *reverse graph* $\bar{G} = (V, \bar{E}, w_n, w_e)$ where the orientation of all edges is changed: $\bar{E} = \{(j, i), (i, j) \in E\}$. Note that vertex and edge weights are kept unchanged. Given a schedule π of G , we may build the reverse schedule $\bar{\pi}$ such that $q \leq_{\bar{\pi}} p$ whenever $p \leq_\pi q$. It is straightforward to check that both schedules have the same peak memory on their respective graph: $\mu_G(\pi) = \mu_{\bar{G}}(\bar{\pi})$.

In this paper, we concentrate on series-parallel graphs, which are defined as follows (see for example [8]).

Definition 1. A two-terminal series-parallel graph, or SP-graph, G with terminals s and t is recursively defined to be either:

Base case: A graph with two vertices s and t , and an edge (s, t) .

Series composition: The series composition of two SP-graphs G_1 with terminals s_1, t_1 and G_2 with terminals s_2, t_2 formed by identifying $s = s_1$, $t = t_2$ and $t_1 = s_2$, and denoted by $\langle G_1, G_2 \rangle$;

Parallel composition: The parallel composition of two SP-graphs G_1 with terminals s_1, t_1 and G_2 with terminals s_2, t_2 formed by identifying $s = s_1 = s_2$ and $t = t_1 = t_2$, and denoted by $\{G_1, G_2\}$.

The vertices s and t are called source and target of the graph.

When the graph is defined this way, the dependencies are from the source vertices to the target vertices. Series-parallel graphs can be recognized and decomposed into a tree of series and parallel combination in linear time [26].

While solving the peak memory minimization problem for sp-graphs, we will need a solution for a sub-family of sp-graphs, which are called fork-joins defined below.

Definition 2. A chain is a two-terminal series-parallel graph obtained without using any parallel composition. A fork-join graph is a two-terminal series-parallel graph obtained by the unique parallel composition of a number of chains. These chains are called the branches of the fork-join graph.

A sample fork-join graph is shown in Fig. 2.

2.2 Related work

There have been several studies on graph traversals for minimizing the peak memory in the domain of numerical linear algebra, where the focus is on tree-shaped task graphs. The elimination tree is a graph theoretical model that represents the storage requirements, and computational dependencies and requirements in Cholesky and LU factorization of sparse matrices. Liu [19] describes how such trees are built, and how the multifrontal method organizes the computations according to elimination trees. In this context, Liu [17] discusses how to find a memory-minimizing postorder traversal when the traversal is required to correspond to a postorder traversal of the tree. In a follow-up study [18], an exact algorithm is shown to solve the problem without the postorder constraint on the traversal (see also [14] for another algorithm). Memory minimization is still a concern in modern multifrontal solvers when dealing with large matrices. Among others, efforts have been made to design dynamic schedulers that takes into account dynamic pivoting (which impacts the weights of edges and vertices) when scheduling elimination trees with strong memory constraints [13], to consider both task and tree parallelism with memory constraints [2], and to incorporate out-of-core storage decisions [14]. While these studies try to optimize memory management in existing parallel solvers, we aim at designing a simple model to study the fundamental underlying scheduling problem.

The problem of scheduling a task graph under memory constraints also appears in the processing of scientific workflows whose tasks require large I/O files. Such workflows arise in many scientific fields, such as image processing, genomics, and geophysical simulations. The problem

of task graphs handling large data has been identified by Ramakrishnan et al. [23] who propose some simple heuristics. Surprisingly, in the context of quantum chemistry computations, Lam et al. [15] have recently rediscovered the algorithm of Liu [18].

On the more theoretical side, this work builds upon many papers that have addressed the pebble game and its variants. Scheduling a graph on one processor with the minimum amount of memory amounts to revisiting the I/O pebble game with pebbles of arbitrary sizes that must be loaded into main memory before *firing* (executing) the task. The pioneering work of Sethi and Ullman [25] deals with a variant of the pebble game that translates into the simplest instance of our problem when all input/output files have weight 1, all execution files have weight 0 and the graph is a rooted tree, where the aim is to minimize the number of registers that are needed to compute an arithmetic expression. The problem of determining whether a general DAG can be executed with a given number of pebbles has been shown NP-complete by Sethi [24] if no vertex is pebbled more than once (the general problem allowing recomputation, that is, re-pebbling a vertex which have been pebbled before, has been proven PSPACE complete [11]). However, this problem has a polynomial complexity for tree-shaped graphs [25], where it corresponds to the MCLA problem for rooted trees. Yannakakis [28] deals with the MCLA problem for (unrooted) trees where the objective is to minimize the number of edges cut, while Monien et al. [20] show that the problem becomes NP-complete in the presence of edge weights.

The work of Blumofe and Leiserson [5] provided the foundations for many studies on multithreaded computations [1, 3, 22]; leading to implementation of compilers for multithreaded parallel computing [4]. Similarly to the series-parallel graphs studied in the present paper, they focus on structured computations which are special classes of DAGs, some of them being close to series-parallel graphs. However, these works are less closely related to the present paper, since most of their effort are devoted to reducing the impact of parallelism on memory. The usual goal is to guarantee that the increase in memory needed for a parallel execution will be limited compared to the memory required by a sequential depth first search traversal of the graph.

3 Solving the peak memory problem on trees and fork-join graphs

We first recall Liu’s algorithm [18] for solving the peak memory minimization problem on trees. We then propose its adaptation for a subclass of series-parallel graphs. The algorithm proposed by Liu applies to *in-trees*, that is trees whose dependencies are directed towards the root (contrarily to *out-trees*, where dependencies are directed towards leaves). Note that if T is an in-tree and π is the schedule of T computed by Liu’s algorithm, then $\bar{\pi}$ is also a peak memory minimizing schedule of the out-tree \bar{T} .

3.1 Liu’s algorithm for trees

Liu [18] proposes an algorithm to find an optimal tree traversal for peak memory minimization. We present this algorithm as it serves as a basis of our strategies for fork-join and general series-parallel graphs. Liu’s original work concerns a different model where there is no edge weights (there are only vertex weights). We present here its immediate adaptation to our model.

Liu’s algorithm is shown in Algorithm 1. To compute an optimal traversal for a tree rooted at vertex k , it first recursively computes an optimal traversal for the subtrees rooted at each of its children, then merges them and finally adds the root at the end. The difficult part is of course how to merge the children traversals. There is no reason to process the subtrees one after the other, and an optimal solution may interleave the processing on one subtree with that of the

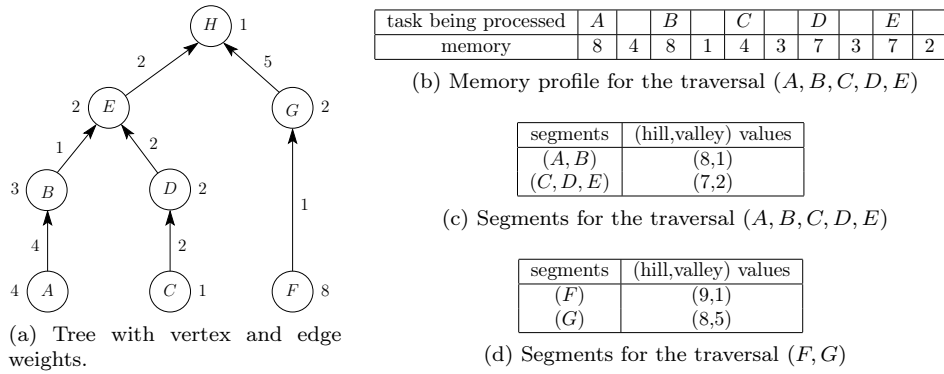


Figure 1: Sample in-tree and decomposition of its schedule in segments.

Algorithm 1 LIU-TREE-SCHEDULE(T)**Require:** $T = (V, E, w_n, w_e)$: tree with vertex- and edge-weights.**Ensure:** π : Schedule with the minimum peak memory $\mu(\pi)$

► Base case

if $V = \{u\}$ **then** **return** $\langle u \rangle$

► General case

Let r be the root of T and T_1, T_2, \dots, T_k its subtrees**for** $i = 1$ to k **do** $\pi_i \leftarrow \text{LIU-TREE-SCHEDULE}(T_i)$ Compute the hill-valleys segments of T_i in schedule π_i Sort all segments of all subtrees in non-increasing (*hill – valley*) valueBased on this segments o, order the vertices in each segment consecutively, followed by the root r , to build π

others. Liu makes the observation that in an optimal traversal the switching points between the subtrees' processing have to be local minima in the memory profile: while processing one subtree T_i , there is no reason to switch to T_j if one can reduce the memory needed for T_i by processing one more task in T_i . This remark leads to slicing the traversal into atomic parts, called *segments*. The end-points of segments (which are local minima in the memory profile) are called *valleys*, while the peak memory vertices of each segment are called *hills*. The memory occupation at valleys and hills are denoted by M^v and M^h and are defined as follows, for a given traversal:

- M_1^h is the peak memory of the whole traversal;
- M_i^v is the minimum amount of memory occurring after the step when M_i^h is (last) attained, for $i \geq 1$
- M_i^h is the peak memory after the step when M_{i-1}^v is (last) attained, for $i \geq 2$.

The sequence of hill-valley ends when the last vertex is reached. The segments consist of the vertices comprised between two valleys. For example, for the tree depicted in Figure 1a, consider the traversal (A, B, C, D, E) of the subtree rooted in E . The memory occupation during and after the processing of the tasks and the segments that are deduced from this memory profile are illustrated in Figure 1c. Similarly, the segments of the traversal (F, G) of the subtree rooted in G are detailed on Figure 1d.

To merge the traversals of the subtrees, the first step is to compute all hill-valley segments. Then the lists of segments are merged using the following criterion: if several segments are available (one for each subtree in the beginning), it is always beneficial to start with the segment with the maximum (*hill – valley*) difference. Intuitively, the residual memory will only increase when processing segments, so it is better (i) to start with the segment with larger peak memory (hill) to avoid large memory consumption later, and (ii) to start with the segment with smaller residual memory (valley) to ensure an increase of memory as small as possible.

In the example of Figure 1, when merging the traversals from the subtrees rooted at E and G at the vertex H , we start by comparing the first segments of each subtree: segment (F) is selected because it has a larger (*hill – valley*) value of $9 - 1 = 8$. This segment is ordered, and then removed from the segments list, and we proceed by comparing segments (A, B) and (G) : (A, B) has larger (*hill – valley*) value of $8 - 1 = 7$, so it is selected. By iterating this process, we end up with the following ordering: $(F), (A, B), (C, D, E), (G)$. The last step is to add the root H of the subtree to the traversal.

Liu proves that Algorithm 1 is correct and its worst-case runtime complexity is $O(n^2)$. We note that even if this algorithm is designed for in-trees, we can compute an optimal schedule of an out-tree by reversing all edges to obtain an in-tree, applying this algorithm, and then reversing the obtained schedule.

3.2 Algorithm for fork-join graphs

The main idea (summarized in Fig 2) is to remove one edge from each branch, so as to disconnect the graph into one out-tree (the part S in the figure) and one in-tree (the part T in the figure). Then, we can reuse Liu's algorithm to compute an optimal traversal for these two trees. The following lemma states that if the removed edges are of minimal weight in each branch, it is possible to first schedule all the vertices that are *before* this minimal cut, and then all the vertices *after* the cut, without increasing the peak memory.

Lemma 1. *Let G be a fork-join graph. For each branch B_i of this graph, let $e_i^{\min} = (u_i^{\min}, v_i^{\min})$ be an edge of B_i with the minimum weight. Let S be the set of ancestors of the u_i^{\min} 's, including them. Let T be the set of successors of the v_i^{\min} 's, including them. Let π be a schedule of G and γ be the schedule obtained from π by scheduling all vertices of S before vertices of T , formally $\gamma = \langle \pi[S], \pi[T] \rangle$. Then, $\mu(\gamma) \leq \mu(\pi)$.*

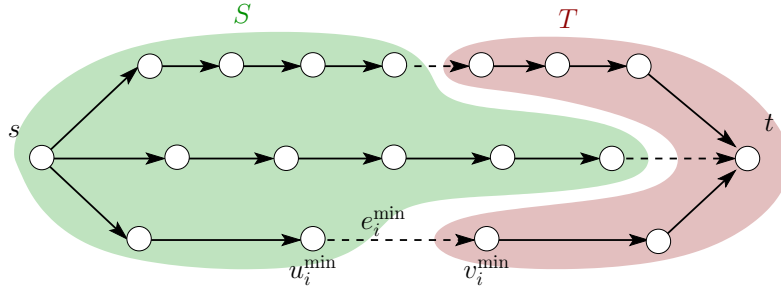


Figure 2: Sample fork-join graph and its decomposition for Lemma 1. If the dashed edges are minimum in the corresponding branches, an optimal traversal can be found by first ordering the vertices in the set S and then the vertices in the set T .

This intuitive result can be proved by using Theorem 2, Corollary 1 and Theorem 1 given below. We do not formalize the proof, since the objective of this section is to give an intuition on the algorithm for general sp-graphs.

