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► **To cite this version:**

Julio Araujo, Guillaume Ducoffe, Nicolas Nisse, Karol Suchan. On interval number in cycle convexity. [Research Report] Inria Sophia Antipolis; I3S. 2016. hal-01394201v1

**HAL Id: hal-01394201**

**<https://inria.hal.science/hal-01394201v1>**

Submitted on 8 Nov 2016 (v1), last revised 24 Apr 2018 (v4)

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# On interval number in cycle convexity\*

Julio Araujo<sup>†</sup>    Guillaume Ducoffe<sup>‡</sup>    Nicolas Nisse<sup>‡</sup>    Karol Suchan<sup>§¶</sup>

## Abstract

Many notions in *graph convexity* have been defined and studied for various applications, such as *geodesic convexity* (generalizing the classical convexity in Euclidean space to graphs), *monophonic convexity* (to model spreading of rumor or disease in a network), etc. Each of the convexity notions led to the study of important graph invariants such as the *hull number* (minimum number of vertices whose *hull set* is the entire graph) or the *interval number* (minimum number of vertices whose *interval* is the whole graph).

Recently, Araujo et al. introduced the notion of Cycle Convexity of graphs for its application in Knot Theory. Roughly, the *tunnel number* of a knot embedded in a plane is equivalent to the hull number of a corresponding planar 4-regular graph in cycle convexity. In this paper, we study the interval number of a graph in cycle convexity. Precisely, given a graph  $G$ , its *interval number* in cycle convexity, denoted by  $in_{cc}(G)$ , is the minimum cardinality of a set  $S \subseteq V(G)$  such that every vertex  $w \in V(G) \setminus S$  has two distinct neighbors  $u, v \in S$  such that  $u$  and  $v$  lie in same connected component of  $G[S]$ .

In this work, first we provide bounds on  $in_{cc}(G)$  and its relations to other graph convexity parameters, and explore its behavior on grids. Then, we present some hardness results by showing that deciding whether  $in_{cc}(G) \leq k$  is NP-complete, even if  $G$  is a split graph or a bounded-degree planar graph, and that the problem is W[1]-hard in bipartite graphs when  $k$  is the parameter. As a consequence, we obtain that it cannot be approximated up to a constant factor in the class of split graphs (unless  $P = NP$ ).

On the positive side, we present polynomial-time algorithms to compute  $in_{cc}(G)$  for outerplanar graphs, cobipartite graphs and interval graphs. We also present FPT algorithms to compute it for  $(q, q - 4)$ -graphs, where  $q$  is the parameter and for bounded treewidth graphs.

## 1 Introduction

Many different notions of convexity have been proposed in the context of graphs [Var76, Jam82, FJ86, Duc88, CM99], depending on what has been modeled. A convexity in graphs can be roughly described as the rules of propagation of an infection process on the vertices. As an example, communities in social networks can be modeled as small-diameter convex subgraphs w.r.t. the standard geodesic convexity (defined in Sec. 1.1) [WF94]. Recently, *cycle convexity* in graphs has been defined for its applications in Knot Theory [ACG<sup>+</sup>16]. This paper is dedicated to the study of the *interval number* of graphs in cycle convexity. Before going into the details of cycle convexity and of the corresponding interval number of a graph, let us recall general definitions of convexity and graph convexity.

A *convexity space* is an ordered pair  $(V, \mathcal{C})$ , where  $V$  is a non-empty finite set and  $\mathcal{C}$  is a family of subsets of  $V$ , called *convex sets*, that satisfies:

**(C1)**  $\emptyset, V \in \mathcal{C}$ ; and

**(C2)**  $C \cap C' \in \mathcal{C}$ , for all  $C, C' \in \mathcal{C}$ .

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\*This work has been partially supported by ANR project Stint (ANR-13-BS02-0007), the associated Inria team AIDyNet, the project ECOS-Sud Chile.

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Given a subset  $C \subseteq V$ , the *convex hull of  $C$*  (with respect to  $(V, \mathcal{C})$ ) is the unique inclusion-minimum  $C' \in \mathcal{C}$  containing  $C$ , and it is denoted by  $\text{hull}_{(V, \mathcal{C})}(C)$ . If  $\text{hull}_{(V, \mathcal{C})}(C) = V$ , then  $C$  is said to be a *hull set* of  $(V, \mathcal{C})$ . The *hull number of  $V$  with respect to  $\mathcal{C}$*  is the size of a minimum hull set, and is denoted by  $hn(V, \mathcal{C})$ .

When studying Euclidean spaces, the set  $V$  is  $\mathbb{R}^n$  and  $\mathcal{C}$  contains all the convex subsets of  $\mathbb{R}^n$ . Recall that a set  $C \subseteq \mathbb{R}^n$  is *convex* if, for any two points  $p_1$  and  $p_2$  in  $C$ , any convex combination of  $p_1$  and  $p_2$  also belongs to  $C$ . In other words,  $C$  is convex if all the points on the line segment between  $p_1$  and  $p_2$  also belong to  $C$ , for every pair of points  $p_1, p_2 \in C$ . Remember that the convex hull of a set of points  $C$  in  $\mathbb{R}^n$  can be computed by recursively augmenting  $C$  with any point that lies on a segment between two points already in  $C$ . Metaphorically, it can be seen as a contagion (or percolation) process starting from the initial set  $C$  that infects (or percolates) the whole convex hull  $\text{hull}_{(V, \mathcal{C})}(C)$ .

These notions were brought to graph theory in the 80's [FJ86, Duc88] by taking the vertex set of an input graph  $G = (V, E)$  as the set  $V$ . The set  $\mathcal{C}$  of convex sets depends on the definition of the ‘‘contagion process’’ that is defined in terms of an *interval function*. Let us now present some formal definitions concerning graph convexities and their parameters.

## 1.1 Related work

**General approach.** Let us first give a general definition of  $f$ -convexity according to some function  $f$ .

Let  $G = (V, E)$  be a finite graph. An *interval function*  $f : 2^V \times V \rightarrow 2^V$  takes a subset  $X \subseteq V$ , the set of *infected* vertices, and a vertex  $v \in V \setminus X$ , a *healthy* vertex, as an input and returns either  $X \cup \{v\}$ , in which case the vertex has been infected or  $X$ , when  $x$  remains healthy. That is, for any  $X \subseteq V$  and  $v \in V \setminus X$ ,  $f(X, v) \in \{X, X \cup \{v\}\}$ . If  $v \in f(X, v)$ ,  $v$  is said to be *generated* by  $X$ .

For any  $X \subseteq V$ , let us define the *interval  $f(X)$*  of  $X$  as  $\bigcup_{v \in V \setminus X} f(X, v)$ , i.e.  $f(X)$  consists of  $X$  plus any vertex that is generated by  $X$  according to  $f$ . We can then define a convexity space  $(V, \mathcal{C}_f)$  with  $\mathcal{C}_f \subseteq 2^V$  containing the fixed points of  $f$ . The interval  $f(X)$  of  $X$  is also called the set *generated* by  $X$ , and  $X$  is a *generator* of  $f(X)$ . We say that a set  $X \subseteq V$  is a *generator* (a.k.a. *interval set*) for  $G$  if  $f(X) = V$ . A generator  $X$  of  $f(X)$  such that no proper subset of  $X$  is a generator of  $f(X)$  is called a *minimal generator*. A minimum cardinality generator is called a *minimum generator*.

Intuitively, a set  $X$  generates  $f(X)$  in one step. The infection process may continue applying recursively  $f$  to the new set of generated vertices. The *convex hull  $f^\infty(X)$*  of a set  $X \subseteq V$  (with respect to  $f$ ) is the set of all vertices that will eventually be generated when  $X$  is the set of initially infected vertices. More formally,  $f^\infty(X) = \lim_{i \rightarrow \infty} f^i(X)$ . A set  $X$  is called a *recursive generator* of  $f^\infty(X)$ . If  $f^\infty(X) = V$ ,  $X$  is called a *recursive generator* (a.k.a. *hull set*) for  $G$ .

For a graph  $G$  and an interval function  $f$ , several natural questions can be asked. For instance:

- (a) What is the minimum size of a generator for  $G$ , i.e., the minimum size of a set  $X \subseteq V$  such that  $f(X) = V$ ? This graph invariant is called the *interval number*  $\square$  of  $G$  in the convexity defined by the function  $f$ , denoted here by  $in_f(G)$ .
- (b) What is the minimum size of a recursive generator for  $G$ , i.e., the minimum size of a set  $X \subseteq V$  whose convex hull is  $V$ ? This graph invariant is called the *hull number*  $\square$  of  $G$  (with respect to  $f$ ), denoted here by  $hn_f(G)$ .

Many other graph-convexity parameters have been considered in the literature like *convexity number*  $\square$ , *Radon number* (e.g., [DRdS<sup>+</sup>12]), *Carathéodory number* (e.g., [DRdS<sup>+</sup>13]), *rank number*  $\square$ , etc.

Several interval functions have been proposed and studied. For instance:

**Geodetic convexity** [DGK<sup>+</sup>09, ACG<sup>+</sup>13]: two contaminated vertices infect any vertex on a shortest path between them. This is similar to the classic notion of convexity in an Euclidean space, if one sees the segment between two points as the shortest path between them. This function is denoted here by *sp*. That is,  $sp(X, v) = X \cup \{v\}$  if and only if  $v$  belongs to a shortest path between two vertices of  $X$ . In Section 2.2, the corresponding hull number (resp., interval number) of a graph  $G$  is denoted by  $hn_{sp}(G)$  (resp.,  $in_{sp}(G)$ ).

**Monophonic convexity** [Jam82, DPS10]: two contaminated vertices infect any vertex on an induced path between them. The function is denoted by  $m$  and  $m(X, v) = X \cup \{v\}$  if and only if  $v$  belongs to an induced path between two vertices of  $X$ . In Section 2.2, the corresponding hull number (resp., interval number) of a graph  $G$  is denoted by  $hn_m(G)$  (resp.,  $in_m(G)$ ).

**$P_3$ -convexity** [Var76, CDD<sup>+</sup>10]: a vertex is contaminated if it has two contaminated neighbors, i.e.  $P_3(X, v) = X \cup \{v\}$  if and only if  $|N(v) \cap X| \geq 2$ . In Section 2.2, the corresponding hull number (resp., interval number) of a graph  $G$  is denoted by  $hn_{P_3}(G)$  (resp.,  $in_{P_3}(G)$ ).

**$P_3^*$ -convexity** []: a vertex is contaminated if it has two contaminated non-adjacent neighbors, i.e.  $P_3^*(X, v) = X \cup \{v\}$  if and only if there exist  $x, y \in X \cap N(v)$ ,  $x \neq y$  and  $\{x, y\} \notin E(G)$ . In Section 2.2, the corresponding hull number (resp., interval number) of a graph  $G$  is denoted by  $hn_{P_3^*}(G)$  (resp.,  $in_{P_3^*}(G)$ ).

Recently, a new graph convexity called *cycle convexity* has been defined, motivated by its relationship with a parameter in Knot Theory [ACG<sup>+</sup>16].

**Definition 1** (Cycle convexity). In cycle convexity, a vertex is contaminated if it has two *distinct* neighbors in a same connected component of the graph induced by the infected vertices. The corresponding interval function is denoted by  $cc$ , and  $cc(X, v) = X \cup \{v\}$  if and only if there is a path  $P$  of length at least 2 in the subgraph  $G[X]$  induced by  $X$ , such that both endpoints of  $P$  are neighbors of  $v$ .

In [ACG<sup>+</sup>16], it is shown that the hull number in cycle convexity of a 4-regular planar graph provides an upper bound for the *tunnel number* of the associated knot or link embedded in the 3-dimensional sphere.<sup>1</sup> A tight upper bound for  $hn_{cc}(G)$  is provided, when  $G$  is a 4-regular planar graph and it is proven that it is NP-complete to determine whether  $hn_{cc}(G) \leq k$ , given a planar graph  $G$  and a positive integer  $k$ .

## 1.2 Our results

In this paper, we further study this new graph convexity and focus on the corresponding interval number.

Section 2 is devoted to some general basic results.

Section 3 mainly covers the interval number  $in_{cc}$  of square grids. We show that, for any grid  $G_n$  with  $n$  vertices,  $n/2 \leq in_{cc}(G_n) \leq n/2 + o(n)$ . This result is mainly based on general non-trivial bounds on the interval number in cycle convexity.

In Section 4, we show that deciding whether  $in_{cc}(G) \leq k$  is NP-hard when  $k$  is part of the input, and  $G$  belongs to the class of split graphs, or bipartite graphs, or planar graphs with maximum degree 6. The results for split graphs and for bipartite graphs are even stronger because both reductions are FPT-reductions from the Dominating Set problem. Thus, determining  $in_{cc}(G)$  for any  $G$  in these subclasses of graphs is  $W[2]$ -hard.

On the positive side, in Section 5, we present polynomial-time algorithms to compute  $in_{cc}(G)$  for outerplanar graphs, cobipartite graphs and interval graphs. Finally, Section 6 is devoted to parameterized algorithms. We present FPT algorithms to compute  $in_{cc}(G)$  for  $(q, q-4)$ -graphs, where  $q$  is the parameter and for bounded treewidth graphs.

## 2 Preliminaries

For basic notions in graph theory, convexity in euclidean spaces, computational complexity and parameterized complexity, the reader is referred to [BM08, FG06, Roc70, GJ79].

All graphs considered in this paper are simple (i.e., without parallel edges nor loops). Unless explicitly stated, graphs considered in this paper are connected.

Throughout the paper, we use the following notations. Let  $G = (V, E)$  be a graph. For any  $v \in V$ ,  $N(v) \subseteq V$  denotes the set of neighbors of  $v$ , and  $N[v] = \{v\} \cup N(v)$  is the closed neighborhood of  $v$ . Let

<sup>1</sup>Formal definitions of tunnel, knot or link are technical and out of the scope of this paper. We refer to [] for the interested reader.

$E_v \subseteq E$  denote the set of edges incident to  $v$ . For any  $X \subseteq V$ ,  $N(X) = \bigcup_{v \in X} N(v) \setminus X$ , i.e. it is the set of neighbors of the vertices in  $X$  that are out of  $X$ , and  $N[X] = N(X) \cup X$ .  $G[X]$  is the subgraph induced by  $X$  in  $G$ . A vertex  $v \in V$  is *universal* if  $N[v] = V$ .

In this paper, we consider *cycle convexity*, where a vertex gets contaminated if it is adjacent to two *distinct* vertices of a same contaminated connected component. More formally, for any  $X \subset V$  and  $v \in V \setminus X$ , let  $cc(X, v) = \{v\} \cup X$  if and only if there is a connected component  $A$  of  $G[X]$  such that  $|A \cap N(v)| \geq 2$ . Otherwise,  $cc(X, v) = X$ . This means that the vertex  $v$  is generated by  $X$  if and only if there is a cycle  $C$  in  $G$  containing  $v$  and such that  $V(C) \subseteq X \cup \{v\}$  (note that such a cycle is not necessarily unique). We say that this cycle  $C$  generates  $v$  for the set  $X$ .

