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# Robust global synchronization of Brockett oscillators\*

Hafiz Ahmed<sup>1</sup>, Rosane Ushirobira<sup>2</sup>, and Denis Efimov<sup>2,3,4</sup>

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<sup>1</sup>Department of Electrical and Computer Engineering, Clemson University, Clemson, SC 29631, USA.

<sup>2</sup>Inria, Non-A team, 40 avenue Halley, 59650 Villeneuve d'Ascq, France.

<sup>3</sup>CRIStAL CNRS UMR 9189, Ecole Centrale de Lille, Avenue Paul Langevin, 59651 Villeneuve d'Ascq, France.

<sup>4</sup>Department of Control Systems and Informatics, University ITMO, 49 avenue Kronverkskiy 197101 Saint Petersburg, Russia.

## Abstract

In this technical report, motivated by a recent work of R. Brockett [1], a robust global synchronization problem of multistable Brockett oscillators has been studied within an Input-to-State Stability (ISS) framework. Following a recent generalization of the classical ISS theory to multistable systems and its application to the synchronization of multistable systems, two synchronization protocols are designed with respect to compact invariant sets of the unperturbed Brockett oscillator. The conditions obtained in our work are global and applicable for families of non-identical oscillators in contrast to the local analysis of [1]. Numerical simulation examples illustrate our theoretical results.

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# 1 Introduction

Over the past decade, considerable attention has been devoted to the problem of coordinated motion of multiple autonomous agents due to its broad applications in various areas. One critical issue arising from multi-agent systems is to develop distributed control policies based on local information that enables all agents to reach an agreement on certain quantities of interest, which is known as the consensus problem. A classic example of distributed coordination/consensus in physics, engineering and biology is the synchronization of arrays of coupled nonlinear oscillators [2–7]. Oscillator synchronization has several potential application domains, for instance power networks [8, 9], smooth operations of micro-grids [10], real-time distributed control in networked systems [11] and so on.

The problem of synchronization has been addressed by researchers from various technical fields like physics, biology, neuroscience, automatic control, *etc.* To have a better insight on the contribution of automatic control community in this area, interested readers may consult [12–18]. In the context of the synchronization of oscillators, R. Brockett has recently introduced the following model [1]:

$$\ddot{x} + \varepsilon \dot{x} (\dot{x}^2 + x^2 - 1) + x = \varepsilon^2 u, \quad x \in \mathbb{R}^n, \quad \varepsilon > 0. \quad (1)$$

In [1], a centralized synchronization protocol has been proposed for the model (1), such that the conventional averaging theory does not predict the existence of a periodic (almost periodic) solution for small  $\varepsilon$ . However, qualitative synchronization together with small amplitude irregular motion can be seen through numerical studies. Following [1], for  $\varepsilon$  sufficiently small, but non-zero, let us consider the set

$$S_\varepsilon = \{(x, \dot{x}) \in \mathbb{R}^2 : \dot{x}^2 + x^2 - 1 + 2\varepsilon^2 x \dot{x} \operatorname{sign}(\dot{x}^2 + x^2 - 1) = \varepsilon\},$$

which contains two smooth closed contours:  $\Gamma_\varepsilon^+$  lies outside the unit circle in the  $(x, \dot{x})$ -space and  $\Gamma_\varepsilon^-$  lies inside the unit circle. Both curves approach the unit circle as  $\varepsilon$  goes to zero. Then the main result of [1] is given below.

**Theorem 1.** *Let  $\Gamma_\varepsilon^\pm$  be as before. Then there exist  $\varepsilon_0 > 0$  such that for all  $0 < \varepsilon < \varepsilon_0$ , the solutions of (1) beginning in the annulus bounded by  $\Gamma_\varepsilon^+$  and  $\Gamma_\varepsilon^-$  remain in this annulus for all time, provided that  $|u| \leq \sqrt{x^2 + \dot{x}^2}$ .*

Theorem 1 provides a local synchronization result which depends on a small parameter  $\varepsilon \neq 0$ . Moreover, the result is applicable to the synchronization of identical oscillators only.

The goal of this work is to extend the result of [1] and to develop a protocol of *global* synchronization in the network of (1), for the case of identical and non-identical models of the agents. It is assumed that the oscillators are connected through a  $N$ -cycle graph<sup>1</sup> [19]. The proposed solution is based on the framework of ISS for multistable systems [20, 21].