Thanks to this result, we know that there exists an optimal schedule which ordered first vertices from S , and then vertices from T . Assume for a moment that the weight of all e_i^{\min} edges is zero ($w_e(e_i^{\min}) = 0$). Then, it is as if the graph was disconnected, and we have two separate trees to schedule. T is an in-tree, and Liu's algorithm can compute an optimal schedule π for it. S is an out-tree, so that \bar{S} is an in-tree: if γ is the optimal schedule computed by Liu's algorithm for \bar{S} , $\bar{\gamma}$ is optimal for S . Then, $\langle \bar{\gamma}, \pi \rangle$ is an optimal schedule of the whole graph. This approach can be generalized to fork-join graphs with non-zero weights on the minimal edges, as stated in Algorithm 2. We will prove that this algorithm computes an optimal schedule of all fork-join graphs (Theorem 8).

Algorithm 2 FORK-JOIN-SCHEDULE(T)

Require: $FJ = (V, E, w_n, w_e)$: fork-join graph with vertex- and edge-weights.

Ensure: π : Schedule with minimal peak memory $\mu(\pi)$

Let B_1, \dots, B_q be the branches of FJ

for $i = 1$ to q **do**

Let $e_i^{\min} = (u_i^{\min}, v_i^{\min})$ be the edge of minimum weight in B_i

Remove the edge e_i^{\min} from the graph

Update the weight of each other edge e of B_i : $w_e(e) \leftarrow w_e(e) - w_e(e_i^{\min})$

Update the weight of each vertex u in B_i (except s and t): $w_n(u) \leftarrow w_n(u) + w_e(e_i^{\min})$

Consider the two trees T^{out}, T^{in} made by disconnecting FJ

Let \bar{T}^{out} be the in-tree obtained by reversing all edges in T^{out}

$\pi_1 \leftarrow \text{LIU-TREE-SCHEDULE}(\bar{T}^{out})$

$\pi_2 \leftarrow \text{LIU-TREE-SCHEDULE}(T^{in})$

return $\langle \bar{\pi}_1, \pi_2 \rangle$

4 Solving the peak memory problem for series-parallel graphs

In this section, we present our main contribution: a polynomial time algorithm to compute a peak memory minimizing schedule of series-parallel graphs. In order to prove the correctness of the proposed algorithm, we rely on a new graph model, and new metrics. We first present the new model (Section 4.1), then introduce a new relation on schedules and a new objective (Sections 4.2 and 4.3). Next we present our algorithm and the (sketch of) proof of optimality (Sections 4.4 and 4.5). For the sake of readability, some of the detailed proofs are given in the Appendix.

4.1 A simpler model with only vertex weights

We introduce here a new task-graph model, called the *cumulative weight* model, which has only vertex weights (no edge weights). Formally, let $G = (V, E, \omega)$, where $\omega : V \rightarrow \mathbb{Z}$ is a weight function on the vertices. For a given set $U \subseteq V$ of vertices, $\omega(U)$ denotes the sum of weights of vertices in U , i.e., $\omega(U) = \sum_{v \in U} \omega(v)$. The vertex-weight function ω should satisfy $\omega(G) = 0$. Let π be a schedule and for each vertex v , consider the cumulative sum of the weights of all vertices scheduled before v :

$$\Sigma(v, \pi) = \omega(\{u \leq_{\pi} v\}). \quad (1)$$

We define the *cutwidth* $\rho(\pi)$ of a schedule π as the maximum of all these sums:

$$\rho(\pi) = \max_{p \in G} \omega(\{q \leq_{\pi} p\}) = \max_{p \in G} \Sigma(p, \pi). \quad (2)$$

Below (Theorem 1), we show that by minimizing the cutwidth in the cumulative weight model, one can minimize the peak memory in the original model. For this purpose we need some definitions. Given an instance $G = (V, E, w_n, w_e)$ of the peak memory problem, we construct an instance of the cumulative weight problem, with a (directed) bipartite graph $G_B = (V_B, E_B, \omega)$ as follows. For each vertex $p \in V$, we introduce a pair of vertices $p_{start}, p_{stop} \in V_B$ such that $(p_{start}, p_{stop}) \in E_B$. The vertex p_{start} represents the beginning of the computation corresponding to the task p , while p_{stop} represents the end of this computation. For each edge $(p, q) \in E$, we add an edge $(p_{stop}, q_{start}) \in E_B$. For each vertex p , we set

$$\omega(p_{start}) = w_n(p) + \sum_{(p,r) \in E} w_e(p, r), \quad (3)$$

to represent the allocation of temporary and output data in the memory at the beginning of the task p , and

$$\omega(p_{stop}) = -w_n(p) - \sum_{(q,p) \in E} w_e(q, p), \quad (4)$$

to represent the deallocation of temporary and input data from the memory at the end of the task p . Note that with these definitions $\omega(G_B) = 0$. Let π_B be a schedule of G_B . It is easy to see that a schedule can be constructed for G using the order of “stop” vertices in π_B . Given π_B , let π be a schedule of G such that $p \leq_{\pi} q$ whenever $p_{stop} \leq_{\pi_B} q_{stop}$. Observe that $\Sigma(v_{stop}, \pi_B) = \sum_{x \leq_{\pi} v <_{\pi} y} w_e(x, y)$, that is the cumulative weight of v_{stop} corresponds to the weight of the edges (in G) which are cut by an imaginary line just after v under the schedule π . We now show how minimizing the cutwidth in G_B minimizes the peak memory in G .

Theorem 1. *Let π_B be a schedule of G_B whose cutwidth $\rho(\pi_B)$ is minimum for G_B and π be the corresponding schedule of G . Then, $\mu(\pi)$ is the minimum peak memory for G .*

Proof. We provide the big lines of the proof by using lemmas proved in Appendix A. By Lemma A.4, we can assume without loss of generality that p_{start} and p_{stop} are consecutive for all p in π_B . This implies that $\mu(\pi) = \rho(\pi_B)$ by Lemma A.3. Suppose for the sake of contradiction that $\mu(\pi)$ is not minimum. In other words, there is a schedule γ of G where $\mu(\gamma) < \mu(\pi)$. Then, we can construct γ_B for G_B from γ by replacing each vertex p with p_{start} and p_{stop} . By Lemma A.3 again, $\mu(\gamma) = \rho(\gamma_B)$ and hence $\rho(\gamma_B) < \rho(\pi_B)$, a contradiction. \square

4.2 A new relation on schedules

We present here a relation on schedules that allows to compare them. We show that this relation subsumes the comparison using the cutwidth: a schedule minimum for this relation has a minimum cutwidth. This relation extends the partial orders introduced by Liu on pebble cost sequences for trees [18].

4.2.1 Definition of the relation on schedules

Given a directed graph G , two vertices p and q of G , and two schedules π and γ of G , we say that the schedule π at p *dominates* the schedule γ at q (and we write $p \xrightarrow{\pi} q$) if and only if

$$\text{C1. } \Sigma(p, \pi) \leq \Sigma(q, \gamma), \text{ and}$$

$$\text{C2. } \min\{\Sigma(s, \pi) : s \geq_{\pi} p\} \leq \min\{\Sigma(r, \gamma) : r \geq_{\gamma} q\}.$$

Note that the second condition (C2) is equivalent to that for any $r \geq_{\gamma} q$, there exists $s \geq_{\pi} p$ such that $\Sigma(s, \pi) \leq \Sigma(r, \gamma)$.

Now, we define the relation \preceq on the set of schedules for G as follows.

Definition 3 (Relation \preceq). *Let π and γ be two schedules of G . We say that $\pi \preceq \gamma$ for G , if for each $p \in G$, there is a vertex $q \in G$ such that $p \xrightarrow{\pi} q$.*

Contrarily to the relation \prec introduced by Liu on pebble cost sequences [18], our relation is not a partial order, as it is not anti-symmetric. However, we are able to prove that it is reflexive and transitive, and thus is a preorder.

Lemma 2. *The relation \preceq is a preorder (reflexive and transitive).*

Proof. We need to show that the relation \preceq is both reflexive and transitive. Let α , β , and θ be schedules of G .

- (i) *Reflexivity.* Since $p \xrightarrow{\alpha} p$ for each $p \in V$, $\alpha \preceq \alpha$, and thus, \preceq is reflexive.
- (ii) *Transitivity.* Assume that $\alpha \preceq \beta$ and $\beta \preceq \theta$. Take any $p \in G$. Let v and q be such that $p \xrightarrow{\alpha} v$ and $v \xrightarrow{\beta} q$. We claim that $p \xrightarrow{\alpha} q$, as well. There are two conditions we should examine.
 - 1) (C1). This follows as $\Sigma(p, \alpha) \leq \Sigma(v, \beta)$ and $\Sigma(v, \beta) \leq \Sigma(q, \theta)$, due to the first conditions of $p \xrightarrow{\alpha} v$ and $v \xrightarrow{\beta} q$.
 - 2) (C2). Take any vertex $r \geq_{\theta} q$. Then, there exists $s \geq_{\alpha} p$ and $w \geq_{\beta} v$ such that $\Sigma(s, \alpha) \leq \Sigma(w, \beta) \leq \Sigma(r, \theta)$, as a consequence of the second conditions of $p \xrightarrow{\alpha} v$ and $v \xrightarrow{\beta} q$.

\square

4.2.2 Properties of the relation on schedules

We now describe a few properties on the preorder \preceq , and especially its connection with the reverse of a graph. We recall and extend the notion of reverse graph from Section 2.1 to the vertex-weighted model: given a graph $G = (V, E, \omega)$, its reverse graph $\bar{G} = (V, \bar{E}, \bar{\omega})$ has the same set of vertices as G , however all edges are reversed $\bar{E} = \{(q, p) : (p, q) \in E\}$ and all vertex weights take the opposite sign of their value in G : $\bar{\omega}(p) = -\omega(p)$ for any $p \in V$. The reverse of a schedule π is defined as in Section 2.1: $q \leq_{\bar{\pi}} p$ whenever $p \leq_{\pi} q$, for any pair of vertices p, q .

Lemma 3. *For any vertex $p \in G$, we have*

$$\Sigma(p, \pi) - \Sigma(p, \bar{\pi}) = \omega(p), \quad (5)$$

where $\Sigma(p, \pi)$ and $\Sigma(p, \bar{\pi})$ are defined over G and \bar{G} , respectively.

Proof. Notice that $\{q \leq_{\bar{\pi}} p\} = \{q \geq_{\pi} p\}$ due to the definition above. Recall that $\omega(G) = 0$, and thus $\bar{\omega}(\bar{G}) = -\omega(G) = 0$. Then,

$$\begin{aligned} \Sigma(p, \bar{\pi}) &= \bar{\omega}(\{q \leq_{\bar{\pi}} p\}) = \bar{\omega}(\{q \geq_{\pi} p\}) \\ &= \bar{\omega}(G) - \bar{\omega}(\{q <_{\pi} p\}) \\ &= -\bar{\omega}(\{q <_{\pi} p\}) \\ &= \omega(\{q <_{\pi} p\}) = \Sigma(p, \pi) - \omega(p). \end{aligned}$$

□

Lemma 4. *For any schedule π of G , $\rho(\pi) = \rho(\bar{\pi})$.*

Proof. Let \bar{p}^* be a vertex such that $\rho(\bar{\pi}) = \Sigma(\bar{p}^*, \bar{\pi})$ and q be its predecessor under π . Then,

$$\begin{aligned} \rho(\pi) \geq \Sigma(q, \pi) &= \Sigma(\bar{p}^*, \pi) - \omega(\bar{p}^*) && \text{holds by the definitions (1) and (1)} \\ &= \Sigma(\bar{p}^*, \bar{\pi}) && \text{by Equation (5)} \\ &= \rho(\bar{\pi}). \end{aligned}$$

By noting that the roles of π and $\bar{\pi}$ can be changed in the above lines, we obtain $\rho(\bar{\pi}) \geq \rho(\pi)$ and hence the equality $\rho(\pi) = \rho(\bar{\pi})$. □

The last property on the \preceq relations shows that it implies a comparison on the cutwidth, not only for the schedules being compared but also for their reverse.

Lemma 5. *If $\pi \preceq \gamma$ then $\rho(\pi) \leq \rho(\gamma)$ and $\rho(\bar{\pi}) \leq \rho(\bar{\gamma})$.*

Proof. Assume that $\pi \preceq \gamma$. Then for any $p \in V$, there exists $q \in V$ such that $p \xrightarrow{\pi} q$, and thus, $\Sigma(p, \pi) \leq \Sigma(q, \gamma)$. Then, by definition, $\rho(\pi) \leq \rho(\gamma)$. Moreover, thanks to Lemma 4, we have $\rho(\bar{\gamma}) = \rho(\gamma) \geq \rho(\pi) = \rho(\bar{\pi})$. □

Corollary 1. *If π is minimal for \preceq on G , then π and $\bar{\pi}$ have the minimum cutwidths (on G and \bar{G} respectively).*

4.3 Min-cut optimality

We are now ready to present the central notion needed to provide an optimal algorithm for series-parallel graphs, namely the min-cut optimality. We show that a min-cut optimal schedule is also cutwidth optimal.

