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Marie-Odile Bristeau, David Froger, Raouf Hamouda, Anne Mangeney, Jacques Sainte-Marie, Martin Vallée

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# Numerical simulations of 3d hydrostatic Navier-Stokes system with free surface.

M.-O. Bristeau, D. Froger, R. Hamouda, A. Mangeney,  
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November 7, 2016

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### Abstract

In this paper we propose a stable and robust discrete model for the simulation of the 3d incompressible and hydrostatic Navier-Stokes system with free surface.

The idea is use a Galerkin type approximation of the velocity field in order to obtain an accurate description of the vertical profile of the horizontal velocity. Such a strategy has several advantages. It allows

- to rewrite the Navier-Stokes equations under the form of a system of conservation laws with source terms,
- the easy handling of the free surface, which does not require moving meshes,
- the possibility to take advantage of robust and accurate numerical techniques developed in extensive amount for Shallow Water type systems.

The proposed numerical scheme is endowed with stability properties such as positivity of the water depth, preservation of the equilibrium at rest and fully discrete entropy inequality. We confront the model and the discrete scheme with some numerical academic and realistic examples.

**Keywords** Free surface flows, Navier-Stokes equations, Free surface, Hydrostatic assumption, Kinetic description, Finite volumes.

## 1 Introduction

The outline of the paper is as follows. In Section 2, we recall the incompressible Navier-Stokes equations, the boundary conditions and the hydrostatic approximation. The layer-averaged system obtained by a vertical discretization of the hydrostatic model is described in Section 3. The kinetic interpretation of the model is given in Section 4 allowing to derive a numerical scheme presented in Section 5. Numerical validations are shown in Section 6.

## 2 The Navier-Stokes system

We consider the classical incompressible Navier-Stokes system

$$\nabla \cdot \mathbf{U} = 0, \quad (1)$$

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot (\mathbf{U} \otimes \mathbf{U}) = \nabla \cdot \boldsymbol{\Sigma} + \mathbf{g}, \quad (2)$$

where  $\mathbf{U}(t, x, y, z) = (u, v, w)^T$  is the velocity,  $\mathbf{u}(t, x, y, z) = (u, v)^T$  is the horizontal velocity,  $p$  is the fluid pressure,  $\mathbf{g} = (0, 0, -g)^T$  represents the gravity forces and  $\mu$  is the viscosity coefficient. The quantity  $\nabla$  denotes

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix},$$

and  $\nabla_{x,y}$  corresponds to the projection of  $\nabla$  on the horizontal plane i.e.

$$\nabla_{x,y} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}.$$

We assume a Newtonian fluid and hence the stress tensor  $\Sigma$  is given by

$$\Sigma = -p\mathbb{I} + \mu[\nabla\mathbf{U} + (\nabla\mathbf{U})^T]. \quad (3)$$

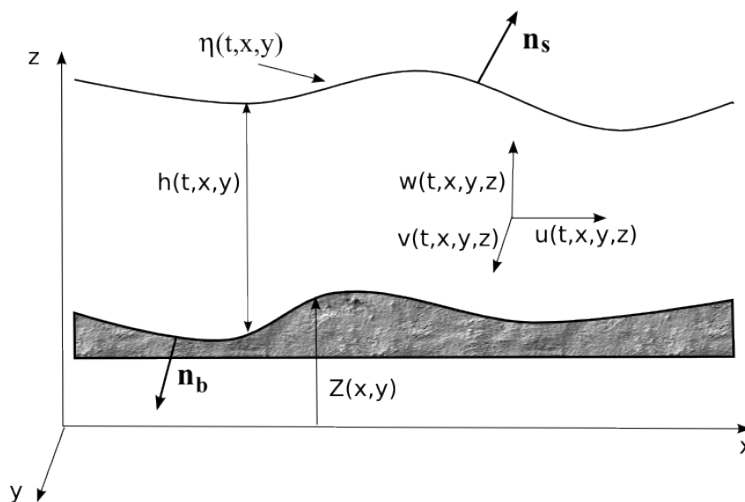


Figure 1: Fluid domain and notations.

We consider a free surface flow (see Fig. 1), therefore we assume

$$z(x,y) \leq z \leq \eta(t,x,y) := h(t,x,y) + z(x,y)$$

with  $z(x,y)$  the bottom elevation and  $h(t,x,y)$  the water depth.

## 2.1 Boundary conditions

### 2.1.1 Bottom and free surface

Let  $\mathbf{n}_b$  and  $\mathbf{n}_s$  be the unit outward normals at the bottom and at the free surface respectively defined by (see Fig 1)

$$\mathbf{n}_b = \frac{1}{\sqrt{1 + |\nabla_{x,y}z|^2}} \begin{pmatrix} -\nabla_{x,y}z \\ 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{n}_s = \frac{1}{\sqrt{1 + |\nabla_{x,y}\eta|^2}} \begin{pmatrix} -\nabla_{x,y}\eta \\ 1 \end{pmatrix}.$$

On the bottom we prescribe an impermeability condition

$$\mathbf{U} \cdot \mathbf{n}_b = 0, \quad (4)$$

and a friction condition given e.g. by a Navier law

$$(\boldsymbol{\Sigma} \cdot \mathbf{n}_b) \cdot \mathbf{t}_i = -\kappa \mathbf{U} \cdot \mathbf{t}_i, \quad i = 1, 2 \quad (5)$$

with  $\kappa$  a Navier coefficient and  $(\mathbf{t}_i, i = 1, 2)$  two tangential vectors.

On the free surface, the kinematic boundary condition is satisfied

$$\frac{\partial \eta}{\partial t} + \mathbf{u}(t, x, y, \eta) \cdot \nabla_{x,y} \eta - w(t, x, y, \eta) = 0 \quad (6)$$

and the no stress condition

$$\boldsymbol{\Sigma} \cdot \mathbf{n}_s = -p^a(t, x, y) \mathbf{n}_s, \quad (7)$$

where  $p^a(t, x, y)$  is a given quantity accounting for the atmospheric pressure. On solid walls, we prescribe a slip condition

$$\mathbf{U} \cdot \mathbf{n} = 0, \quad (8)$$

$\mathbf{n}$  being the outward normal to the considered wall.

### 2.1.2 Fluid boundaries

In this paper we consider only two types of fluid boundaries: solid walls on which we prescribe a slip condition  $\mathbf{U} \cdot \mathbf{n} = 0$  with  $\mathbf{n}$  the unit outward normal to the considered boundary, and fluid boundaries on which we prescribe zero, one or two of the following conditions depending on the type of the flow (fluvial or torrential) : Water level  $h + z(x, y)$  given, flux  $h\mathbf{U}$  given.

The system is completed with some initial conditions

$$h(0, x, y) = h^0(x, y), \quad \mathbf{U}(0, x, y, z) = \mathbf{U}^0(x, y, z),$$

with  $\mathbf{U}^0$  satisfying the divergence free condition (1).

## 2.2 The hydrostatic Navier-Stokes system

The vorticity is defined by

$$\boldsymbol{\Omega} = \begin{pmatrix} \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \end{pmatrix}.$$

Assuming the first two components of  $\mathbf{\Omega}$  vanishes then the viscous terms in the momentum equation along  $z$  of the Navier-Stokes system namely,

$$\frac{\partial}{\partial x} \left( \mu \frac{\partial u}{\partial z} + \mu \frac{\partial w}{\partial x} \right) + \frac{\partial}{\partial y} \left( \mu \frac{\partial v}{\partial z} + \mu \frac{\partial w}{\partial y} \right) + 2\mu \frac{\partial^2 w}{\partial z^2},$$

also vanish. Hence the momentum equation along  $z$  of the Navier-Stokes reduces to

$$\frac{\partial w}{\partial t} + \frac{\partial uw}{\partial x} + \frac{\partial vw}{\partial y} + \frac{\partial w^2}{\partial z} + \frac{\partial p}{\partial z} = -g.$$

The hydrostatic assumption consists in neglecting the vertical acceleration of the fluid

$$\frac{\partial w}{\partial t} + \frac{\partial uw}{\partial x} + \frac{\partial vw}{\partial y} + \frac{\partial w^2}{\partial z} \approx 0,$$

see [14, 21, 24] for justifications of the analysis of hydrostatic models and for their asymptotic derivation [1, 19, 23]. This implies that the pressure writes

$$p = p^a(t, x, y) + g(\eta - z), \quad (9)$$

$p^a(t, x, y)$  being the atmospheric pressure introduced in (7). Therefore, the hydrostatic approximation of the system (1)-(8) consists in the model

$$\nabla \cdot \mathbf{U} = 0, \quad (10)$$

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla_{x,y} \cdot (\mathbf{u} \otimes \mathbf{u}) + \frac{\partial \mathbf{u}w}{\partial z} + g \nabla_{x,y} \eta = -\nabla_{x,y} p^a + \nabla \cdot \hat{\mathbf{T}}, \quad (11)$$

with

$$\hat{\mathbf{T}} = \mu \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} \end{pmatrix},$$

and with the kinematic boundary conditions (4),(6) and the dynamical boundary conditions

$$\hat{T} \cdot \mathbf{n}_b = \kappa \mathbf{u}|_{z=z(x,y)}, \quad (12)$$

$$\hat{T} \cdot \mathbf{n}_s = 0. \quad (13)$$

Notice that  $\nabla \cdot \hat{\mathbf{T}} = \mu \Delta \mathbf{u}$  where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

The smooth solutions of the system (10)-(13) satisfies the energy balance

$$\begin{aligned} \frac{\partial}{\partial t} \int_{z_b}^{\eta} E dz + \nabla_{x,y} \cdot \int_{z(x,y)}^{\eta} \left( \mathbf{u}(E + g(\eta - z)) - \mu \mathbf{u} \cdot \nabla_{x,y} \mathbf{u} \right) dz \\ = -\mu \int_{z(x,y)}^{\eta} \hat{\mathbf{T}}^2 dz - \kappa \mathbf{u}|_{z=z(x,y)}^2, \end{aligned} \quad (14)$$

with  $\hat{\mathbf{T}}^2 = \sum_{i,j} T_{i,j}^2$  and

$$E = E(z, u, v) = \frac{u^2 + v^2}{2} + gz. \quad (15)$$

**Remark 2.1** *The viscous terms in Eq. (11) result from a rewriting of the quantity  $\nabla \cdot \Sigma$  appearing in Eq. (2) using the divergence free condition (1). Therefore, the dynamic boundary conditions (12)-(13) are slightly different from (5),(7).*

In the case of an inviscid fluid, the system (10)-(13) consists in the incompressible and hydrostatic Euler equations with free surface and reads

$$\nabla \cdot \mathbf{U} = 0, \quad (16)$$

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla_{x,y} \cdot (\mathbf{u} \otimes \mathbf{u}) + \frac{\partial \mathbf{u} w}{\partial z} + g \nabla_{x,y} \eta = -\nabla_{x,y} p^a, \quad (17)$$

coupled with the two kinematic boundary conditions (4),(6).

We recall the fundamental stability property related to the fact that the hydrostatic Euler system admits, for smooth solutions, an energy conservation that can be written under

$$\frac{\partial}{\partial t} \int_{z_b}^{\eta} E dz + \nabla_{x,y} \cdot \int_{z_b}^{\eta} \left( \mathbf{u}(E + g(\eta - z)) \right) dz = 0, \quad (18)$$

with  $E$  defined by (15).

**Remark 2.2** *When considering hydrostatic models for incompressible fluids, the pressure  $p$  is no more defined as the Lagrange multiplier of the incompressibility condition but has an expression as a function of the other variables, see (9). is satisfied.*



### 3 The layer-averaged model

We consider a discretization of the fluid domain by layers (see Fig. 2) where the layer  $\alpha$  contains the points of coordinates  $(x, y, z)$  with  $z \in L_\alpha(t, x, y) = (z_{\alpha-1/2}, z_{\alpha+1/2})$  and  $\{z_{\alpha+1/2}\}_{\alpha=1,\dots,N}$  is defined by

$$\begin{cases} z_{\alpha+1/2}(t, x, y) = z(t, x, y) + \sum_{j=1}^{\alpha} h_j(t, x, y), & \alpha \in [0, \dots, N], \\ h_\alpha(t, x, y) = z_{\alpha+1/2}(t, x, y) - z_{\alpha-1/2}(t, x, y) = l_\alpha h(t, x, y), \end{cases} \quad (19)$$

and

$$\sum_{\alpha=1}^N l_\alpha = 1.$$

The layer averaged discretization is fully described in [15] in the 2d case with a general rheology.

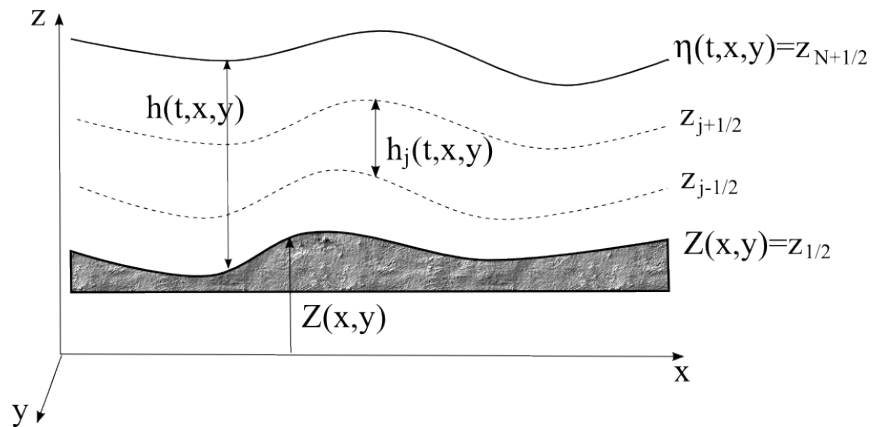


Figure 2: Fluid domain discretized by layers and notations.

#### 3.1 The layer-averaged Euler system

The layer-averaging process for the hydrostatic Euler and Navier-Stokes system is precisely described in the paper (papier MOB,BDM,CG,JSM), the reader can refer to it. In the following, we only recall the main steps.

Let us calculate the energy balance associated with the incompressible hydrostatic Euler equations (16)-(17),(4),(6). Considering enough smooth

solutions, multiplying the first component of Eq. (17) by  $u$ , the second component by  $v$  respectively and integrating the obtained equations over the layer  $\alpha$  gives

$$\begin{aligned} \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \left( \frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y} + \frac{\partial uw}{\partial z} + g \frac{\partial \eta}{\partial x} \right) u \, dz &= 0, \\ \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \left( \frac{\partial v}{\partial t} + \frac{\partial uv}{\partial x} + \frac{\partial v^2}{\partial y} + \frac{\partial vw}{\partial z} + g \frac{\partial \eta}{\partial y} \right) v \, dz &= 0, \end{aligned}$$

and after simple manipulations, we obtain

$$\begin{aligned} \frac{\partial}{\partial t} \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \frac{u^2}{2} \, dz + \frac{\partial}{\partial x} \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} u \left( \frac{u^2}{2} + g\eta \right) \, dz + \frac{\partial}{\partial y} \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} v \frac{u^2}{2} \, dz = \\ g\eta \frac{\partial}{\partial x} \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} u \, dz + \frac{u_{\alpha+1/2}^2}{2} G_{\alpha+1/2} - \frac{u_{\alpha-1/2}^2}{2} G_{\alpha-1/2}, \end{aligned} \quad (20)$$

and

$$\begin{aligned} \frac{\partial}{\partial t} \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} \frac{v^2}{2} \, dz + \frac{\partial}{\partial x} \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} u \frac{v^2}{2} \, dz + \frac{\partial}{\partial y} \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} v \left( \frac{v^2}{2} + g\eta \right) \, dz = \\ g\eta \frac{\partial}{\partial y} \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} v \, dz + \frac{v_{\alpha+1/2}^2}{2} G_{\alpha+1/2} - \frac{v_{\alpha-1/2}^2}{2} G_{\alpha-1/2}, \end{aligned} \quad (21)$$

with  $u_{\alpha+1/2} = u(t, x, y, z_{\alpha+1/2})$ ,  $v_{\alpha+1/2} = v(t, x, y, z_{\alpha+1/2})$  and

$$\begin{aligned} G_{\alpha+1/2} &= \frac{\partial z_{\alpha+1/2}}{\partial t} + u_{\alpha+1/2} \frac{\partial z_{\alpha+1/2}}{\partial x} + v_{\alpha+1/2} \frac{\partial z_{\alpha+1/2}}{\partial y} - w_{\alpha+1/2}, \\ &= \frac{\partial z_{\alpha+1/2}}{\partial t} + \mathbf{u}_{\alpha+1/2} \cdot \nabla_{x,y} z_{\alpha+1/2} - w_{\alpha+1/2}. \end{aligned} \quad (22)$$

Then an integration over the layer  $\alpha$  of the divergence free condition gives

$$\frac{\partial h_{\alpha}}{\partial t} + \frac{\partial}{\partial x} \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} u \, dz + \frac{\partial}{\partial y} \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} v \, dz = G_{\alpha+1/2} - G_{\alpha-1/2},$$

and the sum of the above relation multiplied by  $g\eta$  with Eqs. (20) and (21) gives the energy balance

$$\begin{aligned} \frac{\partial}{\partial t} \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} E dz + \frac{\partial}{\partial x} \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} uE dz + \frac{\partial}{\partial y} \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} vE dz = \\ \left( \frac{u_{\alpha+1/2}^2 + v_{\alpha+1/2}^2}{2} + g\eta \right) G_{\alpha+1/2} - \left( \frac{u_{\alpha-1/2}^2 + v_{\alpha-1/2}^2}{2} + g\eta \right) G_{\alpha-1/2}, \end{aligned} \quad (23)$$

with  $E$  defined by (15).

### 3.2 Closure relations

Let us denote  $\langle f \rangle_\alpha$  the average along the vertical axis in the layer  $\alpha$  of the quantity  $f = f(t, x, y, z)$  i.e.

$$\langle f \rangle_\alpha(t, x, y) = \int_{z_{\alpha-1/2}}^{z_{\alpha+1/2}} f(t, x, y, z) dz. \quad (24)$$

If  $u'_\alpha, v'_\alpha$  are defined as the deviations of  $u, v$  with respect to their layer-averages over the layer  $\alpha$ , then it comes for  $z \in L_\alpha$

$$u = \frac{\langle u \rangle_\alpha}{h_\alpha} + u'_\alpha, \quad v = \frac{\langle v \rangle_\alpha}{h_\alpha} + v'_\alpha, \quad (25)$$

with  $\langle u'_\alpha \rangle = \langle v'_\alpha \rangle = 0$ . Following the moment closure proposed by Levermore [22], we study the minimization problem

$$\min_{u'_\alpha, v'_\alpha} \langle \{E(h, u, v)\} \rangle_\alpha. \quad (26)$$

The energy  $E(z, u, v)$  defined by (15) being quadratic with respect to  $u, v$  we notice

$$\begin{aligned} \langle u^2 \rangle_\alpha &= \frac{\langle u \rangle_\alpha^2}{h_\alpha} + \frac{2\langle u \rangle_\alpha \langle u'_\alpha \rangle_\alpha}{h_\alpha} + \langle (u'_\alpha)^2 \rangle_\alpha \\ &= \frac{\langle u \rangle_\alpha^2}{h_\alpha} + \langle (u'_\alpha)^2 \rangle_\alpha \\ &\geq \frac{\langle u \rangle_\alpha^2}{h_\alpha}, \end{aligned} \quad (27)$$

and the same inequality holds for  $\langle v^2 \rangle_\alpha$ . Therefore the solution of the minimization problem (26) is given by

$$\langle E \left( z, \frac{\langle u \rangle_\alpha}{h_\alpha}, \frac{\langle v \rangle_\alpha}{h_\alpha} \right) \rangle_\alpha = \min_{u'_\alpha, v'_\alpha} \langle \{E(z, u, v)\} \rangle_\alpha. \quad (28)$$

Since the only choices leading to an equality in relations (27) corresponds to

$$u = \frac{\langle u \rangle_\alpha}{h_\alpha}, \quad v = \frac{\langle v \rangle_\alpha}{h_\alpha}, \quad (29)$$

the closure relations (29) motivate the definition of piecewise constant approximation of the velocity field  $u$ ,  $v$  and  $w$  as follows.