<sup>1</sup>A cycle graph  $C_N$  is a graph on  $N$  nodes containing a single cycle through all nodes, or in other words,  $N$  number of vertices connected in a closed chain.

The ISS property provides a natural framework of stability analysis with respect to input perturbations (see [22] and references therein). The classical definition allows the stability properties with respect to arbitrary compact invariant sets (and not simply equilibria) to be formulated and characterized. Nevertheless, the implicit requirement is that these sets should be simultaneously Lyapunov stable and globally attractive, which makes the basic theory not applicable for a global analysis of many dynamical behaviors of interest having multistability [23–25], periodic oscillations [26], just to name a few, and only local analysis remains possible [27]. Some attempts were made to overcome such limitations by introducing the notions of almost global stability [28] and almost input-to-state stability [29], *etc.*

Recently, the authors in [20, 21] have found that a natural way of developing ISS theory for systems with multiple invariant sets consists in relaxing the Lyapunov stability requirement [30] (rather than the global nature of the attractivity property). Using this relatively mild condition, the ISS theory has been generalized in [20, 21], as well as the related literature on time invariant autonomous dynamical systems on compact spaces [31] for multistable systems. Multistability accounts for the possible coexistence of various oscillatory regimes or equilibria in the state space of the system for the same set of parameters. Any system that exhibits multistability is called a multistable system. Frequently, for a given set of initial conditions and inputs it is very difficult to predict the asymptotic regime that a multistable system will attain asymptotically [32]. Following the results of [20, 21], the authors in [33] have provided conditions for the robust synchronization of multistable systems in the presence of external inputs. Readers can consult [34] for an overview of recent developments in the ISS framework, dealing in particular with the extension of the classical concept to systems with multiple invariant sets and possibly evolving on Riemannian manifolds.

The results presented in [21] and [33] can be applied to provide sufficient conditions for the existence of robust synchronization for identical/non-identical Brockett oscillators in the presence of external inputs. In [35], a global synchronization protocol has been proposed for the case  $N = 2$ . In this work, this result is extended to the general case  $N > 2$  and to this end another synchronization control is proposed, which is not based on the theory of [33] and a special Lyapunov function is designed characterizing synchronization conditions for a family of non-identical Brockett oscillators. In opposite to the local results of [1], the conditions obtained in this work are global. The obtained synchronized system may demonstrate phase or anti-phase synchronization phenomena depending on parameters of the oscillators.

The rest of this technical report is organized as follows. Section 2 introduces some preliminaries about decomposable sets, notions of robustness and conditions of robust synchronization of multistable systems. More details about Brockett oscillator (such as the proof that it possesses ISS property) and the synchronization of a family of oscillators (the main results) can be found in Section 3 and 4, respectively. In Section 5, a numerical simulation example is given to illustrate these results. Concluding remarks in Section 6 close this technical

report.

## 2 Preliminaries

### 2.1 Preliminaries on input-to-stability of multistable systems

This section has been taken from [20, 33]. Let  $M$  be an  $n$ -dimensional  $\mathcal{C}^2$  connected and orientable Riemannian manifold without a boundary and  $x \in M$ . Let  $f : M \times \mathbb{R}^m \rightarrow T_x M$  be a map of class  $\mathcal{C}^1$ . Throughout this work, we assume that all manifolds are embedded in a Euclidean space of dimension  $n$ , so they contain 0. Consider a nonlinear system of the following form:

$$\dot{x}(t) = f(x(t), d(t)), \quad (2)$$

where the state  $x(t) \in M$  and  $d(t) \in \mathbb{R}^m$  (the input  $d(\cdot)$  is a locally essentially bounded and measurable signal) for  $t \geq 0$ . We denote by  $X(t, x; d(\cdot))$  the uniquely defined solution of (2) at time  $t$  satisfying  $X(0, x; d(\cdot)) = x$ . Together with (2), we will analyze its unperturbed version:

$$\dot{x}(t) = f(x(t), 0). \quad (3)$$

A set  $S \subset M$  is invariant for the unperturbed system (3) if  $X(t, x; 0) \in S$ , for all  $t \in \mathbb{R}$  and for all  $x \in S$ . For a set  $S \subset M$ , define the distance to  $S$  from a point  $x \in M$  by  $|x|_S = \inf_{a \in S} \delta(x, a)$  where the  $\delta(x_1, x_2)$  denotes the Riemannian distance between  $x_1$  and  $x_2$  in  $M$ . We have  $|x| = |x|_{\{0\}}$  for  $x \in M$ , the usual Euclidean norm of a vector  $x \in \mathbb{R}^n$ . For a signal  $d : \mathbb{R} \rightarrow \mathbb{R}^m$ , the essential supremum norm is defined as  $\|d\|_\infty = \text{ess sup}_{t \geq 0} |d(t)|$ .

A function  $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is said to belong to class  $\mathcal{K}$ , *i.e.*  $\alpha \in \mathcal{K}$ , if it is continuous, strictly increasing and  $\alpha(0) = 0$ . Furthermore,  $\alpha \in \mathcal{K}_\infty$  if  $\alpha \in \mathcal{K}$  and it is unbounded *i.e.*  $\lim_{s \rightarrow \infty} \alpha(s) = \infty$ . For any  $x \in M$ , the  $\alpha$ - and  $\omega$ -limit sets for (3) can be defined as follows:

$$\begin{aligned} \alpha(x) &:= \left\{ y \in M \mid y = \lim_{n \rightarrow -\infty} X(x, t_n) \text{ with } t_n \searrow -\infty \right\}, \\ \omega(x) &:= \left\{ y \in M \mid y = \lim_{n \rightarrow \infty} X(x, t_n) \text{ with } t_n \nearrow \infty \right\}. \end{aligned}$$

For an integer  $N \geq 1$  the symbol  $\overline{1, N}$  denotes the sequence  $1, \dots, N$ .

### 2.2 Decomposable sets

Let  $\Lambda \subset M$  be a compact invariant set for (3).

**Definition 2.** [31] A decomposition of  $\Lambda$  is a finite and disjoint family of compact invariant sets  $\Lambda_1, \dots, \Lambda_k$  such that  $\Lambda = \bigcup_{i=1}^k \Lambda_i$ .

For an invariant set  $\Lambda$ , its attracting and repulsing subsets are defined as follows:

$$\begin{aligned}\mathfrak{A}(\Lambda) &= \{x \in M \mid |X(t, x, 0)|_{\Lambda} \rightarrow 0 \text{ as } t \rightarrow +\infty\}, \\ \mathfrak{R}(\Lambda) &= \{x \in M \mid |X(t, x, 0)|_{\Lambda} \rightarrow 0 \text{ as } t \rightarrow -\infty\}.\end{aligned}$$

Define a relation on invariant sets in  $M$ : for  $\mathcal{W} \subset M$  and  $\mathcal{D} \subset M$ , we write  $\mathcal{W} \prec \mathcal{D}$  if  $\mathfrak{A}(\mathcal{W}) \cap \mathfrak{R}(\mathcal{D}) \neq \emptyset$ .

**Definition 3.** [31] Let  $\Lambda_1, \dots, \Lambda_k$  be a decomposition of  $\Lambda$ , then

1. An  $r$ -cycle ( $r \geq 2$ ) is an ordered  $r$ -tuple of distinct indices  $i_1, \dots, i_r$  such that  $\Lambda_{i_1} \prec \dots \prec \Lambda_{i_r} \prec \Lambda_{i_1}$ .
  - (a) A 1-cycle is an index  $i$  such that  $(\mathfrak{R}(\Lambda_i) \cap \mathfrak{A}(\Lambda_i)) \setminus \Lambda_i \neq \emptyset$ .
  - (b) A filtration ordering is a numbering of the  $\Lambda_i$  so that  $\Lambda_i \prec \Lambda_j \Rightarrow i \leq j$ .

As we can conclude from Definition 3, the existence of an  $r$ -cycle with  $r \geq 2$  is equivalent to the existence of a heteroclinic cycle for (3) [36]. Moreover, the existence of a 1-cycle implies the existence of a homoclinic cycle for (3) [36].