4.3.1 Definition of min-cut optimality

Min-cut optimality is based on the classical notion of *cut* in a graph: a cut (S, T) of G is defined as a bisection of its vertices into two nonempty sets S and T . We say that a cut is *topological* if there exists no edge $(p, q) \in E$ such that $p \in T$ and $q \in S$ (or equivalently, for any edge $(p, q) \in E$, we have either $p \in S$ or $q \in T$). The *width* of a topological cut (S, T) is defined as $c(S, T) = \omega(S)$.

Consider a topological cut (S, T) and a schedule π . If in π all vertices of S appear before all vertices of T , that is, for $p \in S$ and $q \in T$, we have $p \leq_\pi q$, then we say that π is in *compliance* with (S, T) . In this case, we have

$$c(S, T) = \Sigma(p^*, \pi), \text{ where } p^* = \max^\pi S. \quad (6)$$

We say that a topological cut (S, T) is *minimum*, or a *min- ω -cut*, if its width is minimum, that is, $c(S, T) = \min\{c(S', T') \mid (S', T') \text{ is a topological cut}\}$.

We define by $G[S]$ the subgraph of G which contains only the vertices in S and the edges between these vertices. Similarly, for any schedule π of G , $\pi[S]$ is the schedule induced by π on $G[S]$.

Definition 4 (Min-cut optimality). *Let (S, T) be a minimum topological cut. A schedule π is cut-optimal with (S, T) , if π is in compliance with (S, T) , $\bar{\pi}[S]$ is minimal for \preceq on $\bar{G}[S]$, and $\pi[T]$ is minimal for \preceq on $G[T]$. Furthermore, π is called min-cut optimal if π is cut-optimal with some min- ω -cut of G .*

We now state a simple lemma which will be useful in the following.

Lemma 6. *Let π be a schedule of G . Let (S, T) be any minimum topological cut of G . Then,*

$$c(S, T) \leq \Sigma(p, \pi),$$

for any $p <_\pi \max^\pi G$.

Proof. Take any $p <_\pi \max^\pi G$. Then suppose for the sake of contradiction that $\Sigma(p, \pi) < c(S, T)$. Consider the topological cut (S^p, T^p) , where $S^p = \{v \leq_\pi p\}$. Then, by (6), we have $c(S^p, T^p) = \omega(S^p) = \Sigma(p, \pi) < c(S, T)$, which contradicts that $c(S, T)$ is minimum. \square

4.3.2 Properties of min-cut optimality

We present two important properties of min- ω -cuts.

The first property is that transforming a schedule so that it is in compliance with a min- ω -cut will not make it larger in the sense of the \preceq preorder (and thus, thanks to Lemma 5, will not increase its cutwidth). This is shown in the next theorem.

Theorem 2. *Let G be an acyclic graph with a single source vertex and a single sink vertex, and γ be a schedule of G . Let (S, T) be a min- ω -cut of G , and $\pi = \langle \gamma[S], \gamma[T] \rangle$. Then, $\pi \preceq \gamma$ and $\bar{\pi} \preceq \bar{\gamma}$.*

Proof. Since (S, T) is a topological cut of G , π is a schedule of G . Note that γ may not be in compliance with (S, T) .

The proof relies on the two following properties: for each $p \in G$, $p \xrightarrow{\pi \rightarrow \gamma} p$ (which proves that $\pi \preceq \gamma$ according to the definition of this relation) and $p \xrightarrow{\bar{\pi} \rightarrow \bar{\gamma}} p$ (which proves $\bar{\pi} \preceq \bar{\gamma}$). We only provide the proof of the first property, as the second one can be proved using the very same arguments.

Let s^+ be the last scheduled vertex of S in γ . Similarly, let t^- be the first scheduled vertex of T in γ . Formally,

$$s^+ = \max^\gamma S, \quad t^- = \min^\gamma T.$$

Note that s^+ is also the last scheduled vertex of S in π (and t^- the first scheduled vertex of T in π) and $\Sigma(s^+, \pi) = c(S, T)$.

To prove that $p \xrightarrow{\pi \rightarrow \gamma} p$ for $p \in G$, we investigate $\Sigma(p, \pi)$ and $\Sigma(p, \gamma)$. First, for any $p \in V$, we consider s_p (respectively t_p), the last scheduled vertex of S (resp. of T) that does not come after p in γ :

$$s_p = \max^\gamma \{s \in S, s \leq_\gamma p\}, \quad t_p = \max^\gamma \{t \in T, t \leq_\gamma p\}.$$

Note that s_p is always defined since G has a single source vertex. However, t_p may not be defined, as the set $\{t \in T, t \leq_\gamma p\}$ may be empty, which occurs when $p \in S$ and p comes before t^- in γ ; in this case $p = s_p$. If t_p is defined, we have:

$$\begin{aligned} \Sigma(p, \gamma) &= \Sigma(s_p, \gamma[S]) + \Sigma(t_p, \gamma[T]) \\ &= \Sigma(s_p, \pi[S]) + \Sigma(t_p, \pi[T]) \\ &= \Sigma(s_p, \pi) + \Sigma(t_p, \pi) - \Sigma(s^+, \pi) \\ &= \Sigma(s_p, \pi) + \Sigma(t_p, \pi) - c(S, T). \end{aligned} \tag{7}$$

As a result,

$$\Sigma(p, \gamma) = \begin{cases} \Sigma(p, \pi) & \text{if } t_p \text{ is not defined,} \\ \Sigma(t_p, \pi) + \Sigma(s_p, \pi) - c(S, T) & \text{otherwise} \end{cases} \tag{8}$$

We are now ready to show $p \xrightarrow{\pi \rightarrow \gamma} p$, for any $p \in V$. We verify the two conditions defining this statement given in Section 4.2.1 for any $p \in V$, as follows.

1. The first condition to verify is C1: $\Sigma(p, \pi) \leq \Sigma(p, \gamma)$. When t_p is not defined, this condition directly derives from Equation (8). Consider now that t_p is defined. We consider two cases depending on which side of the cut p lies:

- (a) ($p \in S$). Then, $s_p = p$. This implies

$$\Sigma(p, \gamma) = \Sigma(p, \pi) + \Sigma(t_p, \pi) - c(S, T).$$

Note that $t_p <_\gamma p <_\gamma \max^\gamma G$, since $p \in S$. Since $\Sigma(t_p, \pi) \geq c(S, T)$ by Lemma 6, the condition C1 is met.

- (b) ($p \in T$). Then, $t_p = p$. This implies

$$\Sigma(p, \gamma) = \Sigma(p, \pi) + \Sigma(s_p, \pi) - c(S, T).$$

Note that $s_p <_\gamma \max^\gamma G$, since $s_p \in S$. Since $\Sigma(s_p, \pi) \geq c(S, T)$ by Lemma 6, the condition C1 is met.

2. The second condition to verify (C2) writes

$$\min\{\Sigma(s, \pi) : s \geq_\pi p\} \leq \min\{\Sigma(r, \gamma) : r \geq_\gamma p\}$$

(or equivalently, for any $r \geq_\gamma p$, there exists $s \geq_\pi p$ such that $\Sigma(s, \pi) \leq \Sigma(r, \gamma)$). First, note that the last vertex is the same in both schedules: $r = \max^\pi G = \max^\gamma G$ and thus $\Sigma(r, \pi) = \Sigma(r, \gamma)$. Now, consider any $r \geq_\gamma p$ such that $r <_\gamma \max^\gamma G$. We consider the two following cases:

- a) p is in S . Then, $\Sigma(s^+, \pi) = c(S, T) \leq \Sigma(r, \gamma)$ and $s^+ \geq_\pi p$.
- b) p is in T . Let $t_r = \max^\gamma\{t \in T, t \leq_\gamma r\}$. Note that t_r is well defined, since $r \geq_\gamma p$. We also have $t_r \geq_\gamma p$. Then, by Equation (7),

$$\Sigma(r, \gamma) = \Sigma(s_r, \pi) + \Sigma(t_r, \pi) - c(S, T).$$

Since $\Sigma(s_r, \pi) \geq c(S, T)$ by Lemma 6, we have $\Sigma(t_r, \pi) \leq \Sigma(r, \gamma)$. Notice that $t_r \geq_\pi p$ since $t_r \geq_\gamma p$, and, both $p \in T$ and $t_r \in T$.

□

The following property of min- ω -cuts gives the full importance of min-cut optimality. It states that, for a schedule π , being min-cut optimal implies optimality for the \preceq relation both for π and its reverse $\bar{\pi}$.

Theorem 3. *Let G be a directed acyclic graph with a single source vertex and a single sink vertex, and π be a min-cut optimal schedule of G . Then, π is minimal for \preceq on G and $\bar{\pi}$ is minimal for \preceq on \bar{G} .*

Proof. Consider a directed acyclic graph G with single source and sink vertices, and a min-cut optimal schedule π . Since π is min-cut optimal, π is cut-optimal with some min- ω -cut (S, T) . Take any schedule λ of G , and consider the schedule $\gamma = \langle \lambda[S], \lambda[T] \rangle$, which is in compliance with (S, T) . We have $\gamma \preceq \lambda$ and $\bar{\gamma} \preceq \bar{\lambda}$ by Theorem 2. Here, we only show $\pi \preceq \gamma$ for G (which induces $\pi \preceq \lambda$), as with similar arguments one can also show $\bar{\pi} \preceq \bar{\gamma}$ (which induces $\bar{\pi} \preceq \bar{\lambda}$).

By the definition of a min-cut optimal schedule, $\bar{\pi}[S]$ (respectively $\pi[T]$) is minimal for \preceq on S (resp. on T), thus $\bar{\pi}[S] \preceq \bar{\gamma}[S]$ and $\pi[T] \preceq \gamma[T]$. We show that for any vertex $p \in V$, there is a vertex $q \in V$ such that $p \xrightarrow{\pi} q$. We consider a vertex $p \in V$ and distinguish two cases depending where p lies:

1. ($p \in S$). Let q be such that $\Sigma(q, \gamma[S]) = \rho(\gamma[S])$. We show $p \xrightarrow{\pi} q$ by examining the two conditions.

- (a) (C1: $\Sigma(p, \pi) \leq \Sigma(q, \gamma)$). Since $\bar{\pi}[S] \preceq \bar{\gamma}[S]$, by Corollary 1 we have $\rho(\pi[S]) \leq \rho(\gamma[S])$. Hence,

$$\Sigma(p, \pi) \leq \rho(\pi[S]) \leq \rho(\gamma[S]) = \Sigma(q, \gamma).$$

- (b) (C2: $\min\{\Sigma(s, \pi) : s \geq_\pi p\} \leq \min\{\Sigma(r, \gamma) : r \geq_\gamma q\}$). Let t be the sink vertex. Note that $\Sigma(t, \pi) \leq \Sigma(t, \gamma)$ and $t \geq_\pi p$. Take any $r \geq_\gamma q$ and $r <_\gamma t$. For $s^+ = \max^\pi\{s \in S\}$, we have $\Sigma(s^+, \pi) = c(S, T) \leq \Sigma(r, \gamma)$. Notice that $s^+ \geq_\pi p$, as $p \in S$.

2. ($p \in T$). Let q be such that $p \xrightarrow{\pi[T]} q$; such a q exists, because $\pi[T] \preceq \gamma[T]$. We show $p \xrightarrow{\pi} q$ by examining the two conditions:

- (a) (C1: $\Sigma(p, \pi) \leq \Sigma(q, \gamma)$). It holds that

$$\Sigma(p, \pi) = \Sigma(p, \pi[T]) + \Sigma(s^+, \pi[S]) \tag{9}$$

$$\leq \Sigma(q, \gamma[T]) + \Sigma(s^+, \pi[S]) \tag{10}$$

$$= \Sigma(q, \gamma[T]) + \Sigma(s^+, \gamma[S]) = \Sigma(q, \gamma). \tag{11}$$

Equations (9) and (11) come from the fact that π and γ are in compliance with (S, T) . Equation (10) is derived from (9) and $p \xrightarrow{\pi[T]} q$.

- (b) (C2: $\min\{\Sigma(s, \pi) : s \geq_{\pi} p\} \leq \min\{\Sigma(r, \gamma) : r \geq_{\gamma} q\}$). Take any $r \geq_{\gamma} q$. Then, there exists $s \geq_{\pi} p$ such that $\Sigma(s, \pi[T]) \leq \Sigma(r, \gamma[T])$. Then,

$$\begin{aligned} \Sigma(s, \pi) &= \Sigma(s, \pi[T]) + \Sigma(s^+, \pi[S]) \\ &\leq \Sigma(r, \gamma[T]) + \Sigma(s^+, \pi[S]) \\ &= \Sigma(r, \gamma[T]) + \Sigma(s^+, \gamma[S]) = \Sigma(r, \gamma). \end{aligned}$$

□

Note that the previous two theorems only apply to directed graphs with a single source and a single sink, and thus in particular to series-parallel graphs. However, it is easy to transform any directed graph so that it obeys these conditions: simply add two zero-weight artificial vertices, one for the source and one for the sink, connect all sources of the original graph to the new source, and connect the new sink to all original sinks.