Let us put  $cc(X) = \bigcup_{v \in V \setminus X} cc(X, v)$ . The interval number  $in_{cc}(G)$  of  $G$  (in cycle convexity) is the smallest integer  $k$  such that there exists  $X \subseteq V$  with  $|X| \leq k$  and  $cc(X) = V$ . Finally, the hull number  $hn_{cc}(G)$  of  $G$  (with respect to cycle convexity) is the smallest integer  $k$  such that there exists  $X \subseteq V$ ,  $|X| \leq k$  and  $cc^\infty(X) = \lim_{i \rightarrow \infty} cc^i(X) = V$ .

## 2.1 Basic properties

Let us start with some simple observations that are used all along the paper.

The following two claims are straightforward.

**Claim 2.** *Let  $G = (V, E)$  be a simple graph and  $X \subseteq V$ .  $cc(X) = V$  if and only if, for every  $v \in V \setminus X$ , there exist  $x, y \in X \cap N(v)$ ,  $x \neq y$ , and  $x, y$  belong to a same connected component of  $G[X]$ .*

**Claim 3.** *For any  $n$ -node forest  $T$ ,  $in_{cc}(T) = n$ .*

*For any  $n$ -vertex cycle  $C_n$ ,  $n \geq 3$ ,  $in_{cc}(C_n) = n - 1$ .*

*For any complete  $n$ -vertex graph  $K_n$ ,  $n \geq 2$ ,  $in_{cc}(K_n) = 2$ .*

**Theorem 4.** *For any simple  $n$ -vertex  $m$ -edge connected graph  $G$ ,  $in_{cc}(G) = 2$  if and only if  $G$  has two distinct universal vertices. It can be decided in  $\mathcal{O}(n + m)$ -time.*

*Proof.* First, let us assume that  $G = (V, E)$  admits a generator  $X = \{u, v\} \subseteq V$  (with  $u \neq v$ ), i.e.  $cc(X) = V$ . If  $|V| = 2$ , the result follows since  $G$  is connected. Let us assume that  $|V| > 2$ . By Claim 2, for any  $w \in cc(X) \setminus X$ ,  $w$  has two neighbors in a same contaminated connected component. This directly implies that  $\{u, v\} \subseteq N(w)$  and  $\{u, v\} \in E$ . Hence,  $u$  and  $v$  are universal.

Reciprocally, if  $G$  has two (distinct) universal vertices  $u$  and  $v$ , then  $X = \{u, v\}$  is clearly a generator for  $G$ , since any vertex of  $V \setminus X$  is in a triangle with  $u$  and  $v$ . Hence,  $in_{cc}(G) \leq 2$  and the equality follows since  $|V| > 1$ .

Note the existence of two universal vertices can be checked in  $\mathcal{O}(n + m)$ -time, simply by computing the degree distribution.  $\square$

## 2.2 Relationship with other graph convexities

First, let us compare the hull number and interval number in cycle convexity with the corresponding invariants in geodetic and monophonic convexity.

In what follows, we say that two parameters are not comparable if none of the two can be upper-bounded by a function of the other for all graphs. By extension, we say that two convexities are not comparable if their respective hull number and interval number are not comparable.

**Lemma 5.** *Cycle convexity is not comparable to geodetic nor monophonic convexity.*

*Proof.* Let  $C_n$  be an  $n$ -vertex cycle. It is easy to check that:  $hn_m(C_n) = in_m(C_n) = 2$  and  $hn_{sp}(C_n) = in_{sp}(C_n) = 2$  if  $n$  is even, and  $hn_{sp}(C_n) = in_{sp}(C_n) = 3$  otherwise. For any  $n$ -vertex complete graph  $K_n$ ,  $hn_{sp}(K_n) = hn_m(K_n) = in_{sp}(K_n) = in_m(K_n) = n$ . Combining the above equalities with Claim 3, we get that interval and hull numbers in cycle-convexity and geodetic (resp., monophonic) convexity are not comparable.  $\square$

Then, let us compare the hull number and interval number in cycle convexity with the corresponding invariants in  $P_3$  (resp.,  $P_3^*$ )-convexity.

**Lemma 6.** *In any graph  $G$ ,*

$$\begin{aligned} hn_{P_3}(G) &\leq hn_{cc}(G) \leq 2 \cdot hn_{P_3}(G) - 1 \leq 2 \cdot hn_{P_3^*}(G) - 1, \text{ and} \\ in_{P_3}(G) &\leq in_{cc}(G) \leq 2 \cdot in_{P_3}(G) - 1 \leq 2 \cdot in_{P_3^*}(G) - 1. \end{aligned}$$

*Proof.* It directly follows from Claim 2 and the definition of  $P_3$ -convexity that  $in_{P_3}(G) \leq in_{cc}(G)$  for any graph  $G$ . By a similar argument, it is easy to show that, for any  $i \geq 1$  and for any  $X \subseteq V(G)$ ,  $cc^i(X) \subseteq P_3^i(X)$ . Hence,  $hn_{P_3}(G) \leq hn_{cc}(G)$  for any graph  $G$ .

Note that  $hn_{P_3}(G) \leq hn_{P_3^*}(G)$  and  $in_{P_3}(G) \leq in_{P_3^*}(G)$  for any graph  $G$  []. Hence, it only remains to prove the upper bounds for cycle-convexity.

For any  $X \subseteq V(G)$ , let  $P_3(X)$  be the set of vertices obtained by adding to  $X$  all vertices that have at least two neighbors in  $X$ . That is,  $P_3(X)$  is the set of vertices generated by  $X$  in  $P_3$ -convexity. For any  $i \geq 2$ , let  $P_3^i(X) = P_3^{i-1}(P_3(X))$  and  $P_3^1(X) = P_3(X)$ .

Let  $X \subseteq V(G)$  be any recursive generator for  $G$  in  $P_3$ -convexity, i.e., such that  $P_3^\infty(X) = \lim_{i \rightarrow \infty} P_3^i(X) = V(G)$ . Let  $c \geq 1$  be the number of connected components of  $G[X]$ . We prove by induction on  $c$  that there exists  $Y \subseteq V(G) \setminus X$ ,  $|Y| \leq c - 1$  and  $Y \cup X$  recursively generates  $G$  in cycle-convexity. This implies that  $hn_{cc}(G) \leq |X| + c - 1 \leq 2|X| - 1$ . Since it holds for any recursive generator  $X$  in  $P_3$ -convexity, this will imply that  $hn_{cc}(G) \leq 2 \cdot hn_{P_3}(G) - 1$ .

First, let us assume that  $c = 1$ . We prove by induction on  $i$  that, for any  $i \geq 1$ ,  $P_3^i(X) = cc^i(X)$ . Indeed, at each step,  $P_3^i(X)$  induces a connected subgraph of  $G$ . Moreover, any vertex  $v$  that gets generated (in  $P_3$ -convexity) in  $P_3^i(X)$  has at least two neighbors in  $P_3^{i-1}(X)$ , thus, by Claim 2, we also have  $v \in cc(P_3^{i-1}(X))$ . Hence,  $P_3^i(X) \subseteq cc^i(X)$ . By similar arguments,  $cc^i(X) \subseteq P_3^i(X)$ . Hence, if  $c = 1$ , then  $hn_{cc}(G) \leq hn_{P_3}(G)$ .

Now, let us assume that  $G[X]$  has  $c > 1$  connected components. Let us assume that  $X$  does not recursively generate  $G$  in cycle-convexity and let  $y \in V(G) \setminus X$  be one of the first vertices that are not generated. More precisely, let  $i$  be the minimum integer such that  $P_3^i(X) \setminus cc^i(X) \neq \emptyset$  ( $i$  exists since  $X$  recursively generates  $V(G)$  in  $P_3$ -convexity) and let  $y \in P_3^i(X) \setminus cc^i(X)$ . Because  $y$  is generated by  $P_3^{i-1}(X)$  in  $P_3$ -convexity, but not in cycle-convexity, it must have two neighbors in two distinct contaminated connected components of  $G[P_3^{i-1}(X)]$ . Note also that  $G[P_3^{i-1}(X)]$  has at most  $c$  connected components, since new components cannot be created during the infection process. Hence,  $P_3^{i-1}(X) \cup \{y\}$  recursively generates  $G$  in  $P_3$ -convexity and has at most  $c - 1$  connected components. By the induction hypothesis, there is  $Y \subseteq V(G) \setminus (P_3^{i-1}(X) \cup \{y\})$  and  $|Y| \leq c - 2$  such that  $Y \cup (P_3^{i-1}(X) \cup \{y\})$  recursively generates  $G$  in cycle-convexity. Finally, since  $P_3^{i-1}(X) = cc^{i-1}(X)$  (by definition of  $i$  and by similar argument as in paragraph above), we have that  $X \cup Y \cup \{y\}$  recursively generates  $G$  in cycle convexity and  $|Y \cup \{y\}| \leq c - 1$ .

The proof of the fact that  $in_{cc}(G) \leq 2 \cdot in_{P_3}(G) - 1$  is similar. □

Notice that the bounds of Lemma 6 are sharp, since on the one hand we have  $in_{cc}(K_n) = in_{P_3}(K_n) = 2$  for any  $n$ -vertex clique  $K_n$  with  $n \geq 2$ , while on the other hand we have  $in_{cc}(P_{2n+1}) = 2n + 1$  and  $in_{P_3}(P_{2n+1}) = in_{P_3^*}(P_{2n+1}) = n + 1$  for any  $(2n + 1)$ -node path  $P_{2n+1}$  (the same relationships hold for hull number). By the previous lemma, combined with all hardness and inapproximability results on  $P_3$ -convexity [] we have the following.

**Theorem 7.** *The parameters  $hn_{cc}$  and  $in_{cc}$  cannot be approximated up to a constant factor in polynomial-time (unless  $P=NP$ ).*

In the remaining part of the paper, we only consider cycle convexity.

### 3 General bounds and Grids

This section is devoted to general lower bounds on the interval number of a graph (in cycle convexity), depending on its number of vertices and maximum degree. These lower bounds are then improved in the case of grids.

### 3.1 General lower bounds

Let  $X$  be a generator for a graph  $G$ . For any  $v \in V \setminus X$ , let  $g_X(v)$  be two arbitrary distinct vertices in  $N(v) \cap X$  that are in a same connected component of  $G[X]$  (two such vertices must always exist by Claim 2). When  $X$  is clear from the context, we write  $g(v)$  instead of  $g_X(v)$ . Note that the two vertices in  $g(v)$  are the ends of a path in  $X$  which generates the vertex  $v$ .

**Theorem 8.** *For any  $n$ -vertex graph  $G$  with maximum degree  $\Delta$ ,  $in_{cc}(G) \geq \frac{2n}{\Delta+1}$ .*

*Proof.* Let  $X$  be a generator for  $G$ . Let  $x \in X$  be an isolated vertex in  $G[X]$  (i.e.,  $x$  has no neighbors in  $X$ ). Then, for any  $v \in V \setminus X$ ,  $x \notin g(v)$  since  $x$  cannot generate any vertex. For any vertex  $x \in X$  with at least one neighbor in  $X$ , there are at most  $\Delta - 1$  vertices  $v$  in  $V \setminus X$  such that  $x \in g(v)$  (since  $x$  has at most  $\Delta - 1$  neighbors not in  $X$ ).

Hence, for any vertex  $x \in X$ , there are at most  $\Delta - 1$  vertices  $v$  in  $V \setminus X$  such that  $x \in g(v)$ . That is,  $\bigcup_{v \in V \setminus X} g(v) \subseteq X$  and any  $x \in X$  appears in at most  $\Delta - 1$  sets  $g(v)$ . Moreover, recall that  $|g(v)| = 2$  for any  $v \notin X$ .

Therefore,  $2|V \setminus X| = 2(n - |X|) = \sum_{v \in V \setminus X} |g(v)| \leq (\Delta - 1)|X|$ . Hence,  $|X| \geq \frac{2n}{\Delta+1}$ .  $\square$

In the next theorem, the above bound is refined using the structure of the subgraph induced by the generator. It is easy to see that this bound is tight by considering, for instance, any cycle.

**Theorem 9.** *For any  $n$ -vertex graph  $G$  with maximum degree  $\Delta$ , and for any generator  $X$  for  $G$  such that  $G[X]$  has  $r \geq 1$  connected components:  $|X| \geq \frac{2(n-r)}{\Delta}$ .*

*Proof.* Let  $x \in X$  and let  $deg_X(x) \geq 0$  denote its degree in  $G[X]$ . For any  $0 \leq i \leq \Delta$ , let  $X_i$  be the subset of vertices of  $X$  with degree  $i$  in  $G[X]$ , i.e.  $X_i = \{x \in X \mid deg_X(x) = i\}$ . As in the proof of Theorem 8, if  $x \in X_0$ , then  $x$  cannot appear in any set  $g(v)$ ,  $v \in V \setminus X$ . If  $x \in X_i$  for  $i > 0$ , then  $x$  has at most  $\Delta - i$  neighbors in  $V \setminus X$  and, therefore, it appears in at most  $\Delta - i$  sets  $g(v)$ ,  $v \in V \setminus X$ .

Hence, we get that  $2(n - |X|) = \sum_{v \in V \setminus X} |g(v)| \leq \sum_{1 \leq i \leq \Delta} (\Delta - i)|X_i|$ .

Let  $C$  be any connected component of  $G[X]$  and let  $X_i^C = X_i \cap C$ . Moreover, let  $\mathcal{C}_X$  be the set of connected components of  $G[X]$  that are not reduced to a single vertex. The previous inequality can be rewritten as:

$$\sum_{v \in V \setminus X} |g(v)| \leq \sum_{C \in \mathcal{C}_X} \left( \sum_{1 \leq i \leq \Delta} (\Delta - i)|X_i^C| \right). \quad (1)$$

Let us prove that, for any  $C \in \mathcal{C}_X$ ,

$$\sum_{1 \leq i \leq \Delta} (\Delta - i)|X_i^C| \leq 2(\Delta - 1) + (\Delta - 2)(|C| - 2). \quad (2)$$

In order to prove Inequality 2, let us prove more generally that for every subset  $C' \subseteq C$  inducing a connected subgraph in  $G[C]$ .

$$\sum_{1 \leq i \leq \Delta} (\Delta - i)|X_i^{C'}| \leq 2(\Delta - 1) + (\Delta - 2)(|C'| - 2),$$

where  $X_i^{C'}$  denotes the vertices of  $C'$  with degree  $i$  in  $G[C']$  (i.e., with  $i$  neighbors in  $C'$ ). The proof is by induction on  $|X_1^{C'}|$ . If there are at most 2 vertices with degree 1 in  $C'$ , i.e.  $|X_1^{C'}| \leq 2$ , then the result holds since  $\sum_{1 \leq i \leq \Delta} (\Delta - i)|X_i^{C'}| = (\Delta - 1)|X_1^{C'}| + \sum_{2 \leq i \leq \Delta} (\Delta - i)|X_i^{C'}| \leq (\Delta - 1)|X_1^{C'}| + (\Delta - 2) \sum_{2 \leq i \leq \Delta} |X_i^{C'}|$ , and  $\sum_{2 \leq i \leq \Delta} |X_i^{C'}| = |C'| - |X_1^{C'}|$ .