**Definition 4.** Let  $\mathcal{W} \subset M$  be a compact set containing all  $\alpha$ - and  $\omega$ -limit sets of (3). We say that  $\mathcal{W}$  is decomposable if it admits a finite decomposition without cycles,  $\mathcal{W} = \bigcup_{i=1}^k \mathcal{W}_i$ , for some non-empty disjoint compact sets  $\mathcal{W}_i$ , forming a filtration ordering of  $\mathcal{W}$ .

This definition of the compact set  $\mathcal{W}$  will be used all through the article.

### 2.3 Robustness notions

The following robustness notions for systems in (2) have been introduced in [21].

**Definition 5.** We say that the system (2) has the practical asymptotic gain (pAG) property if there exist  $\eta \in \mathcal{K}_{\infty}$  and  $q \in \mathbb{R}$ ,  $q \geq 0$ , such that for all  $x \in M$  and all measurable essentially bounded inputs  $d(\cdot)$ , the solutions are defined for all  $t \geq 0$  and

$$\limsup_{t \rightarrow +\infty} |X(t, x; d)|_{\mathcal{W}} \leq \eta(\|d\|_{\infty}) + q. \quad (4)$$

If  $q = 0$ , then we say that the asymptotic gain (AG) property holds.

**Definition 6.** We say that the system (2) has the limit property (LIM) with respect to  $\mathcal{W}$  if there exists  $\mu \in \mathcal{K}_{\infty}$  such that for all  $x \in M$  and all measurable essentially bounded inputs  $d(\cdot)$ , the solutions are defined for all  $t \geq 0$  and the following holds:

$$\inf_{t \geq 0} |X(t, x; d)|_{\mathcal{W}} \leq \mu(\|d\|_{\infty}).$$

**Definition 7.** We say that the system (2) has the practical global stability (pGS) property with respect to  $\mathcal{W}$  if there exist  $\beta \in \mathcal{K}_\infty$  and  $q \geq 0$  such that for all  $x \in M$  and all measurable essentially bounded inputs  $d(\cdot)$ , the following holds for all  $t \geq 0$ :

$$|X(t, x; d)|_{\mathcal{W}} \leq q + \beta(\max\{|x|_{\mathcal{W}}, \|d\|_\infty\}).$$

To characterize (4) in terms of Lyapunov functions, it has been shown in [21] that the following notion is suitable:

**Definition 8.** We say that a  $\mathcal{C}^1$  function  $V : M \rightarrow \mathbb{R}$  is a practical ISS-Lyapunov function for (2) if there exists  $\mathcal{K}_\infty$  functions  $\alpha_1, \alpha_2, \alpha$  and  $\gamma$ , and scalars  $q, c \geq 0$  such that

$$\alpha_1(|x|_{\mathcal{W}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{W}} + c),$$

the function  $V$  is constant on each  $\mathcal{W}_i$  and the dissipation inequality below holds:

$$DV(x)f(x, d) \leq -\alpha(|x|_{\mathcal{W}}) + \gamma(|d|) + q.$$

If this latter holds for  $q = 0$ , then  $V$  is said to be an ISS-Lyapunov function.

Notice that the existence of  $\alpha_2$  and  $c$  follows (without any additional assumptions) by standard continuity arguments.

The main result of [21] connecting these robust stability properties is stated below:

**Theorem 9.** *Consider a nonlinear system as in (2) and let a compact invariant set  $\mathcal{W}$  containing all  $\alpha$ - and  $\omega$ -limit sets of (3) be decomposable (in the sense of Definition 4). Then the following are equivalent:*

1. The system admits an ISS Lyapunov function;
  - (a) The system enjoys the AG property;
  - (b) The system admits a practical ISS Lyapunov function;
  - (c) The system enjoys the pAG property;
  - (d) The system enjoys the LIM property and the pGS.

A system in (2) that satisfies this list of equivalent properties is called ISS with respect to the set  $\mathcal{W}$  [21].