4.4 Algorithm for general series-parallel graphs

We present here our main contribution, which is a recursive algorithm, called SP-SCHEDULE, that computes a min-cut optimal schedule for a series-parallel graph G in the vertex-weighted model. As presented before, this means that the computed schedule is cut-optimal with some min- ω -cut (S, T) and implies that it also minimizes the cutwidth. The algorithm also outputs this (S, T) cut. We consider here series-parallel graphs as defined in Definition 1, with only vertex weights in accordance with the cumulative weight model presented in Section 4.1.

This algorithm relies on an algorithm to compute a min-cut optimal schedule of a fork-join graph, FJ-SCHEDULE which is presented below. This algorithm for fork-join graphs is similar to Algorithm 2, but applies to our new cumulative weight model. Likewise, this latter algorithm relies on an algorithm for trees, TREE-SCHEDULE, which is the translation of Liu's algorithm for the cumulative weight model.

The base case of the algorithm considers a series-parallel graph with a single edge, and outputs the unique schedule along with the unique topological cut.

In the general case, G is a series or parallel composition of two smaller series-parallel graphs G_1 and G_2 . We first recursively compute schedules π_1 and π_2 that are cut-optimal with topological cuts (S_1, T_1) and (S_2, T_2) for G_1 and G_2 .

If G is a series composition, the final schedule is obtained through a simple concatenation of π_1 and π_2 . This schedule of G is cut-optimal with both topological cuts induced by (S_1, T_1) and (S_2, T_2) on G . Thus, we select the one with lower cutwidth as it is a min- ω -cut of G . Therefore, π is a min-cut-optimal schedule of G .

In case of parallel composition, we transform each subgraphs G_1 and G_2 into a chain using LINEARIZE, based on schedules π_1 and π_2 , respectively: \tilde{G}_1 (resp. \tilde{G}_2) is a chain with the same vertices as G_1 (resp. G_2) such that its unique topological order is π_1 (resp. π_2). The graph obtained by replacing G_1 and G_2 by \tilde{G}_1 and \tilde{G}_2 is a fork-join graph with two branches. We consider the topological-cut (S, T) obtained by unifying the min- ω -cuts of G_1 and G_2 : as we will prove later, it is a min- ω -cut of the fork-join graph, but also of the original graph. We then call the routine FJ-SCHEDULE which finds a schedule π that is cut-optimal with this particular topological cut (S, T) .

Algorithm 4 presents FJ-SCHEDULE, which computes a schedule π that is cut-optimal with the given min- ω -cut (S, T) for a given fork-join graph G . The algorithm simply divides the fork-join graph into two trees according to (S, T) . These trees have a special property: a subtree rooted at every vertex but the root itself has a non-negative total weight. This property follows

Algorithm 3 SP-SCHEDULE(G)

Require: $G = (V, E, \omega)$: series-parallel graph.**Ensure:** π : schedule and (S, T) : min- ω -cut

► Base case

if $|E| = 1$ **then** $(S, T) \leftarrow (\{v\}, \{w\})$ **return** $[\langle v, w \rangle, (S, T)]$ ► G is series or parallel comb. of G_1 and G_2 $[\pi_1, (S_1, T_1)] \leftarrow \text{SP-SCHEDULE}(G_1)$ $[\pi_2, (S_2, T_2)] \leftarrow \text{SP-SCHEDULE}(G_2)$

► Series Composition

if $G = \langle G_1, G_2 \rangle$ **then** $(S, T) \leftarrow \min\{(S_1, T_1 \cup G_2), (G_1 \cup S_2, T_2)\}$ **return** $[\langle \pi_1, \pi_2 \rangle, (S, T)]$

► Parallel Composition

if $G = \{G_1, G_2\}$ **then** $\tilde{G}_1 \leftarrow \text{LINEARIZE}(G_1, \pi_1)$ $\tilde{G}_2 \leftarrow \text{LINEARIZE}(G_2, \pi_2)$ $(S, T) \leftarrow (S_1 \cup S_2, T_1 \cup T_2)$ $\pi \leftarrow \text{FJ-SCHEDULE}(\tilde{G}_1 \cup \tilde{G}_2, (S, T))$ **return** $[\pi, (S, T)]$

Algorithm 4 FJ-SCHEDULE($G, (S, T)$)

Require: $G = (V, E, \omega)$: fork-join directed graph.**Require:** (S, T) : Min- ω -cut of G .**Ensure:** π : Schedule $\bar{\sigma} \leftarrow \text{TREE-SCHEDULE}(\tilde{G}[S])$ $\tau \leftarrow \text{TREE-SCHEDULE}(G[T])$ **return** $\langle \sigma, \tau \rangle$

from the fact that (S, T) is a min- ω -cut of G . The algorithm then schedules the two trees independently using TREE-SCHEDULE routine, which is a translation of Liu's optimal algorithm for trees for the cumulative weight model, and can be found as Algorithm D.1 in Appendix D.

4.5 Correctness of the algorithm

We prove here that SP-SCHEDULE computes a schedule which minimizes the cutwidth (and therefore, thanks to Theorem 1, it also minimizes peak memory). This result is expressed in Theorem 4.

Theorem 4 (Main Theorem). *For any series-parallel graph, SP-SCHEDULE computes a schedule π such the cutwidth of π is optimal.*

Theorem 4 is easily proved below once the following theorem is proved.

Theorem 5. *For any series-parallel graph, SP-SCHEDULE computes a schedule π and a min- ω -cut (S, T) such that π is min-cut optimal with (S, T) .*

The proof of Theorem 5 is decomposed into several steps, following the different cases in the algorithm. These steps are formalized through the following three theorems.

The first step deals with the series composition. The following theorem states that concatenating two min-cut optimal schedules for the subgraphs leads to a min-cut optimal schedule of their series composition.

Theorem 6. *Let G_1 and G_2 be two series-parallel directed graphs. Let π_1 and π_2 each be a min-cut-optimal schedule of G_1 and G_2 , respectively. Then, $\pi = \langle \pi_1, \pi_2 \rangle$ is a min-cut-optimal schedule of their series composition $G = \langle G_1, G_2 \rangle$.*

Proof. The statement follows as a corollary the Lemma B.1 in Appendix B. \square

The second step deals with the parallel composition. The next theorem states that if we have a min- ω -cut of a series-parallel directed graph and an optimal schedule of each part of this directed graph, then there is an optimal schedule of the whole graph which induces these two schedules. This theorem is used to compare the schedule returned by SP-SCHEDULE with an optimal one and show that it is indeed optimal itself.

Theorem 7. *Let G_1 and G_2 be two series-parallel directed graphs, and $G = \{G_1, G_2\}$ be their parallel composition. Let (S, T) be a min- ω -cut of G , and π_1 and π_2 be schedules that are cut-optimal with the topological cuts induced by (S, T) on G_1 and G_2 , respectively. For each schedule γ in compliance with (S, T) of G , there is a schedule π in compliance with (S, T) of G such that*

- (i) π induces π_1 and π_2 ,
- (ii) $\bar{\pi}[S] \preceq \bar{\gamma}[S]$ and $\pi[T] \preceq \gamma[T]$.

Proof. See Appendix C. \square

The final step is to prove that the FJ-SCHEDULE algorithm used to process fork-join graphs produces a min-cut optimal schedule compatible with the provided cut.

Theorem 8. *For any fork-join graph G and any min- ω -cut (S, T) of G , FJ-SCHEDULE computes a schedule π that is cut-optimal with (S, T) .*

Proof. See Appendix D. \square

We are now ready to prove the main theorem by proving Theorem 5.

Proof of Theorem 5. The proof is by induction on the number of edges of G . In the base case, G has a single edge, that connects the source to the sinks. There is a unique schedule of G , which is also cut-optimal with the unique topological cut.

We now assume that the theorem holds for any series-parallel graph with fewer than k edges, where $k > 1$, and consider a series-parallel graph G with exactly k edges. G is made by the (series or parallel) composition of two series-parallel graphs G_1 and G_2 . Both subgraphs have fewer than k edges, thus by induction hypothesis, the recursive calls to SP-SCHEDULE produce two min-cut-optimal schedules π_1 and π_2 together with their respective min- ω -cuts (S_1, T_1) and (S_2, T_2) . We now distinguish between the two possible compositions.

Series composition Theorem 6 shows that the schedule $\pi = \langle \pi_1, \pi_2 \rangle$ computed by SP-SCHEDULE is a min-cut-optimal schedule of G . Besides a min- ω -cut of G can simply be found by selecting one of the min- ω -cut of G_1 and G_2 with minimal cutwidth.

Parallel composition Now, we consider that $G = \{G_1, G_2\}$. The algorithm first compute the cut (S, T) such that $S = S_1 \cup S_2$ and $T = T_1 \cup T_2$. We first prove that (S, T) is a min- ω -cut of G . For the sake of contradiction assume the contrary. It means that there exists a topological cut (S^*, T^*) such that $c(S^*, T^*) < c(S, T)$. Let (S_1^*, T_1^*) and (S_2^*, T_2^*) be the topological cuts induced by (S^*, T^*) on G_1 and G_2 , respectively. Then,

$$\begin{aligned} c(S_1^*, T_1^*) + c(S_2^*, T_2^*) &= c(S^*, T^*) \\ &< c(S, T) \\ &= c(S_1, T_1) + c(S_2, T_2). \end{aligned}$$

This implies $c(S_1^*, T_1^*) < c(S_1, T_1)$ or $c(S_2^*, T_2^*) < c(S_2, T_2)$, which is a contradiction as (S_1, T_1) and (S_2, T_2) both are min- ω -cuts.

In the case of a parallel composition, the algorithm computes the chain graph \tilde{G}_1 (respectively \tilde{G}_2) by sequencing the vertices of G_1 (resp. G_2) in the order of π_1 (resp. π_2) and calls FJ-SCHEDULE on the fork-join graph \tilde{G} made by the parallel composition of \tilde{G}_1 and \tilde{G}_2 and the cut (S, T) . Thanks to Theorem 8, we know that FJ-SCHEDULE outputs a schedule π cut-optimal with (S, T) . Thus, for any schedule γ of \tilde{G} in compliance with (S, T) , we have

$$\bar{\pi}[S] \preceq \bar{\gamma}[S], \text{ and } \pi[T] \preceq \gamma[T]. \quad (12)$$

We now show that π is cut-optimal with (S, T) for G as well, that is, the previous inequalities also holds for any schedule γ of G in compliance with (S, T) . We consider such a schedule γ . Theorem 7 proves that there exists a schedule π^* that (i) induces π_1 and π_2 , (ii) π^* is in compliance with (S, T) , and (iii)

$$\bar{\pi}^*[S] \preceq \bar{\gamma}[S], \text{ and } \pi^*[T] \preceq \gamma[T]. \quad (13)$$

As π^* induces π_1 and π_2 , π^* is a valid schedule of \tilde{G} , we may thus apply Equation (12) to $\gamma = \pi^*$. Combining it with Equation (13), we get

$$\bar{\pi}[S] \preceq \bar{\pi}^*[S] \preceq \bar{\gamma}[S], \text{ and } \pi[T] \preceq \pi^*[T] \preceq \gamma[T].$$

As \preceq is a preorder, it is transitive, and we have $\bar{\pi}[S] \preceq \bar{\gamma}[S]$ and $\pi[T] \preceq \gamma[T]$. \square

Proof of the Main Theorem 4. SP-SCHEDULE computes a schedule π and a min- ω -cut such that π is min-cut optimal with the found cut (see Theorem 5). Theorem 3 implies that π is also minimal for the \preceq relation, which in turn means that π minimizes the cutwidth $\rho(\pi)$ due to Corollary 1. \square

Algorithm complexity

The FJ-SCHEDULE algorithm calls twice Liu’s algorithm for trees, whose worst-case complexity for an input of size n is a $O(n^2)$. Except for the recursive calls, this is the most costly step of the SP-SCHEDULE. Since there are at most $O(n)$ recursive calls, the overall worst-case complexity of SP-SCHEDULE is $O(n^3)$.

5 Conclusion

In this paper, we have proposed an algorithm to schedule a series-parallel task graph with the minimum peak memory. To prove the correctness of the algorithm, we have introduced a new and simpler model, based on a vertex-weighted graph. We have shown how to simulate task graphs with weights on vertices and edges using the new model. The proof of the proposed algorithm is complex, and consists in an extension of Liu’s work on trees [18].

The use of the proposed algorithm is limited to task graphs structured as series-parallel computations, which constitutes an important class of scientific computing applications. However, this algorithm may be applied to general graphs, if we first transform them into series-parallel graphs, for example using the SP-ization process [12], which is also suggested in the literature [6]. This will lead to suboptimal schedules, because some fictitious edges are added to a graph when it is turned into a series-parallel graph. Future research directions include looking for optimal schedules for other regular graph structures that appear in scientific computing (such as 2D or 3D meshes), and for graphs made as a combination of these regular structures.

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Appendix

The appendix is composed of four parts. Appendix A contains lemmas used in the proof of Theorem 1 in which we proved that minimizing the cutwidth in the cumulative weight model minimizes the peak memory in the original model. The sections Appendix B, C, and D contain lemmas used in showing the correctness of SP-SCHEDULE algorithm. These three sections give the proof of Theorems 6, 7, and 8, respectively. The last section Appendix D also contains the adaptation of Liu's algorithm to the cumulative weight model.