### 3.3 The proposed model

Using the notations (19), let us consider the space  $\mathbb{P}_{0,h}^{N,t}$  of piecewise constant functions defined by

$$\mathbb{P}_{0,h}^{N,t} = \left\{ \mathcal{K}_{z \in L_\alpha(t,x,y)}(z), \quad \alpha \in \{1, \dots, N\} \right\},$$

where  $\mathcal{K}_{z \in L_\alpha(t,x,y)}(z)$  is the characteristic function of the layer  $L_\alpha(t, x, y)$ . Using this formalism, the projection of  $u$ ,  $v$  and  $w$  on  $\mathbb{P}_{0,h}^{N,t}$  is a piecewise constant function defined by

$$X^N(t, x, y, z, \{z_\alpha\}) = \sum_{\alpha=1}^N \mathcal{K}_{[z_{\alpha-1/2}, z_{\alpha+1/2}]}(z) X_\alpha(t, x, y), \quad (30)$$

for  $X \in (u, v, w)$ .

**Proposition 3.1** *The solutions of the incompressible and hydrostatic Euler equations (16)-(17),(4),(6) satisfying the closure relations (29) are also solutions of the system*

$$\sum_{\alpha=1}^N \frac{\partial h_\alpha}{\partial t} + \sum_{\alpha=1}^N \nabla_{x,y} \cdot (h_\alpha \mathbf{u}_\alpha) = 0, \quad (31)$$

$$\begin{aligned} & \frac{\partial h_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla_{x,y} \cdot (h_\alpha \mathbf{u}_\alpha \otimes \mathbf{u}_\alpha) + \nabla_{x,y} \left( \frac{g}{2} h h_\alpha \right) \\ & = -g h_\alpha \nabla z + \mathbf{u}_{\alpha+1/2} G_{\alpha+1/2} - \mathbf{u}_{\alpha-1/2} G_{\alpha-1/2}, \quad \alpha = 1, \dots, N, \end{aligned} \quad (32)$$

that is a layer-averaged approximation of the Euler system. The quantity  $G_{\alpha+1/2}$  (resp.  $G_{\alpha-1/2}$ ) corresponds to mass exchange across the interface  $z_{\alpha+1/2}$  (resp.  $z_{\alpha-1/2}$ ) and  $G_{\alpha+1/2}$  is defined by

$$G_{\alpha+1/2} = \sum_{j=1}^{\alpha} \left( \frac{\partial h_j}{\partial t} + \nabla_{x,y} \cdot (h_j \mathbf{u}_j) \right)$$

$$= - \sum_{j=1}^N \left( \sum_{p=1}^j l_p - \mathbb{K}_{j \leq \alpha} \right) \nabla_{x,y}(h_j \mathbf{u}_j). \quad (33)$$

The velocities at the interfaces  $\mathbf{u}_{\alpha+1/2}$  are defined by

$$\mathbf{u}_{\alpha+1/2} = \begin{cases} \mathbf{u}_\alpha & \text{if } G_{\alpha+1/2} \leq 0 \\ \mathbf{u}_{\alpha+1} & \text{if } G_{\alpha+1/2} > 0 \end{cases} \quad (34)$$

The smooth solutions of (31),(32) satisfy an energy balance and we have the following proposition.

**Proposition 3.2** *The system (31),(32) admits, for smooth solutions, the energy balance*

$$\begin{aligned} & \frac{\partial}{\partial t} E_\alpha + \frac{\partial}{\partial x} \left( u_\alpha \left( E_\alpha + \frac{g}{2} h_\alpha h \right) \right) + \frac{\partial}{\partial y} \left( v_\alpha \left( E_\alpha + \frac{g}{2} h_\alpha h \right) \right) \\ &= \left( u_{\alpha+1/2} u_\alpha + v_{\alpha+1/2} v_\alpha - \frac{u_\alpha^2 + v_\alpha^2}{2} + g\eta \right) G_{\alpha+1/2} \\ & \quad - \left( u_{\alpha-1/2} u_\alpha + v_{\alpha-1/2} v_\alpha - \frac{u_\alpha^2 + v_\alpha^2}{2} + g\eta \right) G_{\alpha-1/2}, \end{aligned} \quad (35)$$

with

$$E_\alpha = \frac{h_\alpha u_\alpha^2 + h_\alpha v_\alpha^2}{2} + \frac{g}{2} h_\alpha h + g h_\alpha z. \quad (36)$$

The sum of Eqs. (35) for  $\alpha = 1, \dots, N$  gives the energy balance

$$\begin{aligned} & \frac{\partial}{\partial t} \sum_{\alpha=1}^N E_\alpha + \frac{\partial}{\partial x} \sum_{\alpha=1}^N u_\alpha \left( E_\alpha + \frac{g}{2} h_\alpha h \right) + \frac{\partial}{\partial y} \sum_{\alpha=1}^N v_\alpha \left( E_\alpha + \frac{g}{2} h_\alpha h \right) \\ &= - \sum_{\alpha=1}^N \frac{(u_{\alpha+1/2} - u_\alpha)^2 + (v_{\alpha+1/2} - v_\alpha)^2}{2} |G_{\alpha+1/2}|. \end{aligned} \quad (37)$$

**Remark 3.1** *Equation (37) is a layer discretization of the energy balance (15). The definition of  $\mathbf{u}_{\alpha+1/2}$  given in (34) ensures the right hand side in Eq. (37) is nonpositive. Notice that a centered definition for  $\mathbf{u}_{\alpha+1/2}$  i.e.*

$$\mathbf{u}_{\alpha+1/2} = \frac{\mathbf{u}_\alpha + \mathbf{u}_{\alpha+1}}{2}, \quad (38)$$

instead of (34) leads to a vanishing right hand side in Eq. (37). But the centered choice (38) does not allow to obtain an energy balance in the variable density case and does not give a maximum principle, at the discrete level, see [5]. Simple calculations show that any other choice than (34) or (38) leads to a non negative r.h.s. in (37).

It is noticeable that, thanks to the kinematic boundary condition at each interface, the vertical velocity is no more a variable of the system (32). This is an advantage of this formulation over the hydrostatic model where the vertical velocity is needed in the momentum equation (11) and is deduced from the incompressibility condition (10). Even if the vertical velocity  $w$  no more not appear in the model (31)-(32), it can be obtained as follows.

**Proposition 3.3** *The piecewise constant approximation of the vertical velocity  $w$  defined by Eq. (30) is given by*

$$w_\alpha = k_\alpha - z_\alpha \nabla_{x,y} \cdot \mathbf{u}_\alpha \quad (39)$$

with

$$\begin{aligned} k_1 &= \nabla_{x,y} \cdot (z(x, y) \mathbf{u}_1), \\ k_{\alpha+1} &= k_\alpha + \nabla_{x,y} \cdot (z_{\alpha+1/2} (\mathbf{u}_{\alpha+1} - \mathbf{u}_\alpha)). \end{aligned}$$

The quantities  $\{w_\alpha\}_{\alpha=1}^N$  are obtained only using a post-processing of the variables governing the system (31)-(32).

**Proof of prop. 3.1** *An integration of the divergence free condition over the layer  $\alpha$  gives*

$$\frac{\partial h_\alpha}{\partial t} + \nabla_{x,y} \cdot (h_\alpha \mathbf{u}_\alpha) = G_{\alpha+1/2} - G_{\alpha-1/2}, \quad (40)$$

with  $G_{\alpha\pm 1/2}$  defined by (22). The sum for  $\alpha = 1, \dots, N$  of the above relations gives Eq. (31) where the kinematic boundary conditions (4),(6) corresponding to

$$G_{1/2} = G_{N+1/2} = 0,$$

have been used. Similarly, the sum for  $j = 1, \dots, \alpha$  of the relations (40) gives the expression (33) for  $G_{\alpha+1/2}$ .

Equation (32) is obtained by an integration of Eqs. (17) over the layer  $\alpha$  and using the decomposition (30) for  $u$  and  $v$ . ■

**Proof of prop. 3.2** *In order to obtain (35) we multiply (32) by  $\mathbf{u}_\alpha$  and we perform simple manipulations. More precisely, the momentum equation along the  $x$  axis multiplied by  $u_\alpha$  gives*

$$\left( \frac{\partial}{\partial t} (h_\alpha u_\alpha) + \frac{\partial}{\partial x} \left( h_\alpha u_\alpha^2 + \frac{g}{2} h h_\alpha \right) + \frac{\partial}{\partial y} (h_\alpha u_\alpha v_\alpha) \right) u_\alpha = \left( -g h_\alpha \frac{\partial z}{\partial x} + u_{\alpha+1/2} G_{\alpha+1/2} - u_{\alpha-1/2} G_{\alpha-1/2} \right) u_\alpha.$$

*Considering first the left hand side of the preceding equation excluding the pressure terms, we denote*

$$I_{u,\alpha} = \left( \frac{\partial}{\partial t} (h_\alpha u_\alpha) + \frac{\partial}{\partial x} (h_\alpha u_\alpha^2) + \frac{\partial}{\partial y} (h_\alpha u_\alpha v_\alpha) \right) u_\alpha,$$

*and using (40) we have*

$$I_{u,\alpha} = \frac{\partial}{\partial t} \left( \frac{h_\alpha u_\alpha^2}{2} \right) + \frac{\partial}{\partial x} \left( u_\alpha \frac{h_\alpha u_\alpha^2}{2} \right) + \frac{\partial}{\partial y} \left( v_\alpha \frac{h_\alpha u_\alpha^2}{2} \right) + \frac{u_\alpha^2}{2} (G_{\alpha+1/2} - G_{\alpha-1/2}).$$

*Now we consider the contribution of the pressure terms over the energy balance i.e.*

$$I_{p,u,\alpha} = \left( \frac{\partial}{\partial x} \left( \frac{g}{2} h h_\alpha \right) + g h_\alpha \frac{\partial z}{\partial x} \right) u_\alpha,$$

*and it comes*

$$\begin{aligned} I_{p,u,\alpha} &= g h_\alpha \frac{\partial}{\partial x} (h + z) u_\alpha \\ &= g \frac{\partial}{\partial x} (h_\alpha (h + z) u_\alpha) - g (h + z) \frac{\partial}{\partial x} (h_\alpha u_\alpha) \\ &= \frac{\partial}{\partial x} \left( \left( \frac{g}{2} h_\alpha h + \frac{g}{2} h_\alpha (h + 2z) \right) u_\alpha \right) - g (h + z) \frac{\partial}{\partial x} (h_\alpha u_\alpha). \end{aligned}$$

*Performing the same manipulations over the momentum equation along  $y$  and adding the terms  $I_{u,\alpha}, I_{v,\alpha}, I_{p,u,\alpha}$  and  $I_{p,v,\alpha}$  gives the result. ■*

**Proof of prop. 3.3** *Using the boundary condition (4), an integration from  $z(x, y)$  to  $z$  of the divergence free condition (10) easily gives*

$$w = -\nabla_{x,y} \cdot \int_{z(x,y)}^z \mathbf{u} \, dz.$$

Replacing formally in the above equation  $\mathbf{u}$  (resp.  $w$ ) by  $\mathbf{u}^N$  (resp.  $w^N$ ) defined by (30) and performing an integration over the layer  $L_1$  of the obtained relation yields

$$\begin{aligned} h_1 w_1 &= - \int_{z(x,y)}^{z_{3/2}} \nabla_{x,y} \int_{z_b}^z \mathbf{u}_1 \, dz dz_1 \\ &= h_1 \nabla_{x,y} \cdot (z(x,y) \mathbf{u}_1) - \frac{z_{3/2}^2 - z(x,y)^2}{2} \nabla_{x,y} \cdot \mathbf{u}_1, \end{aligned}$$

*i.e.*

$$w_1 = \nabla_{x,y} \cdot (z(x,y) \mathbf{u}_1) - z_1 \nabla_{x,y} \cdot \mathbf{u}_1,$$

corresponding to (39) for  $\alpha = 1$ . A similar computation for the layers  $L_2, \dots, L_N$  proves the result (39) for  $\alpha = 2, \dots, N$ .

A more detailed version of this proof is given in [15]. ■

### 3.4 Quasilinear form

Introducing  $q_{x,\alpha} = h_\alpha u_\alpha$ ,  $q_{y,\alpha} = h_\alpha v_\alpha$ , the system (31)-(32) can be written under the form of a quasilinear PDE

$$\mathcal{M} \frac{\partial U}{\partial t} + \mathcal{A}_x \frac{\partial U}{\partial x} + \mathcal{A}_y \frac{\partial U}{\partial y} = \mathcal{S}_b(U), \quad (41)$$

with

$$U = (h, q_{x,1}, \dots, q_{x,N}, q_{y,1}, \dots, q_{y,N})^T, \quad (42)$$

and  $\mathcal{M}$ ,  $\mathcal{A}_x$  and  $\mathcal{A}_y$  are three square matrices of size  $2N + 1$  defined by

- matrix  $\mathcal{M}$

$$\begin{array}{ll} \mathcal{M}_{i,i} = 1 & \text{for } i = 1, \dots, 2N + 1 \\ \mathcal{M}_{i+1,1} = -u_{i+1/2} \sum_{j=1}^i l_j + u_{i-1/2} \sum_{j=1}^{i-1} l_j & \text{for } i = 1, \dots, N \\ \mathcal{M}_{i+N+1,1} = -v_{i+1/2} \sum_{j=1}^i l_j + v_{i-1/2} \sum_{j=1}^{i-1} l_j & \text{for } i = 1, \dots, N \\ \mathcal{M}_{i,j} = 0 & \text{else} \end{array}$$



- matrix  $\mathcal{A}_x$

$$\begin{aligned}
\mathcal{A}_{x,1,i} &= 1 && \text{for } i = 2, \dots, N+1 \\
\mathcal{A}_{x,i+1,i+1} &= u_i - u_{i+1/2} && \text{for } i = 1, \dots, N \\
\mathcal{A}_{x,i+1,1} &= gh_i - l_i u_i^2 && \text{for } i = 1, \dots, N \\
\mathcal{A}_{x,i+1,j} &= -(u_{i+1/2} - u_{i-1/2}) && \text{for } i = 1, \dots, N \quad j = 2, \dots, i-1 \\
\mathcal{A}_{x,N+1+i,N+1} &= l_i u_i v_i && \text{for } i = 1, \dots, N \\
\mathcal{A}_{x,N+1+i,N+1+i} &= u_i && \text{for } i = 1, \dots, N \\
\mathcal{A}_{x,1+i,N+1+i} &= u_i && \text{for } i = 1, \dots, N \\
\mathcal{A}_{x,i,j} &= 0 && \text{else}
\end{aligned}$$

- matrix  $\mathcal{A}_y$

$$\mathcal{A}_{y,N+2,i} = 1 \quad \text{for } i = 2, \dots, N+1$$

and  $\mathcal{S}_b(U)$  is a vector defined by

$$\mathcal{S}_b(U) = \left( 0, gh_1 \frac{\partial z}{\partial x}, \dots, gh_N \frac{\partial z}{\partial x}, gh_1 \frac{\partial z}{\partial y}, \dots, gh_N \frac{\partial z}{\partial y} \right)^T. \quad (43)$$

Neglecting the variations along the axis  $y$  i.e. considering the layer-averaged of the 2d  $(x, z)$  of the Navier-Stokes equations, the quasilinear form writes

$$\mathcal{M}^1 \frac{\partial U^1}{\partial t} + \mathcal{A}_x^1 \frac{\partial U^1}{\partial x} = \mathcal{S}_b(U^1), \quad (44)$$

with

$$U^1 = (h, q_{x,1}, \dots, q_{x,N})^T,$$

and  $\mathcal{M}^1$  and  $\mathcal{A}_x^1$  are two square matrices of size  $N+1$  defined by

- matrix  $\mathcal{M}^1$

$$\begin{aligned}
\mathcal{M}_{i,i}^1 &= 1 && \text{for } i = 1, \dots, N+1 \\
\mathcal{M}_{i+1,1}^1 &= -u_{i+1/2} \sum_{j=1}^i l_j + u_{i-1/2} \sum_{j=1}^{i-1} l_j && \text{for } i = 1, \dots, N \\
\mathcal{M}_{i,j}^1 &= 0 && \text{else}
\end{aligned}$$

- matrix  $\mathcal{A}_x^1$

$$\begin{aligned}
\mathcal{A}_{x,1,i}^1 &= 1 && \text{for } i = 2, \dots, N+1 \\
\mathcal{A}_{x,i+1,i+1}^1 &= u_i - u_{i+1/2} && \text{for } i = 1, \dots, N \\
\mathcal{A}_{x,i+1,1}^1 &= gh_i - l_i u_i^2 && \text{for } i = 1, \dots, N \\
\mathcal{A}_{x,i+1,j}^1 &= -(u_{i+1/2} - u_{i-1/2}) && \text{for } i = 1, \dots, N \quad j = 2, \dots, i-1 \\
\mathcal{A}_{x,i,j}^1 &= 0 && \text{else}
\end{aligned}$$

and  $\mathcal{S}_b(U^1)$  is a vector defined by

$$\mathcal{S}_b(U^1) = \left( 0, gh_1 \frac{\partial z}{\partial x}, \dots, gh_N \frac{\partial z}{\partial x} \right)^T.$$

Notice that in the general case, it is difficult to prove whether or not the systems (41) or its simplified version (44) are hyperbolic.

### 3.5 The layer-averaged Navier-Stokes system

In paragraph 3.1, we have applied the layer-averaging to the Euler system, we now use the same process for the hydrostatic Navier-Stokes system (10)-(13). And we have the following proposition.