Else,  $|X_1^{C'}| \geq 3$ , let us assume that the induction hypothesis holds true for every subset  $C'' \subseteq C$  inducing a connected subgraph of  $G[C]$  and such that  $|X_1^{C''}| < |X_1^{C'}|$ . Let  $v_1 \in X_1^{C'}$  have degree 1 in  $G[C']$ . Let  $v_m$  be the closest vertex to  $v_1$  in  $G[C']$  that has degree at least three in  $G[C']$ . Such a vertex exists since  $G[C']$  is connected, and  $|X_1^{C'}| \geq 3$  implies that  $C'$  is not reduced to a path. Furthermore, let  $P = (v_1, \dots, v_m)$ ,  $m \geq 2$ ,

be a shortest  $v_1 v_m$ -path of  $G[C']$ . By the choice of  $v_m$ , all internal vertices of  $P$  have degree 2 in  $C'$ . Let  $C'' = C' \setminus \{v_1, \dots, v_{m-1}\}$  (note that  $C''$  induces a connected subgraph) and let  $d_m = \deg_{C'}(v_m) - 1 \geq 2$  be the degree of  $v_m$  in  $G[C'']$ . Since there is one vertex of degree 1 less in  $G[C'']$  than in  $G[C']$ , i.e.  $|X_1^{C''}| = |X_1^{C'}| - 1$ , we may apply the induction hypothesis on  $C''$  and  $\sum_{1 \leq i \leq \Delta} (\Delta - i) |X_i^{C''}| \leq 2(\Delta - 1) + (\Delta - 2)(|C''| - 2)$ .

Moreover,  $\sum_{1 \leq i \leq \Delta} (\Delta - i) |X_i^{C'}| = (\Delta - 1) + (\Delta - 2)(m - 2) - 1 + \sum_{1 \leq i \leq \Delta} (\Delta - i) |X_i^{C''}|$ . In the last sum, the first term  $\Delta - 1$  comes from  $v_1$ , the term  $(\Delta - 2)(m - 2)$  comes from  $v_2, \dots, v_{m-1}$  and the  $-1$  comes from the fact that  $v_m$  has degree one more in  $G[C']$  than in  $G[C'']$ .

Altogether, we get that:

$$\sum_{1 \leq i \leq \Delta} (\Delta - i) |X_i^{C'}| \leq (\Delta - 1) + (\Delta - 2)(m - 2) - 1 + 2(\Delta - 1) + (\Delta - 2)(|C''| - 2) = 2(\Delta - 1) + (\Delta - 2)(|C'| - 2).$$

This concludes the proof of Inequality 2.

Let  $Y \subseteq X$  be the set of isolated vertices in  $X$  (i.e., the vertices in  $X$  that have no neighbors in  $X$ ). Let  $r' = |Y|$  and let  $r''$  be the number of connected components of  $X \setminus Y$ . Let  $r = r' + r''$  be the number of connected components of  $X$ . From Inequalities 1 and 2, we get that

$$2(n - |X|) = \sum_{v \in V \setminus X} |g(v)| \leq 2r''(\Delta - 1) + (\Delta - 2)(|X \setminus Y| - 2r'').$$

Note that  $Z = 2r'(\Delta - 1) + (\Delta - 2)(|Y| - 2r') \geq 0$  (since  $|Y| = r'$ ).

Therefore,

$$2(n - |X|) = \sum_{v \in V \setminus X} |g(v)| \leq 2r''(\Delta - 1) + (\Delta - 2)(|X \setminus Y| - 2r'') + Z.$$

Since  $2r''(\Delta - 1) + (\Delta - 2)(|X \setminus Y| - 2r'') + Z = 2(r' + r'')(\Delta - 1) + (\Delta - 2)(|X \setminus Y| + |Y| - 2r'' - 2r') = 2r(\Delta - 1) + (\Delta - 2)(|X \setminus Y| - 2r)$ , we get that

$$2(n - |X|) = \sum_{v \in V \setminus X} |g(v)| \leq 2r(\Delta - 1) + (\Delta - 2)(|X| - 2r).$$

Hence,  $\frac{2(n-r)}{\Delta} \leq |X|$ . □

### 3.2 Application to the grids

In the proof of Theorem 9, the Inequality 2 can be refined in some cases. In particular, we now show that it is the case for any generator of a grid.

**Corollary 10.** *For any grid  $G$  with  $n$  vertices,  $in_{cc}(G) \geq n/2$ .*

*Proof.* Notations are the same as for Theorem 9. Let  $X$  be a fixed generator for  $G$  of minimum size. Consider Inequality 1. For the upper-bound, we assume the worst-case when (i) all vertices in  $X$  have degree  $\Delta = 4$ , and (ii) for all  $x \in X$ ,  $x \in g(v)$  for every of its neighbors  $v$  that is not in  $X$ . So, in the particular case when there is a vertex in  $X$  that has degree strictly less than  $\Delta$ , the upper-bound can be decreased by 1. Similarly, in the particular case when there exists  $x \in X$  such that  $x \notin g(v)$  for some of its neighbors  $v \notin X$ , the upper-bound can also be decreased by 1.

We now claim that for every (non isolated) component  $C$  of  $X$ , we fall in one of the two above cases: precisely, there exist two vertices in  $C$  that have degree strictly less than  $\Delta$  or that have a neighbor that they do not generate. Indeed, the leftmost (resp., rightmost, topmost, downmost) vertex of  $C$  has either the first property (if it is at the border of the grid) or the second one (it cannot generate the vertex that is left, resp., right, top, down to it). As a result, the upper-bound of Inequality 1 can be decreased by  $2r$ , with  $r$  being the number of components in  $X$ . Combined with Inequality 2, one obtains:

$$2(n - |X|) = \sum_{v \in V \setminus X} |g(v)| \leq 2r(\Delta - 1) + (\Delta - 2)(|X| - 2r) - 2r = (\Delta - 2)|X|.$$



Hence,  $in_{cc}(G) = |X| \geq \frac{2n}{\Delta} = n/2$ . □

Since the set that consists of one row plus one column every two columns (plus, if the number of columns is even, one vertex every three rows for the last column) is a generator, we get that

**Theorem 11.** *For any square grid  $G$  with  $n$  vertices,  $n/2 \leq in_{cc}(G) = n/2 + O(\sqrt{n})$ .*

## 4 Hardness for restricted cases

In this section, we prove that the problem is (at least) NP-hard in many graph classes such as split graphs, bipartite graphs or planar graphs with bounded maximum degree.

### 4.1 Relationship with dominating set, and case of split graph

Given a graph  $G = (V, E)$ , a set  $D \subseteq V$  is a *dominating set* of  $G$  if  $N[D] = V$ . Let  $\gamma(G)$  denote the minimum size of a dominating set of  $G$ . Let  $G^*$  be the graph obtained from  $G$  by adding a universal vertex  $u$ , i.e.,  $V(G^*) = V \cup \{u\}$  and  $E(G^*) = E \cup \{\{v, u\} \mid v \in V\}$ .

**Theorem 12.** *For any graph  $G$ ,  $in_{cc}(G^*) = \gamma(G) + 1$ .*

*Proof.* Let  $u$  be the universal vertex added to obtain  $G^*$ .

Let us first prove  $in_{cc}(G^*) \leq \gamma(G) + 1$ . Let  $D$  be a dominating set of  $G$ . We claim that  $D \cup \{u\}$  is a generator for  $G^*$ . Indeed, for any  $v \notin D \cup \{u\}$ , there is a vertex  $w \in D$  that dominates  $v$ . Hence,  $v$  is in the triangle  $\{v, w, u\}$  and it is generated by  $D \cup \{u\}$  in  $G^*$ .

For the opposite direction, let  $X$  be a generator for  $G^*$ . If  $u \notin X$ , let  $C$  be any cycle that generates  $u$  for  $X$  and let  $v \in X$  be any vertex of  $C$ . We claim that  $X' = X \cup \{u\} \setminus \{v\}$  is a generator for  $G^*$ . Clearly,  $C$  generates  $v$  for  $X'$ . Moreover, for any vertex  $w$  in  $V(G) \setminus X$ , there is a cycle  $C_w$  that generates  $w$  for  $X$ . If  $v \notin C_w$ , then  $w$  is still generated by  $C_w$  for  $X'$ . Otherwise, let  $C'_w$  be the cycle obtained from  $C_w$  by replacing  $v$  by  $u$  ( $C'_w$  exists since  $u$  is universal). Then  $C'_w$  generates  $w$  for  $X'$ . Hence, we may assume that  $X$  contains  $u$ . Because  $X$  is a generator for  $G^*$  containing  $u$ , any vertex in  $V(G) \setminus X$  has at least one neighbor in  $X \setminus \{u\}$  (by Claim 2). Hence,  $X \setminus \{u\}$  is a dominating set of  $G$ . □

Recall that a split graph is a graph whose vertex-set can be partitioned into a clique and an independent set. Clearly, any graph obtained by adding a universal vertex to a split graph is a split graph. Moreover, the problem of minimum dominating set in split graph can trivially be reduced from Set Cover Problem. Therefore, from Theorem 12 and the following result, we get the next corollary.

**Theorem 13.** *[] Set Cover is NP-hard and cannot be approximated up to a constant factor unless  $P=NP$ .*

**Corollary 14.** *Computing  $in_{cc}$  in split graphs is NP-hard and cannot be approximated up to a constant factor unless  $P=NP$ .*

### 4.2 Bipartite graphs

We consider the MINIMUM DOMINATING SET Problem (MDS) which takes a graph  $G = (V, E)$  and an integer  $k$  as inputs and asks whether there exists a set  $X \subseteq V$ ,  $|X| \leq k$  and  $N[X] = V$ . It is well known that this problem is NP-hard, and that:

**Theorem 15.** *[FG06] MDS is NP-hard and  $W[2]$ -hard in bipartite graphs.*

**Theorem 16.** *Computing  $in_{cc}$  is NP-hard and  $W[2]$ -hard in bipartite graphs.*

*Proof.* To prove the theorem, we propose an FPT-reduction from MDS (i.e., a reduction preserving both the size of the problem and the parameter).

Let  $G = (A \cup B, E)$  and  $k \in \mathbb{N}$  be a bipartite instance of MDS.

Let  $G^*$  be obtained from  $G$  by adding a vertex  $a$  adjacent to all vertices of  $A$ , a vertex  $b$  adjacent to all vertices in  $B$  and the edge  $\{a, b\}$ . Clearly,  $G^*$  is bipartite. We claim that there is a dominating set of size  $k$  in  $G$  if and only if  $\text{in}_{cc}(G^*) \leq k + 2$ .

Let  $X$  be any dominating set of  $G$  with size  $k$ . Let  $X^* = X \cup \{a, b\}$ . It is easy to check that  $cc(X^*) = V(G^*)$  and therefore,  $\text{in}_{cc}(G^*) \leq k + 2$ .

On the other hand, let  $X^*$  be a minimum generator for  $G^*$  (i.e.,  $cc(X^*) = V(G^*)$ ) such that  $|X^*| = k + 2$ . If  $a, b \in X^*$ , then it is easy to check that  $X^* \setminus \{a, b\}$  dominates  $V(G)$ . Indeed, suppose that  $v \in V(G) \setminus X^*$  (w.l.o.g.,  $v \in A$ ) and  $N(v) \cap X^* \subseteq \{a, b\}$ . Then, by construction of  $G^*$ ,  $N(v) \cap X^* = \{a\}$  and  $X^*$  cannot generate  $v$  by Claim 2.

It remains to prove that, if there exists a set of size  $k + 2$  that generates  $V(G^*)$ , then there is such a set that contains  $a$  and  $b$ . Indeed, assume that  $a \notin X^*$ . Therefore, there is a cycle  $C$  in  $G^*$  where  $a$  is the single vertex not in  $X^*$ . In particular, there is a vertex  $c \in C \cap X^* \cap A$ . We show that  $X' = X^* \cup \{a\} \setminus \{c\}$  is a generator for  $G^*$ . Indeed,  $c$  is clearly generated by  $C$  in  $X'$ . Moreover, for any cycle  $C'$  containing  $c$  that generates a vertex  $v$  for  $X^*$  (notice that  $C'$  cannot contain  $a$ ), the cycle  $C''$  obtained by replacing  $c$  with  $a$  in  $C'$  (it can be done since  $N(c) \subset A$  and  $A \subset N(a)$ ) generates  $v$  for  $X'$ . Hence,  $X'$  will generate the same vertices as  $X^*$ .  $\square$

### 4.3 Planar graphs with bounded degree

**Theorem 17.** *The problem of computing  $\text{in}_{cc}(G)$  in planar graphs  $G$  with degree at most 6 is NP-hard.*

*Proof.* Let  $G = (V, E)$  be a graph with  $n = |V|$  vertices and  $m = |E|$  edges. The graph  $G'$  is obtained from  $G$  by adding  $3m$  new vertices and corresponding edges as follows. For every  $e \in E$ , we add a triangle  $(x_e, y_e, z_e)$ , then we make the vertex  $x_e$  adjacent to both vertices incident to  $e$ . Observe that if  $G$  has maximum degree  $\Delta$  then  $G'$  has maximum degree at most  $\max\{2\Delta, 4\}$ , and furthermore, if  $G$  is planar then so is  $G'$ . Let us now prove that  $\text{in}_{cc}(G') = \gamma(G) + 2m$ , where  $\gamma(G)$  is the domination number of  $G$ . Since it is NP-hard to compute the domination number of planar graphs with maximum degree three [GJ79], the latter will prove that it is NP-hard to compute  $\text{in}_{cc}(G')$  even when  $G$  is a planar graph of maximum degree six.

Let us prove first that  $\text{in}_{cc}(G') \leq \gamma(G) + 2m$ . Consider a minimum-size dominating set  $D$  of  $G$ . The set  $S$  is constructed from  $D$  by adding the two vertices  $x_e, y_e$  for every  $e \in E$ . We claim that  $S$  is a generator for  $G'$ . Indeed, since for every  $e \in E$ , vertex  $z_e$  has its two adjacent neighbors  $x_e, y_e$  in  $S$ , the  $3m$  vertices  $x_e, y_e, z_e$ ,  $e \in E$  are all generated by  $S$ . Furthermore, for every vertex  $v \in V \setminus D$ , since  $D$  is a dominating set there is  $u \in D$  that is adjacent to vertex  $v$ . Let  $e = \{u, v\} \in E$ . The two adjacent vertices  $x_e, u$  are neighbors of vertex  $v$  in  $S$ , hence  $v$  is generated by  $S$ . As a result,  $\text{in}_{cc}(G') \leq |S| = 2m + \gamma(G)$ .