A Problem transformation

Lemmas A.1–A.4 are used in the proof of Theorem 1 and assume the same notation. In particular, the peak memory minimization problem is given by a graph $G = (V, E, w_n, w_e)$, and the corresponding cutwidth instance has the bipartite graph $G_B = (V_B, E_B, \omega)$.

Lemma A.1. *Let π_B be a schedule of G_B . There exists a vertex $p \in G$ such that the cutwidth of π_B is reached on p_{start} , i.e., $\rho(\pi_B) = \Sigma(p_{start}, \pi_B)$.*

Proof. For the sake of contradiction, assume that $\rho(\pi_B) > \Sigma(p_{start}, \pi_B)$ for every $p \in G$. Consider a vertex q of G such that the cutwidth of π_B is reached on q_{stop} , i.e., $\rho(\pi) = \Sigma(q_{stop}, \pi_B)$.

By construction (4), $\omega(v_{stop}) \leq 0$, for any vertex v . We consider p_{start} , the last “start” vertex scheduled before q_{stop} in π_B : $p_{start} = \max^\pi(\{v_{start} \leq_{\pi_B} q_{stop}\})$. Note that there exists such a p_{start} since q_{start} is scheduled before q_{stop} . Then,

$$\Sigma(q_{stop}, \pi_B) = \Sigma(p_{start}, \pi_B) + \sum_{p_{start} <_{\pi_B} v_{stop} \leq_{\pi_B} q_{stop}} \omega(v_{stop}) \leq \Sigma(p_{start}, \pi_B).$$

Then, we have

$$\rho(\pi_B) = \Sigma(q_{stop}, \pi_B) \leq \Sigma(p_{start}, \pi_B),$$

which contradicts that $\rho(\pi_B) > \Sigma(p_{start}, \pi_B)$. \square

Lemma A.2. *Let π be a schedule of G . Then, for any $p \in G$,*

$$\mu(p, \pi) = \omega(p_{start}) + \sum_{v <_{\pi} p} (\omega(v_{start}) + \omega(v_{stop})).$$

Proof. For this proof, we extend the definition of the edge weights in such a way that $w_e(q, r) = 0$ if there is no edge (q, r) in E . With this extension, we may write

$$\begin{aligned} \omega(v_{start}) &= w_n(v) + \sum_{v \leq_{\pi} r} w_e(v, r), \\ \omega(v_{stop}) &= -w_n(v) - \sum_{q \leq_{\pi} v} w_e(q, v). \end{aligned}$$

Now, consider any vertex $p \in G$. We have

$$\begin{aligned}
\mu(p, \pi) &= w_n(p) + \sum_{q \leq \pi p \leq \pi r} w_e(q, r) \\
&= w_n(p) + \sum_{p \leq \pi r} w_e(p, r) + \sum_{q < \pi p \leq \pi r} w_e(q, r) \\
&= \omega(p_{start}) + \sum_{q < \pi p \leq \pi r} w_e(q, r) + \sum_{q \leq \pi r < \pi p} w_e(q, r) - \sum_{q \leq \pi r < \pi p} w_e(q, r) \\
&= \omega(p_{start}) + \sum_{q < \pi p} \left(\sum_{p \leq \pi r} w_e(q, r) + \sum_{q \leq \pi r < \pi p} w_e(q, r) \right) - \sum_{q \leq \pi r < \pi p} w_e(q, r) \\
&= \omega(p_{start}) + \sum_{q < \pi p} \sum_{q \leq \pi r} w_e(q, r) - \sum_{r < \pi p} \sum_{q \leq \pi r} w_e(q, r) \\
&= \omega(p_{start}) + \sum_{v < \pi p} (\omega(v_{start}) - w_n(v)) - \sum_{v < \pi p} (-\omega(v_{stop}) - w_n(v)) \\
&= \omega(p_{start}) + \sum_{v < \pi p} (\omega(v_{start}) - w_n(v) + \omega(v_{stop}) + w_n(v)) \\
&= \omega(p_{start}) + \sum_{v < \pi p} (\omega(v_{start}) + \omega(v_{stop})).
\end{aligned}$$

□

Lemma A.3. *Let π_B be a schedule of G_B such that p_{start} and p_{stop} are consecutive in π_B , for each $p \in G$. Consider the schedule π of G such that $p \leq \pi q$ whenever $p_{stop} \leq_{\pi_B} q_{stop}$. Then, $\mu(\pi) = \rho(\pi_B)$.*

Proof. As p_{start} and p_{stop} are consecutive in π_B , for each $p \in G$, we have

$$\Sigma(p_{start}, \pi_B) = \omega(p_{start}) + \sum_{v < \pi p} (\omega(v_{start}) + \omega(v_{stop}))$$

and, due to Lemma A.2, $\Sigma(p_{start}, \pi_B) = \mu(p, \pi)$, for any $p \in G$. We consider the vertex p which reaches the cutwidth of π_B , as defined by Lemma A.1: $\rho(\pi_B) = \Sigma(p_{start}^*, \pi_B)$. Then,

$$\rho(\pi_B) = \Sigma(p_{start}, \pi_B) = \mu(p, \pi) \leq \mu(\pi). \quad (\text{A.1})$$

Let $q \in G$ be such that $\mu(\pi) = \mu(q, \pi)$. Then,

$$\mu(\pi) = \mu(q, \pi) = \Sigma(q_{start}, \pi_B) \leq \rho(\pi_B). \quad (\text{A.2})$$

By combining Equations A.1 and A.2, we have $\mu(\pi) = \rho(\pi_B)$. □

Lemma A.4. *Let π_B be a schedule of G_B . There exists a schedule π'_B of G_B such that (i) p_{start} and p_{stop} are consecutive in π'_B , for each $p \in G$, and (ii) $\rho(\pi'_B) \leq \rho(\pi_B)$.*

Proof. Let $p \in G$ be such that p_{start} and p_{stop} are not consecutive in π_B . We construct a schedule π''_B from π_B , where p_{start} is placed right before p_{stop} . We show that π''_B has a cutwidth that is smaller than or equal to that of π_B .

Let $q \in G$ be such that $\rho(\pi''_B) = \Sigma(q_{start}, \pi''_B)$, which exists due to Lemma A.1. We investigate the following two cases of q :

- q_{start} is scheduled before p_{start} in π_B . Then, moving p_{start} forward does not influence the cutwidth.
- q_{start} is scheduled after p_{start} in π_B .

We first show that $\Sigma(q_{start}, \pi_B'') \leq \Sigma(q_{start}, \pi_B)$ as follows. If $q_{start} >_{\pi_B} p_{stop}$, then $\Sigma(q_{start}, \pi_B'') = \Sigma(q_{start}, \pi_B)$. Otherwise, since $\omega(p_{stop}) \leq 0$, we have $\Sigma(q_{start}, \pi_B'') = \Sigma(q_{start}, \pi_B) + \omega(p_{stop}) \leq \Sigma(q_{start}, \pi_B)$. Thus, we have $\rho(\pi_B'') = \Sigma(q_{start}, \pi_B'') \leq \Sigma(q_{start}, \pi_B) \leq \rho(\pi_B)$.

We repeat this process until p_{start}, p_{stop} vertices are scheduled consecutively, which does not degrade the cutwidth. \square

B Series composition (used for Theorem 6)

Lemma B.1. *Let G_1 and G_2 be two series-parallel directed graphs. Let (S_1, T_1) and (S_2, T_2) be min- ω -cuts, and π_1 and π_2 be schedules cut-optimal with (S_1, T_1) and (S_2, T_2) for G_1 and G_2 , respectively. Let $\pi = \langle \pi_1, \pi_2 \rangle$ be the schedule of the series composition $G = \langle G_1, G_2 \rangle$ inducing π_1 and π_2 . Then,*

- π is cut-optimal with both $(S_1, T_1 \cup G_2)$ and $(G_1 \cup S_2, T_2)$,
- (S, T) is a min- ω -cut of G , where

$$(S, T) = \min\{(S_1, T_1 \cup G_2), (G_1 \cup S_2, T_2)\}.$$

Proof. **We first concentrate on item (i) of the lemma.** We only show that π is cut-optimal with $(S_1, T_1 \cup G_2)$, as the other can be proven similarly. Let $(S, T) = (S_1, T_1 \cup G_2)$. We need to show $\bar{\pi}[S]$ and $\pi[T]$ both are minimal for \preceq . First consider $\bar{\pi}[S]$. Since π_1 is cut-optimal with (S_1, T_1) for G_1 , $\bar{\pi}_1[S_1]$ is minimal for \preceq on G_1 . Then, $\bar{\pi}[S]$ is minimal for \preceq on G .

Now consider $\tau = \pi[T]$. For any schedule κ of $G[T]$, we need to show $\tau \preceq \kappa$. Take any schedule κ of $G[T]$. We show that there is a vertex $q \in T$ for each vertex $p \in T$ such that $p \xrightarrow{\tau} \kappa q$.

Take any $p \in T$. We consider two scenarios of $p \in T_1$ and $p \in G_2$, separately.

- $(p \in T_1)$. Let $\tau_1 = \tau[T_1] = \pi_1[T_1]$ and $\kappa_1 = \kappa[T_1]$. Notice that π_1 is min-cut-optimal for G_1 . Then, $\pi_1[T_1]$, and equivalently τ_1 , is minimal for \preceq by definition. Then, $\tau_1 \preceq \kappa_1$. Then, there is a vertex $q^* \in T_1$ such that $p \xrightarrow{\tau_1} \kappa_1 q^*$. Note that $q^* \in T_1 \subseteq T$, and we show $p \xrightarrow{\tau} \kappa q^*$, as follows.

- (C1: $\Sigma(p, \tau) \leq \Sigma(q^*, \kappa)$). $\Sigma(p, \tau) = \Sigma(p, \tau_1) \leq \Sigma(q^*, \kappa_1) = \Sigma(q^*, \kappa)$.

- (C2: $\min\{\Sigma(s, \tau) : s \geq_{\tau} p\} \leq \min\{\Sigma(r, \kappa) : r \geq_{\kappa} q^*\}$). We examine a vertex $r \geq_{\kappa} q^*$, in relation to the vertex $t^o = \max^{\kappa} T = \max^{\tau} T$, which is the single sink vertex of G .

Case 1. ($r \in T_1$). Since $p \xrightarrow{\tau_1} \kappa_1 q^*$, there is a vertex $s \in T_1$ and $s \geq_{\tau_1} p$ such that $\Sigma(s, \tau_1) \leq \Sigma(r, \kappa_1)$. Then, $s \geq_{\tau} p$ and

$$\Sigma(s, \tau) = \Sigma(s, \tau_1) \leq \Sigma(r, \kappa_1) = \Sigma(r, \kappa).$$

Case 2. ($r \in G_2, r <_{\kappa} t^o$). Consider $s \in G_2$ such that $\Sigma(s, \pi_2) = c(S_2, T_2)$. Then, $\Sigma(s, \pi_2) \leq \Sigma(r, \kappa_2)$, where $\kappa_2 = \kappa[G_2]$, due to Lemma 6. Then, $s \geq_{\tau} p$ and

$$\begin{aligned} \Sigma(s, \tau) &= \Sigma(s, \pi_2) + \omega(T_1) \\ &\leq \Sigma(r, \kappa_2) + \omega(T_1) = \Sigma(r, \kappa). \end{aligned}$$

Case 3. ($r = t^o$). Consider $s = t^o$. Then, $s \geq_\tau p$, and

$$\Sigma(s, \tau) = \Sigma(t^o, \tau) = \Sigma(t^o, \kappa) = \Sigma(r, \kappa).$$

- b) ($p \in G_2$). Let $\tau_2 = \tau[G_2] = \pi_2$ and $\kappa_2 = \kappa[G_2]$. Notice that π_2 is min-cut-optimal for G_2 . Then, π_2 , and equivalently τ_2 , is minimal for \preceq on G_2 , due to Theorem 3. Then, $\tau_2 \preceq \kappa_2$. Then, there is a vertex $q^* \in G_2$ such that $p \xrightarrow{\tau_2} \kappa_2 q^*$. Notice that $\Sigma(v, \tau) = \Sigma(v, \tau_2) + \omega(T_1)$ and $\Sigma(v, \kappa) = \Sigma(v, \kappa_2) + \omega(T_1)$, for any vertex $v \in G_2$. Since both $p \in G_2$ and $q^* \in G_2$, and $p \xrightarrow{\tau_2} \kappa_2 q^*$, it holds $p \xrightarrow{\tau} \kappa q^*$, and $q^* \in G_2 \subseteq T$.

We now concentrate on item (ii) of the lemma. We recall that (S_1, T_1) and (S_2, T_2) are both min- ω -cuts. Now, for the sake of contradiction, suppose that none of $(S_1, T_1 \cup G_2)$ and $(G_1 \cup S_2, T_2)$ is a min- ω -cut of G . Take any min- ω -cut (S^*, T^*) of G . Consider the vertex, say v , that is shared by G_1 and G_2 . We analyze the cases of $v \in S^*$ and $v \in T^*$, as follows.