**Proposition 3.4** *The layer-averaged hydrostatic Navier-Stokes system is given by*

$$\begin{aligned} \sum_{\alpha=1}^N \frac{\partial h_\alpha}{\partial t} + \sum_{\alpha=1}^N \nabla_{x,y} \cdot (h_\alpha \mathbf{u}_\alpha) &= 0, \\ \frac{\partial h_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla_{x,y} \cdot (h_\alpha \mathbf{u}_\alpha \otimes \mathbf{u}_\alpha) + \nabla_{x,y} \left( \frac{g}{2} h h_\alpha \right) &= -gh_\alpha \nabla z \\ + \mathbf{u}_{\alpha+1/2} G_{\alpha+1/2} - \mathbf{u}_{\alpha-1/2} G_{\alpha-1/2} + \nabla_{x,y} \cdot (h_\alpha \mathbf{T}_\alpha) \\ - \nabla_{x,y} z_{\alpha+1/2} \mathbf{T}_{\alpha+1/2} + \nabla_{x,y} z_{\alpha-1/2} \mathbf{T}_{\alpha-1/2} \\ + 2\mu_{\alpha+1/2} \frac{\mathbf{u}_{\alpha+1} - \mathbf{u}_\alpha}{h_{\alpha+1} + h_\alpha} - 2\mu_{\alpha-1/2} \frac{\mathbf{u}_\alpha - \mathbf{u}_{\alpha-1}}{h_\alpha + h_{\alpha-1}} - \kappa_\alpha \mathbf{u}_\alpha, \quad \alpha = 1, \dots, N, \end{aligned} \quad (45)$$

with

$$\begin{aligned} \mathbf{T}_{\alpha+1/2} &= \begin{pmatrix} T_{xx,\alpha+1/2} & T_{xy,\alpha+1/2} \\ T_{yx,\alpha+1/2} & T_{yy,\alpha+1/2} \end{pmatrix}, \\ T_{xx,\alpha+1/2} &= \frac{\mu_{\alpha+1/2}}{h_{\alpha+1} + h_\alpha} \left( h_\alpha \frac{\partial u_\alpha}{\partial x} + h_{\alpha+1} \frac{\partial u_{\alpha+1}}{\partial x} \right) - 2\mu_{\alpha+1/2} \frac{\partial z_{\alpha+1/2}}{\partial x} \frac{u_{\alpha+1} - u_\alpha}{h_{\alpha+1} + h_\alpha}, \\ T_{xy,\alpha+1/2} &= \frac{\mu_{\alpha+1/2}}{h_{\alpha+1} + h_\alpha} \left( h_\alpha \frac{\partial u_\alpha}{\partial y} + h_{\alpha+1} \frac{\partial u_{\alpha+1}}{\partial y} \right) - 2\mu_{\alpha+1/2} \frac{\partial z_{\alpha+1/2}}{\partial y} \frac{u_{\alpha+1} - u_\alpha}{h_{\alpha+1} + h_\alpha}, \\ \mathbf{T}_\alpha &= \begin{pmatrix} T_{xx,\alpha} & T_{xy,\alpha} \\ T_{yx,\alpha} & T_{yy,\alpha} \end{pmatrix} = \frac{\mathbf{T}_{\alpha+1/2} + \mathbf{T}_{\alpha-1/2}}{2}, \end{aligned} \quad (47)$$

and

$$\kappa_\alpha = \begin{cases} \kappa & \text{if } \alpha = 1 \\ 0 & \text{if } \alpha \neq 1 \end{cases} \quad \mu_{\alpha+1/2} = \begin{cases} 0 & \text{if } \alpha = 0 \\ \mu & \text{if } \alpha = 1, \dots, N-1 \\ 0 & \text{if } \alpha = N \end{cases} \quad (51)$$

The vertical velocities  $\{w_\alpha\}_{\alpha=1}^N$  are defined by (39).

For smooth solutions, the system (45)-(46) admits the energy balance

$$\begin{aligned} \frac{\partial}{\partial t} \sum_{\alpha=1}^N E_\alpha + \nabla_{x,y} \cdot \sum_{\alpha=1}^N \mathbf{u}_\alpha \left( E_\alpha + \frac{g}{2} h_\alpha h - h_\alpha \mathbf{T}_\alpha \right) \\ - \mu_{\alpha+1/2} (\mathbf{u}_{\alpha+1} + \mathbf{u}_\alpha) \cdot \frac{\mathbf{u}_{\alpha+1} - \mathbf{u}_\alpha}{h_{\alpha+1} + h_\alpha} - \mu_{\alpha-1/2} (\mathbf{u}_\alpha + \mathbf{u}_{\alpha-1}) \cdot \frac{\mathbf{u}_\alpha - \mathbf{u}_{\alpha-1}}{h_\alpha + h_{\alpha-1}} \\ = - \sum_{\alpha=1}^N \frac{(u_{\alpha+1/2} - u_\alpha)^2 + (v_{\alpha+1/2} - v_\alpha)^2}{2} |G_{\alpha+1/2}| \\ - \sum_{\alpha=1}^{N-1} \frac{h_{\alpha+1} + h_\alpha}{2\mu} \mathbf{T}_{\alpha+1/2}^2 - \sum_{\alpha=1}^{N-1} 2\mu \frac{(\mathbf{u}_{\alpha+1} - \mathbf{u}_\alpha)^2}{h_{\alpha+1} + h_\alpha} \\ - \kappa(u_1^2 + v_1^2), \end{aligned} \quad (52)$$

with  $E_\alpha$  defined by (36) and  $\mathbf{T}_{\alpha+1/2}^2 = \sum_{i,j} T_{\alpha+1/2,i,j}^2$ . Relation (52) is consistent with a layer-averaged discretisation of the equation (14).

**Remark 3.2** Notice in the definition (50), since we consider viscous terms we use a centered approximation.

**Remark 3.3** Using simple manipulations, the viscous terms in the layer-averaged Navier-Stokes can be rewritten and Eq. (46) becomes

$$\begin{aligned} \frac{\partial h_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla_{x,y} \cdot (h_\alpha \mathbf{u}_\alpha \otimes \mathbf{u}_\alpha) + \nabla_{x,y} \left( \frac{g}{2} h h_\alpha \right) = -g h_\alpha \nabla z \\ + \mathbf{u}_{\alpha+1/2} G_{\alpha+1/2} - \mathbf{u}_{\alpha-1/2} G_{\alpha-1/2} \\ + h_\alpha \nabla_{x,y} \cdot \mathbf{T}_\alpha - \nabla_{x,y} z_\alpha (\mathbf{T}_{\alpha+1/2} - \mathbf{T}_{\alpha-1/2}) \\ + 2\mu_{\alpha+1/2} \frac{\mathbf{u}_{\alpha+1} - \mathbf{u}_\alpha}{h_{\alpha+1} + h_\alpha} - 2\mu_{\alpha-1/2} \frac{\mathbf{u}_\alpha - \mathbf{u}_{\alpha-1}}{h_\alpha + h_{\alpha-1}} - \kappa_\alpha \mathbf{u}_\alpha, \quad \alpha = 1, \dots, N, \end{aligned}$$

**Remark 3.4** Using simple manipulations, the viscous terms in the layer-averaged Navier-Stokes can be rewritten and Eq. (46) becomes

$$\frac{\partial h_\alpha \mathbf{u}_\alpha}{\partial t} + \nabla_{x,y} \cdot (h_\alpha \mathbf{u}_\alpha \otimes \mathbf{u}_\alpha) + \nabla_{x,y} \left( \frac{g}{2} h h_\alpha \right) = -g h_\alpha \nabla z$$

$$\begin{aligned}
& +\mathbf{u}_{\alpha+1/2}G_{\alpha+1/2} - \mathbf{u}_{\alpha-1/2}G_{\alpha-1/2} + \nabla_{x,y} \cdot (h_\alpha \mathbf{T}_\alpha^1) + \mathbf{T}_\alpha^2 \\
& + \Lambda_{\alpha+1/2}(\mathbf{u}_{\alpha+1} - \mathbf{u}_\alpha) - \Lambda_{\alpha-1/2}(\mathbf{u}_\alpha - \mathbf{u}_{\alpha-1}) - \kappa_\alpha \mathbf{u}_\alpha, \quad \alpha = 1, \dots, N,
\end{aligned}$$

with

$$T_{xx,\alpha+1/2}^1 = \frac{\mu_{\alpha+1/2}}{h_{\alpha+1} + h_\alpha} \left( h_\alpha \frac{\partial u_\alpha}{\partial x} + h_{\alpha+1} \frac{\partial u_{\alpha+1}}{\partial x} \right), \quad (53)$$

$$T_{xy,\alpha+1/2}^1 = \frac{\mu_{\alpha+1/2}}{h_{\alpha+1} + h_\alpha} \left( h_\alpha \frac{\partial u_\alpha}{\partial y} + h_{\alpha+1} \frac{\partial u_{\alpha+1}}{\partial y} \right), \quad (54)$$

$$T_{xx,\alpha+1/2}^2 = 2\mu_{\alpha+1/2} \frac{\partial z_{\alpha+1/2}}{\partial x} \frac{\partial u_{\alpha+1}}{\partial x}, \quad (55)$$

$$T_{xy,\alpha+1/2}^2 = 2\mu_{\alpha+1/2} \frac{\partial z_{\alpha+1/2}}{\partial y} \frac{\partial u_{\alpha+1}}{\partial y}, \quad (56)$$

$$\Lambda_{\alpha+1/2} = \mu_{\alpha+1/2} \frac{2 + \nabla_{x,y}^2 z_{\alpha+1/2}}{h_{\alpha+1} + h_\alpha} - \mu_{\alpha+1/2} \nabla_{x,y} \left( \frac{h_\alpha \nabla_{x,y} z_{\alpha+1/2}}{h_{\alpha+1} + h_\alpha} \right),$$

$$\mathbf{T}_{\alpha+1/2}^i = \begin{pmatrix} T_{xx,\alpha+1/2}^i & T_{xy,\alpha+1/2}^i \\ T_{yx,\alpha+1/2}^i & T_{yy,\alpha+1/2}^i \end{pmatrix}, \quad (57)$$

$$\mathbf{T}_\alpha^i = \begin{pmatrix} T_{xx,\alpha}^i & T_{xy,\alpha}^i \\ T_{yx,\alpha}^i & T_{yy,\alpha}^i \end{pmatrix} = \frac{\mathbf{T}_{\alpha+1/2}^i + \mathbf{T}_{\alpha-1/2}^i}{2}, \quad (58)$$

for  $i = 1, 2$ .

**Proof of prop. 3.4** Using the results of prop. 3.1, it remains to obtain the expression for the layer-averaged viscous terms.

The layer-averaging of the viscous terms appearing in (11) gives

$$\begin{aligned}
\int_{z_{\alpha+1/2}}^{z_{\alpha+1/2}} \mu \Delta u \, dz &= \int_{z_{\alpha+1/2}}^{z_{\alpha+1/2}} \left( \frac{\partial \hat{T}_{xx}}{\partial x} + \frac{\partial \hat{T}_{yx}}{\partial y} + \frac{\partial \hat{T}_{zx}}{\partial z} \right) dz \\
&= \frac{\partial}{\partial x} \left( h_\alpha \hat{T}_{xx,\alpha} \right) + \frac{\partial}{\partial y} \left( h_\alpha \hat{T}_{yx,\alpha} \right) \\
&\quad + \hat{T}_{zx,\alpha+1/2} - \hat{T}_{zx,\alpha-1/2} \\
&\quad - \frac{\partial z_{\alpha+1/2}}{\partial x} \hat{T}_{xx,\alpha+1/2} - \frac{\partial z_{\alpha+1/2}}{\partial y} \hat{T}_{yx,\alpha+1/2} \\
&\quad + \frac{\partial z_{\alpha-1/2}}{\partial x} \hat{T}_{xx,\alpha-1/2} + \frac{\partial z_{\alpha-1/2}}{\partial y} \hat{T}_{yx,\alpha-1/2}, \quad (59)
\end{aligned}$$

with

$$h_\alpha \hat{T}_{xx,\alpha} = \int_{z_{\alpha-1/2}}^{z_{\alpha-1/2}} \hat{T}_{xx} dz.$$

The boundary conditions at the bottom (12) and at the free surface (13) imply that in (59)

$$\hat{T}_{zx,N+1/2} - \frac{\partial \eta}{\partial x} \hat{T}_{xx,N+1/2} - \frac{\partial \eta}{\partial y} \hat{T}_{yx,N+1/2} = 0,$$

and

$$\hat{T}_{zx,1/2} - \frac{\partial z_b}{\partial x} \hat{T}_{xx,1/2} - \frac{\partial z_b}{\partial y} \hat{T}_{yx,1/2} = \kappa u_1.$$

The above expression for  $\hat{T}_{xx,\alpha}$  and relation (59) have been obtained using the following formal computation

$$\begin{aligned} \int_{z_{\alpha+1/2}}^{z_{\alpha+1/2}} \frac{\partial \hat{T}_{xx}}{\partial x} dz &= \frac{\partial}{\partial x} \int_{z_{\alpha+1/2}}^{z_{\alpha+1/2}} \hat{T}_{xx} dz - \frac{\partial z_{\alpha+1/2}}{\partial x} \hat{T}_{xx,\alpha+1/2} + \frac{\partial z_{\alpha-1/2}}{\partial x} \hat{T}_{xx,\alpha-1/2} \\ &= \frac{\partial}{\partial x} \left( h_\alpha \hat{T}_{xx,\alpha} \right) - \frac{\partial z_{\alpha+1/2}}{\partial x} \hat{T}_{xx,\alpha+1/2} + \frac{\partial z_{\alpha-1/2}}{\partial x} \hat{T}_{xx,\alpha-1/2}. \end{aligned}$$

The definitions (48),(49) are motivated by the following computation

$$\begin{aligned} \frac{h_{\alpha+1} + h_\alpha}{2} \hat{T}_{xx,\alpha+1/2} &= \mu_{\alpha+1/2} \int_{z_\alpha}^{z_{\alpha+1}} \frac{\partial u}{\partial x} dz \\ &= \mu_{\alpha+1/2} \frac{\partial}{\partial x} \int_{z_\alpha}^{z_{\alpha+1}} u dz - \mu_{\alpha+1/2} \frac{\partial z_{\alpha+1}}{\partial x} u_{\alpha+1} + \mu_{\alpha+1/2} \frac{\partial z_\alpha}{\partial x} u_\alpha \\ &= \mu_{\alpha+1/2} \frac{\partial}{\partial x} \left( \frac{h_\alpha}{2} u_\alpha + \frac{h_{\alpha+1}}{2} u_{\alpha+1} \right) - \mu_{\alpha+1/2} \frac{\partial z_{\alpha+1}}{\partial x} u_{\alpha+1} + \mu_{\alpha+1/2} \frac{\partial z_\alpha}{\partial x} u_\alpha \\ &= \mu_{\alpha+1/2} \left( \frac{h_\alpha}{2} \frac{\partial u_\alpha}{\partial x} + \frac{h_{\alpha+1}}{2} \frac{\partial u_{\alpha+1}}{\partial x} \right) - \mu_{\alpha+1/2} \frac{\partial z_{\alpha+1/2}}{\partial x} (u_{\alpha+1} - u_\alpha). \end{aligned}$$

In order to prove the energy balance (52) we use the results of prop. 3.2 obtained for the layer-averaged Euler system and it remains to consider the visous and frictions terms multiplied by  $\mathbf{u}_\alpha$ .

Let us define  $\mathbf{R}_\alpha$

$$\mathbf{R}_\alpha = \begin{pmatrix} R_{x,\alpha} \\ R_{y,\alpha} \end{pmatrix} = \nabla_{x,y} \cdot (h_\alpha \mathbf{T}_\alpha) - \nabla_{x,y} z_{\alpha+1/2} \mathbf{T}_{\alpha+1/2} + \nabla_{x,y} z_{\alpha-1/2} \mathbf{T}_{\alpha-1/2} \\ + \mu_{\alpha+1/2} \frac{\mathbf{u}_{\alpha+1} - \mathbf{u}_\alpha}{h_{\alpha+1} + h_\alpha} - \mu_{\alpha-1/2} \frac{\mathbf{u}_\alpha - \mathbf{u}_{\alpha-1}}{h_\alpha + h_{\alpha-1}} - \kappa_\alpha \mathbf{u}_\alpha.$$

We write

$$R_{x,\alpha} u_\alpha = \frac{\partial(u_\alpha h_\alpha T_{xx,\alpha})}{\partial x} + \frac{\partial(u_\alpha h_\alpha T_{yx,\alpha})}{\partial y} \\ + \mu_{\alpha+1/2} \frac{u_{\alpha+1} + u_\alpha}{2} \frac{u_{\alpha+1} - u_\alpha}{h_{\alpha+1} + h_\alpha} - \mu_{\alpha-1/2} \frac{u_\alpha + u_{\alpha-1}}{2} \frac{u_\alpha - u_{\alpha-1}}{h_\alpha + h_{\alpha-1}} \\ - \frac{\partial z_{\alpha+1/2}}{\partial x} T_{xx,\alpha+1/2} u_\alpha - \frac{\partial z_{\alpha+1/2}}{\partial y} T_{xy,\alpha+1/2} u_\alpha \\ + \frac{\partial z_{\alpha-1/2}}{\partial x} T_{xx,\alpha-1/2} u_\alpha + \frac{\partial z_{\alpha-1/2}}{\partial y} T_{xy,\alpha-1/2} u_\alpha \\ - h_\alpha T_{xx,\alpha} \frac{\partial u_\alpha}{\partial x} - h_\alpha T_{xy,\alpha} \frac{\partial u_\alpha}{\partial y} \\ - \frac{\mu_{\alpha+1/2}}{2} \frac{(u_{\alpha+1} - u_\alpha)^2}{h_{\alpha+1} + h_\alpha} - \frac{\mu_{\alpha-1/2}}{2} \frac{(u_\alpha - u_{\alpha-1})^2}{h_\alpha + h_{\alpha-1}} - \kappa_\alpha u_\alpha^2,$$

and using the closure relation (50), it comes for  $\alpha = 2, \dots, N-1$

$$R_{x,\alpha} u_\alpha = \frac{\partial(u_\alpha h_\alpha T_{xx,\alpha})}{\partial x} + \frac{\partial(u_\alpha h_\alpha T_{xy,\alpha})}{\partial y} \\ + \mu_{\alpha+1/2} \frac{u_{\alpha+1} + u_\alpha}{2} \frac{u_{\alpha+1} - u_\alpha}{h_{\alpha+1} + h_\alpha} - \mu_{\alpha+1/2} \frac{u_\alpha + u_{\alpha-1}}{2} \frac{u_\alpha - u_{\alpha-1}}{h_\alpha + h_{\alpha-1}} \\ - T_{xx,\alpha+1/2} \left( \frac{h_\alpha}{2} \frac{\partial u_\alpha}{\partial x} + \frac{\partial z_{\alpha+1/2}}{\partial x} u_\alpha \right) - T_{xy,\alpha+1/2} \left( \frac{h_\alpha}{2} \frac{\partial u_\alpha}{\partial y} + \frac{\partial z_{\alpha+1/2}}{\partial y} u_\alpha \right) \\ + T_{xx,\alpha-1/2} \left( -\frac{h_\alpha}{2} \frac{\partial u_\alpha}{\partial x} + \frac{\partial z_{\alpha-1/2}}{\partial x} \frac{u_\alpha}{2} \right) + T_{xy,\alpha-1/2} \left( -\frac{h_\alpha}{2} \frac{\partial u_\alpha}{\partial y} + \frac{\partial z_{\alpha-1/2}}{\partial y} \frac{u_\alpha}{2} \right) \\ - \frac{\mu_{\alpha+1/2}}{2} \frac{(u_{\alpha+1} - u_\alpha)^2}{h_{\alpha+1} + h_\alpha} - \frac{\mu_{\alpha-1/2}}{2} \frac{(u_\alpha - u_{\alpha-1})^2}{h_\alpha + h_{\alpha-1}} - \kappa_\alpha u_\alpha^2.$$

An analogous relation can be easily obtained for  $R_{y,\alpha} v_\alpha$  and computing the sum over the layers of the obtained quantity, we get

$$\sum_{\alpha=1}^N \mathbf{R}_\alpha \cdot \mathbf{u}_\alpha = \sum_{\alpha=1}^N \left( \frac{\partial(u_\alpha h_\alpha T_{xx,\alpha})}{\partial x} + \frac{\partial(u_\alpha h_\alpha T_{yx,\alpha})}{\partial y} + \frac{\partial(v_\alpha h_\alpha T_{xy,\alpha})}{\partial x} + \frac{\partial(v_\alpha h_\alpha T_{yy,\alpha})}{\partial y} \right)$$

$$\begin{aligned}
& - \sum_{\alpha=1}^{N-1} \frac{h_{\alpha+1} + h_{\alpha}}{2\mu} (T_{xx,\alpha+1/2}^2 + T_{xy,\alpha+1/2}^2 + T_{yx,\alpha+1/2}^2 + T_{yy,\alpha+1/2}^2) \\
& - \sum_{\alpha=1}^{N-1} \mu \frac{(\mathbf{u}_{\alpha+1} - \mathbf{u}_{\alpha})^2}{h_{\alpha+1} + h_{\alpha}} - \kappa u_1^2,
\end{aligned}$$

proving the result. ■

## 4 Kinetic description

In this section we give a kinetic interpretation of the system (31)-(35). The numerical scheme for the system (31)-(32),(39) will be deduced from the kinetic description.