Conversely, let us prove that  $\text{in}_{cc}(G') \geq \gamma(G) + 2m$ . Let  $S$  be a minimum-size generator for  $G'$ . For every  $e \in E$ , if  $y_e \notin S$  then since  $S$  is a generator for  $G'$ , the only two neighbors  $x_e, z_e$  of vertex  $y_e$  must be in  $S$ . The same holds for vertex  $z_e$ . As a result, for every  $e \in E$ , there must be at least two of  $x_e, y_e, z_e$  in  $S$ . Now, let  $D = S \cap V$ . In order to prove the lower-bound, it suffices to prove that  $D$  is a dominating set of  $G$ . By contradiction, suppose existence of a vertex  $v \in V \setminus D$  with no neighbor in  $D$ . Since  $S$  is a generator for  $G'$ , by Claim 2, vertex  $v$  has two neighbors in some connected component of  $G[S]$ . Furthermore, since we assume that  $v$  has no neighbor in  $D = V \cap S$ , there must be  $x_e, x_{e'} \in S$  such that there is an  $x_e x_{e'}$ -path in  $G'[S]$  and  $e, e' \in E$  are two edges incident to vertex  $v$ . Let  $e = \{u, v\}, e' = \{u', v\}$ . By construction of  $G'$ , any  $x_e x_{e'}$ -path in  $S$  must pass by the two vertices  $u, u'$ , thereby contradicting that  $v$  has no neighbors in  $V \cap S$ . As a result,  $D$  is a dominating set of  $V$ , and so,  $\text{in}_{cc}(G') = |S| \geq |D| + 2m \geq \gamma(G) + 2m$ .  $\square$

## 5 Polynomial-time algorithms

This section is devoted to polynomial-time algorithms to compute the interval number in several graph classes.

## 5.1 Cobipartite graphs

Recall that a graph is cobipartite if it is the complement of a bipartite graph. In other words, its vertex set can be partitioned into two cliques.

**Theorem 18.** *Let  $G = (V, E)$  be a  $n$ -vertex cobipartite graph. Then  $in_{cc}(G) \leq 4$ , and:*

- $in_{cc}(G) = 2$  if and only if it contains two universal vertices;
- otherwise,  $in_{cc}(G) = 3$  if and only if one of the two following conditions hold:
  - either there exists  $u \in V$  such that  $in_{cc}(G \setminus u) = 2$ ;
  - or for any partition of  $V$  into two cliques  $A$  and  $B$ , one of them contains two vertices that dominate the other one.

Moreover,  $in_{cc}(G)$  can be computed in  $\mathcal{O}(n^3)$ -time.

*Proof.* Let  $G = (A \cup B, E)$  where  $A$  and  $B$  induce disjoint cliques. Given a co-bipartite graph  $G$ ,  $A$  and  $B$  can be computed in time  $\mathcal{O}(n^2)$  by taking the complement of  $G$ , which is bipartite, then bicolouring this complement (in this situation,  $A$  and  $B$  can be chosen as the two colour classes for the complement).

Clearly, taking two vertices in each clique (or one vertex if a clique is reduced to a single vertex) generates the whole graph. Hence  $in_{cc}(G) \leq 4$ . Furthermore, we have  $in_{cc}(G) > 1$ , and by Theorem 4,  $in_{cc}(G) = 2$  if and only if there are two universal vertices in  $G$  (which can be checked in time  $\mathcal{O}(n^2)$ ). So, we are left to characterize when  $in_{cc}(G) = 3$ . In what follows, let us assume that  $in_{cc}(G) \in \{3, 4\}$ .

Suppose that  $in_{cc}(G) \geq 3$ . First, for any  $u \in V$ , by Theorem 4, it can be checked in time  $\mathcal{O}(n^2)$  whether  $in_{cc}(G \setminus u) = 2$ . If such a vertex is found, then  $in_{cc}(G) = 3$ . The case is solved.

From now on, assume that  $in_{cc}(G) \geq 3$  and, for every vertex  $u \in A \cup B$ ,  $in_{cc}(G \setminus u) > 2$ . We will show that in this case  $in_{cc}(G) = 3$  if and only if one of the cliques,  $A$  or  $B$ , contain two vertices that dominate the other clique.

Suppose  $in_{cc}(G) = 3$ , and let  $S$  be a generator of size three that contains vertices of both cliques. W.l.o.g., say that  $|S \cap A| = 2$ , and let  $S \cap B = \{u\}$  and  $S' = S \cap A$ . Let us show that  $S'$  dominates  $B$ . Indeed,  $u$  must have some neighbor in  $S'$ . Otherwise, by Claim 2, each vertex in  $cc(S) \setminus S$  would have to be adjacent to both vertices in  $S'$ , and  $S'$  would be a generator for  $G \setminus u$  of size 2. This is a contradiction. On the other hand, since  $S \cap B$  contains only one vertex, by Claim 2, every vertex in  $B \setminus S$  must be adjacent to at least one of the vertices in  $S'$ . Hence,  $S'$  dominates  $B$  and  $|S'| = 2$ .

Now, let  $S$  be a generator of size three that is included in one of the cliques, w.l.o.g., say  $S \subseteq A$ . By Claim 2, all vertices in  $B$  are adjacent to at least two vertices in  $S$ . So we may pick any  $v \in S$  and  $S' = S \setminus \{v\}$  still dominates  $B$  and  $|S'| = 2$ .

For the opposite direction, w.l.o.g. suppose that there are  $x, y \in A$  such that  $B \subset N(\{x, y\})$ . Pick any vertex  $z \in B$  and let  $S = \{x, y, z\}$ . It is easy to check that  $S$  generates  $V$  and, by previous remarks,  $in_{cc}(G) = 3$ .

The condition for  $in_{cc}(G) = 3$  can be verified in  $\mathcal{O}(n^3)$ -time by considering all the possible pairs  $x, y$  in  $A$  and  $B$ . □

## 5.2 Interval graphs

This section is devoted to a dynamic programming algorithm to compute a minimum size generator for the cycle convexity in interval graphs.

An *interval* graph is the intersection graph of a set of segments of the line. In other words, each of its vertices corresponds to a segment in  $\mathbb{R}$  and two vertices are adjacent if and only if the corresponding segments intersect [LB62]. A graph is *chordal* if it has no induced cycle of size at least 4 [BP93]. Interval graphs are chordal. In particular, it is well known that any interval graph  $G$  admits a clique-path, that is an ordering of its maximal cliques such that, for any vertex  $v$  of  $G$ , the maximal cliques containing  $v$  are contiguous in the ordering [BP93].

This section is devoted to the proof of the following theorem.

**Theorem 19.** *If  $G$  is an interval graph, then  $in_{cc}(G)$  and a minimum generator can be computed in  $\mathcal{O}(n^3)$ -time.*

Since the proof is technical we have divided it into several parts. First, we present some structural results on the generators for interval graphs, that will be the cornerstone for the algorithm of Theorem 19. These properties can be exploited in order to design a dynamic programming algorithm that computes  $in_{cc}(G)$ , in interval graphs, in exponential time (Lemma 24). Then, we explain how to reduce its time-complexity using the structure of interval graphs.

**Notations.** Let us introduce additional notations for the proof. Given an interval graph  $G = (V, E)$ , let  $P = (X_1, X_2, \dots, X_l)$  be a clique-path of  $G$ , that is, the set of maximal cliques of  $G$  ordered in such a way that, for any vertex  $v$  of  $G$ , the maximal cliques containing  $v$  form an interval (subpath)  $P_v$  in  $P$ . By convention  $X_0 = X_{l+1} = \emptyset$ . Let us set  $Y_i = \bigcup_{j=1}^i X_j$ ,  $V_i = Y_i \setminus X_{i+1}$ . Note that  $V_i \setminus V_{i-1} = X_i \setminus X_{i+1}$ .

The following properties are the cornerstone of our algorithm.

Recall that interval graphs are chordal. As a warm up, we prove a useful claim on generators for chordal graphs.

**Claim 20.** *For any generator  $S$  of a chordal graph  $G$ , every vertex  $x \notin S$  has two adjacent neighbors in  $S$ .*

*Proof.* Since  $x \notin S$ , it is generated by a cycle  $C$  where it is the unique vertex not in  $S$ . Let  $C$  be the shortest among such cycles, and  $u, v$  be the neighbors of  $x$  on  $C$ . Notice that  $C$  must be an induced cycle, otherwise there would exist a shorter cycle  $C'$  that also generates  $x$ . Since  $G$  is a chordal graph,  $C$  must be a triangle, and therefore  $u$  and  $v$  are adjacent. That proves the claim.  $\square$

We will often use the above claim in what follows.

**Claim 21.** *Let  $G = (V, E)$  be an interval graph and  $P = (X_1, X_2, \dots, X_l)$  be a clique-path of  $G$ . For any generator  $S$  for  $G$ , the vertices in  $V_i$  are generated by  $S \cap Y_i$ .*

*Proof.* Since  $G$  is an interval graph, and so, chordal, by Claim 20 every vertex of  $V \setminus S$  has two adjacent neighbours in  $S$ . In particular,  $V_i$  is generated by  $S \cap N[V_i] = S \cap Y_i$ .  $\square$

Next, we describe in Claim 22 a procedure which given a generator  $S$  intersecting some ordered subset  $T$ , ensures the existence (under some conditions) of a generator  $S^*$  of the same size as  $S$  and containing the largest element in  $T$ . Applications to interval graphs are then discussed in the subsequent claims.

**Claim 22.** *Let  $G = (V, E)$  be an interval graph,  $P = (X_1, X_2, \dots, X_l)$  be a clique-path of  $G$  and  $S$  be a generator for  $G$ . Suppose that there exist vertices  $v_i^1, \dots, v_i^k \in S$  and  $u_i^1, \dots, u_i^k \notin S \cup V_{i-1}$  such that the following hold:*

- *the vertices of  $Y_{i-1} \cup \{v_i^1, \dots, v_i^k\}$  are generated by  $S^* = (S \setminus \{v_i^1, \dots, v_i^k\}) \cup \{u_i^1, \dots, u_i^k\}$ ;*
- *and for every  $1 \leq j \leq k$   $N[v_i^j] \setminus V_{i-1} \subseteq N[u_i^j] \setminus V_{i-1}$ .*

*Then,  $S^*$  is also a generator for  $G$ .*

*Proof.* By the hypothesis the set  $S^*$  is a generator for  $Y_{i-1} \cup \{v_i^1, \dots, v_i^k\}$ . Let us prove that every vertex  $x \in V \setminus (Y_{i-1} \cup \{v_i^1, \dots, v_i^k\})$  can be generated with  $S^*$ . If  $x \in S^*$  then we are done. Otherwise,  $x \notin S^*$ , and so,  $x \notin S$ . By Claim 20, since  $G$  is chordal,  $x \notin S$  implies the existence of two adjacent neighbors of  $x$  in  $S$ , denoted by  $y$  and  $z$ . Note that  $y, z \notin V_{i-1}$  since  $x \notin Y_{i-1}$ , and for every  $j$ ,  $u_i^j \notin \{y, z\}$  since  $u_i^j \notin S$ . Let us define  $y^* = y$  if  $y \in S^*$ ; otherwise,  $y = v_i^j$  for some  $j$  and we define  $y^* = u_i^j$ . We define  $z^*$  in a similar fashion. By the hypothesis,  $N[y] \setminus V_{i-1} \subseteq N[y^*] \setminus V_{i-1}$  and similarly  $N[z] \setminus V_{i-1} \subseteq N[z^*] \setminus V_{i-1}$ . Therefore, since  $x \notin V_{i-1}$  and by construction  $y^*, z^* \notin V_{i-1}$  are distinct, vertex  $x$  has two adjacent neighbours  $y^*, z^* \in S^*$ . Hence every vertex  $x \in V \setminus (Y_{i-1} \cup \{v_i^1, \dots, v_i^k\})$  can be generated with  $S^*$ . As a result,  $S^*$  is a generator for  $G$ .  $\square$

Our algorithm for Theorem 19 is based on a technical consequence of the above claim. We describe it next.

**Claim 23.** *Let  $G = (V, E)$  be an interval graph,  $P = (X_1, X_2, \dots, X_l)$  be a clique-path of  $G$  and  $S$  be a generator for  $G$ . Furthermore, let  $T \subseteq Y_i$  be such that  $|T \cap S| \geq k$ . Let  $u_i^1, \dots, u_i^k \in T$  be such that for every  $j$ ,  $u_i^j$  maximizes  $N[u_i^j] \setminus V_{i-1}$  in  $T \setminus \{u_i^{j'} \mid j' < j\}$ . If  $(S \setminus T) \cup \{u_i^1, \dots, u_i^k\}$  generates  $Y_{i-1} \cup T$  and  $u_i^1, \dots, u_i^k \notin V_{i-1}$  then there exists a generator  $S^*$  for  $G$  such that  $|S^*| \leq |S|$  and  $u_i^1, \dots, u_i^k \in S^*$ .*

*Proof.* We assume w.l.o.g. (up to removing  $S \cap \{u_i^1, \dots, u_i^k\}$  from  $T$ ) that  $u_i^1, \dots, u_i^k \notin S$ . Let  $v_i^1, \dots, v_i^k \in T \cap S$ , that exist by the hypothesis. First note that since  $T \subseteq Y_i$  and  $G$  is interval, the subsets in  $\{N[v] \setminus V_{i-1} \mid v \in T\}$  can be ordered by inclusion. Therefore, let us assume w.l.o.g. that  $N[v_i^j] \setminus V_{i-1} \subseteq N[v_i^{j+1}] \setminus V_{i-1}$  for any  $j < k$ , by the choices  $u_i^1, \dots, u_i^k$  we have  $N[v_i^j] \setminus V_{i-1} \subseteq N[u_i^j] \setminus V_{i-1}$  for any  $j$ . Therefore, the result follows from Claim 22 applied to  $S$ ,  $v_i^1, \dots, v_i^k$  and  $u_i^1, \dots, u_i^k$ .  $\square$

Equipped with the above claims, we now describe a first (exponential-time) algorithm for computing a minimum-size generator for interval graphs.

**Lemma 24.** *If  $G$  is an interval graph, then there exists a dynamic programming algorithm that computes  $in_{cc}(G)$  and a minimum generator (in exponential-time).*

*Proof.* Let  $P = (X_1, X_2, \dots, X_l)$  be a clique-path of  $G$ . Note that  $P$  can be computed in  $\mathcal{O}(n + m)$ -time  $\square$ . Our purpose is to compute  $in_{cc}(G)$  by dynamic programming on the clique-path.