- a) ($v \in S^*$). Let (S_2^*, T_2^*) be the topological cut induced by (S^*, T^*) on G_2 . Then,

$$\begin{aligned} c(S_2^*, T_2^*) &= c(G_1 \cup S_2^*, T_2^*) - \omega(G_1) \\ &= c(S^*, T^*) - \omega(G_1) \\ &< c(G_1 \cup S_2, T_2) - \omega(G_1) = c(S_2, T_2). \end{aligned}$$

This contradicts with the fact that (S_2, T_2) is a min- ω -cut of G_2 .

- b) ($v \in T^*$). Let (S_1^*, T_1^*) be the topological cut induced by (S^*, T^*) on G_1 . Then,

$$\begin{aligned} c(S_1^*, T_1^*) &= c(S_1^*, T_1^* \cup G_2) \\ &= c(S^*, T^*) \\ &< c(S_1, T_1 \cup G_2) = c(S_1, T_1). \end{aligned}$$

This contradicts with the fact that (S_1, T_1) is a min- ω -cut of G_1 .

□

C Compatible orders are sufficient (Theorem 7)

In this section, we prove Theorem 7 which allows to transform subgraphs into chains using LINEARIZE before their parallel composition in Algorithm 3.

Definition C.1 (Segmentation into blocks). *Let G be an directed acyclic graph, and κ be a schedule of G . We say $\mathcal{Q}_\kappa = \{Q_1, Q_2, \dots\}$ is a segmentation into blocks on G for κ , if \mathcal{Q}_κ is a partition of the vertices of G , and for any two vertices $p \in Q_j$ and $q \in Q_k$, $p \leq_\kappa q$ implies $j \leq k$. Each $Q_j \in \mathcal{Q}_\kappa$ is called a block of \mathcal{Q}_κ . We define the highest value $H(Q_j)$ of a block $Q_j \in \mathcal{Q}_\kappa$ as*

$$H(Q_j) = \max_{q \in Q_j} \Sigma(q, \kappa).$$

Similar to that of Liu [18], we define the hill-valley segmentation of a schedule of a directed acyclic graph, as follows.

Definition C.2 (Hill-valley segmentation). *Let G be a directed acyclic graph, and τ be a schedule of G . The hill-valley segmentation $\mathcal{P}_\tau = \{P_1, P_2, \dots\}$ is a particular block segmentation into blocks on G for τ , which is defined as follows. Let $v_0 = \min^\tau G$ and*

$$h_1 = \max^\tau \{h \in G : \Sigma(h, \tau) = \rho(\tau)\}.$$

Define v_i and h_i , recursively, as

$$v_i = \max^\tau \{\arg \min_{v \geq_\tau h_i} \Sigma(v, \tau)\},$$

and

$$h_i = \max^\tau \{\arg \max_{h >_\tau v_{i-1}} \Sigma(h, \tau)\}.$$

Finally, define $P_1 = \{v : v_0 \leq_\tau v \leq_\tau v_1\}$, and $P_i = \{v : v_{i-1} < v \leq v_i\}$, for $i > 1$. We say that each $P_i \in \mathcal{P}_\tau$ is a segment of \mathcal{P}_τ . The hill-value H_i and the valley-value V_i of a segment $P_i \in \mathcal{P}_\tau$ are defined as

$$H_i = \Sigma(h_i, \tau), \text{ and } V_i = \Sigma(v_i, \tau).$$

Note that $\Sigma(p, \tau) \leq H_i$, for any $p \in P_i$.

The following property, similar to that given by Liu [18, Lemma 5.1], of the hill-valley segmentation is clear from the definition.

Lemma C.1. *Let G be a directed acyclic graph, τ be a schedule of G , and $\mathcal{P}_\tau = \{P_1, P_2, \dots, P_r\}$ be the hill-valley segmentation of G for τ . Then,*

$$H_1 > H_2 > \dots > H_r \geq V_r > \dots > V_2 > V_1.$$

Hereafter, we assume that \mathcal{P}_τ implies a hill-valley segmentation of G for a schedule τ , implicitly. Similarly, \mathcal{Q}_κ implies any arbitrary segmentation, unless explicitly stated, of G into blocks for a schedule κ .

Definition C.3 (Segment indicates block). *Let G be a directed acyclic graph, τ and κ be schedules of G . Let $P_i \in \mathcal{P}_\tau$ be a segment and $Q_j \in \mathcal{Q}_\kappa$ be a block. We write $P_i \rightarrow Q_j$, and read P_i indicates Q_j , if there is a vertex $q_j^* \in Q_j$ such that $h_i \xrightarrow{\tau} q_j^*$.*

We now demonstrate some properties of the hill-valley segmentation. For Propositions C.1–C.4, we assume G is any arbitrary directed acyclic, and τ and κ are schedules of G .

Proposition C.1. *Let $P_i \in \mathcal{P}_\tau$ be a segment and $Q_j \in \mathcal{Q}_\kappa$ be a block. If $P_i \rightarrow Q_j$, then*

- (i) $H_i \leq H(Q_j)$,
- (ii) $V_i \leq \min\{\Sigma(r, \kappa) : r \geq_\kappa q_j^o\}$, where $q_j^o = \max^\kappa Q_j$.

Proof. Assume $P_i \rightarrow Q_j$. Then, there is a $q_j^* \in Q_j$ such that $h_i \xrightarrow{\tau} q_j^*$, following the definition of dominance given in Section 4.2.1. We prove the two results separately as follows.

- (i) By the first condition (C1) of $h_i \xrightarrow{\tau} q_j^*$, we conclude

$$H_i = \Sigma(h_i, \tau) \leq \Sigma(q_j^*, \kappa) \leq H(Q_j).$$

- (ii) By the second condition (C2) of $h_i \xrightarrow{\tau} q_j^*$, we have

$$\begin{aligned} V_i &= \min\{\Sigma(s, \tau) : s \geq_\tau h_i\} \\ &\leq \min\{\Sigma(r, \kappa) : r \geq_\kappa q_j^*\} \leq \min\{\Sigma(r, \kappa) : r \geq_\kappa q_j^o\}. \end{aligned}$$

□

Proposition C.2. *Let $P_i \in \mathcal{P}_\tau$ be a segment and $Q_j \in \mathcal{Q}_\kappa$ be a block. Let $Q_k \in \mathcal{Q}_\kappa$ be another block with $k < j$. If $P_i \rightarrow Q_k$ and $P_i \not\rightarrow Q_j$, then $H(Q_j) < H_i$.*

Proof. Suppose, for the sake of contradiction, that $P_i \rightarrow Q_k$ and $P_i \not\rightarrow Q_j$ but $H(Q_j) \geq H_i$. Since $P_i \rightarrow Q_k$, there is $q_k^* \in Q_k$ such that $h_i \xrightarrow{\tau \rightarrow \kappa} q_k^*$. Then,

$$\min\{\Sigma(s, \tau) : s \geq_\tau h_i\} \leq \min\{\Sigma(r, \kappa) : r \geq_\kappa q_k^*\}.$$

Consider $q_j^* \in Q_j$ such that $\Sigma(q_j^*, \kappa) = H(Q_j)$. Then, $q_k^* \leq_\kappa q_j^*$, and thus,

$$\min\{\Sigma(s, \tau) : s \geq_\tau h_i\} \leq \min\{\Sigma(r, \kappa) : r \geq_\kappa q_j^*\}. \quad (\text{C.1})$$

Since $H_i \leq H(Q_j)$, as we have supposed for the sake of contradiction, we have

$$\Sigma(h_i, \tau) = H_i \leq H(Q_j) = \Sigma(q_j^*, \kappa). \quad (\text{C.2})$$

Then, Equations (C.1) and (C.2) together imply $h_i \xrightarrow{\tau \rightarrow \kappa} q_j^*$, which arises a contradiction with $P_i \not\rightarrow Q_j$. □

Proposition C.3. *Let $\tau \preceq \kappa$. For any segment $P_i \in \mathcal{P}_\tau$, there is a block $Q_j \in \mathcal{Q}_\kappa$ such that $P_i \rightarrow Q_j$.*

Proof. Take any $P_i \in \mathcal{P}_\tau$. Since $\tau \preceq \kappa$, we have $h_i \xrightarrow{\tau \rightarrow \kappa} q_j^*$, for some $q_j^* \in G$. For $q_j^* \in Q_j$, this implies $P_i \rightarrow Q_j$, by Definition C.3 □

Definition C.4 (Monotonic segments-to-blocks function). *A function $g : \mathcal{P}_\tau \rightarrow \mathcal{Q}_\kappa$ is called monotonic if*

- (i) $P_i \rightarrow g(P_i)$, for any $P_i \in \mathcal{P}_\tau$, and
- (ii) $Q_j = g(P_i)$ and $Q_k = g(P_{i+1})$ implies $j \leq k$.

Proposition C.4, together with Proposition C.3, proves that there is a monotonic segments-to-blocks function $g_{\tau \rightarrow \kappa}$ whenever $\tau \preceq \kappa$.

Proposition C.4. *Let $\tau \preceq \kappa$. For any consecutive segments $P_i, P_{i+1} \in \mathcal{P}_\tau$ and any block $Q_j \in \mathcal{Q}_\kappa$ with $P_i \rightarrow Q_j$, there is a block $Q_k \in \mathcal{Q}_\kappa$ such that $P_{i+1} \rightarrow Q_k$ and $j \leq k$.*

Proof. We prove by contradiction. Suppose the contrary that, $P_{i+1} \not\rightarrow Q_k$ for any $k \geq j$. Take any $Q_k \in \mathcal{Q}_\kappa$ such that $P_{i+1} \rightarrow Q_k$, which exists due to Proposition C.3. Then we have, $P_i \rightarrow Q_j$, $P_{i+1} \not\rightarrow Q_j$, $P_{i+1} \rightarrow Q_k$, and $k < j$.

- (1) $H_{i+1} < H_i$, due to Lemma C.1,
- (2) $H_i \leq H(Q_j)$, due to Proposition C.1,
- (3) $H(Q_j) < H_{i+1}$, due to Proposition C.2.

Then, we obtain $H_{i+1} < H_{i+1}$, which arises a contradiction. □

Hereafter, we focus on a particular kind of directed acyclic graph, so called half-series-parallel. We use half-series-parallel directed graphs and Lemma C.4 as building blocks of the proof of Theorem 7.

Definition C.5 (Half-series-parallel directed graph). *A directed acyclic graph H is called half-series-parallel, if there is a series-parallel directed graph G and a min- ω -cut (S, T) of G such that $H = G[T]$.*

Lemma C.2. *Let H be a half-series-parallel directed graph, and t° be the single sink vertex of H . Let τ be a schedule of H . Then, $\Sigma(p, \tau) \geq 0$, for any $p \in H \setminus \{t^\circ\}$.*

Proof. Consider a series-parallel directed graph G , a min- ω -cut (S, T) of G such that $G[T] = H$. Take any $p \in H \setminus \{t^\circ\}$. For the sake of contradiction, suppose $\Sigma(p, \tau) < 0$. Consider the set $P = \{v \in T : v \leq_\tau p\}$. Then, $(S \cup P, T/P)$ is topological and $\omega(P) = \Sigma(p, \tau) < 0$. Therefore, $c(S \cup P, T/P) = \omega(S) + \omega(P) < \omega(S) = c(S, T)$, which contradicts with the minimality of (S, T) . \square

Lemma C.3. *Let H_1 and H_2 be two half-series-parallel directed graphs, and H be the half-series-parallel directed graph obtained from H_1 and H_2 by unifying their unique sink vertex. Let λ be a schedule of H and $p^* \in H$. Consider $p_i^* = \max^\lambda\{p \in H_i : p \leq_\lambda p^*\}$, for $i = 1, 2$. If p_1^* and p_2^* both are defined, then*

$$\Sigma(p^*, \lambda) = \Sigma(p_1^*, \lambda[H_1]) + \Sigma(p_2^*, \lambda[H_2]).$$

Proof. Assume p_1^* and p_2^* are defined. Then,

$$\begin{aligned} \Sigma(p^*, \lambda) &= \omega(\{p \in H : p \leq_\lambda p^*\}) \\ &= \omega(\{p \in H_1 : p \leq_\lambda p^*\}) + \omega(\{p \in H_2 : p \leq_\lambda p^*\}) \\ &= \omega(\{p \in H_1 : p \leq_{\lambda[H_1]} p_1^*\}) + \omega(\{p \in H_2 : p \leq_{\lambda[H_2]} p_2^*\}) \\ &= \Sigma(p_1^*, \lambda[H_1]) + \Sigma(p_2^*, \lambda[H_2]). \end{aligned}$$

\square

Lemma C.4. *Let H_1 and H_2 be two half-series-parallel directed graphs, and H be the half-series-parallel directed graph obtained from H_1 and H_2 by unifying their unique sink vertex. Let τ be a schedule of H_1 minimal for \preceq . For any schedule λ of H , there is a schedule η of H such that (i) η induces τ , and (ii) $\eta \preceq \lambda$.*

Before proving Lemma C.4, we state and prove Propositions C.5–C.6, which are used in the proof of Lemma C.4. For Propositions C.5–C.6, we use the definitions in Lemma C.4. That is, H_1 and H_2 are half-series-parallel directed graphs, and H is the half-series-parallel directed graph obtained from H_1 and H_2 by unifying their sink vertex. Accordingly, λ is a schedule of H and induces κ on H_1 ($\kappa = \lambda[H_1]$), and \mathcal{Q}_κ is the segmentation of H_1 into blocks induced by κ . This particular segmentation is built such that for any $p, q \in H_1$ with $p \leq_\kappa q$, where $p \in Q_j$ and $q \in Q_k$, it holds that $j = k$ if and only if there is no $r \in H_2$ with $p \leq_\lambda r \leq_\lambda q$. We assume τ is a schedule of H_1 such that $\tau \preceq \kappa$, and $g_{\tau \rightarrow \kappa} : \mathcal{P}_\tau \rightarrow \mathcal{Q}_\kappa$ is a monotonic segments-to-block function (see Definition C.4 and the discussion after it). We introduce a schedule η which is *built upon* $g_{\tau \rightarrow \kappa}$ over λ , as follows. We replace in λ each block $Q_j \in \mathcal{Q}$ with the segments of

$$g_{\tau \rightarrow \kappa}^{-1}(Q_j) = \{P_i \in \mathcal{P}_\tau : g_{\tau \rightarrow \kappa}(P_i) = Q_j\}.$$

By virtue of the monotonic character of $g_{\tau \rightarrow \kappa}$, we end up with a schedule η that induces τ by the above-mentioned replacement. Note that η is equivalent to λ on H_2 , i.e., $\eta[H_2] = \lambda[H_2]$. Figure C.1 and C.2 illustrate the construction of η and the notations used in the following propositions. Then, there only remains to show that $\eta \preceq \lambda$ in order to prove Lemma C.4. To do so, we need Propositions C.5–C.7, concerned with the schedule η of H that is built upon $g_{\tau \rightarrow \kappa}$ over schedule λ of H .