### 4.1 The 1d Saint-Venant system

We begin this section by recalling the classical kinetic approach – used in [26] for example – for the 1d Saint-Venant system

$$\begin{aligned}
\partial_t h + \partial_x(hu) &= 0, \\
\partial_t(hu) + \partial_x(hu^2 + g\frac{h^2}{2}) + gh\partial_x z &= 0,
\end{aligned} \tag{60}$$

with the water depth  $h(t, x) \geq 0$ , the water velocity  $u(t, x) \in \mathbb{R}$  and a slowly varying topography  $z(x)$ .

The kinetic Maxwellian is given by

$$M(U, \xi) = \frac{1}{g\pi} \left( 2gh - (\xi - u)^2 \right)_+^{1/2}, \tag{61}$$

where  $U = (h, hu)^T$ ,  $\xi \in \mathbb{R}$  and  $x_+ \equiv \max(0, x)$  for any  $x \in \mathbb{R}$ . It satisfies the following moment relations,

$$\begin{aligned}
\int_{\mathbb{R}} \begin{pmatrix} 1 \\ \xi \end{pmatrix} M(U, \xi) d\xi &= U, \\
\int_{\mathbb{R}} \xi^2 M(U, \xi) d\xi &= hu^2 + g\frac{h^2}{2}.
\end{aligned}$$

These definitions allow us to obtain a *kinetic representation* of the Saint-Venant system.

**Lemma 4.1** *If the topography  $z(x)$  is Lipschitz continuous, the pair of functions  $(h, hu)$  is a weak solution to the Saint-Venant system (60) if and only if  $M(U, \xi)$  satisfies the kinetic equation*

$$\partial_t M + \xi \partial_x M - g(\partial_x z) \partial_\xi M = Q, \quad (62)$$

for some “collision term”  $Q(t, x, \xi)$  that satisfies, for a.e.  $(t, x)$ ,

$$\int_{\mathbb{R}} Q d\xi = \int_{\mathbb{R}} \xi Q d\xi = 0. \quad (63)$$

**Proof** *If (62) and (63) are satisfied, we can multiply (62) by  $(1, \xi)^T$ , and integrate with respect to  $\xi$ . Using (67) and (63) and integrating by parts the term in  $\partial_\xi M$ , we obtain (60). Conversely, if  $(h, hu)$  is a weak solution to (60), just define  $Q$  by (62); it will satisfy (63) according to the same computations. ■*

The standard way to use Lemma 4.1 is to write a kinetic relaxation equation [3, 9, 10], like

$$\partial_t f + \xi \partial_x f - g(\partial_x z) \partial_\xi f = \frac{M - f}{\epsilon}, \quad (64)$$

where  $f(t, x, \xi) \geq 0$ ,  $M = M(U, \xi)$  with  $U(t, x) = \int (1, \xi)^T f(t, x, \xi) d\xi$ , and  $\epsilon > 0$  is a relaxation time. In the limit  $\epsilon \rightarrow 0$  we recover formally the formulation (62), (63). We refer to [9] for general considerations on such kinetic relaxation models without topography, the case with topography being introduced in [26]. Note that the notion of *kinetic representation* as (62), (63) differs from the so called *kinetic formulations* where a large set of entropies is involved, see [25]. For systems of conservation laws, these kinetic formulations include non-advective terms that prevent from writing down simple approximations. In general, kinetic relaxation approximations can be compatible with just a single entropy. Nevertheless this is enough for proving the convergence as  $\epsilon \rightarrow 0$ , see [8].



## 4.2 Kinetic interpretation

In this paragraph, we give a kinetic interpretation of the model (31)-(32),(37).

To build the Gibbs equilibria, we choose the function

$$\chi_0(z_1, z_2) = \frac{1}{4\pi} \mathbb{1}_{z_1^2 + z_2^2 \leq 4}. \quad (65)$$

This choice corresponds to the 2d version of the kinetic maxwellian used in 1d (see remark 4.2) and we have

$$M_\alpha = M(U_\alpha, \xi, \gamma) = \frac{h_\alpha}{c^2} \chi_0 \left( \frac{\xi - u_\alpha}{c}, \frac{\gamma - v_\alpha}{c} \right), \quad (66)$$

with

$$c = \sqrt{\frac{g}{2}h},$$

and where  $(\xi, \gamma) \in \mathbb{R}^2$ . In other words, we have

$$M_\alpha = \frac{l_\alpha}{g\pi} \mathbb{1}_{(\xi - u_\alpha)^2 + (\gamma - v_\alpha)^2 \leq 2gh}.$$

**Remark 4.2** *Starting from the 2d maxwellian in the single layer case i.e.*

$$M = \frac{1}{g\pi} \mathbb{1}_{(\xi - u)^2 + (\gamma - v)^2 \leq 2gh},$$

and computing its integral w.r.t. the variable  $v$  yields

$$\int_{\mathbb{R}} M d\gamma = \int_{v - \sqrt{2gh - (\xi - u)^2}}^{v + \sqrt{2gh - (\xi - u)^2}} \frac{1}{g\pi} d\gamma = \frac{2}{g\pi} \sqrt{2gh - (\xi - u)^2},$$

that is exactly the expression (61).

The quantity  $M_\alpha$  satisfies the following moment relations,

$$\begin{aligned} \int_{\mathbb{R}^2} \begin{pmatrix} 1 \\ \xi \\ \gamma \end{pmatrix} M(U_\alpha, \xi, \gamma) d\xi d\gamma &= \begin{pmatrix} h_\alpha \\ h_\alpha u_\alpha \\ h_\alpha v_\alpha \end{pmatrix}, \\ \int_{\mathbb{R}^2} \begin{pmatrix} \xi^2 \\ \xi\gamma \\ \gamma^2 \end{pmatrix} M(U_\alpha, \xi, \gamma) d\xi d\gamma &= \begin{pmatrix} h_\alpha u_\alpha^2 + g \frac{h_\alpha h}{2} \\ h_\alpha u_\alpha v_\alpha \\ h_\alpha v_\alpha^2 + g \frac{h_\alpha h}{2} \end{pmatrix}. \end{aligned} \quad (67)$$

The interest of the function  $\chi_0$  and hence the particular form (66) lies in its link with a kinetic entropy. Consider the kinetic entropy

$$H(f, \xi, \gamma, z) = \frac{\xi^2 + \gamma^2}{2} f + gz f, \quad (68)$$

where  $f \geq 0$ ,  $(\xi, \gamma) \in \mathbb{R}^2$ ,  $z \in \mathbb{R}$ . Then one can check the relations

$$\int_{\mathbb{R}^2} \begin{pmatrix} 1 \\ \xi \\ \gamma \end{pmatrix} H(M_\alpha, \xi, \gamma) d\xi d\gamma = \begin{pmatrix} E_\alpha = \frac{h_\alpha}{2}(u_\alpha^2 + v_\alpha^2) + \frac{g}{2}h_\alpha(h + 2z) \\ u_\alpha(E_\alpha + \frac{g}{2}h_\alpha h) \\ v_\alpha(E_\alpha + \frac{g}{2}h_\alpha h) \end{pmatrix} \quad (69)$$

Let us introduce the Gibbs equilibria  $N_{\alpha+1/2}$  defined by for  $\alpha = 0, \dots, N$  by

$$\begin{aligned} N_{\alpha+1/2} &= N(U_{\alpha+1/2}, \xi) = \frac{G_{\alpha+1/2}}{c^2} \chi_0 \left( \frac{\xi - u_{\alpha+1/2}}{c}, \frac{\gamma - v_{\alpha+1/2}}{c} \right) \\ &= \frac{G_{\alpha+1/2}}{g\pi h} \mathcal{K}_{(\xi - u_{\alpha+1/2})^2 + (\gamma - v_{\alpha+1/2})^2 \leq 2gh}, \end{aligned} \quad (70)$$

where  $G_{\alpha+1/2}$  is defined by (33) and  $u_{\alpha+1/2}, v_{\alpha+1/2}$  are given by (34). Notice that from (33), we can give a kinetic interpretation on the exchange terms under the form

$$G_{\alpha+1/2} = \sum_{j=1}^N \left( \sum_{p=1}^j l_p - \mathcal{K}_{j \leq \alpha} \right) \int_{\mathbb{R}^2} \left( \xi \frac{\partial M_j}{\partial x} + \gamma \frac{\partial M_i}{\partial y} \right) d\xi d\gamma. \quad (71)$$

Then we have the two following results.

**Proposition 4.1** *The functions  $\mathbf{u}^N$  defined by (30) and  $h$  are strong solutions of the system (31)-(32) if and only if the sets of equilibria  $\{M_\alpha\}_{\alpha=1}^N, \{N_{\alpha+1/2}\}_{\alpha=0}^N$  are solutions of the kinetic equations defined by*

$$\begin{aligned} (\mathcal{B}_\alpha) \quad \frac{\partial M_\alpha}{\partial t} + \xi \frac{\partial M_\alpha}{\partial x} + \gamma \frac{\partial M_\alpha}{\partial y} - g \frac{\partial z}{\partial x} \frac{\partial M_\alpha}{\partial \xi} - g \frac{\partial z}{\partial x} \frac{\partial M_\alpha}{\partial \gamma} \\ - N_{\alpha+1/2} + N_{\alpha-1/2} = Q_\alpha, \end{aligned} \quad (72)$$

for  $\alpha = 1, \dots, N$ . The quantities  $Q_\alpha = Q_\alpha(t, x, y, \xi, \gamma)$  are ‘‘collision terms’’ equal to zero at the macroscopic level, i.e. they satisfy a.e. for values of  $(t, x, y)$

$$\int_{\mathbb{R}^2} Q_\alpha d\xi d\gamma = \int_{\mathbb{R}^2} \xi Q_\alpha d\xi d\gamma = \int_{\mathbb{R}^2} \gamma Q_\alpha d\xi d\gamma = 0. \quad (73)$$

**Proposition 4.2** *The solutions of (72) are entropy solutions if*

$$\frac{\partial H(M_\alpha)}{\partial t} + \xi \frac{\partial H(M_\alpha)}{\partial x} + \gamma \frac{\partial H(M_\alpha)}{\partial y} - g \frac{\partial z}{\partial x} \frac{\partial H(M_\alpha)}{\partial \xi} - g \frac{\partial z}{\partial y} \frac{\partial H(M_\alpha)}{\partial \gamma} \leq (H(N_{\alpha+1/2}) - H(N_{\alpha-1/2})), \quad (74)$$

with the notation  $H(M) = H(M, \xi, \gamma, z)$  and  $H$  defined by (68). The integration in  $\xi, \gamma$  of relation (74) gives

$$\begin{aligned} \frac{\partial E_\alpha}{\partial t} + \frac{\partial}{\partial x} u_\alpha (E_\alpha + \frac{g}{2} h_\alpha h) + \frac{\partial}{\partial y} v_\alpha (E_\alpha + \frac{g}{2} h_\alpha h) &\leq l_\alpha \left( \frac{u_{\alpha+1/2}^2 + v_{\alpha+1/2}^2}{2} + gz \right) G_{\alpha+1/2} \\ &- l_\alpha \left( \frac{u_{\alpha-1/2}^2 + v_{\alpha-1/2}^2}{2} + gz \right) G_{\alpha-1/2}. \end{aligned}$$

**Proof of proposition 4.1** *The proof relies on averages w.r.t the variables  $\xi, \gamma$  of Eq. (72) using relations (67), (69) and*

$$\begin{aligned} \int_{\mathbb{R}^2} \begin{pmatrix} 1 \\ \xi \\ \gamma \end{pmatrix} N_{\alpha+1/2} d\xi d\gamma &= \begin{pmatrix} G_{\alpha+1/2} \\ u_{\alpha+1/2} G_{\alpha+1/2} \\ v_{\alpha+1/2} G_{\alpha+1/2} \end{pmatrix}, \\ \int_{\mathbb{R}^2} \begin{pmatrix} \frac{\xi^2}{2} \\ \frac{\gamma^2}{2} \end{pmatrix} N_{\alpha+1/2} d\xi d\gamma &= \begin{pmatrix} G_{\alpha+1/2} \\ \left( \frac{u_{\alpha+1/2}^2}{2} + \frac{g}{4} h \right) G_{\alpha+1/2} \\ \left( \frac{v_{\alpha+1/2}^2}{2} + \frac{g}{4} h \right) G_{\alpha+1/2} \end{pmatrix}. \end{aligned}$$

Then using (73), the quantities

$$\sum_1^N \int_{\mathbb{R}^2} (\mathcal{B}_\alpha) d\xi d\gamma, \quad \int_{\mathbb{R}^2} \xi (\mathcal{B}_\alpha) d\xi d\gamma, \quad \text{and} \quad \int_{\mathbb{R}^2} \gamma (\mathcal{B}_\alpha) d\xi d\gamma,$$

respectively gives (31) and (32) that completes the proof. ■

**Proof of prop. 4.2** *The proof is obtained multiplying (72) by  $H'_\alpha(\overline{M}_\alpha, \xi, \gamma, z)$ . Indeed, it is easy to see that*

$$H'_\alpha(M_\alpha, \xi, \gamma, z) \frac{\partial M_\alpha}{\partial v} = \frac{\partial}{\partial v} H_\alpha(M_\alpha, \xi, \gamma, z),$$

for  $v = t, x, y, \xi, \gamma$ . Likewise for the quantity  $H'_\alpha(M_\alpha, \xi, \gamma, z)N_{\alpha+1/2}$ , we have

$$H'_\alpha(M_\alpha, \xi, \gamma, z)N_{\alpha+1/2} = H(N_{\alpha+1/2}, \xi, \gamma, z).$$

So finally, Eq. (72) multiplied by  $H'_\alpha(M_\alpha, \xi, \gamma, z)$  gives

$$\begin{aligned} \frac{\partial H_\alpha}{\partial t} + \xi \frac{\partial H_\alpha}{\partial x} + \gamma \frac{\partial H_\alpha}{\partial y} - g \frac{\partial z}{\partial x} \frac{\partial H_\alpha}{\partial \xi} - g \frac{\partial z}{\partial y} \frac{\partial H_\alpha}{\partial \gamma} \leq \\ \left( \frac{\xi^2 + \gamma^2}{2} + gz \right) (N_{\alpha+1/2} - N_{\alpha-1/2}). \end{aligned}$$

It remains to calculate the sum of the preceding relations from  $\alpha = 1, \dots, N$  and to integrate the obtained relation in  $\xi, \gamma$  over  $\mathbb{R}^2$  that completes the proof. ■

## 5 Numerical scheme

The numerical scheme proposed in this section extend the results presented by some of the authors in [4, 6, 5, 3]. The objective is to propose – based on the kinetic interpretation obtained in prop. 4.1 –, an accurate, robust and efficient numerical approximation of the system (31)-(32).

Notice that, as a consequence of the layer-averaged discretization, the system (31)-(32) and the Boltzmann type equation (72) are only 2d  $(x, y)$  partial differential equations with source terms. Hence, the spacial approximation of the considered PDEs is performed on a 2d planar mesh.

The layer-averaged system obtained in Prop. 3.4 has the form

$$\frac{\partial U}{\partial t} + \frac{\partial F_x(U)}{\partial x} + \frac{\partial F_y(U)}{\partial y} = \mathcal{S}_b(U) + S_e(U, \partial_t U, \partial_x U) + S_{v,f}(U), \quad (75)$$

with  $U$ ,  $\mathcal{S}_b(U)$  defined by (42),(43). We denote with  $F_x(U)$ ,  $F_y(U)$  the fluxes of the conservative part, and with  $S_e(U, \partial_t U, \partial_x U)$  and  $S_{v,f}(U)$  the source terms, representing respectively the momentum exchanges and the viscous and friction effects.

In a first step, we focus on the numerical approximation of the layer-averaged Euler system. The discretization of the viscous and friction terms will be tackled in paragraph 5.5.

We introduce a  $(2N + 1) \times N$  matrix  $\mathcal{K}(\xi, \gamma)$  defined by

$$\begin{aligned} \mathcal{K}_{1,j} &= 1, & \text{for } j = 1, \dots, N \\ \mathcal{K}_{i+1,j} &= \xi \delta_{i,j} & \text{for } i, j = 1, \dots, N \\ \mathcal{K}_{i+N+1,j+N} &= \gamma \delta_{i,j} & \text{for } i, j = 1, \dots, N \end{aligned}$$

with  $\delta_{i,j}$  the Kronecker symbol. Then, using Prop. 4.1, we can write

$$U = \int_{\mathbb{R}^2} \mathcal{K}(\xi, \gamma) M(\xi, \gamma) d\xi d\gamma, \quad F_x(U) = \int_{\mathbb{R}^2} \xi \mathcal{K}(\xi, \gamma) M(\xi, \gamma) d\xi d\gamma, \quad (76)$$

$$F_y(U) = \int_{\mathbb{R}^2} \gamma \mathcal{K}(\xi, \gamma) M(\xi, \gamma) d\xi d\gamma, \quad S_e(U) = \int_{\mathbb{R}^2} \mathcal{K}(\xi, \gamma) N(\xi, \gamma) d\xi d\gamma, \quad (77)$$

with  $M(\xi, \gamma) = (M(U_1, \xi, \gamma), \dots, M(U_N, \xi, \gamma))^T$  and

$$N(\xi, \gamma) = \begin{pmatrix} N_{3/2}(\xi, \gamma) - N_{1/2}(\xi, \gamma) \\ \vdots \\ N_{N+1/2}(\xi, \gamma) - N_{N-1/2}(\xi, \gamma) \end{pmatrix}.$$

Hence, using the above notations, the layer-averaged Euler system (31)-(32) can be written under the form

$$\int_{\mathbb{R}^2} \mathcal{K}(\xi, \gamma) \left( \frac{\partial M(\xi, \gamma)}{\partial t} + \xi \frac{\partial M(\xi, \gamma)}{\partial x} + \gamma \frac{\partial M(\xi, \gamma)}{\partial y} g \nabla_{x,y,z} \cdot \nabla_{\xi,\gamma} M - N(\xi, \gamma) \right) d\xi d\gamma = 0.$$

## 5.1 Semi-discrete (in time) scheme

We consider discrete times  $t^n$  with  $t^{n+1} = t^n + \Delta t^n$ . For the time discretisation of the layer-averaged Navier-Stokes system (45)-(46), we adopt a explicit scheme

$$h^{n+1} = h - \Delta t^n \sum_{\alpha=1}^N \nabla_{x,y} \cdot (h_\alpha \mathbf{u}_\alpha) = 0. \quad (78)$$

$$\begin{aligned} (h_\alpha \mathbf{u}_\alpha)^{n+1} &= h_\alpha \mathbf{u}_\alpha - \Delta t^n \left( \nabla_{x,y} \cdot (h_\alpha \mathbf{u}_\alpha \otimes \mathbf{u}_\alpha) - \nabla_{x,y} \cdot \left( \frac{g}{2} h h_\alpha \right) - g h_\alpha \nabla z \right. \\ &\quad + \mathbf{u}_{\alpha+1/2} G_{\alpha+1/2} - \mathbf{u}_{\alpha-1/2} G_{\alpha-1/2} + \mu \nabla_{x,y} \cdot (h_\alpha \mathbf{T}_\alpha) \\ &\quad - \mathbf{T}_{\alpha+1/2} \nabla_{x,y} z_{\alpha+1/2} + \mathbf{T}_{\alpha-1/2} \nabla_{x,y} z_{\alpha-1/2} \\ &\quad \left. + \mu_{\alpha+1/2} \frac{\mathbf{u}_{\alpha+1} - \mathbf{u}_\alpha}{h_{\alpha+1} + h_\alpha} - \mu_{\alpha-1/2} \frac{\mathbf{u}_\alpha - \mathbf{u}_{\alpha-1}}{h_\alpha + h_{\alpha-1}} - \kappa_\alpha \mathbf{u}_\alpha \right), \quad (79) \end{aligned}$$

$$G_{\alpha+1/2} = \sum_{j=1}^N \left( \sum_{p=1}^{\alpha} l_p - \mathbb{K}_{j \leq \alpha} \right) \nabla_{x,y} \cdot (h_j \mathbf{u}_j),$$

for  $\alpha = 1, \dots, N$  and where the superscript  $n$  has been omitted and the vertical velocities  $\{w_\alpha\}_{\alpha=1}^N$  are defined by (39).