Iteratively, for  $i = 1$  to  $l$ , the algorithm computes a set  $\mathcal{S}_i$  of subsets of vertices such that, for any  $S \in \mathcal{S}_i$ ,  $S \subseteq Y_i$  and  $S$  generates  $V_i$ . Initially,  $\mathcal{S}_1$  consists of all possible subsets of  $Y_1 = X_1$  that generates  $V_1$ . Then, assuming that  $\mathcal{S}_i$  has been built, the next set  $\mathcal{S}_{i+1}$  is computed as follows. For any set  $S \in \mathcal{S}_i$ , the sub-procedure CONVEX( $i, S$ ) (described below) is applied. It returns one or two subsets, each of which are added to  $\mathcal{S}_{i+1}$ . Finally, the algorithm outputs the minimum size of a set in  $\mathcal{S}_l$  and a corresponding set.

The correctness of the above algorithm will directly follows from the properties and the correctness of the sub-procedure CONVEX that we detail below. Moreover, as it will be clear, the sub-procedure CONVEX performs in polynomial time. Therefore, the exponential-time complexity of the algorithm only relies on the potentially exponential size of the sets  $\mathcal{S}_i$ . This drawback will be handled in the next lemma.

The sub-procedure CONVEX takes as input  $i \in \{1, \dots, l\}$ , and any subset  $S_{i-1} \subseteq Y_{i-1}$  that is a generator for  $V_{i-1}$ . It returns one (or possibly two) superset(s)  $S_i$  of  $S_{i-1}$  that is (are) a generator(s) for  $V_i$ . We will show that at least one subset  $S_i$  that is output by the sub-procedure is contained in a generator for  $G$  that is of minimum size among all those containing  $S_{i-1}$ . In particular, this will prove that if we start with  $S_0 = \emptyset$  then the algorithm will compute in  $l$  steps a minimum-size generator for  $G$ .

We both describe the algorithm and prove its correctness simultaneously.

Let  $i \leq l$  be fixed. Several cases and sub-cases have to be considered.

- if  $V_i$  is generated by  $S_{i-1}$ , then since  $S_i = S_{i-1}$  generates  $V_i$ , we set CONVEX( $i, S_{i-1}$ ) =  $\{S_i\} = \{S_{i-1}\}$  and it satisfies all the desired requirements for the sub-procedure;
- else,  $S_{i-1}$  does not generate  $V_i \setminus V_{i-1} = X_i \setminus X_{i+1}$ . Since  $X_i$  is a clique, it implies that  $|S_{i-1} \cap X_i| \leq 1$ . Indeed, otherwise there would be  $u, v \in S_{i-1} \cap X_i$  and any vertex of  $X_i \setminus S_{i-1}$  would belong to a triangle with  $u$  and  $v$  and, hence, would be generated. Two cases must be considered.
  - Let us first assume that  $|S_{i-1} \cap X_i| = 1$ . In that case, let  $u_i \in X_i \setminus S_{i-1}$  maximizing  $|N_G[u_i] \setminus V_{i-1}|$  and let  $S_i = S_{i-1} \cup \{u_i\}$ . Note that  $S_i$  generates  $V_i$ . We set CONVEX( $i, S_{i-1}$ ) =  $\{S_i\}$ . In order to prove that this choice satisfies all the requirements for the sub-procedure, it is sufficient to prove that there is a generator  $S^*$  of  $G$ , containing  $S_{i-1}$  and with minimum size for this property, such that  $S^*$  contains  $S_i$ , i.e. such that  $u_i \in S^*$ . Indeed, let  $S$  be any generator for  $G$ , containing  $S_{i-1}$  and with minimum size for this property. By Claim 21,  $S \cap Y_i$  generates  $V_i$ , and so, since  $S_{i-1}$  does not generate  $V_i$ , there must be a vertex

$v_i \in Y_i \setminus S_{i-1}$  that is in  $S$ . Furthermore,  $S_i$  generates  $V_i$ , and so,  $Y_{i-1}$ . Therefore, the existence of the desired generator  $S^*$  follows directly from Claim 23, by taking  $T = Y_i \setminus S_{i-1}$  and  $u_i^1 = u_i$ .

- Second, let us assume that  $S_{i-1} \cap X_i = \emptyset$ . Let  $u_i \in X_i$  be a vertex maximizing  $|N_G[u_i] \setminus V_{i-1}|$  and let  $u'_i \in X_i \setminus \{u_i\}$  be a vertex maximizing  $|N_G[u'_i] \setminus V_{i-1}|$  (recall that  $X_i \cap S_{i-1} = \emptyset$ , so,  $u_i, u'_i \notin S_{i-1}$ ). Ideally, we would like to set  $\text{CONVEX}(i, S_{i-1}) = \{S_i\}$  in this case, where this time  $S_i = S_{i-1} \cup \{u_i, u'_i\}$ . However, we were not able to prove that it is always possible, and so, there are several subcases to be considered.

More precisely, suppose that there exists a generator  $S$  for  $G$  that is of minimum-size among all those containing  $S_{i-1}$  and that contains at least two vertices of  $Y_i \setminus S_{i-1}$ . In this situation, it can be found by applying Claim 23 to  $S$ ,  $T = Y_i \setminus S_{i-1}$  and  $u_i^1 = u_i, u_i^2 = u'_i$  a generator  $S^*$  of the same size as  $S$  such that  $S_{i-1} \subseteq S^*$  and  $u_i, u'_i \in S^*$ . Choosing  $\text{CONVEX}(i, S_{i-1}) = \{S_i\}$  in this case indeed satisfies all the requirements of the sub-procedure. However, if the existence of such generator  $S$  cannot be guaranteed, then another set  $S'_i$  than  $S_i$  will also need to be output.

- \* First, let us assume that there are at least two vertices  $v_i, v'_i \in X_i \setminus X_{i+1}$  that are not generated by  $S_{i-1}$ . However, assume that at least one of  $v_i$  and  $v'_i$  has no neighbor in  $S_{i-1}$ . We claim that in this subcase, there always exists a generator  $S$  as defined above. The latter will imply that we can choose  $\text{CONVEX}(i, S_{i-1}) = \{S_i\}$  in this subcase.

Indeed, w.l.o.g., say  $N_G(v_i) \cap S_{i-1} = \emptyset$ . Let  $S$  be any generator for  $G$ , containing  $S_{i-1}$  and with minimum size for these properties. By Claim 20, and since  $N_G(v_i) \cap S_{i-1} = \emptyset$ , either  $v_i \in S \setminus S_{i-1}$  or it has two adjacent neighbors in  $S \setminus S_{i-1}$ . Furthermore, there must be  $w'_i \in N_G[v'_i] \setminus S_{i-1}$  such that  $w'_i \in S$  (in order to generate  $v'_i$ ), and in particular, since  $N_G(v_i) \cap S_{i-1} = \emptyset$  and so, no two adjacent vertices in  $S_{i-1} \cup \{v_i\}$  are common neighbors of vertex  $v'_i$ , we may assume by the above claim that  $w'_i \neq v_i$ . Consequently, there must be two distinct vertices  $w_i \in N_G[v_i] \setminus S_{i-1}$  and  $w'_i \in N_G[v'_i] \setminus S_{i-1}$  such that  $w_i, w'_i \in S$  that proves the claim.

- \* Second, assume that every vertex of  $X_i \setminus X_{i+1}$  that is not generated by  $S_{i-1}$  has a neighbor in  $S_{i-1}$ . Note that since  $G$  is an interval graph, there is a vertex of  $S_{i-1}$  that is a common neighbor to all vertices in  $X_i$  with a neighbor in  $S_{i-1}$ . Amongst all vertices in  $X_i$  with one neighbor in  $S_{i-1}$ , choose one vertex  $u''_i$  maximizing  $|N_G[u''_i] \setminus V_{i-1}|$  (possibly,  $u_i = u''_i$  or  $u'_i = u''_i$ ). In this subcase, we will consider the two possibilities  $S'_i = S_{i-1} \cup \{u''_i\}$  or  $S_i = S_{i-1} \cup \{u_i, u'_i\}$  and we set  $\text{CONVEX}(i, S_{i-1}) = \{S_i, S'_i\}$ . Note that any of the two choices  $S_i$  or  $S'_i$  will generate the set  $V_i$ . Indeed, this is clear for the case  $S_i = S_{i-1} \cup \{u_i, u'_i\}$ . In the case  $S'_i = S_{i-1} \cup \{u''_i\}$ , it is sufficient to notice that every vertex of  $X_i \setminus X_{i+1}$  can be generated with  $u''_i$  and any vertex of  $S_{i-1}$  which is a common neighbor of all vertices of  $X_i \setminus X_{i+1}$ . Furthermore, in order to prove that our choice for  $\text{CONVEX}(i, S_{i-1}) = \{S_i, S'_i\}$  satisfies all the requirements for the sub-procedure, it is sufficient to prove that there is a generator  $S^*$  of  $G$ , containing  $S_{i-1}$  and with minimum size for this property, such that  $u_i, u'_i \in S^*$  or  $u''_i \in S^*$ .

Let  $S$  be any generator for  $G$ , containing  $S_{i-1}$  and with minimum size for this property. As said before, if  $S$  contains at least two vertices of  $Y_i \setminus S_{i-1}$  then we are done, as in this situation there exists a generator  $S^*$  of  $G$ , containing  $S_{i-1}$  and with minimum size for this property, such that  $u_i, u'_i \in S^*$ . So, let us assume that  $S$  contains no more than one vertex of  $Y_i \setminus S_{i-1}$ . Since by Claim 21,  $S \cap Y_i$  generates  $V_i$  and  $v_i$  is not generated by  $S_{i-1}$ , there is exactly one vertex  $v_i \in Y_i \setminus S_{i-1}$  contained in  $S$ . We claim that  $v_i$  has a neighbour in  $S_{i-1}$ . The latter will imply by Claim 23 (applied to  $S$ ,  $T = Y_i \cap N_G(S_{i-1})$ ,  $u_i^1 = u''_i$ ) the existence of a generator  $S^*$  containing  $u''_i$ . In order to prove the claim, let  $w_i \in X_i \setminus X_{i+1}$  be a vertex not generated by  $S_{i-1}$ . By the hypothesis,  $w_i$  has a neighbour in  $S_{i-1}$ . In particular if  $w_i \in S$  then we are done as in this situation,  $w_i = u''_i$  has a neighbour in  $S_{i-1}$ . Otherwise, by Claims 20 and 21  $w_i$  has two adjacent neighbours  $y, z \in S \cap Y_i$ , at most one of which is in  $S_{i-1}$  (else,  $w_i$  would be generated by  $S_{i-1}$ ). Since  $v_i$  is the unique vertex of  $Y_i \setminus S_{i-1}$  that is in  $S$ , it implies that  $v_i$  is one of  $y$  and  $z$ , say w.l.o.g.  $y = v_i$ , and  $z \in S_{i-1}$  is a neighbour of  $v_i$ , that proves the

claim.

- \* Else, in all other cases, there is a unique vertex  $v_i \in X_i \setminus X_{i+1}$  that cannot be generated with  $S_{i-1}$ , and this vertex has no neighbor in  $S_{i-1}$ . In this subcase, we will consider the two possibilities  $S'_i = S_{i-1} \cup \{v_i\}$  and  $S_i = S_{i-1} \cup \{u_i, u'_i\}$  and we will choose  $\text{CONVEX}(i, S_{i-1}) = \{S_i, S'_i\}$ . Again, note that both choices of  $S_i$  or  $S'_i$  will generate  $V_i$ . Furthermore, and as in the previous sub-cases, in order to prove that our choice for  $\text{CONVEX}(i, S_{i-1}) = \{S_i, S'_i\}$  satisfies all the requirements for the sub-procedure, it is sufficient to prove that there is a generator  $S^*$  of  $G$ , containing  $S_{i-1}$  and with minimum size for these properties, such that  $u_i, u'_i \in S^*$  or  $v_i \in S^*$ . Indeed, let  $S$  be any generator for  $G$ , containing  $S_{i-1}$  and with minimum size for this property. By the Claim 20,  $v_i \in S$  (in such case, it is trivially generated), or it has two adjacent neighbors in  $S$ . Moreover, in the latter case, these two neighbours are in  $Y_i \setminus S_{i-1}$ , which as explained before implies by Claim 22 the existence of a generator  $S^*$  such that  $u_i, u'_i \in S^*$ .

From the properties of sub-procedure CONVEX, the dynamic programming algorithm presented at the beginning of the proof actually computes a minimum-size generator of  $G$ .  $\square$

Having explained the guiding principles of the algorithm with Lemma 24, we finally explain how to turn it into a cubic-time algorithm.

*Proof of Theorem 19.* Computing  $\text{CONVEX}(i, S_{i-1})$  for all possible pairs  $i, S_{i-1}$  would result in a complexity that is exponential in  $n$ . In order to obtain the desired polynomial-time complexity, we will partition the family of sets  $S_{i-1}$  into classes such that, when  $S_{i-1}$  and  $S'_{i-1}$  are in the same class, the set of vertices added by the algorithm is the same. Formally,  $S_{i-1}$  and  $S'_{i-1}$  will be said equivalent whenever  $\{T_i \subseteq Y_i \setminus S_{i-1} \mid S_{i-1} \cup T_i \in \text{CONVEX}(i, S_{i-1})\} = \{T'_i \subseteq Y_i \setminus S'_{i-1} \mid S'_{i-1} \cup T'_i \in \text{CONVEX}(i, S'_{i-1})\}$ . By doing so, we will show that we only need to consider polynomially many classes. Let us define  $\alpha(S_{i-1}) = \langle j_v, j_e \rangle$ , where  $j_v$ , respectively  $j_e$ , is the maximum index  $j$  such that  $X_j$  contains a vertex, respectively two adjacent vertices (an edge), of  $S_{i-1}$ . We set  $j_v = 0$  if  $S_{i-1} = \emptyset$ , similarly  $j_e = 0$  if  $G[S_{i-1}]$  is edgeless.

In what follows, we shall prove that all decisions made by the algorithm CONVEX only depend on  $\alpha(S_{i-1})$ . In other words, for every pair  $S_{i-1}, S'_{i-1}$ , generators of  $V_{i-1}$ , such that  $\alpha(S_{i-1}) = \alpha(S'_{i-1})$  we have  $\{T_i \subseteq Y_i \setminus S_{i-1} \mid S_{i-1} \cup T_i \in \text{CONVEX}(i, S_{i-1})\} = \{T'_i \subseteq Y_i \setminus S'_{i-1} \mid S'_{i-1} \cup T'_i \in \text{CONVEX}(i, S'_{i-1})\} = \mathbf{generator}(i, \alpha(S_{i-1}))$ , where  $\mathbf{generator}$  is a function defined in what follows.