## 2.4 Robust synchronization of multistable systems

This section summarizes the result on robust synchronization of multistable systems obtained in [33]. The following family of nonlinear systems is considered in this section:

$$\dot{x}_i(t) = f_i(x_i(t), u_i(t), d_i(t)), \quad i = \overline{1, N}, \quad N > 1, \quad (5)$$

where the state  $x_i(t) \in M_i$ , with  $M_i$  an  $n_i$ -dimensional  $\mathcal{C}^2$  connected and orientable Riemannian manifold without a boundary, the control  $u_i(t) \in \mathbb{R}^{m_i}$  and the external disturbance  $d_i(t) \in \mathbb{R}^{p_i}$  ( $u_i(\cdot)$  and  $d_i(\cdot)$  are locally essentially bounded and measurable signals) for  $t \geq 0$  and the map  $f_i : M_i \times \mathbb{R}^{m_i} \times \mathbb{R}^{p_i} \rightarrow T_{x_i}M_i$  is  $\mathcal{C}^1$ ,  $f_i(0, 0, 0) = 0$ . Denote the common state vector of (5) as  $x = [x_1^T, \dots, x_N^T]^T \in M = \prod_{i=1}^N M_i$ , so  $M$  is the corresponding Riemannian manifold of dimension  $n = \sum_{i=1}^N n_i$  where the family (5) evolves and  $d = [d_1^T, \dots, d_N^T]^T \in \mathbb{R}^p$  with  $p = \sum_{i=1}^N p_i$  is the exogenous input.

**Assumption 1.** For all  $i = \overline{1, N}$ , each system in (5) has a compact invariant set  $\mathcal{W}_i$  containing all  $\alpha$ - and  $\omega$ -limit sets of  $\dot{x}_i(t) = f_i(x_i(t), 0, 0)$ ,  $\mathcal{W}_i$  is decomposable in the sense of Definition 4, and the system is ISS with respect to the set  $\mathcal{W}_i$  and inputs  $u_i$  and  $d_i$ .

This assumption implies that family (5) is composed of robustly stable nonlinear systems.

Let a  $\mathcal{C}^1$  function  $\psi(x) : M \rightarrow \mathbb{R}^q$ ,  $\psi(0) = 0$  be a synchronization measure for (5). We say that the family (5) is synchronized (or reached the consensus) if  $\psi(x(t)) \equiv 0$  for all  $t \geq 0$  on the solutions of the network under properly designed control actions

$$u_i(t) = \varphi_i(\psi(x(t))) \quad (6)$$

( $\varphi_i : \mathbb{R}^q \rightarrow \mathbb{R}^{m_i}$  is a  $\mathcal{C}^1$  function,  $\varphi_i(0) = 0$ ) for  $d(t) \equiv 0$ ,  $t \geq 0$ . In this case, the set  $\mathcal{A} = \{x \in \mathcal{W} \mid \psi(x) = 0\}$  contains the synchronous solutions of the unperturbed family in (5) and the problem of synchronization of “natural” trajectories is considered since  $\mathcal{A} \subset \mathcal{W}$  (due to  $\varphi_i(0) = 0$  in (6), the convergence of  $\psi$  (synchronization/consensus) implies that the solutions belong to  $\mathcal{W}$ ).

By selecting the shapes of  $\varphi_i$ , it is possible to guarantee robust synchronization of (5) for any measurable and essentially bounded input  $d$ .

**Proposition 10.** Let Assumption 1 be satisfied for (5). Then there exist  $\varphi_i$ ,  $i = \overline{1, N}$  in (6) such that the interconnection (5), (6) has pGS property with respect to the set  $\mathcal{W}$ .

A simple and effective choice of any bounded functions  $\varphi_i$ ,  $i = \overline{1, N}$  in (6) makes the result of Proposition 10 to be valid.

**Assumption 2.** The set  $\mathcal{A}$  is compact, it contains all  $\alpha$ - and  $\omega$ -limit sets of (5), (6) for  $d = 0$ , and it is decomposable.

Therefore, it is assumed that the controls  $\varphi_i(\psi)$  ensure the network global synchronization, while decomposability in general follows from Assumption 1.

**Theorem 11.** Let conditions of Proposition 10 be satisfied together with Assumption 2, then the interconnection (5), (6) is ISS with respect to  $\mathcal{A}$ .