Notice that an implicit discretization can be adopted for the visous and friction terms with classical advantages and limitations compared to the explicit in time scheme that are not detailed here.

## 5.2 Finite volume formalism

We recall here the general formalism of finite volumes on unstructured meshes. Let  $\Omega$  denote the computational domain with boundary  $\Gamma$ , which we assume is polygonal. Let  $T_h$  be a triangulation of  $\Omega$  for which the vertices are denoted by  $P_i$  with  $S_i$  the set of interior nodes and  $G_i$  the set of boundary nodes. The dual cells  $C_i$  are obtained by joining the centers of mass of the triangles surrounding each vertex  $P_i$ . We use the following notations (see Fig. 3):

- $K_i$ , set of subscripts of nodes  $P_j$  surrounding  $P_i$ ,
- $|C_i|$ , area of  $C_i$ ,
- $\Gamma_{ij}$ , boundary edge between the cells  $C_i$  and  $C_j$ ,
- $L_{ij}$ , length of  $\Gamma_{ij}$ ,
- $\mathbf{n}_{ij}$ , unit normal to  $\Gamma_{ij}$ , outward to  $C_i$  ( $\mathbf{n}_{ji} = \mathbf{n}_{ij}$ ).

We define the piecewise constant functions  $U^n(x, y)$  corresponding to time  $t^n$  and  $z(x, y)$  as

$$U^n(x, y) = U_i^n, \quad z(x, y) = z_i, \quad \text{for } (x, y) \in C_i^2, \quad (80)$$

with  $U_i^n = (h_i^n, q_{x,1,i}^n, \dots, q_{x,N,i}^n, q_{y,1,i}^n, \dots, q_{y,N,i}^n)^T$  i.e.

$$U_i^n \approx \frac{1}{|C_i|} \int_{C_i} U(t^n, x, y) dx dy, \quad z_i \approx \frac{1}{|C_i|} \int_{C_i} z(x, y) dx dy.$$

A finite volume scheme for solving the classical 2d Saint-Venant system is a formula of the form

$$U_i^{n+1} = U_i^n - \sum_{j \in K_i} \sigma_{i,j} \mathcal{F}_{i,j} - \sigma_i \mathcal{F}_{i,e}, \quad (81)$$

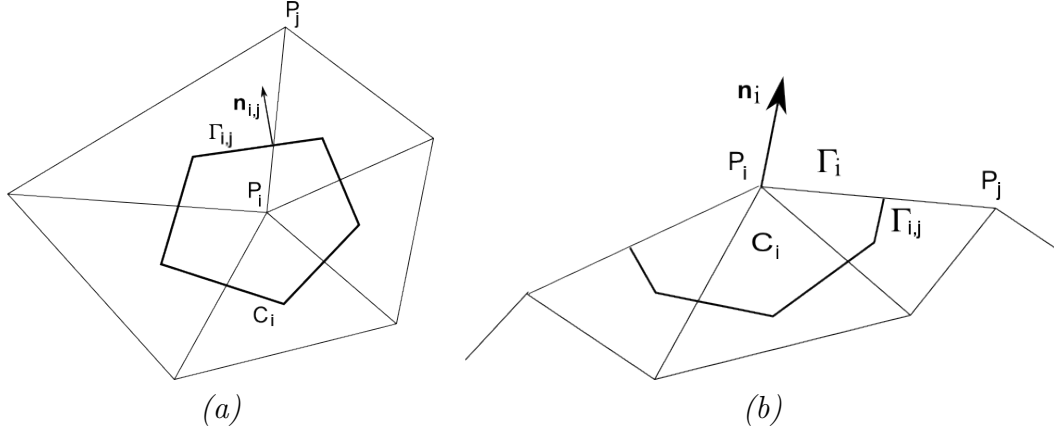


Figure 3: (a) Dual cell  $C_i$  and (b) Boundary cell  $C_i$ .

with

$$\sigma_{i,j} = \frac{\Delta t^n L_{i,j}}{|C_i|} \quad \sigma_i = \sum_{j \in K_i} \frac{\Delta t^n L_{i,j}}{|C_i|}.$$

Here we consider first-order explicit schemes where

$$\mathcal{F}_{i,j} = F(U_i^n, U_j^n, z_i - z_j, \mathbf{n}_{i,j}). \quad (82)$$

and

$$\mathcal{F}_{i,j} = F(U_i, U_j, z_i - z_j, \mathbf{n}_{i,j}) = \begin{pmatrix} F(U_{1,i}, U_{1,j}, z_i - z_j, \mathbf{n}_{i,j}) \\ \vdots \\ F(U_{N,i}, U_{N,j}, z_i - z_j, \mathbf{n}_{i,j}) \end{pmatrix}, \quad (83)$$

$$\mathcal{F}_{i,e} = F(U_i, U_{i,e}, \mathbf{n}_i) = \begin{pmatrix} F(U_{1,i}, U_{1,j}, \mathbf{n}_i) \\ \vdots \\ F(U_{N,i}, U_{N,j}, \mathbf{n}_i) \end{pmatrix}.$$

Relation (81) tells how to compute the values  $U_i^{n+1}$  knowing  $U_i^n$  and discretized values  $z_i$  of the topography. In (81) the term  $\mathcal{F}_{i,j}$  denotes an interpolation of the normal component of the flux  $F(U) \cdot \mathbf{n}_{i,j}$  along the edge  $C_{i,j}$ . The functions  $F(U_l, U_r, \Delta z, \mathbf{n}) \in \mathbb{R}^{2N+1}$  are the numerical fluxes, see [12].

In the next paragraph we define  $\mathcal{F}(U_i^n, U_j^n, z_i - z_j, \mathbf{n}_{i,j})$  using the kinetic interpretation of the system. The computation of the value  $U_{i,e}$ , which denotes a value outside  $C_i$  (see Fig. 3-(b)) defined such that the boundary conditions are satisfied, and the definition of the boundary flux  $F(U_i^n, U_{i,e}^n, \mathbf{n}_i)$

are described paragraph 5.6. Notice we assume a flat topography on the boundaries i.e.  $z_i = z_{i,e}$ .

We shall often omit the superscript  $n$  and e.g. denote  $U_i$  instead of  $U_i^n$ , whenever there is no ambiguity.

### 5.3 Discrete kinetic equation

The choice of a kinetic scheme is motivated by several arguments. First, the kinetic interpretation is a suitable starting point for building a stable numerical scheme. We will prove in paragraph 5.3 that the proposed kinetic scheme preserves positivity of the water depth and ensures a discrete local maximum principle for the tracer concentration (temperature, salinity...). Second, the construction of the kinetic scheme does not need the computation of the system eigenvalues. This point is very important here since these eigenvalues are not available in explicit analytical form, and they are hardly accessible even numerically. Furthermore, as previously mentioned, hyperbolicity of the multilayer model may not hold, and the kinetic scheme allows overcoming this difficulty.

#### 5.3.1 Without topography

In a first step we consider a situation with flat bottom. Following lemma 4.1, the model (45)-(46) reduces to the classical Saint-Venant system with source terms (see Eq. (72)) and its kinetic interpretation is given by

$$\frac{\partial M_\alpha}{\partial t} + \xi \frac{\partial M_\alpha}{\partial x} + \gamma \frac{\partial M_\alpha}{\partial y} - N_{\alpha+1/2} + N_{\alpha-1/2} = Q_\alpha,$$

with the notations defined in paragraph 4.2.

Let  $C_i$  be a cell with boundary  $\Gamma_i$ , see Fig. 3. The integral over  $C_i$  of the kinetic equation gives

$$\int_{C_i} \left( \frac{\partial M_\alpha}{\partial t} + \xi \frac{\partial M_\alpha}{\partial x} + \gamma \frac{\partial M_\alpha}{\partial y} \right) dx dy = |C_i| \frac{\partial M_{\alpha,i}}{\partial t} + \int_{\Gamma_i} M_\alpha \begin{pmatrix} \xi \\ \gamma \end{pmatrix} \cdot \mathbf{n}_i dl, \quad (84)$$

$\mathbf{n}_i$  being the outward normal to the cell  $C_i$ . Denoting

$$\zeta_i = \begin{pmatrix} \xi \\ \gamma \end{pmatrix} \cdot \mathbf{n}_i,$$



and with the notations of Fig. 3, we write

$$\int_{\Gamma_i} M_\alpha \left( \begin{smallmatrix} \xi \\ \gamma \end{smallmatrix} \right) \cdot \mathbf{n}_i dl = \sum_{i \in K_i} \int_{\Gamma_{i,j}} M_\alpha \left( \begin{smallmatrix} \xi \\ \gamma \end{smallmatrix} \right) \cdot \mathbf{n}_{i,j} dl = \sum_{i \in K_i} L_{i,j} M_\alpha \left( \begin{smallmatrix} \xi \\ \gamma \end{smallmatrix} \right) \cdot \mathbf{n}_{i,j}.$$

Now we use the classical kinetic upwinding leading to

$$\sum_{i \in K_i} L_{i,j} M_\alpha \left( \begin{smallmatrix} \xi \\ \gamma \end{smallmatrix} \right) \cdot \mathbf{n}_{i,j} = \sum_{j \in K_i} L_{i,j} M_{\alpha,i} \zeta_{i,j} \mathbb{1}_{\zeta_{i,j} \geq 0} + \sum_{j \in K_i} L_{i,j} M_{\alpha,j} \zeta_{i,j} \mathbb{1}_{\zeta_{i,j} \leq 0},$$

with

$$\zeta_{i,j} = \left( \begin{smallmatrix} \xi \\ \gamma \end{smallmatrix} \right) \cdot \mathbf{n}_{i,j}.$$

Therefore, the kinetic scheme is given by

$$\begin{aligned} f_{\alpha,i}^{n+1-} &= M_i - \frac{\Delta t^n}{|C_i|} \sum_{j \in K_i} L_{i,j} M_{\alpha,i} \zeta_{i,j} \mathbb{1}_{\zeta_{i,j} \geq 0} - \frac{\Delta t^n}{|C_i|} \sum_{j \in K_i} L_{i,j} M_{\alpha,j} \zeta_{i,j} \mathbb{1}_{\zeta_{i,j} \leq 0} \\ &\quad + \Delta t^n \left( N_{\alpha+1/2,i} - N_{\alpha-1/2,i} \right) \\ &= \left( 1 - \frac{\Delta t^n}{|C_i|} \sum_{j \in K_i} L_{i,j} \zeta_{i,j} \mathbb{1}_{\zeta_{i,j} \geq 0} \right) M_{\alpha,i} - \frac{\Delta t^n}{|C_i|} \sum_{j \in K_i} L_{i,j} M_{\alpha,j} \zeta_{i,j} \mathbb{1}_{\zeta_{i,j} \leq 0} \\ &\quad + \Delta t^n \left( N_{\alpha+1/2,i} - N_{\alpha-1/2,i} \right), \end{aligned} \quad (85)$$

with the exchange terms  $\{N_{\alpha+1/2,i}\}_{\alpha=0}^N$

$$\Delta t^n N_{\alpha+1/2,i}(\xi, \gamma) = \frac{G_{\alpha+1/2,i}}{g\pi h_i} \mathbb{1}_{(\xi - u_{\alpha+1/2,i})^2 + (\gamma - v_{\alpha+1/2,i})^2 \leq 2gh_i}, \quad (86)$$

and from (71)

$$\begin{aligned} G_{\alpha+1/2,i} &= -\frac{1}{|C_i|} \sum_{k=1}^N \left( \sum_{p=1}^{\alpha} l_p - \mathbb{1}_{k \leq \alpha} \right) \\ &\quad \sum_{j \in K_i} L_{i,j} \int_{\mathbb{R}^2} \left( M_{k,i} \zeta_{i,j} \mathbb{1}_{\zeta_{i,j} \geq 0} - M_{k,j} \zeta_{i,j} \mathbb{1}_{\zeta_{i,j} \leq 0} \right) d\xi d\gamma. \end{aligned}$$

By analogy with the computations in (67), we can recover the macroscopic quantities  $U_{\alpha,i}^{n+1}$  at time  $t^{n+1}$  by integration of the relation (85) since

$$U_{\alpha,i}^{n+1} = \int_{\mathbb{R}^2} \left( \begin{smallmatrix} 1 \\ \xi \\ \gamma \end{smallmatrix} \right) f_{\alpha,i}^{n+1-} d\xi d\gamma. \quad (87)$$

The scheme (85) and the definition (87) allow to complete the definition of the macroscopic scheme (81),(83) with the numerical flux given by the flux vector splitting formula [10]

$$\begin{aligned}\mathcal{F}_{i,j} &= \mathbf{F}^+(\mathbf{U}_i^n, \mathbf{n}_{i,j}) + \mathbf{F}^-(\mathbf{U}_j^n, \mathbf{n}_{i,j}) \\ &= \int_{\mathbb{R}^2} \begin{pmatrix} 1 \\ \xi \\ \gamma \end{pmatrix} M_i \zeta_{i,j} \mathcal{K}_{\zeta_{i,j} \geq 0} d\xi d\gamma + \int_{\mathbb{R}^2} \begin{pmatrix} 1 \\ \xi \\ \gamma \end{pmatrix} M_j \zeta_{i,j} \mathcal{K}_{\zeta_{i,j} \leq 0} d\xi d\gamma. \quad (88)\end{aligned}$$

### 5.3.2 With topography

The hydrostatic reconstruction scheme (HR scheme for short) for the Saint-Venant system has been introduced in [2] in the 1d case and described in 2d for unstructured meshes in [4]. The HR in the context of the kinetic description for the Saint-Venant system has been studied in [17].

In order to take into account the topography source and to preserve relevant equilibria, the HR leads to a modified version of (81) under the form

$$U_i^{n+1} = U_i^n - \sum_{j \in K_i} \sigma_{i,j} \mathcal{F}_{i,j}^* - \sigma_i \mathcal{F}_{i,e} + \sum_{j \in K_i} \sigma_{i,j} \mathcal{S}_{i,j}^*, \quad (89)$$

where

$$\mathcal{F}_{i,j}^* = F(U_{i,j}^*, U_{j,i}^*, \mathbf{n}_{i,j}), \quad \mathcal{S}_{i,j} = S(U_i, U_{i,j}^*, \mathbf{n}_{i,j}) = \begin{pmatrix} 0 \\ \frac{g}{2}(h_{i,j}^{*2} - h_i^2) \mathbf{n}_{i,j} \end{pmatrix}, \quad (90)$$

with

$$z_{i,j}^* = \max(z_i, z_j), \quad h_{i,j}^* = \max(h_i + z_i - z_{i,j}^*, 0), \quad U_{i,j}^* = (h_{i,j}^*, h_{i,j}^* u_i, h_{i,j}^* v_i)^T. \quad (91)$$

We would like here to propose a kinetic interpretation of the HR scheme, which means to interpret the above numerical fluxes as averages with respect to the kinetic variables of a scheme written on a kinetic function  $f$ . More precisely, we would like to approximate the solution to (72) by a kinetic scheme such that the associated macroscopic scheme is exactly (89)-(90) with homogeneous numerical flux  $\mathcal{F}$  given by (88). We denote  $M_{\alpha,i,j}^* = M(U_{\alpha,i,j}^*, \xi, \gamma)$  for any  $\alpha = 1, \dots, N$  and we consider the scheme

$$f_{\alpha,i}^{n+1-} = M_{\alpha,i} - \frac{\Delta t^n}{|C_i|} \sum_{j \in K_i} L_{i,j} \zeta_{i,j} \mathcal{K}_{\zeta_{i,j} \geq 0} M_{\alpha,i,j}^* - \frac{\Delta t^n}{|C_i|} \sum_{j \in K_i} L_{i,j} M_{\alpha,j,i}^* \zeta_{i,j} \mathcal{K}_{\zeta_{i,j} \leq 0}$$

$$+\Delta t^n \left( N_{\alpha+1/2,i}^* - N_{\alpha-1/2,i}^* \right) - \frac{\Delta t^n}{|C_i|} \sum_{j \in K_i} L_{i,j} (M_{\alpha,i} - M_{\alpha,i,j}^*) \theta_{\alpha,i,j}, \quad (92)$$

where

$$\theta_{\alpha,i,j} = \begin{pmatrix} \xi - u_{\alpha,i} \\ \gamma - v_{\alpha,i} \end{pmatrix} \cdot \mathbf{n}_{i,j}.$$

For the exchange terms, we have from (70)

$$\Delta t^n N_{\alpha+1/2,i}^*(\xi, \gamma) = \frac{G_{\alpha+1/2,i}^*}{g\pi h_i} \mathbb{K}_{(\xi - u_{\alpha+1/2,i})^2 + (\gamma - v_{\alpha+1/2,i})^2 \leq 2gh_i}, \quad (93)$$

and using (71) we get

$$G_{\alpha+1/2,i}^* = -\frac{1}{|C_i|} \sum_{k=1}^N \left( \sum_{p=1}^{\alpha} l_p - \mathbb{K}_{k \leq \alpha} \right) \sum_{j \in K_i} L_{i,j} \int_{\mathbb{R}^2} (M_{k,i,j}^* \zeta_{i,j} \mathbb{K}_{\zeta_{i,j} \geq 0} - M_{k,i,j}^* \zeta_{i,j} \mathbb{K}_{\zeta_{i,j} \leq 0}) d\xi d\gamma.$$

It is easy to see that in the previous formula, we have the moment relations

$$\int_{\mathbb{R}^2} (M_{\alpha,i} - M_{\alpha,i,j}^*) \theta_{\alpha,i,j} d\xi d\gamma = 0, \quad (94)$$

$$\int_{\mathbb{R}^2} \begin{pmatrix} \xi \\ \gamma \end{pmatrix} (M_{\alpha,i} - M_{\alpha,i,j}^*) \theta_{\alpha,i,j} d\xi d\gamma = \frac{g}{2} l_{\alpha} (h_{i,j}^{*2} - h_i^2) \mathbf{n}_{i,j}, \quad (95)$$

Using again (87), the integration of (92) multiplied by  $\begin{pmatrix} 1 \\ \xi \\ \gamma \end{pmatrix}$  with respect to  $\xi, \gamma$  then gives the HR scheme (89)-(90) with (88),(91). Thus as announced, (92) is a kinetic interpretation of the HR scheme in 3d for an unstructured mesh.