By doing so, we will be able to modify the exponential-time algorithm of Lemma 24 as follows: iteratively, for  $i = 1$  to  $l$ , the new algorithm computes a set  $\mathcal{S}_i$  of subsets of vertices such that: as before for any  $S \in \mathcal{S}_i$ ,  $S \subseteq Y_i$  and  $S$  generates  $V_i$ , and in addition  $\alpha(S) \neq \alpha(S')$  for every  $S, S' \in \mathcal{S}_i$ . In more details, at first all possible subsets  $S$  of  $Y_1 = X_1$  that generates  $V_1$  are considered, and their respective class  $\alpha(S)$  is stored. For every such a class  $\langle j_v, j_e \rangle$ , exactly *one* minimum-size subset  $S$  is placed in  $\mathcal{S}_1$  such that  $\alpha(S) = \langle j_v, j_e \rangle$  and  $V_1$  is generated by  $S$ . Then, assuming that  $\mathcal{S}_i$  has been built, the next set  $\mathcal{S}_{i+1}$  is computed as follows. For any set  $S \in \mathcal{S}_i$ , the sub-procedure  $\text{CONVEX}(i, S)$  is applied and it returns one or two subsets, stored in an intermediate set  $\mathcal{S}'_i$ . Finally, for every class  $\langle j_v, j_e \rangle$  of a subset in  $\mathcal{S}'_i$ , exactly *one* minimum-size subset  $S \in \mathcal{S}'_i$  is placed in  $\mathcal{S}_i$  such that  $\alpha(S) = \langle j_v, j_e \rangle$ . Finally, the algorithm outputs the minimum size of a set in  $\mathcal{S}_l$  and a corresponding set.

The remaining of the proof is devoted to the description of the sub-procedure  $\mathbf{generator}$ . As before, we prove its correctness simultaneously, *i.e.*, for any  $S_{i-1}$  generator of  $V_{i-1}$  we have  $\text{CONVEX}(i, S_{i-1}) = \{T_i \subseteq Y_i \setminus S_{i-1} \mid T_i \in \mathbf{generator}(i, \alpha(S_{i-1}))\}$ .

Let  $\alpha(S_{i-1}) = \langle j_v, j_e \rangle$  be defined as above. Note that since  $G$  is an interval graph, any vertex of  $X_i$  with some neighbor in  $S_{i-1}$  must have a neighbor in  $S_{i-1} \cap X_{j_v}$ , and similarly every vertex of  $X_i$  with two adjacent neighbors in  $S_{i-1}$  must have two neighbors in  $S_{i-1} \cap X_{j_e}$ . In what follows, we will rely on the two above observations.

For every vertex  $v \in V(G)$ , let  $js_v, jl_v$  be respectively the smallest and the largest index  $j$  such that  $v \in X_j$ . We will show that we can base the decisions of the algorithm on the  $js_v$  (*i.e.*, which case or subcase applies), while we can pick the desired vertices  $u_i, u'_i, u''_i$  (as defined earlier in the proof of Lemma 24) by

using the  $jl_v$ . W.l.o.g. (up to a preprocessing step in  $\mathcal{O}(n+m)$ -time), the vertices of  $X_i$ , resp. the vertices of  $X_i \setminus X_{i+1}$ , are ordered by increasing value of  $js_v$ . Furthermore, for every  $1 \leq i \leq l$ , we store a vertex  $f_i$  such that  $js_{f_i} \leq i$  and  $jl_{f_i}$  is maximized. Similarly, we store a vertex  $f'_i \neq f_i$  such that  $js_{f'_i} \leq i$  and  $jl_{f'_i}$  is maximized. In this situation, the vertices  $u_i, u'_i, u''_i$  (as defined earlier in the proof of Lemma 24) can be chosen respectively as  $f_i, f'_i, f_{j_v}$ , and accessed in constant-time.

Now let us revisit the three main cases of the sub-procedure CONVEX, as follows.

- First, we need to verify whether  $S_{i-1}$  generates  $V_i$ . If  $j_e \geq i$  then there are two adjacent vertices of  $S_{i-1}$  in  $X_i$ , and so,  $V_i$  is generated by  $S_{i-1}$ . Otherwise, since  $S_{i-1}$  is assumed to generate  $V_{i-1}$ , we have that  $V_i$  is generated by  $S_{i-1}$  if and only if for every  $v_i \in X_i \setminus X_{i+1}$ ,  $j_e \in [js_{v_i}, jl_{v_i}]$ . Altogether,  $V_i$  is generated by  $S_{i-1}$  if and only if for all  $v_i \in X_i \setminus X_{i+1}$ ,  $js_{v_i} \leq j_e$ . It can be verified in constant-time by taking  $v_i \in X_i \setminus X_{i+1}$  maximizing  $js_{v_i}$ .

We recall that in this first case of the sub-procedure,  $\text{CONVEX}(i, S_{i-1}) = \{S_{i-1}\}$ . Therefore, we set  $\text{generator}(i, \alpha(S_{i-1})) = \{\emptyset\}$ .

- Else,  $V_i$  is not generated by  $S_{i-1}$ , and so,  $|X_i \cap S_{i-1}| \leq 1$ . We need to decide whether  $X_i \cap S_{i-1} \neq \emptyset$ . In this situation,  $|X_i \cap S_{i-1}| = 1$  if and only if  $i \leq j_v$ .

In this case, we have  $\text{CONVEX}(i, S_{i-1}) = \{S_i\}$  with  $S_i = S_{i-1} \cup \{u_i\}$ . In particular, we can set  $\text{generator}(i, \alpha(S_{i-1})) = \{\{u_i\}\}$ . By the choice of  $u_i$  (maximizing  $|N_G(u_i) \setminus V_{i-1}|$ ),  $jl_{u_i} \geq j_v$ . Furthermore,  $u_i$  has a neighbor in  $S_{i-1}$ , and so, in  $S_{i-1} \cap X_{j_v}$ . Hence,  $\alpha(S_i) = \langle jl_{u_i}, j_v \rangle$  and the class of the new subset  $S_i$  can be computed in constant-time.

- Else,  $X_i \cap S_{i-1} = \emptyset$  and we need to distinguish between three subcases. As stated above, the vertices  $v_i \in X_i \setminus X_{i+1}$  that are not generated by  $V_{i-1}$  are precisely those satisfying  $j_e \leq js_{v_i}$ . By considering the two vertices of  $X_i \setminus X_{i+1}$  maximizing  $js_{v_i}$ , we can check in constant-time whether there are at least two vertices not generated by  $S_{i-1}$ . Finally, for every vertex  $v_i \in X_i \setminus X_{i+1}$ , it has no neighbor in  $S_{i-1}$  if and only if  $js_{v_i} \geq j_v$ . Again, this can be tested by taking  $v_i \in X_i \setminus X_{i+1}$  maximizing  $js_{v_i}$ .

In the first subcase, since  $\text{CONVEX}(i, S_{i-1}) = \{S_i\}$  with  $S_i = S_{i-1} \cup \{u_i, u'_i\}$ , we have  $\alpha(S_i) = \langle \max\{j_v, jl_{u_i}\}, \max\{j_e, jl_{u'_i}\} \rangle = \langle jl_{u_i}, jl_{u'_i} \rangle$ . In particular, we set  $\text{generator}(i, \alpha(S_i)) = \{\{u_i, u'_i\}\}$ .

In the second subcase, we test for  $S_i = S_{i-1} \cup \{u_i, u'_i\}$  and  $S'_i = S_{i-1} \cup \{u''_i\}$  and we have  $\text{CONVEX}(i, S_{i-1}) = \{S_i, S'_i\}$ . In particular, we set  $\text{generator}(i, \alpha(S_i)) = \{\{u_i, u'_i\}, \{u''_i\}\}$ . Furthermore, we have  $\alpha(S'_i) = \langle jl_{u''_i}, j_v \rangle$  since  $u''_i$  has a neighbor in  $S_{i-1}$ , and so, in  $S_{i-1} \cap X_{j_v}$ .

In the third and final subcase, we test for  $S_i = S_{i-1} \cup \{u_i, u'_i\}$  and  $S'_i = S_{i-1} \cup \{v_i\}$  and we have  $\text{CONVEX}(i, S_{i-1}) = \{S_i, S'_i\}$ , where  $v_i \in X_i \setminus X_{i+1}$  is the only vertex that is not generated by  $S_{i-1}$ . In particular, we set  $\text{generator}(i, \alpha(S_i)) = \{\{u_i, u'_i\}, \{v_i\}\}$ . Furthermore, we have  $\alpha(S'_i) = \langle jl_{v_i}, j_e \rangle$  since  $v_i$  has no neighbor in  $S_{i-1}$ .

Overall, there are  $\mathcal{O}(n^2)$  possibilities for  $\alpha(S_{i-1})$ . In other words, the main algorithm only need to consider at most  $\mathcal{O}(n^2)$  subsets in each of the sets  $S_i$ . As a result,  $\text{in}_{cc}(G)$  and a minimum-size generator for  $G$  can be computed using dynamic programming in  $\mathcal{O}(n^3)$  time and space.  $\square$

## 6 FPT algorithms

### 6.1 Bounded treewidth graphs

This section is devoted to the proof that computing the interval number of a graph admits an FPT algorithm parameterized by the treewidth.

Let  $G = (V, E)$  be a graph. A *tree-decomposition*  $(T, \mathcal{X})$  of  $G$  consists of a tree  $T$  and a set  $\mathcal{X} = (X_t)_{t \in V(T)}$  of subsets of  $V$  indexed by the vertices of  $T$ , satisfying the following properties.

- $\bigcup_{t \in V(T)} X_t = V$ ;



- for any edge  $\{u, v\} \in E$ , there exists  $t \in V(T)$  with  $u, v \in X_t$ ;
- for any vertex  $v \in V$ , the set  $\{t \in V(T) \mid v \in X_t\}$  induces a subtree of  $T$ .

The *width* of  $(T, \mathcal{X})$  equals  $\max_{t \in V(T)} |X_t| - 1$  and the *treewidth* of  $G$ , denoted by  $tw(G)$ , is the minimum width over all tree-decompositions of  $G$ .

A tree-decomposition  $(T, \mathcal{X})$  is *nice* if  $T$  is rooted in some vertex  $r \in V(T)$ , any vertex of  $T$  has at most two children and, for any  $t \in V(T)$ , one of the following cases holds:

- $t$  is a leaf of  $T$  and  $|X_t| = 1$  (*Leaf vertex*);
- $t$  has one child  $u$  and there exists  $v \in V$  such that  $X_u = X_t \cup \{v\}$  (*Forget vertex*);
- $t$  has one child  $u$  and there exists  $v \in V$  such that  $X_t = X_u \cup \{v\}$  (*Introduce vertex*);
- $t$  has two children  $u$  and  $w$  and  $X_u = X_w = X_t$  (*Join vertex*).

**Theorem 25.** *If an  $n$ -vertex graph  $G$  has tree-width at most  $k$ , then  $in_{cc}(G)$  can be computed in time  $2^{\mathcal{O}(k \log k)} n$ .*

*Proof.* Let  $(T, \mathcal{X})$  be a nice tree-decomposition of an  $n$ -vertex graph  $G$  with width  $k = \mathcal{O}(tw(G))$  and  $\mathcal{O}(n)$  vertices. Such a decomposition exists and can be computed in time  $2^{\mathcal{O}(k)} n$  [BDD<sup>+</sup>16].

Our algorithm proceeds by dynamic programming from the leaves of  $T$  to its root  $r$ . For any  $t \in V(T)$ , let  $T_t$  be the subtree of  $T$  rooted in  $t$  (i.e., if  $t = r$  is the root, then  $T_t = T$ , otherwise  $T_t$  is the component of  $T \setminus e$  containing  $t$ , where  $e$  is the edge between  $t$  and its parent). Let  $G_t$  be the subgraph induced by  $\bigcup_{u \in V(T_t)} X_u$ .

**Representation of a generator  $S$  in a bag  $X_t$ .** First, let us consider a generator  $S \subseteq V(G)$  and  $t \in V(T)$ . We aim at describing how  $S$  is “represented” in  $X_t$ . Let  $n_t = |S \cap V(G_t)|$ .

Let  $S_1, S_2, \dots, S_\ell$  be the connected components of  $S$ . Because  $S$  is a generator and by the properties of tree-decompositions, for any vertex  $v \in V(G_t) \setminus (X_t \cup S)$ , there exists  $i \leq \ell$  such that  $v$  has two neighbors in  $S_i \cap V(G_t)$ .

For any  $i \leq \ell$ , let  $S_{i,1}, \dots, S_{i,c_i}$  be the connected components induced by  $S_i \cap V(G_t)$  in  $G_t$ . Then, for any  $i \leq \ell$  and  $j \leq c_i$ , let  $S_{i,j}^t = S_{i,j} \cap X_t$ . Note that  $S_{1,1}^t, \dots, S_{1,c_1}^t, S_{2,1}^t, \dots, S_{\ell,c_\ell}^t$  are disjoint subsets of  $X_t$  and there are no edges between any two of them.

Let  $S^t = \bigcup_{1 \leq i \leq \ell} S_i \cap X_t = S \cap X_t$  and let  $N^t = X_t \setminus S^t$  be the set of vertices in  $X_t$  that do not belong to the generator  $S$ . The set  $N^t$  can be partitioned in two subsets. Let  $A^t$  be the vertices  $v$  of  $N^t$  such that, there exists  $i \leq \ell$  such that  $v$  has two neighbors in  $S_i \cap V(G_t)$  (i.e., they are the vertices that are “already” generated). Finally, let  $B^t = N^t \setminus A^t$  be the set of vertices that “still need to be generated”.

The tuple  $(n_t, A^t, B^t, (S_{1,1}^t, \dots, S_{1,c_1}^t), \dots, (S_{\ell,1}^t, \dots, S_{\ell,c_\ell}^t))$  fully characterises the generator  $S$  in  $X_t$ . This tuple is called the *representative* of  $S$  in bag  $X_t$ .

**Partial solutions.** Let  $t \in V(T)$ . From the previous paragraph, we define a *partial solution at bag  $X_t$*  as any tuple  $(n_t, A^t, B^t, (S_{1,1}^t, \dots, S_{1,c_1}^t), \dots, (S_{\ell,1}^t, \dots, S_{\ell,c_\ell}^t))$  such that:

- $(A^t, B^t, S_{1,1}^t, \dots, S_{1,c_1}^t, \dots, S_{\ell,1}^t, \dots, S_{\ell,c_\ell}^t)$  is a partition of  $X_t$ ;
- for any four integers  $a, b, c, d$ ,  $(a, b) \neq (c, d)$ , no edge has an end in  $S_{a,b}^t$  and its other end in  $S_{c,d}^t$ ;
- for any vertex  $v \in B^t$ , there exists no  $i \leq \ell$  such that  $v$  has two neighbors in  $\bigcup_{j \leq c_i} S_{i,j}^t$ .
- $n_t \geq |S^t|$ .

**The algorithm.** Note that, for any  $t \in V(T)$ , there are at most  $2^{\mathcal{O}(k \log k)}$  such partial solutions to be considered (indeed, for each possible partition, we only keep the corresponding tuple for which  $n_t$  is minimum).