### 3 The Brockett Oscillator

Let us consider the Brockett oscillator [1]:

$$\ddot{\xi} + b\dot{\xi}(\xi^2 + \xi^2 - 1) + \xi = au, \quad (7)$$

where  $\xi \in \mathbb{R}$ ,  $\dot{\xi} \in \mathbb{R}$  are the states variables,  $a, b > 0$  are parameters and  $u$  is the control input. By considering  $x_1 = \xi$ ,  $\dot{x}_1 = x_2 = \dot{\xi}$ ,  $x = [x_1, x_2]^T$  and  $|x| = \sqrt{x_1^2 + x_2^2}$  equation (7) can be written in the state-space form as:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + au - bx_2(|x|^2 - 1), \end{aligned} \quad (8)$$

where the states of the system (8), *i.e.*  $x$ , evolve in the manifold  $M = \mathbb{R}^2$ . By analyzing equation (8) it can be seen that the unperturbed system admits two invariant sets: namely, the origin  $\mathcal{W}_1 = \{0\}$  and the limit cycle  $\mathcal{W}_2 = \Gamma = \{x \in M : |x|^2 = 1\}$ . So, the invariant set for the trajectories of (8) can be defined as:

$$\mathcal{W} := \mathcal{W}_1 \cup \mathcal{W}_2 = \{0\} \cup \Gamma. \quad (9)$$

In order to verify the decomposability of the invariant set  $\mathcal{W}$ , we need to know the nature of the equilibrium  $\mathcal{W}_1$  and the limit cycle  $\mathcal{W}_2 = \Gamma$ . This information can be obtained by analyzing the Lyapunov stability of the unperturbed system (8).

#### 3.1 Stability of the autonomous Brockett oscillator

Since,  $\mathcal{W}$  is invariant for the trajectories of (9), then the following proposition provides the stability of the unforced Brockett oscillator with respect to  $\mathcal{W}$ .

**Proposition 12.** *For the unperturbed Brockett oscillator defined in (8) with  $u = 0$ , the limit cycle  $\Gamma$  is almost globally asymptotically stable and the origin is unstable.*

*Proof.* The instability of the origin of the unperturbed system (8) can be verified for a linearized version of the system. The eigenvalues of the linearized system  $\lambda_{1,2} = \frac{1}{2}(b \pm \sqrt{b^2 - 4})$  have always positive real parts for any  $b > 0$ . Alternatively, this fact can also be checked through LMI formulation which is given in Remark 13.

To analyze the stability of the limit cycle  $\mathcal{W}_2$ , let us consider the following Lyapunov function:

$$U(x) = \frac{1}{2}(|x|^2 - 1)^2,$$

which is zero on the set  $\mathcal{W}_2$  and positive otherwise. Evaluating the total derivative of  $U$  along the solutions of (8), we obtain:

$$\begin{aligned}\dot{U} &= (|x|^2 - 1) \{2aux_2 - 2bx_2^2 (|x|^2 - 1)\} \\ &= -2bx_2^2 (|x|^2 - 1)^2 + 2aux_2 (|x|^2 - 1) \\ &\leq -2bx_2^2 (|x|^2 - 1)^2 + bx_2^2 (|x|^2 - 1)^2 + \frac{a^2}{b}u^2 \\ &\leq -bx_2^2 (|x|^2 - 1)^2 + \frac{a^2}{b}u^2.\end{aligned}$$

Then for  $u = 0$  we have  $\dot{U} \leq 0$  and all trajectories are globally bounded. By LaSalle's invariance principle [37], all trajectories of the system converge to the set where  $\dot{U} = 0$ . Note that  $\{x \in M : \dot{U} = 0\} = \mathcal{W}_2 \cup \{x \in M : x_2 = 0\}$  and on the line  $x_2$  there is the only invariant solution at the origin (in  $\mathcal{W}_1$ ), therefore  $\dot{U} = 0$  for all  $x \in \mathcal{W}$ , which contains all invariant solutions of the system. Since the origin is unstable, it can be concluded that the limit cycle  $\mathcal{W}_2$  is almost globally asymptotically stable.  $\square$