We consider a velocity  $v_m \geq 0$  such that for all  $\alpha, i$ ,

$$M(U_{\alpha,i}, \xi, \gamma) > 0 \Rightarrow |\xi| \leq v_m \text{ and } |\gamma| \leq v_m. \quad (96)$$

This means equivalently that  $|u_{\alpha,i}| + |v_{\alpha,i}| + \sqrt{2gh_i} \leq v_m$ . We consider a CFL condition strictly less than one,

$$\sigma_i v_m \leq \beta < \frac{1}{2} \quad \text{for all } i, \quad (97)$$

where  $\sigma_i = \Delta t^n \sum_{j \in K_i} L_{i,j} / |C_i|$ , and  $\beta$  is a given constant.

And the following proposition holds.

**Proposition 5.1** *Under the CFL condition (97), the scheme (92) verifies the following properties.*

- (i) *The macroscopic scheme derived from (92) using (87) is a consistent discretization of the layer-averaged Euler system (31)-(32).*
- (ii) *The kinetic function remains nonnegative i.e.*

$$\sum_{\alpha=1}^N f_{\alpha,i}^{n+1-} \geq 0, \quad \forall(\xi, \gamma) \in \mathbb{R}^2, \forall i, \forall \alpha.$$

- (iii) *The scheme (92) is kinetic well balanced i.e. at rest*

$$f_{\alpha,i}^{n+1-} = M_{\alpha,i}, \quad \forall(\xi, \gamma) \in \mathbb{R}^2, \forall i, \forall \alpha = 1, \dots, N. \quad (98)$$

**Proof of prop. 5.1** (i) *Since the Boltzmann type equations (72) are almost linear transport equations with source terms, the discrete kinetic scheme (92) is clearly a consistent discretization (72). And therefore using the kinetic interpretation given in prop. 4.1, the macroscopic scheme obtained from (92) using (87) is a consistent discretization of the layer-averaged Euler system (31)-(32).*

(ii) *In (92) we have*

$$\frac{\Delta t^n}{|C_i|} \sum_{j \in K_i} L_{i,j} M_{\alpha,j,i}^* \zeta_{i,j} \mathbb{1}_{\zeta_{i,j} \leq 0} \leq 0,$$

and the HR (91) ensures

$$M_{\alpha,i,j}^* \leq M_{\alpha,i}, \quad \forall(\xi, \gamma) \in \mathbb{R}^2, \forall \alpha$$

leading to

$$f_{\alpha,i}^{n+1-} \geq \left( 1 - \frac{\Delta t^n}{|C_i|} \sum_{j \in K_i} L_{i,j} (\zeta_{i,j} \mathbb{1}_{\zeta_{i,j} \geq 0} + \theta_{\alpha,i,j} \mathbb{1}_{\theta_{\alpha,i,j} \geq 0}) \right) M_{\alpha,i} + \Delta t^n (N_{\alpha+1/2,i}^* - N_{\alpha-1/2,i}^*),$$

and since  $G_{1/2,i}^* = G_{N+1/2,i}^* = 0$  it comes  $N_{1/2,i}^* = N_{N+1/2,i}^* = 0$  and hence

$$\sum_{\alpha=1}^N f_{\alpha,i}^{n+1-} \geq \sum_{\alpha=1}^n \left( 1 - \frac{\Delta t^n}{|C_i|} \sum_{j \in K_i} L_{i,j} (\zeta_{i,j} \mathbb{1}_{\zeta_{i,j} \geq 0} + \theta_{\alpha,i,j} \mathbb{1}_{\theta_{\alpha,i,j} \geq 0}) \right) M_{\alpha,i}.$$

But  $\zeta_{i,j} \mathbb{1}_{\zeta_{i,j} \geq 0} \leq \max\{|\xi|, |\gamma|\}$ ,  $\theta_{\alpha,i,j} \mathbb{1}_{\theta_{\alpha,i,j} \geq 0} \leq \max\{|\xi - u_{\alpha,i}|, |\gamma - v_{\alpha,i}|\}$  and therefore

$$\begin{aligned} \frac{\Delta t^n}{|C_i|} \sum_{j \in K_i} L_{i,j} (\zeta_{i,j} \mathbb{1}_{\zeta_{i,j} \geq 0} + \theta_{\alpha,i,j} \mathbb{1}_{\theta_{\alpha,i,j} \geq 0}) \\ \leq \sigma_i (\max\{|\xi|, |\gamma|\} + \max\{|\xi - u_{\alpha,i}|, |\gamma - v_{\alpha,i}|\}) \leq 1, \end{aligned}$$

where (96),(97) have been used.

(iii) Considering the situation at rest i.e.  $u_{\alpha,i} = v_{\alpha,i} = 0 \forall \alpha, i$  and  $h_i + z_i = h_j + z_j \forall i, j$  we have

$$M_{\alpha,i} = M_{\alpha,i,j}^*, \quad \forall \alpha, i, j.$$

From (92), this gives (98). ■

## 5.4 Macroscopic scheme

Following (84)-(85), the macroscopic scheme requires to calculate flux having the form

$$F(U) = \begin{pmatrix} F_h \\ F_{hu} \\ F_{hv} \end{pmatrix} = \int_{n_x \xi + n_y \gamma \geq 0} \begin{pmatrix} 1 \\ \xi \\ \gamma \end{pmatrix} (n_x \xi + n_y \gamma) M d\xi d\gamma.$$

$n_x$  and  $n_y$  being the components of a normal unit vector we have  $n_x^2 + n_y^2 = 1$ . For an interface with unit normal  $\mathbf{n} = (n_x, n_y)^T$ , we define a local basis  $(\mathbf{n}, \tau)$  associated to the normal direction and to the tangential one and the associated rotation matrix

$$R_{\mathbf{n}} = \begin{pmatrix} n_x & -n_y \\ n_y & n_x \end{pmatrix}.$$

We can write

$$F(U) = h \int_{n_x(cz_1+u)+n_y(cz_2+v) \geq 0} (n_x(cz_1+u)+n_y(cz_2+v)) \begin{pmatrix} 1 \\ u + cz_1 \\ v + cz_2 \end{pmatrix} \chi_0(z_1, z_2) dz_1 dz_2,$$

where  $\chi_0$  is defined by (65) and using the change of variables induced by the rotation  $R_{\mathbf{n}}$  it comes

$$F(U) = h \int_{\{y_1 \geq -\frac{\tilde{u}}{c}\} \times \mathbb{R}} (\tilde{u} + cy_1) \begin{pmatrix} 1 \\ u + c(n_x y_1 - n_y y_2) \\ v + c(n_y y_1 + n_x y_2) \end{pmatrix} \chi_0(y_1, y_2) dy_1 dy_2$$

$$= h \int_{\{y_1 \geq -\frac{\tilde{u}}{c}\} \times \mathbb{R}} (\tilde{u} + cy_1) \begin{pmatrix} 1 \\ u + cn_x y_1 \\ v + cn_y y_1 \end{pmatrix} \chi_0(y_1, y_2) dy_1 dy_2, \quad (99)$$

with

$$y_1 = n_x z_1 + n_y z_2, \quad y_2 = n_x z_2 - n_y z_1, \quad \tilde{u} = n_x u + n_y v.$$

And simple computations give

$$\begin{aligned} F_h &= h \int_{\{y_1 \geq -\frac{\tilde{u}}{c}\} \times \mathbb{R}} (\tilde{u} + cy_1) \chi_0(y_1, y_2) dy_1 dy_2 \\ &= h \int_{y_1 = -\frac{\tilde{u}}{c}}^{+\infty} (\tilde{u} + cy_1) \left( \int_{-\infty}^{+\infty} \chi_0(y_1, y_2) dy_2 \right) dy_1 \\ &= \frac{h}{\pi} \int_{y_1 = -\frac{\tilde{u}}{c}}^{+\infty} (\tilde{u} + cy_1) \sqrt{1 - \frac{y_1^2}{4}} dy_1, \end{aligned} \quad (100)$$

$$\begin{aligned} F_{hu} &= h \int_{\{y_1 \geq -\frac{\tilde{u}}{c}\} \times \mathbb{R}} (\tilde{u} + cy_1)(u + cn_x y_1) \chi_0(y_1, y_2) dy_1 dy_2 \\ &= h \int_{y_1 = -\frac{\tilde{u}}{c}}^{+\infty} (\tilde{u} + cy_1)(u + cn_x y_1) \left( \int_{-\infty}^{+\infty} \chi_0(y_1, y_2) dy_2 \right) dy_1 \\ &= \frac{h}{\pi} \int_{y_1 = -\frac{\tilde{u}}{c}}^{+\infty} (\tilde{u} + cy_1)(u + cn_x y_1) \sqrt{1 - \frac{y_1^2}{4}} dy_1, \end{aligned}$$

and likewise for  $F_{hv}$  we have

$$F_{hv} = \frac{h}{\pi} \int_{y_1 = -\frac{\tilde{u}}{c}}^{+\infty} (\tilde{u} + cy_1)(v + cn_y y_1) \sqrt{1 - \frac{y_1^2}{4}} dy_1.$$

It is possible to obtain explicit formula for the expression of  $F_h$ ,  $F_{hu}$  and  $F_{hv}$  since defining

$$\begin{aligned} I_1(z) &= \int^z (u + cz) \sqrt{1 - \frac{z^2}{4}} dz, \\ I_2(z) &= \int^z (u + cz)(v + cnz) \sqrt{1 - \frac{z^2}{4}} dz, \end{aligned}$$

we have

$$I_1(z) = -\frac{4c}{3} \left(1 - \frac{z^2}{4}\right)^{3/2} + u \left( \frac{z}{2} \sqrt{1 - \frac{z^2}{4}} + \arcsin\left(\frac{z}{2}\right) \right),$$

and

$$I_2(z) = -c^2 n z \left(1 - \frac{z^2}{4}\right)^{3/2} + \frac{c^2 n + uv}{2} z \sqrt{1 - \frac{z^2}{4}} + (c^2 n + uv) \arcsin\left(\frac{z}{2}\right) - \frac{4c(nu + v)}{3} \left(1 - \frac{z^2}{4}\right)^{3/2}.$$

Therefore, it comes

$$\begin{aligned} F_h &= \frac{h}{\pi} (I_1(\max\{-u/c, 2\}) - I_1(\max\{-u/c, -2\})), \\ F_{hu} &= \frac{h}{\pi} (I_2(\max\{-u/c, 2\}) - I_2(\max\{-u/c, -2\})), \\ F_{hv} &= \frac{hv}{\pi} (I_1(\max\{-u/c, 2\}) - I_1(\max\{-u/c, -2\})). \end{aligned}$$

## 5.5 The discrete layer-averaged Navier-Stokes system

Rewriting (79) under the form

$$(h_\alpha \mathbf{u}_\alpha)^{n+1} = \widetilde{h_\alpha \mathbf{u}_\alpha} + \Delta t^n S_{v,f}(h, \{(h \mathbf{u}_\alpha)\}), \quad (101)$$

with  $S_{v,f}(h, \{(h \mathbf{u}_\alpha)\}) = (S_{v,f,1}, \dots, S_{v,f,N})^T$  and

$$\begin{aligned} \widetilde{h_\alpha \mathbf{u}_\alpha} &= h_\alpha \mathbf{u}_\alpha - \Delta t^n \left( \nabla_{x,y} \cdot (h_\alpha \mathbf{u}_\alpha \otimes \mathbf{u}_\alpha) - \nabla_{x,y} \left( \frac{g}{2} h h_\alpha \right) - g h_\alpha \nabla z \right. \\ &\quad \left. + \mathbf{u}_{\alpha+1/2} G_{\alpha+1/2} - \mathbf{u}_{\alpha-1/2} G_{\alpha-1/2} \right), \\ S_{v,f,\alpha} &= \nabla_{x,y} \cdot (h_\alpha \mathbf{T}_\alpha) + \mu_{\alpha+1/2} \frac{\mathbf{u}_{\alpha+1} - \mathbf{u}_\alpha}{h_{\alpha+1} + h_\alpha} - \mu_{\alpha-1/2} \frac{\mathbf{u}_\alpha - \mathbf{u}_{\alpha-1}}{h_\alpha + h_{\alpha-1}} \\ &\quad - \mathbf{T}_{\alpha+1/2} \nabla_{x,y} z_{\alpha+1/2} + \mathbf{T}_{\alpha-1/2} \nabla_{x,y} z_{\alpha-1/2} - \kappa_\alpha \mathbf{u}_\alpha, \end{aligned}$$

with the definitions (47)-(50) for  $\mathbf{T}_\alpha$ ,  $\mathbf{T}_{\alpha\pm 1/2}$ , it remains to give a fully discrete scheme for the visous and friction terms  $S_{v,f,\alpha}$ .

The discretization of (101) is done using a finite element / finite difference approximation. Notice that, compared to the advection and pressure terms, the discretization of the viscous terms raises less difficulties and we propose a stable scheme that will be extended to more general rheology terms [16] and more completely analysed in a forthcoming paper.

Using a classical  $\mathbb{P}_1$  finite element approximation of Eq. (101), we get

$$\begin{aligned} \mathcal{M}_\alpha \mathbf{U}_\alpha^{n+1} &= \mathcal{M}_\alpha \mathbf{U}_\alpha - \Delta t^n \left( \mathcal{K}_{\alpha+1} \mathbf{U}_{\alpha+1} + \mathcal{K}_\alpha \mathbf{U}_\alpha + \mathcal{K}_{\alpha-1} \mathbf{U}_{\alpha-1} \right) \\ &\quad + \Delta t^n \mathcal{L}_{\alpha+1/2} (\mathbf{U}_{\alpha+1} - \mathbf{U}_\alpha) - \Delta t^n \mathcal{L}_{\alpha-1/2} (\mathbf{U}_\alpha - \mathbf{U}_{\alpha-1}) \\ &\quad - \Delta t^n \mathcal{Z}_{\alpha+1/2} \mathbf{U}_{\alpha+1} - \Delta t^n \mathcal{Z}_{\alpha-1/2} \mathbf{U}_{\alpha-1} - \Delta t^n \kappa_\alpha \mathcal{M} \mathbf{U}_\alpha \end{aligned} \quad (102)$$

with the matrices

$$\begin{aligned} \mathcal{M} &= \int_{\Omega} \varphi_i \varphi_j \, dx dy, \\ \mathcal{M}_\alpha &= \int_{\Omega} h_{\alpha,j} \varphi_i \varphi_j \, dx dy, \\ \mathcal{K}_\alpha &= \frac{1}{2} \int_{\Omega} \left( \frac{h_{\alpha,j}}{h_{\alpha+1,j} + h_{\alpha,j}} + \frac{h_{\alpha,j}}{h_{\alpha,j} + h_{\alpha-1,j}} \right) \nabla_{x,y} \varphi_i \nabla_{x,y} \varphi_j \, dx dy, \\ \mathcal{K}_{\alpha\pm 1} &= \frac{\mu_{\alpha\pm 1/2}}{2} \int_{\Omega} \frac{h_{\alpha\pm 1,j}}{h_{\alpha+1,j} + h_{\alpha,j}} \nabla_{x,y} \varphi_i \nabla_{x,y} \varphi_j \, dx dy, \\ \mathcal{L}_{\alpha+1/2} &= \mu_{\alpha+1/2} \int_{\Omega} \left( \frac{1 + \nabla_{x,y}^2 z_{\alpha+1/2}}{h_{\alpha+1} + h_\alpha} - \nabla_{x,y} \left( \frac{h_\alpha \nabla_{x,y} z_{\alpha+1/2}}{h_{\alpha+1} + h_\alpha} \right) \right) \varphi_i \varphi_j \, dx dy, \\ \mathcal{Z}_{\alpha+1/2} &= \mu_{\alpha+1/2} \int_{\Omega} \nabla_{x,y} z_{\alpha+1/2} \varphi_i \nabla_{x,y} \varphi_j \, dx dy, \end{aligned}$$

where the expression of the viscous terms given in Remark 3.4 is used.

Using a mass-lumping strategy, Eq. (102) becomes

$$\begin{aligned} \mathcal{M} h_\alpha^{n+1} \mathbf{U}_\alpha^{n+1} &= \mathcal{M} h_\alpha^{n+1} \mathbf{U}_\alpha - \frac{\Delta t^n}{2} \mathcal{K} \left( \frac{h_{\alpha+1}}{h_{\alpha+1} + h_\alpha} \mathbf{U}_{\alpha+1} + \left( \frac{h_\alpha}{h_{\alpha+1} + h_\alpha} \right. \right. \\ &\quad \left. \left. + \frac{h_\alpha}{h_\alpha + h_{\alpha-1}} \right) \mathbf{U}_\alpha + \frac{h_{\alpha-1}}{h_\alpha + h_{\alpha-1}} \mathbf{U}_{\alpha-1} \right) \\ &\quad + \Delta t^n \mathcal{M} r_{\alpha+1/2}^d (\mathbf{U}_{\alpha+1} - \mathbf{U}_\alpha) - \Delta t^n \mathcal{M} r_{\alpha-1/2}^d (\mathbf{U}_\alpha - \mathbf{U}_{\alpha-1}) \\ &\quad - \Delta t^n \mu_{\alpha+1/2} \mathcal{Z} \nabla_{x,y}^d z_{\alpha+1/2} \mathbf{U}_{\alpha+1} - \Delta t^n \mu_{\alpha-1/2} \mathcal{Z} \nabla_{x,y}^d z_{\alpha-1/2} \mathbf{U}_{\alpha-1} \\ &\quad - \Delta t^n \kappa_\alpha \mathcal{M} \mathbf{U}_\alpha, \end{aligned} \quad (103)$$

with

$$\begin{aligned} \mathcal{K} &= \frac{1}{2} \int_{\Omega} \nabla_{x,y} \varphi_i \nabla_{x,y} \varphi_j \, dx dy, \\ \mathcal{Z} &= \int_{\Omega} \varphi_i \nabla_{x,y} \varphi_j \, dx dy, \end{aligned}$$



$$r_{\alpha+1/2} = \mu_{\alpha+1/2} \left( \frac{1 + \nabla_{x,y}^2 z_{\alpha+1/2}}{h_{\alpha+1} + h_{\alpha}} - \nabla_{x,y} \left( \frac{h_{\alpha} \nabla_{x,y} z_{\alpha+1/2}}{h_{\alpha+1} + h_{\alpha}} \right) \right),$$

and where the superscript  $d$  corresponds to a finite difference approximation of the considered quantity. Multiplying (103) by  $\mathcal{M}^{-1}$  gives

$$\begin{aligned} \mathbf{U}_{\alpha}^{n+1} &= \mathbf{U}_{\alpha} - \frac{\Delta t^n}{2h_{\alpha}^{n+1}} \mathcal{M}^{-1} \mathcal{K} \left( \frac{h_{\alpha+1}}{h_{\alpha+1} + h_{\alpha}} \mathbf{U}_{\alpha+1} + \left( \frac{h_{\alpha}}{h_{\alpha+1} + h_{\alpha}} \right. \right. \\ &\quad \left. \left. + \frac{h_{\alpha}}{h_{\alpha} + h_{\alpha-1}} \right) \mathbf{U}_{\alpha} + \frac{h_{\alpha-1}}{h_{\alpha} + h_{\alpha-1}} \mathbf{U}_{\alpha-1} \right) \\ &\quad + \Delta t^n \frac{r_{\alpha+1/2}^d}{h_{\alpha}^{n+1}} (\mathbf{U}_{\alpha+1} - \mathbf{U}_{\alpha}) - \Delta t^n \frac{r_{\alpha-1/2}^d}{h_{\alpha}^{n+1}} (\mathbf{U}_{\alpha} - \mathbf{U}_{\alpha-1}) \\ &\quad - \Delta t^n \frac{\mu_{\alpha+1/2}}{h_{\alpha}^{n+1}} \mathcal{M}^{-1} \mathcal{Z} \nabla_{x,y}^d z_{\alpha+1/2} \mathbf{U}_{\alpha+1} - \Delta t^n \frac{\mu_{\alpha-1/2}}{h_{\alpha}^{n+1}} \mathcal{M}^{-1} \mathcal{Z} \nabla_{x,y}^d z_{\alpha-1/2} \mathbf{U}_{\alpha-1} \\ &\quad - \Delta t^n \frac{\kappa_{\alpha}}{h_{\alpha}^{n+1}} \mathcal{M} \mathbf{U}_{\alpha}, \end{aligned} \tag{104}$$

that can be solved using an implicit or explicit – with the corresponding CFL condition – technique.