For any leaf  $t$  of  $T$ , computing the partial solutions in  $t$  (each one is a partition of  $X_t$ ) can be done in constant time (by brute force), imposing that, in a leaf  $t$ ,  $n_t = |S^t|$ .

It remains to prove that, in constant time, we can compute the partial solutions for an internal node  $t$  using the partial solutions of its children. There are three cases.

**$t$  is a Forget vertex** Let  $u \in V(T)$  be its child and  $v \in V$  such that  $X_u = X_t \cup \{v\}$ .

Let  $P_u = (n_u, A^u, B^u, (S_{1,1}^u, \dots, S_{1,c_1}^u), \dots, (S_{\ell,1}^u, \dots, S_{\ell,c_\ell}^u))$  be any partial solution computed for bag  $X_u$ .

- if  $v \in B^u$ , then the solution is discarded. Indeed, vertex  $v$  will never be generated by a solution extending  $P_u$ .
- If there exists  $i \leq \ell$  such that  $c_i > 1$  and there exists  $j \leq c_i$  such that  $S_{i,j}^u = \{v\}$  then the solution is discarded. Indeed, the sets  $S_{i,1}^u, \dots, S_{i,c_i}^u$  will never be subsets of a connected component of a generator (there is no way to “reconnect”  $S_{i,j}^u$ ).
- Otherwise, the tuple  $(n_u, A^u \setminus \{v\}, B^u, (S_{1,1}^u \setminus \{v\}, \dots, S_{1,c_1}^u \setminus \{v\}), \dots, (S_{\ell,1}^u \setminus \{v\}, \dots, S_{\ell,c_\ell}^u \setminus \{v\}))$  is a partial solution for bag  $X_t$ .

Then, for each partial solution with same partition of  $X_t$ , we keep the corresponding tuple that has the minimum  $n_t$ .

**$t$  is an Introduce vertex** Let  $u \in V(T)$  be its child and  $v \in V$  such that  $X_t = X_u \cup \{v\}$ .

Let  $P_u = (n_u, A^u, B^u, (S_{1,1}^u, \dots, S_{1,c_1}^u), \dots, (S_{\ell,1}^u, \dots, S_{\ell,c_\ell}^u))$  be any partial solution computed for bag  $X_u$ . There are two ways to include  $v$  in a partial solution. That is, any partial solution for bag  $X_u$  leads to one or more partial solutions for bag  $X_t$ .

First,  $v$  may not be part of the generator.

- If there is no  $i \leq \ell$  such that  $v$  has two neighbors in  $\bigcup_{1 \leq j \leq c_i} S_{i,j}^u$ , then the tuple  $(n_u, A^u, B^u \cup \{v\}, (S_{1,1}^u, \dots, S_{1,c_1}^u), \dots, (S_{\ell,1}^u, \dots, S_{\ell,c_\ell}^u))$  is a partial solution for bag  $X_t$ .
- Otherwise, the tuple  $(n_u, A^u \cup \{v\}, B^u, (S_{1,1}^u, \dots, S_{1,c_1}^u), \dots, (S_{\ell,1}^u, \dots, S_{\ell,c_\ell}^u))$  is a partial solution for bag  $X_t$ .

Second, it may be possible to include  $v$  in the generator.

- If there is a unique  $i \leq \ell$  such that  $v$  has neighbors in  $\bigcup_{1 \leq j \leq c_i} S_{i,j}^u$ . In that case, let  $I \subseteq \{1, \dots, c_i\}$  be such that  $v$  has neighbors in  $S_{i,j}^u$  if and only if  $j \in I$ . W.l.o.g. (up to reorder the sets),  $i = 1$  and  $I = \{1, \dots, h\}$  for some  $h \leq c_1$  and let  $S_{1,1}^t = \{v\} \cup \bigcup_{j \leq h} S_{1,j}^u$ .

Moreover, let  $X \subseteq B^u$  be the set of neighbors of  $v$  that have another neighbor in  $\bigcup_{1 \leq j \leq c_1} S_{1,j}^u$

Then, the tuple  $(n_u+1, A^u \cup X, B^u \setminus X, (S_{1,1}^t, S_{1,h+1}^u, \dots, S_{1,c_1}^u), (S_{2,1}^u, \dots, S_{2,c_2}^u) \dots, (S_{\ell,1}^u, \dots, S_{\ell,c_\ell}^u))$  is a partial solution for bag  $X_t$ .

- Otherwise, if  $v$  has no neighbors in  $S^t = \bigcup_{1 \leq i \leq \ell} S_i \cap X_t$ , then there are several partial solutions that are created:
  - $(n_u+1, A^u, B^u, (S_{1,1}^u, \dots, S_{1,c_1}^u), \dots, (S_{\ell,1}^u, \dots, S_{\ell,c_\ell}^u), (\{v\}))$  is a partial solution for bag  $X_t$ ;
  - for any  $i \leq \ell$ ,  $(n_u+1, A^u \cup X, B^u \setminus X, (S_{1,1}^u, \dots, S_{1,c_1}^u), \dots, (S_{i,1}^u, \dots, S_{i,c_i}^u, \{v\}), \dots, (S_{\ell,1}^u, \dots, S_{\ell,c_\ell}^u))$  is a partial solution for bag  $X_t$ , where  $X$  is the set of vertices in  $B^u \cap N(v)$  that have a neighbor in  $\bigcup_{1 \leq j \leq c_i} S_{i,j}^u$ .

Then, for each partial solution with same partition of  $X_t$ , we keep the corresponding tuple that has the minimum  $n_t$ .

$t$  is a **Join vertex** Let  $u, w \in V(T)$  be its children.

Let  $P_u = (n_u, A^u, B^u, (S_{1,1}^u, \dots, S_{1,c_1^u}^u), \dots, (S_{\ell_u,1}^u, \dots, S_{\ell_u,c_{\ell_u}^u}^u))$  be any partial solution computed for bag  $X_u$  and let  $P_w = (n_w, A^w, B^w, (S_{1,1}^w, \dots, S_{1,c_1^w}^w), \dots, (S_{\ell_w,1}^w, \dots, S_{\ell_w,c_{\ell_w}^w}^w))$  be any partial solution computed for bag  $X_w$  such that

- $\ell_u = \ell_w = \ell$ ;
- for any  $i \leq \ell$ ,  $\bigcup_{j \leq c_i^u} S_{i,j}^u = \bigcup_{j \leq c_i^w} S_{i,j}^w$ .

Let  $i \leq \ell$ . Recall that  $(S_{i,1}^u, \dots, S_{i,c_i^u}^u)$  is supposed to represent some connected component  $S_i$  of some expected generator  $S$  for  $G$ . Precisely,  $\bigcup_{j \leq c_i^u} S_{i,j}^u = S \cap X_u$  and, for any  $j \leq c_i^u$ ,  $S_{i,j}^u$  is the intersection of a connected component of  $S_i \cap V(G_u)$  with  $X_u = X_t$ . Similarly, for any  $j \leq c_i^w$ ,  $S_{i,j}^w$  is the intersection of a connected component of  $S_i \cap V(G_w)$  with  $X_w = X_t$ . To obtain a partial solution in bag  $X_t$ , we need to “merge” some components  $S_{i,j}^u$  and  $S_{i,j'}^w$  if they are part of the same connected component of  $S_i \cap V(G_t)$ . We proceed as follows.

For any  $i \leq \ell$ , let us consider the bipartite graph  $G_i$  with vertices  $\{S_{i,1}^u, \dots, S_{i,c_i^u}^u, S_{i,1}^w, \dots, S_{i,c_i^w}^w\}$  and, for any  $j \leq c_i^u$  and  $j' \leq c_i^w$ , add an edge between  $S_{i,j}^u$  and  $S_{i,j'}^w$  if both sets intersect.

Let  $C_1, \dots, C_{c_i^t}$  be the connected components of  $G_i$ . For any  $j \leq c_i^t$ , let  $S_{i,j}^t = \bigcup_{S_{i,h}^u \in C_j} S_{i,h}^u \cup \bigcup_{S_{i,h}^w \in C_j} S_{i,h}^w$ .

Finally, let  $n_t = n_u + n_w - |S^u|$ .

Then,  $(n_t, A^u \cup A^w, B^u \cap B^w, (S_{1,1}^t, \dots, S_{1,c_1^t}^t), \dots, (S_{\ell,1}^t, \dots, S_{\ell,c_\ell^t}^t))$  is a partial solution for bag  $X_t$ .

Then, for each partial solution with same partition of  $X_t$ , we keep the corresponding tuple that has the minimum  $n_t$ .

**Correctness.** First, we need to prove that, for any minimum-size generator  $S$  for  $G$ , the algorithm returns its representative as a partial solution of the root  $r$  of the tree-decomposition. For this purpose, we can prove the following more general claim by a classical induction (from the leaves to the root).

**Claim 26.** *Let  $S$  be a minimum-size generator for  $G$ . For any  $t \in V(T)$ , the algorithm computes a representative of  $S$  as a partial solution of bag  $X_t$ .*

On the other hand, we can prove the following claim by considering the sets of partial solutions (for each bag  $X_t$ ) that lead to some partial solution computed for the root.

**Claim 27.** *Any partial solution  $(n_r, A^r, B^r, (S_{1,1}^r, \dots, S_{1,c_1^r}^r), \dots, (S_{\ell,1}^r, \dots, S_{\ell,c_\ell^r}^r))$  of the root  $r$  such that  $B^r = \emptyset$  and  $c_i = 1$  for all  $i \leq \ell$  is the representative to a generator of size  $n_r$  for the entire graph.*

The first constraint ensures that all vertices of  $G$  are well generated and the latter constraint ensures that the “connected components” of the expected generator have “actually” been connected.  $\square$

Since outer-planar graphs have treewidth at most 2 [Bod98], we get the following corollary:

**Corollary 28.** *The interval number (in cycle convexity) can be computed in linear-time in the class of outer-planar graphs.*

For purpose of completeness, we present a weaker (but simpler to prove) theorem.

**Theorem 29.** *Let  $k, t$  be fixed integers.  $in_{cc}(G) \leq k$  can be decided in time  $O(n)$  in the class of graphs  $G$  with treewidth at most  $t$ .*

*Proof.* According to Courcelle's Theorem [CM93], deciding if  $in_{cc}(G) \leq k$  ( $k$  being a fixed integer) is FPT in  $t + k$  (and linear in  $|V(G)|$ ) in the class of graphs of treewidth at most  $t$  ( $t$  being also fixed) if the problem is expressible in MSOL.

We propose the following logical formula to express the problem.

$\exists v_1, \dots, v_k \in V, \exists X_1, \dots, X_k \subseteq V$   
*// the vertices  $v_i$  are the ones of the generator and the sets  $X_j$  are the connected components of the generator.*

- $\forall i \leq k, \exists j \leq k$ , such that  $v_i \in X_j$  *// all  $v_i$ 's are in one set  $X_j$*
- $\forall j \leq k, \forall v \in X_j, \exists i \leq k$  such that  $v = v_i$  *// the  $X_j$ 's contains only vertices of the generator*
- $\forall j \leq k, \forall Y \subset X_j, \exists v \in Y$  and  $\exists u \in X_j \setminus Y$  such that  $\{u, v\} \in E$  *// each  $X_j$ 's is connected*
- $\forall v \in V$ , if  $v \notin X_j$  for all  $j \leq k$ , then  $\exists j \leq k, \exists u \in X_j, \exists w \in X_j$ , such that  $\{u, v\} \in E$  and  $\{w, v\} \in E$  and  $w \neq u$ . *// all vertices can be generated*

□

## 6.2 $(q, q - 4)$ -graphs

Let  $q \geq 4$ . A  $(q, q - 4)$ -graph  $G = (V, E)$  is such that for any  $S \subseteq V, |S| \leq q$ ,  $S$  induces at most  $q - 4$  paths on four vertices  $\square$ . In this section,  $q \geq 4$  being fixed, we present a linear-time algorithm in order to compute  $ihn(G)$  and a minimum-size generator for  $G$ , for any  $(q, q - 4)$ -graph  $G$ . The algorithm we present uses a well-known decomposition of  $(q, q - 4)$ -graphs.

Let  $G_1$  and  $G_2$  be two graphs.  $G_1 \cup G_2$  denotes the disjoint union of  $G_1$  and  $G_2$ .  $G_1 \oplus G_2$  denotes the complete join of  $G_1$  and  $G_2$ , i.e. the graph obtained from  $G_1 \cup G_2$  by adding an edge between any two vertices  $v \in V(G_1)$  and  $w \in V(G_2)$ . A *spider*  $G = (S \cup K \cup R, E)$  is a graph with vertex set  $V = S \cup K \cup R$  and edge set  $E$  such that:

1.  $(S, K, R)$  is a partition of  $V$  and  $R$  may be empty;
2. the subgraph  $G[K \cup R]$  induced by  $K$  and  $R$  is the complete join  $K \oplus R$ , and  $K$  separates  $S$  and  $R$ , i.e. any path from a vertex in  $S$  and a vertex in  $R$  contains a vertex in  $K$ ;
3.  $S$  is a stable set,  $K$  is a clique,  $|S| = |K| \geq 2$ , and there exists a bijection  $f : S \rightarrow K$  such that, for all vertices  $s \in S$ , either  $N(s) \cap K = K - \{f(s)\}$  or  $N(s) \cap K = \{f(s)\}$ . Roughly speaking, the edges between  $S$  and  $K$  are either a matching or an anti-matching. In the latter case or if  $|S| = |K| \leq 2$ ,  $G$  is called *thin*, otherwise  $G$  is *thick*.

A graph  $G = (S \cup K \cup R, E)$  is a *pseudo spider* if it satisfies only the first two properties of a spider. A graph  $G = (S \cup K \cup R, E)$  is a *q-pseudo spider* if it is a pseudo spider and, moreover,  $|S \cup K| \leq q$ . Note that  $q$ -pseudo spiders and spiders are pseudo spiders.

**Theorem 30** ([BO98]). *Let  $q \geq 4$  and let  $G$  be a  $(q, q - 4)$ -graph. Then, one of the following holds:*

1.  $G = G_1 \cup G_2$  is the disjoint union of two  $(q, q - 4)$ -graphs  $G_1$  and  $G_2$ , or
2.  $G = G_1 \oplus G_2$  is the join of two  $(q, q - 4)$ -graphs  $G_1$  and  $G_2$ , or
3.  $G$  is a spider  $(S \cup K \cup R, E)$  where  $G[R]$  is a  $(q, q - 4)$ -graph, or
4.  $G$  is a  $q$ -pseudo spider  $(S \cup K \cup R, E)$  where  $G[R]$  is a  $(q, q - 4)$ -graph.