*Remark 13.* To check the instability of the origin in an alternative way, let us consider a small closed ball  $B(\rho)$  with the radius  $\rho$  around the origin with  $B(\rho) = \{x \in \mathbb{R}^2 : |x|^2 \leq \rho\}$ . Inside this ball, by imposing the parameter  $b = 1$  without losing generality, the unperturbed system of (8) can be written as the following uncertain linear system:

$$\dot{x} = Ax, A = \begin{bmatrix} 0 & 1 \\ -1 & -(\tilde{\rho} - 1) \end{bmatrix}, \tilde{\rho} \in [0, \rho], \quad (10)$$

where the matrix  $A \in \mathbb{R}^{2 \times 2}$  belongs to the domain  $\mathcal{D}_A$  defined as:

$$\mathcal{D}_A \triangleq \left\{ A : A = \beta_1 A_1 + \beta_2 A_2, \beta_1, \beta_2 > 0, \sum_{i=1}^2 \beta_i = 1 \right\} \quad (11)$$

with  $A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$  and  $A_2 = \begin{bmatrix} 0 & 1 \\ -1 & -(\rho - 1) \end{bmatrix}$ . Then, by applying Chetaev instability theorem [38], it can be concluded that the origin is unstable if there exist  $P > 0, Q > 0$  such that for  $i = 1, 2$

$$A_i^T P + P A_i \succeq Q. \quad (12)$$

The LMI (12) can be easily verified by using any standard solvers like Yalmip [39].

In our case, let us select  $\rho = 0.2$ . With this value of  $\rho$ , we obtain the following values for  $P$  and  $Q$ ,

$$P = \begin{bmatrix} 21.4643 & -6.8278 \\ \star & 17.8390 \end{bmatrix}, Q = \begin{bmatrix} 6.1040 & -1.3080 \\ \star & 7.8838 \end{bmatrix}.$$

With these values of  $P$  and  $Q$ , LMI (12) is satisfied. As a result, it can be concluded that the origin is unstable.

### 3.2 Stability of the non-autonomous Brockett oscillator

In the previous section, we have proved the stability of the unperturbed system. In this section, we will analyze the stability of the Brockett oscillator in the presence of input. As it was shown in the previous section, the limit cycle  $\Gamma$  is almost globally asymptotically stable. So, any solution of the unperturbed Brockett oscillator converges to  $\Gamma$ , except for the one initiated at 0, which is unstable. So, it can be concluded that  $\mathcal{W}$  contains all  $\alpha$ - and  $\omega$ -limit sets of the unperturbed systems of (8) and it admits a decomposition without cycles. Consequently the result of [20,21] can be applied for our case to show the robust stability of the Brockett oscillator in (8) with respect to  $\mathcal{W}$ :

**Proposition 14.** *The system (8) is ISS with respect to the set  $\mathcal{W}$ .*

*Proof.* To prove the ISS property, let us introduce two new variables  $y$  and  $h$  as,

$$\begin{aligned} y(x) &= |x|^2 - 1, \quad \dot{y} = -2bx_2^2y + 2ax_2u, \\ h(x) &= (x_1 + x_2)y, \quad \dot{h} = a[y + 2x_2(x_1 + x_2)]u \\ &\quad - (h - 2x_2y + bx_2y^2 + 2bx_2^2h). \end{aligned}$$

Next, let us consider the following Lyapunov function for the unperturbed system of (8) for  $c, d > 0$ :

$$W(x) = \frac{1}{2} \left( h^2(x) + cy^2(x) + \frac{1}{2}dy^4(x) \right). \quad (13)$$

Notice that  $W(x) = 0$  for all  $x \in \mathcal{W}_2$  and positive otherwise. Therefore, there exist  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$  such that the first condition of Definition 8 is satisfied for all  $x \in M$  for the above function  $W(x)$ . Evaluating the total derivative of  $W$ , along the solutions of (8), we obtain

$$\begin{aligned} \dot{W} &= cy\dot{y} + h\dot{h} + dy^3\dot{y} \\ &= 2au[h(x_2^2 + x_1x_2 + 0.5y) + x_2y(c + dy^2)] \\ &\quad - h^2 - 2bx_2^2[h^2 + y^2(c + dy^2)] + hx_2y(2 - by). \end{aligned}$$

Next by applying Young's inequality, we can derive the series of relations:

$$\begin{aligned} 2hx_2y &\leq \frac{1}{2}h^2 + 2x_2^2y^2, \quad hx_2y^2 \leq \frac{h^2}{4b} + bx_2^2y^4, \\ x_2^2hu &\leq x_2^2 \left( \frac{b}{4a}h^2 + \frac{a}{b}u^2 \right), \\ hux_1x_2 &\leq \frac{b}{4a}h^2x_2^2 + \frac{a}{b}x_1^2u^2, \quad huy \leq \frac{h^2}{16a} + 4ay^2u^2, \\ (c + dy^2)ux_2y &\leq (c + dy^2) \left( \frac{b}{2a}x_2^2y^2 + \frac{a}{2b}u^2 \right). \end{aligned}$$