The main purpose of this paper is to propose a stable and robust numerical approximation of the incompressible Euler system with free surface. Voluntarily, we give few details concerning the numerical approximation of the dissipative terms:

- the viscous and friction terms are dissipative and hence a reasonable approximation leads to a stable numerical scheme.
- As far as the authors knows, if analytical solutions exist for the Euler system, they are hardly accessible for the Navier-Stokes system.
- In this paper, we consider a simplified newtonian rheology for the fluid, the numerical approximation of the general (layer-averaged) rheology will be studied [15] will be studied in a forthcoming paper.

## 5.6 Boundary conditions

The contents of this paragraph slightly differ from previous works of one of the authors [18] and valid for the classical Saint-Venant system. First, we focus on the boundary conditions for the layer averaged Euler system i.e. the system (78)-(79) for  $\mu = 0$ ,  $\kappa = 0$  and then for the viscous part.

### 5.6.1 Layer averaged Euler system

In this paragraph we detail the computation of the boundary flux  $\mathcal{F}(\mathbf{U}_i, \mathbf{U}_{e,i}, \mathbf{n}_i)$  appearing in (81),(83). The variable  $\mathbf{U}_{i,e}^n$  can be interpreted as an approximation of the solution in a ghost cell adjacent to the boundary. As before we introduce the vector

$$\mathbf{U}_{i,e} = (h_{i,e}^n, (hu)_{1,i,e}, \dots, (hu)_{N,i,e}, (hv)_{1,i,e}, \dots, (hv)_{N,i,e})^T,$$

and we will use the flux vector splitting form associated to the kinetic formulation (88)

$$\mathcal{F}(\mathbf{U}_i, \mathbf{U}_{i,e}, \mathbf{n}_i) = \mathbf{F}^+(\mathbf{U}_i, \mathbf{n}_i) + \mathbf{F}^-(\mathbf{U}_{i,e}, \mathbf{n}_i) \quad (105)$$

with  $\mathbf{U}_{i,e}^n$  defined according to the type of the boundary.

**Solid wall** If we consider a node  $i_0$  belonging to a solid wall, we prescribe a slip condition written

$$\mathbf{u}_\alpha \cdot \mathbf{n}_{i_0} = 0, \quad (106)$$

for  $\alpha = 1, \dots, N$ . We assume the continuity of the water depth  $h_{i_0,e} = h_{i_0}$  and of the tangential component of velocity i.e.

$$\mathbf{u}_{\alpha,i_0,e} = \mathbf{u}_{\alpha,i_0} - 2(\mathbf{u}_{\alpha,i_0} \cdot \mathbf{n}_{i_0})\mathbf{n}_{i_0},$$

and from (99) we obtain

$$F_h^+(\mathbf{U}_{\alpha,i_0}) + F_h^-(\mathbf{U}_{\alpha,i_0,e}) = 0,$$

and

$$\begin{aligned} \begin{pmatrix} F_{hu}^+(\mathbf{U}_{\alpha,i_0}) + F_{hu}^-(\mathbf{U}_{\alpha,i_0,e}) \\ F_{hv}^+(\mathbf{U}_{\alpha,i_0}) + F_{hv}^-(\mathbf{U}_{\alpha,i_0,e}) \end{pmatrix} \cdot \mathbf{n}_{i_0} &= \frac{gh_{\alpha,i_0}h_{i_0}}{2}, \\ \begin{pmatrix} F_{hu}^+(\mathbf{U}_{\alpha,i_0}) + F_{hu}^-(\mathbf{U}_{\alpha,i_0,e}) \\ F_{hv}^+(\mathbf{U}_{\alpha,i_0}) + F_{hv}^-(\mathbf{U}_{e,\alpha,i_0}) \end{pmatrix} \cdot \mathbf{t}_{i_0} &= 0, \end{aligned}$$

for  $\alpha = 1, \dots, N$  for a vector  $\mathbf{t}_{i_0}$  orthogonal to  $\mathbf{n}_{i_0}$ . The condition (106) is therefore prescribed weakly.

**Fluid boundary** Even the considered model is more complex than the Shallow water system, we can consider that the type of a flow depends on the value of the Froud number  $Fr_\alpha = \frac{|\mathbf{u}_\alpha|}{\sqrt{gh}}$ , a flow is said torrential, for  $|\mathbf{u}_\alpha| > \sqrt{gh}$  and fluvial, for  $|\mathbf{u}_\alpha| < \sqrt{gh}$ .

Generally, for the fluid boundaries, the conditions prescribed by the user depend on the type of the flow defined by this criterium.

We have also to notice that with  $\mathbf{n}$  the outward unit normal to the boundary edge, an inflow boundary corresponds to  $\mathbf{u}_\alpha \cdot \mathbf{n} < 0$  and an outflow one to  $\mathbf{u}_\alpha \cdot \mathbf{n} > 0$ .

We will treat the following cases: for a fluvial flow boundary, we distinguish the cases where the flux or the water depth are given, while for a torrential flow we distinguish the inflow or outflow boundaries.

**Fluvial boundary. Flux given** We consider first a fluvial boundary, so we assume that

$$|\mathbf{u}_\alpha| < \sqrt{gh}. \quad (107)$$

If for each layer, the flux  $\mathbf{q}_{\alpha,g}$  is given, then we wish to impose

$$\begin{aligned} (F_h(\mathbf{U}_{\alpha,i}) + F_h(\mathbf{U}_{e,\alpha,i})) \cdot \mathbf{n}_i &= \mathbf{q}_{g,\alpha} \cdot \mathbf{n}_i, \\ F_h(\mathbf{U}_{e,\alpha,i}) \cdot \mathbf{t}_i &= q_{g,\alpha} \cdot \mathbf{t}_i, \end{aligned} \quad (108)$$

with  $\mathbf{n}_i \cdot \mathbf{t}_i = 0$ .

To impose this directly leads to instabilities (especially because the numerical values are not necessarily in the regime of validity of this condition). We propose to discretize it in a weak form. We denote

$$a_1 = \mathbf{q}_{g,\alpha} \cdot \mathbf{n}_i - F_h(\mathbf{U}_{\alpha,i}) \cdot \mathbf{n}_i. \quad (109)$$

If  $a_1 \geq 0$ , we prescribe

$$F_h(\mathbf{U}_{e,\alpha,i}) = 0, \quad F_{hu}(\mathbf{U}_{e,\alpha,i}) = 0, \quad \text{and} \quad F_{hv}(\mathbf{U}_{e,\alpha,i}) = 0.$$

If  $a_1 < 0$ , we have to write a third equation to be able to compute the three components of  $\mathbf{U}_{e,\alpha}$  and by analogy with what is done for the Saint-Venant system – where the Riemann invariant related to the outgoing characteristic is preserved – we assume the quantity  $\mathbf{u}_\alpha \cdot \mathbf{n}$  is constant along the interface, i.e.

$$\mathbf{u}_{e,\alpha,i} \cdot \mathbf{n}_i - 2\sqrt{gh_{e,i}} = \mathbf{u}_{\alpha,i} \cdot \mathbf{n}_i - 2\sqrt{gh_i}. \quad (110)$$

As (107) is satisfied, the eigenvalue  $u_{e,\alpha,i}\cdot\mathbf{n}_i - 2\sqrt{gh_{e,i}}$  is positive.

We use the equations (108) and (110) to compute  $h_{e,i}$  and  $\mathbf{u}_{e,\alpha,i}\cdot\mathbf{n}_i$ . We denote

$$a_2 = \mathbf{u}_{\alpha,i}\cdot\mathbf{n}_i - 2\sqrt{gh_i},$$

and

$$m = \frac{u_{e,\alpha,i}\cdot\mathbf{n}_i}{\sqrt{gh_{e,i}}}. \quad (111)$$

Then the equation (110) gives

$$\sqrt{gh_{e,i}} (m - 2) = a_2 \quad (112)$$

and using the definition of  $F_h$  (see (100)) with (108),(109) we have

$$\frac{h_{e,i}}{\pi} \int_{z \leq \frac{-u_{e,\alpha,i}\cdot\mathbf{n}_i}{\sqrt{\frac{gh_{e,i}}{2}}}} \left( u_{e,\alpha,i}\cdot\mathbf{n}_i + \sqrt{\frac{gh_{e,i}}{2}} z \right) \sqrt{1 - \frac{z^2}{4}} dz = a_1, \quad (113)$$

or using (110)

$$\psi(h_{e,i}) = a_1, \quad (114)$$

with

$$\psi(h_{e,i}) = \frac{h_{e,i}}{\pi} \int_{z \leq \frac{-(2\sqrt{gh_{e,i}} + a_2)}{\sqrt{\frac{gh_{e,i}}{2}}}} \left( 2\sqrt{gh_{e,i}} + a_2 + \sqrt{\frac{gh_{e,i}}{2}} z \right) \sqrt{1 - \frac{z^2}{4}} dz.$$

It is easy to see that  $h \mapsto \psi(h)$  is a growing function of  $h$  with  $\psi(0) = 0$  and  $\psi(+\infty) = +\infty$  and therefore, Eq. (113) admits a unique solution for any  $a_1 > 0$ . Using (112), Eq. (114) is equivalent to solve for  $m$

$$\Psi(m) = a_2, \quad (115)$$

with

$$\Psi(m) = K \frac{(m - 2)}{\phi(m)^{\frac{1}{3}}},$$

and

$$K = (\sqrt{2}g a_1)^{\frac{1}{3}},$$

$$\phi(m) = \frac{1}{\pi} \int_{z \leq -\sqrt{2}m} (\sqrt{2}m + z) \sqrt{1 - \frac{z^2}{4}} dz.$$

In practice, we use a Newton-Raphson algorithm to solve an equivalent form of Eq. (115), namely

$$m - 2 - \frac{a_2}{K} \phi(m)^{1/3} = 0.$$

Once the above equation has been solved, from (111),(112) we deduce

$$h_{e,i} = \frac{1}{g} \left( \frac{a_2}{m-2} \right)^2,$$

and

$$\mathbf{u}_{e,\alpha,i} \cdot \mathbf{n}_i = \frac{a_2 m}{m-2}.$$

**Remark 5.1** Notice that in the procedure proposed to calculate  $\mathbf{u}_{e,\alpha,i}$ ,  $h_{e,i}$ , a different value of  $h_{e,i}$  is calculated for each layer  $\alpha$ .

**Fluvial boundary. Water depth given** We verify that the flow is actually fluvial, i.e.

$$(\mathbf{u}_{\alpha,i} \cdot \mathbf{n}_i - \sqrt{gh_i})(\mathbf{u}_{\alpha,i} \cdot \mathbf{n}_i + \sqrt{gh_i}) \leq 0. \quad (116)$$

Since the water depth is given, we write

$$h_{e,i} = h_{g,i}. \quad (117)$$

We assume the continuity of the tangential component

$$(h\mathbf{u})_{e,\alpha,i} \cdot \mathbf{t}_i = (h\mathbf{u})_{\alpha,i} \cdot \mathbf{t}_i, \quad (118)$$

with  $\mathbf{t}_i \cdot \mathbf{n}_i = 0$ . To define completely  $\mathbf{u}_{e,\alpha,i}$ , we assume, as in the previous case, that the Riemann invariant is constant along the outgoing characteristic (110), so we obtain

$$\mathbf{u}_{e,\alpha,i} \cdot \mathbf{n}_i = \mathbf{u}_{\alpha,i} \cdot \mathbf{n}_i + 2\sqrt{g}(\sqrt{h_i} - \sqrt{h_{g,i}}). \quad (119)$$

Sometimes it appears that the numerical values do not satisfy the condition (116), then the flow is in fact torrential and

- if  $\mathbf{u}_{\alpha,i} \cdot \mathbf{n}_i > 0$ , the condition (117) cannot be satisfied (see Sec. 6.2.4),
- if  $\mathbf{u}_{\alpha,i} \cdot \mathbf{n}_i < 0$ , one condition is missing and we prescribe

$$\mathbf{u}_{e,\alpha,i} \cdot \mathbf{n}_i = \mathbf{u}_{\alpha,i} \cdot \mathbf{n}_i.$$

**Torrential inflow boundary** For a torrential inflow boundary we assume that the water depth and the flux are given, then we prescribe

$$\begin{aligned} h_{e,i} &= h_{g,i}, \\ (h\mathbf{u})_{e,\alpha,i} \cdot \mathbf{t}_i &= (h\mathbf{u})_{g,\alpha,i} \cdot \mathbf{t}_i, \end{aligned}$$

and

$$(F_h(\mathbf{U}_{\alpha,i}) + F_h(\mathbf{U}_{e,\alpha,i})) \cdot \mathbf{n}_i = \mathbf{q}_{g,\alpha} \cdot \mathbf{n}_i = (h\mathbf{u})_{g,\alpha,i} \cdot \mathbf{n}_i.$$

In this case we have to compute  $(h\mathbf{u})_{e,\alpha,i} \cdot \mathbf{n}_i$  or  $\mathbf{u}_{e,\alpha,i} \cdot \mathbf{n}_i$ . We consider an inflow boundary, so  $(h\mathbf{u})_{g,\alpha,i} \cdot \mathbf{n}_i < 0$  therefore using the notation (109) we have  $a_1 < 0$ . By analogy with the previous section we denote

$$m = \frac{\mathbf{u}_{e,\alpha,i} \cdot \mathbf{n}_i}{\sqrt{gh_{g,i}}},$$

then the equation for  $m$  is (see (111)-(113))

$$\phi(m) = \sqrt{\frac{2}{g}} \frac{a_1}{h_{g,i}^{3/2}}.$$

As in the paragraph entitled *Flux given*, the above equation has a unique solution  $m < 2$  for  $a_1 < 0$ .

**Torrential outflow boundary** In the case of a torrential outflow boundary, we do not prescribe any condition. We assume that the two Riemann invariants are constant along the outgoing characteristics leading to

$$\begin{aligned} \mathbf{u}_{e,\alpha,i} \cdot \mathbf{n}_i - 2\sqrt{gh_{e,i}} &= \mathbf{u}_{\alpha,i} \cdot \mathbf{n}_i - 2\sqrt{gh_i}, \\ \mathbf{u}_{e,\alpha,i} \cdot \mathbf{n}_i + 2\sqrt{gh_{e,i}} &= \mathbf{u}_{\alpha,i} \cdot \mathbf{n}_i + 2\sqrt{gh_i}, \end{aligned}$$

and we deduce

$$h_{e,i} = h_i, \quad \mathbf{u}_{e,\alpha,i} \cdot \mathbf{n}_i = \mathbf{u}_{\alpha,i} \cdot \mathbf{n}_i.$$

We assume that we also have also  $(h\mathbf{u})_{e,\alpha,i} \cdot \mathbf{t}_i = (h\mathbf{u})_{\alpha,i} \cdot \mathbf{t}_i$ .

### 5.6.2 Layer averaged Navier-Stokes system

Because of the fractional step we use, the boundary conditions for the layer-averaged Euler system are, to some extent, independent from the one used for the rheology terms.

For the resolution of Eq. (101), boundary conditions associated with the operator

$$\nabla_{x,y} \cdot (h_\alpha \mathbf{T}_\alpha) - \nabla_{x,y} z_{\alpha+1/2} \mathbf{T}_{\alpha+1/2} + \nabla_{x,y} z_{\alpha-1/2} \mathbf{T}_{\alpha-1/2},$$

have to be specified and we prescribe homogeneous Neumann boundary conditions. Of course, in particular cases, Dirichlet or Robin type boundary conditions can also be considered.

## 5.7 Second order schemes

In order to improve the accuracy of the results the first-order scheme defined in paragraphs 5.2-5.4 can be extended to a formally second-order one using a MUSCL like extension (see [28]).

### 5.7.1 Second order reconstruction

In the definition of the flux (90), we replace the piecewise constant values  $\mathbf{U}_{i,j}$ ,  $\mathbf{U}_{j,i}$  by more accurate reconstructions deduced from piecewise linear approximations, namely the values  $\tilde{\mathbf{U}}_{i,j}$ ,  $\tilde{\mathbf{U}}_{j,i}$  reconstructed on both sides of the interface. The reconstruction procedure is similar to the one used and described in [4, paragraph. 5.1].

### 5.7.2 Modified Heun scheme

The explicit time scheme (78)-(79) used in the previous paragraphs corresponds to a first order explicit Euler scheme. The second-order accuracy in time is usually recovered by the Heun method [11] that is a slight modification of the second order Runge-Kutta method. More precisely, for a dynamical system written under the form

$$\frac{\partial y}{\partial t} = f(y), \tag{120}$$

the Heun scheme consists in defining  $y^{n+1}$  by

$$y^{n+1} = y(t^n + \Delta t^n) = \frac{y^n + \tilde{y}^{n+2}}{2}, \tag{121}$$

with

$$\tilde{y}^{n+1} = y^n + \Delta t^n f(y^n, t^n),$$

$$\tilde{y}^{n+2} = \tilde{y}^{n+1} + \Delta t^n f(\tilde{y}^{n+1}, t^{n+1}). \quad (122)$$

But the scheme defined by (121) does not preserve the invariant domains. Indeed, the time step being given by a CFL condition,  $\Delta t^n$  in the relation (122) should be replaced by  $\tilde{\Delta t}^{n+1}$  i.e. the time step satisfying the CFL condition and calculated using  $\tilde{y}^{n+1}$ . Thus in situations where the time step strongly varies from one iteration to another, the Heun scheme does not preserve the positivity of the scheme.

To overcome this difficulty, we propose an improvement of the Heun scheme

**Proposition 5.2** *The scheme defined by*

$$y^{n+1} = \alpha y^n + \gamma \tilde{y}^{n+2},$$

with

$$\begin{aligned} \Delta t^n &= \frac{2\Delta t_1^n \Delta t_2^n}{\Delta t_1^n + \Delta t_2^n}, \\ \tilde{y}^{n+1} &= y^n + \Delta t_1^n f(y^n), \\ \tilde{y}^{n+2} &= \tilde{y}^{n+1} + \Delta t_2^n f(\tilde{y}^{n+1}), \\ \gamma &= \frac{(\Delta t^n)^2}{2\Delta t_1^n \Delta t_2^n}, \\ \alpha &= 1 - \gamma, \end{aligned}$$

is second order and compatible with a CFL constraint. Since  $\alpha, \gamma \geq 0$ ,  $y^{n+1}$  is a convex combination of  $y^n$  and  $\tilde{y}^{n+2}$  so the scheme preserves the positivity. For the previous relations  $\Delta t_1^n$  and  $\Delta t_2^n$  respectively satisfy the CFL conditions associated with  $y^n$  and  $\tilde{y}^{n+1}$ .

When  $\Delta t_1^n = \Delta t_2^n = \Delta t^n$ , the scheme reduces to the classical Heun scheme with  $\alpha = \gamma = \frac{1}{2}$ .