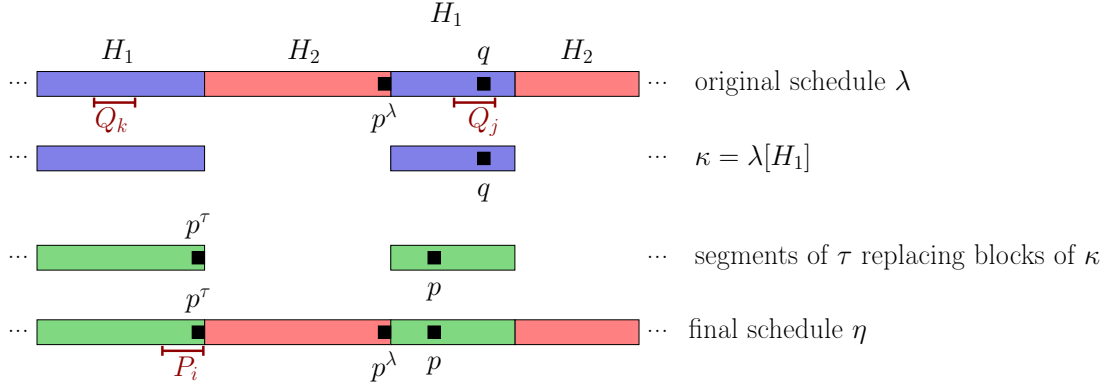


Figure C.1: Illustration of a schedule η built upon $g_{\tau \rightarrow \kappa}$ over λ , and notations used in Propositions C.5 and C.6.

Proposition C.5. Let $P_i \in \mathcal{P}_\tau$ be a segment and $Q_j \in \mathcal{Q}_\kappa$ be a block such that $g_{\tau \rightarrow \kappa}(P_i) = Q_j$. Recall from above that τ is minimal for \preceq on H_1 and λ is any schedule of H . Let κ be the schedule induced by λ on H_1 , and η be the schedule of H built upon $g_{\tau \rightarrow \kappa}$ over λ . For any $p \in P_i$ and any $q \in Q_j$, if $\Sigma(p, \tau) \leq \Sigma(q, \kappa)$, then $\Sigma(p, \eta) \leq \Sigma(q, \lambda)$.

Proof. Assume $\Sigma(p, \tau) \leq \Sigma(q, \kappa)$. Let $p^\lambda = \max^\lambda\{w \in H_2, w \leq_\lambda p\}$. Then, we have two cases to consider.

1. (p^λ is not defined). Then, $\Sigma(p, \eta) = \Sigma(p, \tau) \leq \Sigma(q, \kappa) = \Sigma(q, \lambda)$.
2. (p^λ is defined). Then, by the use of Lemma C.3, we have

$$\begin{aligned} \Sigma(p, \eta) &= \Sigma(p, \tau) + \Sigma(p^\lambda, \eta[H_2]) \\ &\leq \Sigma(q, \kappa) + \Sigma(p^\lambda, \lambda[H_2]) = \Sigma(q, \lambda). \end{aligned}$$

Note that by construction of η , $p^\lambda = \max^\lambda\{w \in H_2, w \leq_\lambda q\}$ which justifies the last equality. □

Proposition C.6. Recall that η and λ are schedules of H where η is built upon $g_{\tau \rightarrow \kappa}$ over λ using the same τ and κ as before. Let $p \in H_1$ and $p^\lambda = \max^\lambda\{w \in H_2, w \leq_\lambda p\}$. If p^λ is defined, then $\Sigma(p^\lambda, \eta) \leq \Sigma(p, \lambda)$.

Proof. Assume p^λ is defined. Let $p^\tau = \max^\tau\{w \in H_1, w \leq_\tau p\}$. We consider the following two cases, separately.

1. (p^τ is not defined). Since H_1 is half-series-parallel, $\Sigma(p, \kappa) \geq 0$, as suggested by Lemma C.2. Then, by the use of Lemma C.3, we have

$$\begin{aligned} \Sigma(p^\lambda, \eta) &= \Sigma(p^\lambda, \eta[H_2]) \\ &\leq \Sigma(p^\lambda, \lambda[H_2]) + \Sigma(p, \kappa) = \Sigma(p, \lambda). \end{aligned}$$

2. (p^τ is defined). Let $P_i \in \mathcal{P}_\tau$ such that $p^\tau \in P_i$, and notice that $p^\tau = v_i$. Similarly, let $Q_j \in \mathcal{Q}_\kappa$ such that $p \in Q_j$. Let $Q_k = g_{\tau \rightarrow \kappa}(P_i)$, and $q_k^o = \max^\kappa Q_k$. By definition of p^λ and

p^τ , we have $k < j$, and thus, $p \geq_\kappa q_k^o$. Note that $P_i \rightarrow Q_k$. Then, due to Proposition C.1, we have

$$\Sigma(p^\tau, \tau) = V_i \leq \min\{\Sigma(r, \kappa) : r \geq_\kappa q_k^o\} \leq \Sigma(p, \kappa).$$

Then, by Lemma C.3, we have

$$\begin{aligned} \Sigma(p^\lambda, \eta) &= \Sigma(p^\tau, \tau) + \Sigma(p^\lambda, \eta[H_2]) \\ &\leq \Sigma(p, \kappa) + \Sigma(p^\lambda, \lambda[H_2]) = \Sigma(p, \lambda). \end{aligned}$$

□

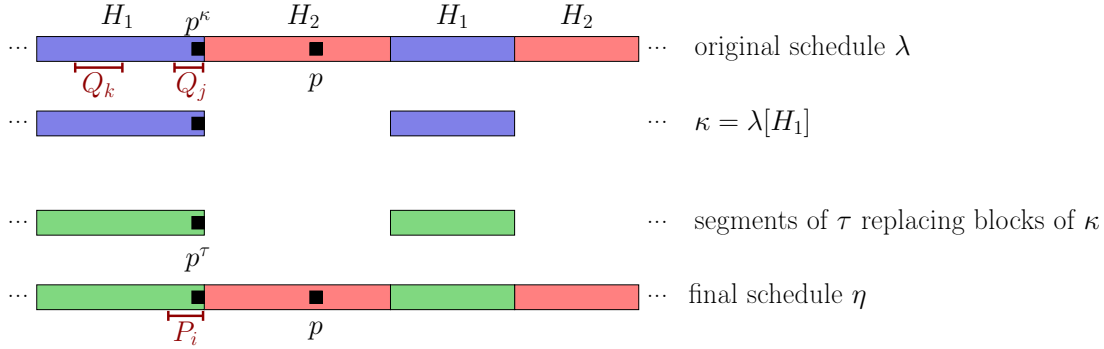


Figure C.2: Illustration of a schedule η built upon $g_{\tau \rightarrow \kappa}$ over λ , and notations used in Proposition C.7.

Proposition C.7. Recall that η and λ are schedules of H where η is built upon $g_{\tau \rightarrow \kappa}$ over λ using the same τ and κ as before. For any $p \in H_2$, we have $\Sigma(p, \eta) \leq \Sigma(p, \lambda)$.

Proof. Let $p^\kappa = \max^\lambda\{w \in H_1, w \leq_\lambda p\}$ and $p^\tau = \max^\eta\{w \in H_1, w \leq_\eta p\}$. Note that when p^κ is not defined, then p^τ is not defined either. Then, we have the following three cases to consider.

1. (Either p^κ or p^τ is not defined). Then,

$$\Sigma(p, \eta) = \Sigma(p, \eta[H_2]) = \Sigma(p, \lambda[H_2]) = \Sigma(p, \lambda).$$

2. (p^κ is defined, but p^τ is not). Since H_1 is half-series-parallel, $\Sigma(p^\kappa, \kappa) \geq 0$, as Lemma C.2 suggests. Thus, by the use of Lemma C.3, we have

$$\begin{aligned} \Sigma(p, \eta) &= \Sigma(p, \lambda[H_2]) \\ &\leq \Sigma(p, \lambda[H_2]) + \Sigma(p^\kappa, \kappa) = \Sigma(p, \kappa). \end{aligned}$$

3. (Both p^κ and p^τ are defined). Let $P_i \in \mathcal{P}_\tau$ such that $p^\tau \in P_i$, and notice that $p^\tau = v_i$. Similarly, let $Q_j \in \mathcal{Q}_\kappa$ such that $p^\kappa \in Q_j$. Let $Q_k = g_{\tau \rightarrow \kappa}(P_i)$, and $q_k^o = \max^\kappa Q_k$. By definition of p^κ and p^τ , we have $j \geq k$, and thus, $p^\kappa \geq_\kappa q_k^o$. Note that $P_i \rightarrow Q_k$. Then, due to Proposition C.1, we have

$$\Sigma(p^\tau, \tau) = V_i \leq \min\{\Sigma(r, \kappa) : r \geq_\kappa q_k^o\} \leq \Sigma(p^\kappa, \kappa).$$

Then, by Lemma C.3, we have

$$\begin{aligned} \Sigma(p, \eta) &= \Sigma(p^\tau, \tau) + \Sigma(p, \eta[H_2]) \\ &\leq \Sigma(p^\kappa, \kappa) + \Sigma(p, \lambda[H_2]) = \Sigma(p, \lambda). \end{aligned}$$

□

Now, we are ready to prove Lemma C.4.

Proof of Lemma C.4. Recall that τ is a schedule of H_1 which is minimal for \preceq , and λ is an arbitrary schedule of H that induces κ on H_1 . Since τ is minimal for \preceq , $\tau \preceq \kappa$. Then, let $g_{\tau \rightarrow \kappa}$ be a monotonic segments-to-blocks function, which exists due to Propositions C.3 and C.4. Let η be a schedule of H built upon $g_{\tau \rightarrow \kappa}$ over λ .

With the above construction (i) is proved, as η induces τ in H_1 .

We now establish (ii) $\eta \preceq \lambda$. For this, we show that for each $p \in H$, there is $q^* \in H$ such that $p \xrightarrow{\eta} q^*$. For any p , either $p \in H_1$ or $p \in H_2$. We explore these two cases separately.

a) ($p \in H_1$). Let $p \in P_i$ and $Q_j = g_{\tau \rightarrow \kappa}(P_i)$. Since $P_i \rightarrow Q_j$, there is a vertex $q_j^* \in Q_j$ such that $p \xrightarrow{\tau} q_j^*$. Now, we show that it also holds $p \xrightarrow{\eta} q_j^*$.

- 1) (C1: $\Sigma(p, \eta) \leq \Sigma(q_j^*, \lambda)$). Since $p \xrightarrow{\tau} q_j^*$, we have $\Sigma(p, \tau) \leq \Sigma(q_j^*, \kappa)$. Then, Proposition C.5 suggests that C1 is met.
- 2) (C2: $\min\{\Sigma(s, \eta) : s \geq_{\eta} p\} \leq \min\{\Sigma(r, \lambda) : r \geq_{\lambda} q_j^*\}$). We show that for each $r \geq_{\lambda} q_j^*$, there is a vertex $s \geq_{\eta} p$ such that $\Sigma(s, \eta) \leq \Sigma(r, \lambda)$. We examine three cases: $r \in Q_j$, or $r \in H_1 \setminus Q_j$, or $r \in H_2$.