In [], the authors have reinforced the above characterization of Theorem 30 as follows. A module in a graph  $G$  is any subset  $\mathcal{M} \subseteq V(G)$  such that for any  $v \in V(G) \setminus \mathcal{M}$ , either  $\mathcal{M} \subseteq N_G(v)$  or  $\mathcal{M} \cap N_G(v) = \emptyset$ . A module  $\mathcal{M}$  is strong if it does not overlap any other module, *i.e.*, for any module  $\mathcal{M}'$  of  $G$ , either one of  $\mathcal{M}, \mathcal{M}'$  is contained in the other or  $\mathcal{M}$  and  $\mathcal{M}'$  do not intersect. The quotient graph of  $G$  is the graph with vertex-set the inclusion wise maximal strong modules of  $G$  that are proper subsets of  $V(G)$  such that there is an edge between  $\mathcal{M}$  and  $\mathcal{M}'$  when there is an edge of  $E(G)$  with an end in each of  $\mathcal{M}, \mathcal{M}'$ . By [], if  $G$  is a  $(q, q-4)$ -graph that is a  $q$ -pseudo-spider  $(S \cup K \cup R, E)$  then either  $R = \emptyset$  or the quotient graph of  $G[K \cup S]$  is a split graph.

In the proof of next Theorem, we use the following easy Proposition:

**Proposition 31.** *If  $G = (V, E)$  has a 1-vertex  $v \in V(G)$ , then  $in_{cc}(G) = in_{cc}(G - v) + 1$ .*

*Proof.* Since  $G$  is simple and  $v$  does not have two distinct neighbors, then  $v$  must belong to any generator for  $G$ .  $\square$

**Theorem 32.** *For a fixed  $q$ , if  $G$  is a  $(q, q-4)$ -graph, then  $ihn$  can be computed in  $f(q) \cdot (n+m)$ -time.*

*Proof.* Let  $G$  be a  $(q, q-4)$ -graph. Without loss of generality,  $G$  is connected (else, we consider the connected components of  $G$  separately). Furthermore, if  $G$  has a universal vertex  $u$  then by Theorem 12,  $in_{cc}(G) = \gamma(G \setminus u) + 1$ . Since  $G \setminus u$  is a  $(q, q-4)$ -graph (because this class is closed by induced subgraph) and so, its dominating number  $\gamma(G \setminus u)$  can be computed in linear-time [], the minimum-size  $ihn(G)$  of a generator for  $G$  can also be computed in linear-time in this case. Therefore, let us assume for the remaining of the proof that there is no universal vertex of  $G$ . In particular,  $G$  has  $n \geq 2$  vertices, and by Theorem 4, it is easy to check whether  $in_{cc}(G) = 2$  in time  $\mathcal{O}(n+m)$ . Hence, from now on, we may assume that  $in_{cc}(G) \geq 3$ . We will consider all the cases of Theorem 30 (but the first one since  $G$  is connected).

- Suppose that  $G = G_1 \oplus G_2$  is the complete join of the two  $(q, q-4)$ -graphs  $G_1$  and  $G_2$ . Note that since  $G$  has no universal vertex, the graphs  $G_1$  and  $G_2$  have  $n_1 \geq 2$  and  $n_2 \geq 2$  vertices, respectively. In such case, since every vertex of  $G_1$ , resp. of  $G_2$ , is adjacent to every vertex of  $G_2$ , resp. of  $G_1$ , two vertices on each side are enough in order to generate  $G$ , hence  $in_{cc}(G) \leq 4$ . Therefore, we are left to characterize when  $in_{cc}(G) = 3$ . We will prove the following claim.

**Claim 33.**  *$in_{cc}(G) = 3$  if and only if  $\min\{\gamma(G_1), \gamma(G_2)\} = 2$ .*

In order to do so, assume the existence of a generator  $S$  of size three for  $G$ . Clearly,  $|S \cap V(G_i)| \geq 2$  for some  $i \in \{1, 2\}$ . W.l.o.g., let us assume that  $S$  has at least two vertices in  $G_1$ . Furthermore, since  $S$  is a generator for  $G$ , every vertex of  $V(G) \setminus S$  has at least two neighbors in  $S$ .

We aim at proving that there exists  $D \subseteq S \cap V(G_1)$  of size two that dominates  $G_1$ . There are three cases to be considered.

- Case  $|S \cap V(G_1)| = 2$ . Since  $|S \cap V(G_2)| = 1$ , each vertex of  $G_1 \setminus S$  must have at least one neighbor in  $S \cap V(G_1)$  to be generated (otherwise it would have a unique neighbor in  $S$ ). Hence,  $S \cap V(G_1)$  dominates  $G_1$ .
- Case  $|S \cap V(G_1)| = 3$  (*i.e.*,  $S \subseteq V(G_1)$ ) and  $S$  is connected. Let  $Y \subset S$  be any subset of size two of  $S$ . Every vertex of  $G_1 \setminus S$  must have at least two neighbors in  $S$  (to be generated). Therefore, every vertex of  $G_1 \setminus S$  has at least one neighbor in  $Y$ . Hence,  $Y$  dominates  $G_1$ .
- Case  $|S \cap V(G_1)| = 3$  and  $S$  is not connected. Since  $S$  must have at least one connected component with at least 2 vertices, let  $x$  and  $y$  be two adjacent vertices of  $S$  and let  $\{z\} = S \setminus \{x, y\}$ . Let  $w \in V(G_1) \setminus S$ . The only way that  $w$  is generated is that  $x$  and  $y$  are neighbors of  $w$ . Hence,  $x$  dominates  $V(G_1) \setminus \{z\}$ . Finally,  $\{x, z\}$  dominates of  $G_1$ .

Conversely, suppose that there exists  $i \in \{1, 2\}$  such that  $\gamma(G_i) = 2$  (observe that  $\gamma(G_i) \geq \gamma(G) > 1$  because  $G$  has no universal vertex). W.l.o.g., let us assume that  $\gamma(G_1) = 2$ . Let  $S_1 = \{u, v\}$  be a

dominating set of  $G_1$ , let  $x \in V(G_2)$  be arbitrary and let  $S = S_1 \cup \{x\}$ . By construction, every vertex of  $G$  has at least two neighbors in  $S$ , furthermore  $S$  induces a connected subgraph of  $G$ , hence  $S$  is a generator for  $G$  by Claim 2.

So, the claim is proved and  $in_{cc}(G) = 3$  if and only if  $\min\{\gamma(G_1), \gamma(G_2)\} = 2$ , which can be decided in linear-time  $\square$ .

- Now suppose that  $G = (S \cup K \cup R, E)$  is a spider. There are two subcases to be considered.
  - If  $G$  is a thin spider then every vertex in  $S$  is a 1-vertex, and so, by Proposition 31,  $in_{cc}(G) = |S| + in_{cc}(G \setminus S)$ . Since  $G \setminus S$  is still a  $(q, q - 4)$ -graph, one can apply recursively our algorithm on  $G \setminus S$ . Precisely, since every vertex  $u \in K$  is a universal vertex of  $G \setminus S$ , by Theorem 12,  $in_{cc}(G \setminus S) = \gamma(G \setminus (S \cup \{u\})) + 1$ , that can be computed in linear-time  $\square$ .
  - Else,  $G$  is a thick spider. Let us prove that  $in_{cc}(G) = 3$  (recall that we have assumed that  $in_{cc}(G) \geq 3$ ). Indeed, since  $G$  is a thick spider,  $|K| = |S| \geq 3$ , and we pick any three vertices in  $K$  in order to form the subset  $X$  that we will prove to be a generator for  $G$ . In order to prove it, since  $X$  induces a triangle, it is sufficient to prove that every vertex of  $G \setminus X$  has at least two adjacent neighbors in  $X$ . First, since  $X \subseteq K$  and  $K$  is a clique, the latter holds for every vertex of  $K \setminus X$ . Furthermore, since there is a complete join between  $K$  and  $R$ , it also holds for every vertex of  $R$ . Finally, since every vertex of  $S$  has  $|K| - 1$  neighbors in  $K$ , there is at most one vertex in  $X$  that is nonadjacent to a given vertex in  $S$ , and so, since  $|X| \geq 3$ , every vertex of  $S$  has at least two neighbors in  $X$ . Consequently,  $X$  is a generator for  $G$ .
- Else,  $G = (S \cup K \cup R, E)$  is a  $q$ -pseudo-spider. Let us assume that  $R \neq \emptyset$  (else, since  $|K \cup S| \leq q$ , the problem can be solved by using exhaustive search). By  $\square$ the quotient graph  $\mathcal{Q}$  of  $G[K \cup S]$  is a split graph, with the strong modules in  $K$  and the strong modules in  $S$  forming respectively a clique and a stable set in  $\mathcal{Q}$ . In particular if  $K$  is a strong module (i.e., there is a join between  $S$  and  $K$ ) then  $G = G[K] \oplus G[S \cup R]$ , and so, we go back to a previous case. So, we assume the existence of two strong modules  $K_1, K_2 \subseteq K$ . Note that there exists a complete join between  $K_1$  and  $K_2$  (because the modules of  $K$  induce a clique in  $\mathcal{Q}$ ).

Roughly, we will show that in order to compute a minimum-size generator for  $G$ , it suffices to “guess” its part in  $S \cup K$ . Precisely, we address the following subproblem: given  $X_0 \subseteq S \cup K$ , does there exist a subset  $R^*$  such that  $X_0 \cup R^*$  is a generator for  $G$ , and if so, what is the minimum-size of such a subset? We will show the following claim.

**Claim 34.** *Given  $X_0 \subseteq S \cup K$ , it can be decided in linear-time whether it exists a set  $R^*$  such that  $X_0 \cup R^*$  is a generator for  $G$ . Moreover, if it exists, such a set  $R^*$  with minimum size can be computed in linear-time.*

Overall, since  $|S \cup K| \leq q$ , there are at most  $2^q$  subsets  $X_0$  to be considered, and so, since  $q$  is fixed,  $in_{cc}(G)$  can be computed in linear-time in this case.

In order to prove the claim, consider a minimum-size generator  $X^*$  for  $G$ . Let  $X_0 = X^* \setminus R$ . Suppose that  $X_0 \neq X^*$  (else, we are done).

First we claim that  $X_0 \cup \{r\}$  has to generate  $S$  for any choice of vertex  $r \in R$ . Indeed, let  $s \in S \setminus X_0$  be arbitrary. By Claim 2 there is a path in  $X^*$  whose both ends are distinct and adjacent to  $s$ . In particular, if this path lies onto  $X_0$  then we are done. Else, as  $K$  is a  $SR$ -separator, the path goes by two vertices in  $K$ , that can be linked through vertex  $r$  since there is a complete join between  $K$  and  $R$ . Therefore,  $s$  is generated by  $X_0 \cup \{r\}$ , that proves the claim.

Furthermore, we claim that it can be assumed that  $|X^* \cap R| \leq 2$ . Indeed, if it is not the case, then let  $r_0 \in X^* \cap R$  be arbitrary, and let  $u_1 \in K_1, u_2 \in K_2$ . We claim that  $X = X_0 \cup \{u_1, u_2, r_0\}$  is a generator for  $G$  (note that  $|X| \leq |X^*|$ ). Indeed, by the previous claim it generates  $S$ . Moreover, since there is a complete join between  $K$  and  $R$ , and the strong modules in  $K$  form a clique of  $\mathcal{Q}$ , vertices

$u_1, u_2, r_0$  form a triangle and every vertex of  $R$ , resp. of  $K$ , is adjacent to  $u_1$  and  $u_2$ , resp. to  $r_0$  and at least one of  $u_1$  or  $u_2$  (because they are in different modules). Therefore, these three vertices generate  $K \cup R$  by Claim 2, that finally proves the claim.

Thus we are left to distinguish between the subcases  $|X^* \cap R| = 2$  and  $|X^* \cap R| = 1$ .

- Suppose that  $|X^* \cap R| = 2$ . We claim that it can be assumed that  $X_0 \cap K = \emptyset$ . Indeed, if it not the case then assume w.l.o.g. that  $X_0 \cap K_1 \neq \emptyset$ , and let  $u_1 \in K_1 \cap X_0$  and let  $r_0, r_1 \in X^* \cap R$ . Let  $u_2 \in K_2$ , it can be shown similarly as for the previous claim that  $X = X_0 \cup \{u_2, r_0\}$  is a generator for  $G$ . Since  $|X| \leq |X^*|$  and  $|X \cap R| = 1$ , the latter proves the claim, and so, from now on let us assume that  $X_0 \cap K = \emptyset$ . In this subcase, since  $K$  is a  $SR$ -separator, the two vertices in  $X^* \cap R$  generate  $R$ , i.e.,  $in_{cc}(G[R]) \leq 2$ . By Theorem 4, it implies that  $|R| = 2$  or the two vertices in  $X^* \cap R$  are two universal vertices of  $G[R]$ . Note that any choice of two universal vertices of  $G[R]$  will generate  $R \cup K$  in this subcase.
- Else,  $X^* \cap R = \{r_0\}$ . By Claim 2, every vertex of  $R \setminus r_0$  has two neighbors in  $X^*$  (in order to be generated). So,  $R = \{r_0\}$ , or  $|X_0 \cap K| \geq 2$ , or  $r_0$  is a universal vertex of  $G[R]$ . Note that if  $|X_0 \cap K| = 2$  then  $X_0 \cup \{r\}$  is a generator for  $G$  for any choice of  $r \in R$ , furthermore for any choice of a universal vertex  $r_0$  of  $G[R]$ ,  $X_0 \cup \{r_0\}$  is a generator for  $G$ .

To sum up, given  $X_0$  fixed a candidate for the subset  $R^*$  can be chosen as follows. Assume that  $|R| \geq 3$ , or else since  $q$  is fixed, the above subproblem can be solved in constant-time by exhaustive search. If  $X_0$  is a generator for  $G$  then  $R^* = \emptyset$ . Else if  $|X_0 \cap K| \geq 2$  then  $R^* = \{r_0\}$  with  $r_0 \in R$  being arbitrary. Else if  $|X_0 \cap K| = 1$  then we search for a universal vertex in  $R$ ; if no such vertex exists then we discard  $X_0$ , else  $R^* = \{r_0\}$  with  $r_0$  being universal to  $R$ . Else,  $X_0 \cap K = \emptyset$ , we search for the existence of two universal vertices in  $R$ ; if no such two vertices exist then we discard  $X_0$ , else  $R^* = \{r_0, r_1\}$  with  $r_0, r_1$  being two universal vertices of  $R$ . In all cases, we discard  $X_0$  when  $X_0 \cup R^*$  does not generate  $G$ . If it does then by the above characterization  $R^*$  is a minimum-size subset of  $R$  such that  $X_0 \cup R^*$  is a generator for  $G$ .

□

Cographs are the graphs without any induced  $P_4$  (path on four vertices). In other words, they are exactly the  $(q, q - 4)$ -graphs for  $q = 4$ . Therefore, the following corollary directly follows.

**Corollary 35.** *If  $G$  is a cograph, then  $in_{cc}(G)$  can be computed in linear-time.*

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