**Proof of proposition 5.7.2** *Using (120), a Taylor expansion of  $y(t^n + \Delta t^n)$  gives*

$$y(t^n + \Delta t^n) = y^n + \Delta t^n f(y^n) + \frac{(\Delta t^n)^2}{2} f(y^n) f'(y^n) + \mathcal{O}((\Delta t^n)^3).$$



Using the definitions given in the proposition, we have

$$\tilde{y}^{n+2} = y^n + (\Delta t_1^n + \Delta t_2^n) f(y^n) + \Delta t_1^n \Delta t_2^n f(y^n) f'(y^n) + \mathcal{O}(\Delta t_2^n (\Delta t_1^n)^2),$$

and a simple calculus gives

$$\alpha y^n + \beta \tilde{y}^{n+1} + \gamma \tilde{y}^{n+2} - y(t^n + \Delta t^n) = \mathcal{O}((\Delta t^n)^3),$$

that completes the proof. ■

## 6 Numerical validations

### 6.1 Thacker' analytical solution

The proposed model is an approximation of the 3d Navier-Stokes and hence it is more general than the Saint-Venant system [7, 20]. But when considering flows with

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0,$$

the solutions of the model depicted in this paper should corresponds to the solutions of the Saint-Venant system. Thus the Thacker' analytical solution [27] for the 2d Saint-Venant system is also an analytical solution for the 3d incompressible and hydrostatic Euler system.

From [27], considering the parameters  $(h_0, \eta_0, a) \in \mathbb{R}_+^* \times \mathbb{R}^2$

$$z(x, y) = -h_0 \left( 1 - \frac{x^2 + y^2}{a^2} \right),$$

then

$$\begin{cases} h = \frac{\eta_0 h_0}{a^2} \left( 2 \left( x - \frac{L}{2} \right) \cos \omega t + 2 \left( y - \frac{L}{2} \right) \sin \omega t - \eta_0 \right) - z(x, y) \\ u = -\eta_0 \omega \sin \omega t \\ v = \eta_0 \omega \cos \omega t \\ w = \mathbf{u} \cdot \nabla_{x,y} z(x, y) \end{cases}$$

with  $\omega = \sqrt{2gh_0}/a$ .

In order to evaluate the convergence rate of the simulated solution  $H_{sim}$  towards the analytical one  $H_{anal}$ , we plot the error rate versus the space discretization. Over the geometrical domain defined by  $(x, y) \in [-L/2, L/2]^2$ , with the parameter values  $L = 4$ ,  $h_0 = 1$  m,  $a = 1.5$  m and  $\eta_0 = 0.5$  m, and

for five unstructured meshes with 1273, 1104, 30441, 59473 and 98137 nodes, we have plotted (see Fig. 4) the  $\log(L^2 - error)$  over the water depth at time  $T = 2\pi/\omega$  seconds versus  $\log(h_0/h_i)$  for the first and second-order schemes and they are compared to the theoretical order. Notice that over Fig. 4, we

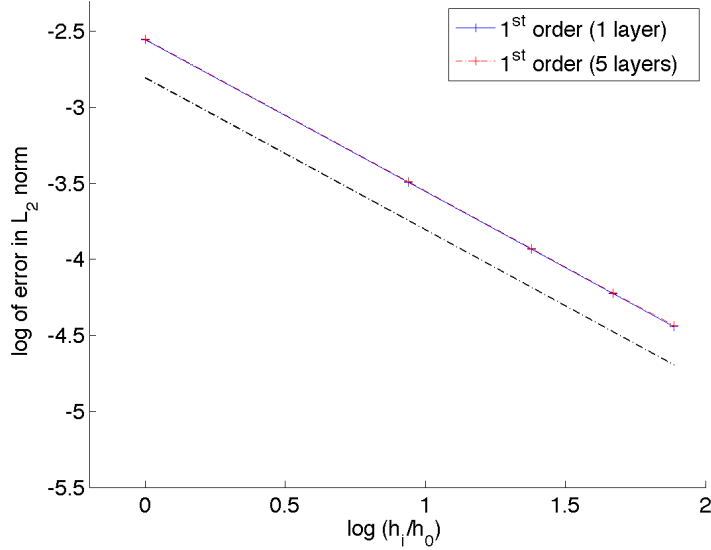


Figure 4: Parabolic bowl: error between the analytical solution and the simulated one with the five unstructured meshes, first order '+' (space and time) and second order extension (space and time) schemes.

have plotted the simulated solution obtained with 1 layer – that corresponds to the classical Saint-Venant system – and the simulated solution with 5 layers. The obtained convergence curves coincide proving the stability of the numerical scheme for the layers averaged system.

## 6.2 Stationary analytical solutions

We now confront our numerical model with stationary analytical solutions for the free surface Euler system proposed by some of the authors in [13].

### 6.2.1 Smooth solution

The analytical solution given in [13, Prop. 3.1] and defined by

$$z_b = \bar{z}_b - H_0 - \frac{\alpha^2 \beta^2}{2g \sin^2(\beta H_0)}, \quad (123)$$

$$u_{\alpha,\beta} = \frac{\alpha\beta}{\sin(\beta H_0)} \cos(\beta(z - z_b)), \quad (124)$$

$$v_{\alpha,\beta} = 0,$$

$$w_{\alpha,\beta} = \alpha\beta \left( \frac{\partial z_b}{\partial x} \frac{\cos(\beta(z - z_b))}{\sin(\beta H_0)} + \frac{\partial H_0}{\partial x} \frac{\sin(\beta(z - z_b)) \cos(\beta H_0)}{\sin^2(\beta H)} \right),$$

with  $\alpha = 1 \text{ m}^2 \cdot \text{s}^{-1}$ ,  $\beta = 1 \text{ m}^{-1}$ ,  $\bar{z}_b = cst$ ,  $x_{max} = 20 \text{ m}$  and

$$H_0(x, y) = \frac{1}{2} + \frac{3}{2} \frac{1}{1 + (x - \frac{1}{2}x_{max})^2} - \frac{1}{2} \frac{1}{2 + (x - \frac{2}{3}x_{max})^2}, \quad (125)$$

is a stationary regular analytical solution of the incompressible and hydrostatic Euler system with free surface (16)-(17),(4),(6) with  $p^a = 0$ .

In order to obtain the simulated solution, we consider the topography defined by (123),(125) and we impose the following boundary conditions

- solid wall for the two boundaries  $y = 0$  and  $y = 2$ ,
- given water depth  $H_0(x_{max}, y)$  at  $x = x_{max} = 20$ ,
- given flux defined by (124) at  $x = 0$ .

We have performed the simulations for several unstructured meshes of the geometrical domain  $(x, y) \in [0, x_{max}] \times [0, 2]$  and an adapted number of layers so that each 3d element of the mesh can be approximatively considered as a regular polyhedron (see remark 6.1)

- 81 nodes and 1 layer (81 elements),
- 275 nodes and 2 layers (550 elements),
- 1005 nodes and 4 layers (4020 elements),
- 3833 nodes and 8 layers (30664 elements),
- 14961 nodes and 17 layers (254337 elements),

- 36201 nodes and 26 layers (944226 elements).

Over Fig. 5, we have depicted the features of the analytical solution we use for the convergence test, it clearly appears over Fig. 5 that the velocity profile of chosen analytical solution varies along the  $z$  axis. Figure 6) gives the convergence curve towards the analytical solution i.e. the  $\log(L^2 - error)$  over the water depth – at time  $T = 300$  seconds when the stationary regime is reached – versus  $\log(h_0/h_i)$  for the first and second-order schemes and they are compared to the theoretical order.

**Remark 6.1** *In the case of a flat topography and a flat free surface, the 3d elements composing the mesh are triangular prisms. But in more complex situations, the lower and upper triangles are not necessarily parallel and the 3d elements of the mesh are a portion of a vertical cylinder with a triangular basis.*

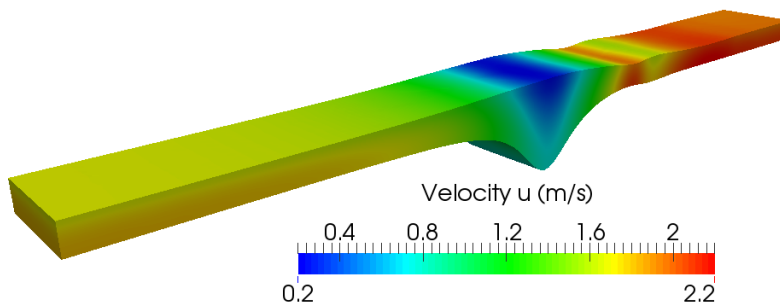


Figure 5: Surface level of the continuous analytical solution and horizontal velocity.

### 6.2.2 Analytical solution with a shock

We still consider the topography defined by (123) and (125) but now we assume the velocity profiles have the form

$$u_{\alpha,\beta} = \frac{\alpha\beta}{\sin(\beta H)} \cos(\beta(z - z_b)),$$

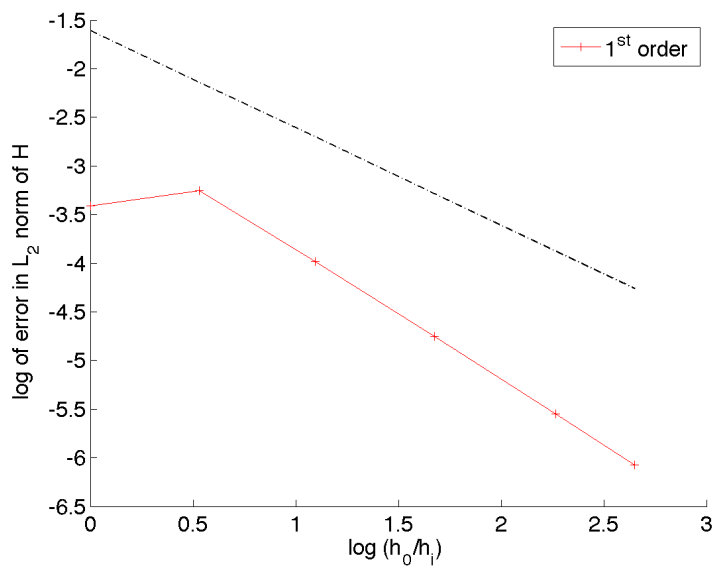


Figure 6: Stationary analytic continuous solution: error between the analytical solution and the simulated one with the six meshes, first order '+' (space and time) and second order extension (space and time) schemes.

$$w_{\alpha,\beta} = \alpha\beta \left( \frac{\partial z_b \cos(\beta(z - z_b))}{\partial x \sin(\beta H)} + \frac{\partial H \sin(\beta(z - z_b)) \cos(\beta H)}{\partial x \sin^2(\beta H)} \right),$$

where  $H$  and  $H_0$  differ. Following the results given in [13, paragraph 3.4], the quantities  $(H, u_{\alpha,\beta}, w_{\alpha,\beta})$  are solutions of the incompressible and hydrostatic Euler system with free surface (16)-(17),(4),(6) with  $p^a = 0$  and  $z_b$  defined by (123),(125) if  $H$  is solution of the ODE

$$\frac{\partial}{\partial x} \left( \frac{g}{2} H^2 + \frac{\alpha^2 \beta^2 H}{2 \sin(\beta H)^2} + \frac{\alpha^2 \beta \cos(\beta H)}{2 \sin(\beta H)} \right) = 0, \quad (126)$$

completed with the boundary condition  $H(x_{max}) = H_{lim}$ .

Depending on the choice of the parameters  $\alpha, \beta, H_{lim}$  the solution of Eq. (126) can be discontinuous with an entropic shock, see [13, paragraph 3.4]. The particular choice  $\alpha = 0.4 \text{ m}^2 \cdot \text{s}^{-1}$   $\beta = 1 \text{ m}^{-1}$  and  $H_{lim} = 0.5 \text{ m}$  actually leads to a discontinuous solution whose features are depicted over Fig. 7. Figure 8) gives the convergence curve towards the analytical solution i.e. the  $\log(L^2 - error)$  over the water depth – at time  $T = 300$  seconds when the stationary regime is reached – versus  $\log(h_0/h_i)$  for the first and second-order schemes and they are compared to the theoretical order.

**Remark 6.2** *Because of the amplitude of the velocity between the boundaries and the shock, the variations of the velocity profile along the  $z$  axis does not appear clearly over Fig. 7 but is similar to the one observed over Fig. 7.*

### 6.3 Simulation of a tsunami

Now we confront our discrete model to a tsunami propagation. The 2014 Iquique earthquake struck off the coast of Chile on 1 April, with a moment magnitude of 8.2, at 20:46 local time (23:46 UTC). The epicenter of the earthquake was approximately 95 kilometres (59 mi) northwest of Iquique, see Fig. 9.

We have carried out simulations of the tsunami induced by the earthquake using

- a topography obtained from the NOAA (National centers for environmental informations, <https://www.ngdc.noaa.gov/mgg/global/global.html> using the ETOPO1 data (1-arc minute global relief model),

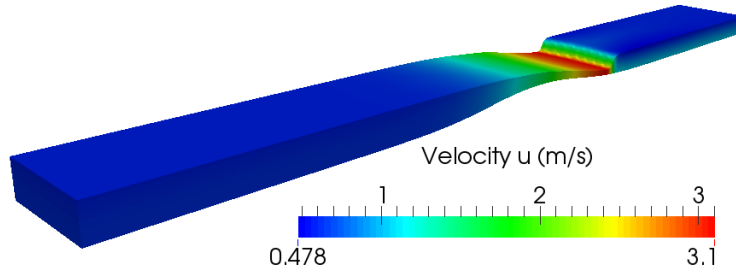


Figure 7: Surface level of the discontinuous analytical solution and horizontal velocity.

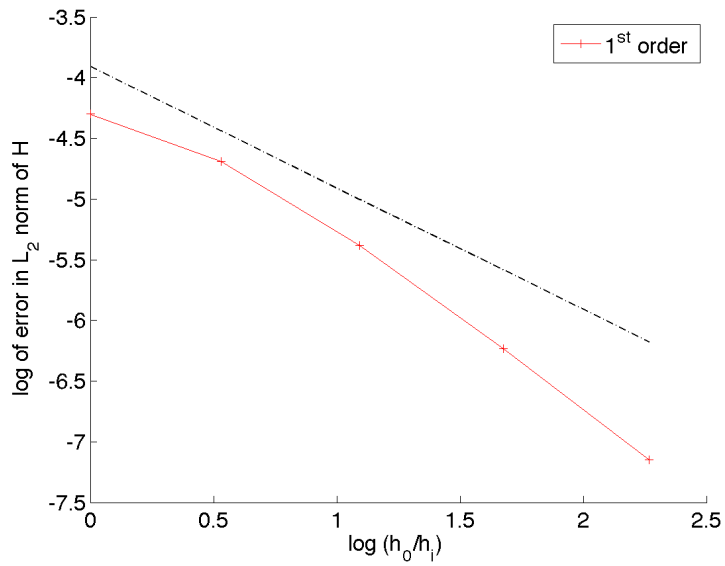


Figure 8: Stationary analytic solution with a shock: error between the analytical solution and the simulated one with the six meshes, first order '+' (space and time) and second order extension (space and time) schemes.

- an unstructured mesh whose dimensions – a square of 2224.2 km<sup>2</sup> – corresponds to the domain covered by Fig. 9,
- a source (displacement of the bottom due to the seism) obtained at IPGP by M. Vallée) TO COMPLETE

The simulation have been performed using several meshes and we present “converged” results in the sense that a finer mesh would give the same results. The results given over Fig. 10 have been obtained with a mesh containing 545821 nodes and 5 layers. We confront the numerical resolution – the first order scheme (space and time) and the second order scheme (space and time) – with the DART measurements.

For the initial conditions at time  $t_0$ , we start from an ocean at rest and the seism occur at time  $t_0$ . In the simulation, we do not consider the Coriolis force, the tides and the ocean currents.

The simulation has been carried out with a Mac book air 1.7 GHz Intel core i7 and takes 35 minutes.

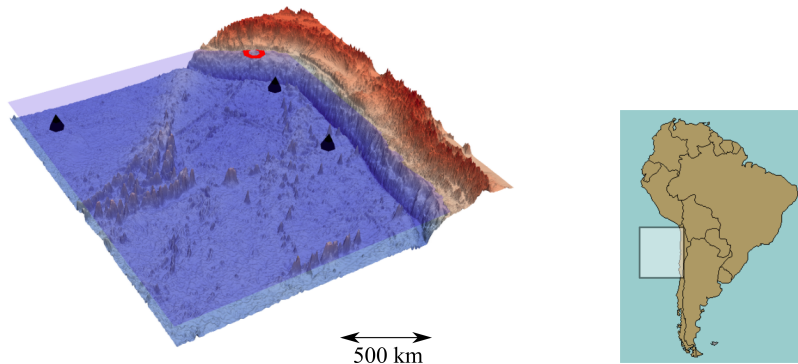


Figure 9: The chilean coast with the epicenter (red cone), the pacific ocean and the location of the two DART buoys (black boxes).

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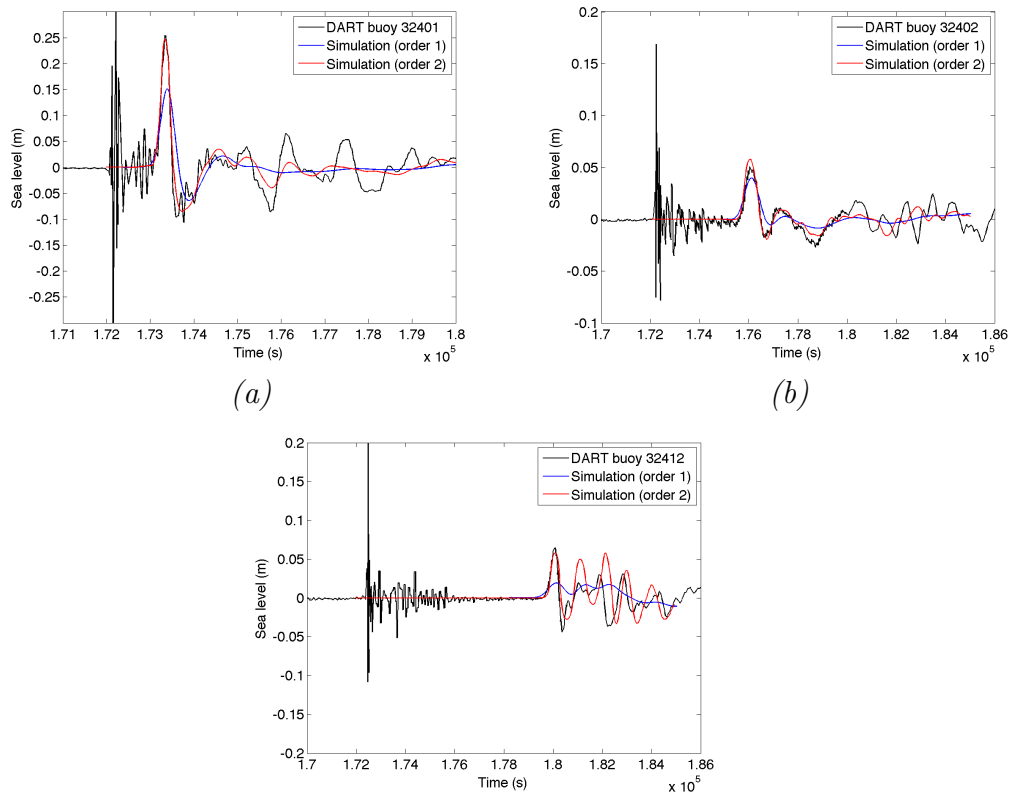


Figure 10: The Chilean coast with the epicenter (red disk) and the location of the three nearest DART buoys (black cones) - obtained for the moment with a coarse mesh.

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