Case 1. ($r \in Q_j$). Since $p \xrightarrow{\tau} q_j^*$, there is a vertex $s \geq_{\tau} p$ such that $\Sigma(s, \tau) \leq \Sigma(r, \kappa)$. We consider two scenarios of $s \in P_i$ and $s \notin P_i$.

i) ($s \in P_i$). Then, Proposition C.5 suggests

$$\Sigma(s, \eta) \leq \Sigma(r, \lambda),$$

and notice that $s \geq_{\eta} p$, as $s \geq_{\tau} p$.

ii) ($s \notin P_i$). Then, $s \geq_{\tau} v_i \geq_{\tau} p$. Since $\Sigma(v_i, \tau) \leq \Sigma(s, \tau)$, by the definition of v_i , we have $\Sigma(v_i, \tau) \leq \Sigma(r, \kappa)$. Then, Proposition C.5 suggests

$$\Sigma(v_i, \eta) \leq \Sigma(r, \lambda),$$

and notice that $v_i \geq_{\eta} p$, as $v_i \geq_{\tau} p$.

Case 2. ($r \in H_1 \setminus Q_j$). Let $r \in Q_k$ and $r^\lambda = \max^\lambda\{w \in H_2, w \leq_{\lambda} r\}$. Since $r \geq_{\kappa} q_j^*$ and $r \notin Q_j$, it holds $k > j$, and thus, r^λ is defined and $r^\lambda \geq_{\lambda} q_j^*$. Then, Proposition C.6 suggests

$$\Sigma(r^\lambda, \eta) \leq \Sigma(r, \lambda),$$

and notice that $r^\lambda \geq_{\eta} p$, as $r^\lambda \geq_{\lambda} q_j^*$.

Case 3. ($r \in H_2$). Then, Proposition C.7 suggests

$$\Sigma(r, \eta) \leq \Sigma(r, \lambda),$$

and notice that $r \geq_{\eta} p$, as $r \geq_{\lambda} q_j^*$.

b) ($p \in H_2$). We show $p \xrightarrow{\eta} p$. The first condition (C1) directly holds due to Proposition C.7. Considering the second condition (C2), we show, for any $r \geq_{\lambda} p$, there is $s \geq_{\eta} p$, such that $\Sigma(s, \eta) \leq \Sigma(r, \lambda)$. We explore the two cases of $r \in H_1$ and $r \in H_2$, separately.

Case 1. ($r \in H_1$). Let $r^\lambda = \max^\lambda\{w \in H_2, w \leq_\lambda r\}$. Since $r \geq_\lambda p$ and $p \in H_2$, r^λ is defined and $r^\lambda \geq_\lambda p$. Due to Proposition C.6, we have

$$\Sigma(r^\lambda, \eta) \leq \Sigma(r, \lambda),$$

and notice that $r^\lambda \geq_\eta p$, as $r^\lambda \geq_\lambda p$.

Case 2. ($r \in H_2$). Proposition C.7 suggests

$$\Sigma(r, \eta) \leq \Sigma(r, \lambda),$$

and notice that $r \geq_\eta p$, as $r \geq_\lambda p$.

□

We first recall Theorem 7, and show the proof as a consequence of Lemma C.4.

Theorem C.1 (Restatement of Theorem 7). *Let G_1 and G_2 be two series-parallel directed graphs, and $G = \{G_1, G_2\}$ be their parallel composition. Let (S, T) be a min- ω -cut of G , and π_1 and π_2 be schedules that are cut-optimal with the topological cuts induced by (S, T) on G_1 and G_2 , respectively. For each schedule γ in compliance with (S, T) of G , there is a schedule π in compliance with (S, T) of G such that*

- (i) π induces π_1 and π_2 ,
- (ii) $\bar{\pi}[S] \preceq \bar{\gamma}[S]$ and $\pi[T] \preceq \gamma[T]$.

Proof of Theorem 7. Take any schedule γ in compliance with (S, T) of G . Let (S_1, T_1) be topological cut induced by (S, T) on G_1 . Consider $\lambda^t = \gamma[T]$ and $\tau_1^t = \pi_1[T_1]$. Then, τ_1^t is minimal for \preceq , since π_1 is cut-optimal with (S_1, T_1) . Then, we have a schedule η^t of $G[T]$ inducing τ_1^t such that $\eta^t \preceq \lambda^t$, as suggested by Lemma C.4. Now, consider $\bar{\lambda}^s = \bar{\gamma}[S]$ and $\bar{\tau}_1^s = \bar{\pi}_1[S_1]$. Then, $\bar{\tau}_1^s$ is minimal for \preceq , since π_1 is cut-optimal with (S_1, T_1) . Then, we have a schedule $\bar{\eta}^s$ of $G[S]$ inducing $\bar{\tau}_1^s$ such that $\bar{\eta}^s \preceq \bar{\lambda}^s$, as suggested by Lemma C.4. We consider $\hat{\pi} = \langle \eta^s, \eta^t \rangle$. We notice that (i) $\hat{\pi}$ induces π_1 and $\gamma[G_2]$ on G_1 and G_2 , respectively. It also holds that (ii) $\hat{\pi}[S] \preceq \bar{\gamma}[S]$ and $\hat{\pi}[T] \preceq \gamma[T]$.

Now, apply the procedure given above on $\hat{\pi}$ (instead of γ) for π_2 (instead of π_1). Then, we obtain $\pi = \langle \hat{\eta}^s, \hat{\eta}^t \rangle$ that (i) induces $\hat{\pi}[G_1]$ and π_2 on G_1 and G_2 , respectively. It also holds that (ii) $\bar{\pi}[S] \preceq \hat{\pi}[S]$ and $\pi[T] \preceq \hat{\pi}[T]$.

As a result, (i) π induces π_1 and π_2 . Secondly, since \preceq is transitive, it holds that (ii) $\bar{\pi}[S] \preceq \bar{\gamma}[S]$ and $\pi[T] \preceq \gamma[T]$. □

D FJ-SCHEDULE is min-cut-optimal (Theorem 8)

FJ-SCHEDULE, presented in Algorithm 4, relies on TREE-SCHEDULE to compute schedules for trees that are minimal for \preceq . This latter algorithm, which we describe below, in turn relies on Liu's algorithm for trees. However, Liu's algorithm uses another model, namely the generalized pebble game: each node is provided with a number of pebbles which should be used to pebble it. As usual in pebble games, the objective is to pebble the tree up to the root using a minimal number of pebbles. In order to prove that his algorithm is optimal, Liu's relies on the notion of *Pcost* sequences. In the following, we prove that a schedule minimal for *Pcost* sequences is also minimal for our relation \preceq . The proof of Theorem D.1 thus largely relies on notations and concepts borrowed from [18]. We first introduce a restriction on the trees on which Tree-Schedule may be applied.

Definition D.1 (Liu-compatible tree). *Let T be a directed in-tree and r be its root vertex. We say T is Liu-compatible if $\omega(T[v]) \geq 0$ for each vertex $v \in T - \{r\}$, where $T[v]$ is the subtree rooted at v .*

Algorithm D.1 TREE-SCHEDULE($G, (S, T)$)

Require: $G = (V, E, \omega)$: Liu-compatible vertex-weighted tree with root r

Ensure: π : A minimal schedule for \preceq

for $v \in V$ **do**

if $v = r$ **then**

$\tau(v) \leftarrow \omega(T - \{r\})$

else

$\tau(v) \leftarrow \omega(T - \{r\}[v])$

 Call Algorithm 4.1 in [18] (“Pebble-Ordering”) on T , with “Combine” procedure from Algorithm 6.1 (ibid.) to compute schedule π

return π

Theorem D.1. *For any Liu-compatible tree T , Algorithm D.1 computes a schedule which is minimal for \preceq .*

Proof. For this proof, we use the notations given in [18], and we assume that the reader is familiar with its results. In this article, Liu considers for any vertex v of a tree a non-negative value $\tau(v)$ which represents the number of pebbles required to satisfy this node. We consider here the following pebbling function as defined in Algorithm D.1, namely:

$$\tau(v) = \begin{cases} \omega(T[v]) & v \in T - \{r\} \\ \omega(T - \{r\}) & v = r. \end{cases} \quad (\text{D.1})$$

Notice that $\tau(v) \geq 0$ for each $v \in T$, as T is Liu-compatible. Let $\text{PARENT}(v)$ represent the parent vertex of v , for a vertex $v \in T - \{r\}$. As in [18], we consider for any schedule π of T the value $\text{peb}_\pi(v)$ which represents the “total number of pebbles used during the pebbling of the vertex v ” while following schedule π .

We first prove that with the previous pebbling function, $\text{peb}_\pi(v) = \Sigma(v, \pi)$, for each $v \in T - \{r\}$. Take any $v \in T - \{r\}$. Let $F_v = \{r \in T : r \leq_\pi v <_\pi \text{PARENT}(r)\}$, that is, the set populated by the root vertices of the pruned forest. Then,

$$\text{peb}_\pi(v) = \sum_{r \in F_v} \tau(r) = \sum_{r \in F_v} \omega(T[r]) = \omega(\{u \leq_\pi v\}) = \Sigma(v, \pi).$$

We then consider the $P\text{cost}(\pi)$, the cost sequence of a schedule π , as defined in [18]. We prove that if $P\text{cost}(\pi) \prec P\text{cost}(\gamma)$ then $\pi \preceq \gamma$, for two schedules π and γ of T .

Assume $P\text{cost}(\pi) \prec P\text{cost}(\gamma)$. Now, for each vertex $p \in T$, we need to exhibit a vertex $q^* \in T$ such that $p \xrightarrow{\pi \rightarrow \gamma} q^*$. Since $r = \max^\pi T = \max^\gamma T$, we have $r \xrightarrow{\pi \rightarrow \gamma} r$. Now, take any vertex $p \in T - \{r\}$. Let p reside in the i^{th} hill-valley segment with respect to π . By definition of \prec in [18, Section 5.2], there exists j such that $\tilde{H}_i \leq H_j$, and $\tilde{V}_i \leq V_j$, where $(\tilde{H}_i, \tilde{V}_i)$ and (H_j, V_j) are the hill-valley values for i^{th} and j^{th} segments with respect to schedules π and γ , respectively. Consider a vertex $q^* \in T - \{r\}$ so that q^* resides in the j^{th} hill-valley segment with respect to γ , and $\text{peb}_\gamma(q^*) = H_j$, where such q^* exists due to Equation D.1. Now, we show $p \xrightarrow{\pi \rightarrow \gamma} q^*$ as follows.

- 1) (C1: $\Sigma(p, \pi) \leq \Sigma(q^*, \gamma)$). Recall that $peb_\pi(p) = \Sigma(p, \pi)$ and $peb_\gamma(q^*) = \Sigma(q^*, \gamma)$. Then, we have:

$$\Sigma(p, \pi) = peb_\pi(p) \leq \tilde{H}_i \leq H_j = peb_\gamma(q^*) = \Sigma(q^*, \gamma).$$

- 2) (C2: $\min\{\Sigma(s, \pi) : s \geq_\pi p\} \leq \min\{\Sigma(r, \gamma) : r \geq_\gamma q^*\}$). Recall that $peb_\pi(v) = \Sigma(v, \pi)$ and $peb_\gamma(v) = \Sigma(v, \gamma)$, for each $v \in T - \{r\}$. By definition of \tilde{V}_i and V_j ,

$$\min\{\Sigma(s, \pi) : s \geq_\pi p\} = \tilde{V}_i \leq V_j = \min\{\Sigma(r, \gamma) : r \geq_\gamma q^*\}.$$

Thanks to [18, Theorem 6.4], we know that Algorithm 4.1 in [18] with Combine procedure 6.1 (ibid.) computes a schedule π of T so that $Pcost(\pi) \prec Pcost(\gamma)$, for any schedule γ of T . As a corollary, π is minimal for \preceq on T . \square

Theorem D.2 (Restatement of Theorem 8). *For any fork-join graph G and any min- ω -cut (S, T) of G , FJ-SCHEDULE computes a schedule π that is cut-optimal with (S, T) .*

Proof of Theorem 8. Let G be a fork-join directed graph and (S, T) be a min- ω -cut of G . We first check that both $\tilde{G}[S]$ and $G[T]$ are Liu-compatible trees.

We only show that $G[T]$ is Liu-compatible, as the other can be proven similarly. For the sake of contradiction, suppose that $G[T]$ is not Liu-compatible. Then, there is a vertex, say v , such that $\omega(T[v]) < 0$. Consider the topological cut (S', T') , where $S' = S \cup T[v]$ and $T' = G - S'$. Then, $c(S', T') = \omega(S') = \omega(S) + \omega(T[v]) < \omega(S) = c(S, T)$. This contradicts the fact that (S, T) is a min- ω -cut of G .

The first steps of FJ-SCHEDULE as described in Algorithm 4 is to compute schedules $\bar{\sigma}$ and τ for $\tilde{G}[S]$ and $G[T]$ using TREE-SCHEDULE. Thanks to Theorem D.1, we know that these schedules are minimal for \preceq .

Then, by definition, $\pi = \langle \bar{\sigma}, \tau \rangle$ is cut-optimal with (S, T) , as π is in compliance with (S, T) , $\bar{\sigma}$ is minimal for \preceq on $\tilde{G}[S]$, and τ is minimal for \preceq on $G[T]$. \square

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