By substituting these inequalities for  $c = \frac{3}{b}$ ,  $d = 2b$  and after simplification, we obtain

$$\begin{aligned} \dot{W} &\leq -h^2 \left( \frac{1}{8} + bx_2^2 \right) - x_2^2 y^2 (1 + b^2 y^2) \\ &\quad + \frac{a^2}{b^2} [2b|x|^2 + 10b^2 y^2 + 3] u^2. \end{aligned}$$

From the properties of the functions  $h$  and  $y$  we can substantiate that  $W$  is a practical ISS Lyapunov function for (8). Consequently, using Theorem 9 it can be concluded that the system (8) is ISS with respect to the set  $\mathcal{W}$  from the input  $u$ .  $\square$

*Remark 15.* It is straightforward to check that there exists a function  $\alpha \in \mathcal{K}_\infty$  such that for all  $x \in M$  and  $u = 0$  we have  $\dot{W} \leq -\alpha(|x|_{\mathcal{W}})$ . Thus  $W$  is a global Lyapunov function establishing multistability of (8) with respect to  $\mathcal{W}$  for  $u = 0$ .

## 4 Synchronization of Brockett oscillators

The following family of Brockett oscillators is considered in this section for some  $N > 1$ :

$$\begin{aligned} \dot{x}_{1i} &= x_{2i}, \\ \dot{x}_{2i} &= a_i u_i - x_{1i} - b_i x_{2i} (|x_i|^2 - 1), \quad i = \overline{1, N}, \end{aligned} \quad (14)$$

where  $a_i, b_i > 0$  are the parameters of an individual oscillator, the state  $x_i = [x_{1i} \ x_{2i}]^T \in M_i = \mathbb{R}^2$ , the control  $u_i \in \mathbb{R}$  ( $u_i : \mathbb{R}_+ \rightarrow \mathbb{R}$  is locally essentially bounded and measurable signal). Denote the common state vector of (5) as  $x = [x_1^T, \dots, x_N^T]^T \in M = \prod_{i=1}^N M_i$ , so  $M$  is the corresponding Riemannian manifold of dimension  $n = 2N$  where the family (5) behaves and  $u = [u_1, \dots, u_N]^T \in \mathbb{R}^N$  is the common input. Through propositions 12 and 14, it has been shown that each member of family (14) is robustly stable with respect to the set  $\mathcal{W}_i = \{x_i \in M_i : |x_i|^2 = 1\} \cup \{0\}$ . Consequently, the family (14) is a robustly stable nonlinear system. As a result, Assumption 1 is satisfied for the case of the family of Brockett oscillators (14).

The synchronization problem is then to find a protocol  $u$  that makes the family (14) synchronized. There are several works devoted to synchronization and design of consensus protocols for such a family or oscillatory network [40–42].

### 4.1 Problem statement

Let a  $\mathcal{C}^1$  function  $\psi : M \rightarrow \mathbb{R}^q$ ,  $\psi(0) = 0$  be a synchronization measure for (14). We say that the family (14) is synchronized (or reached the consensus) if

$\psi(x(t)) \equiv 0$  for all  $t \geq 0$  on the solutions of the network under properly designed control actions

$$u_i(t) = \varphi_i[\psi(x(t))], \quad (15)$$

where  $\varphi_i : \mathbb{R}^q \rightarrow \mathbb{R}$  is a  $\mathcal{C}^1$  function,  $\varphi_i(0) = 0$ . Due to the condition  $\varphi_i(0) = 0$ , the convergence of  $\psi$  (synchronization/consensus) implies that the solutions of the interconnection belong to  $\mathcal{W} = \prod_{i=1}^N \mathcal{W}_i$ . In this case the set  $\mathcal{A} = \{x \in \mathcal{W} \mid \psi(x) = 0\}$  contains the synchronous solutions of the family in (15) and the problem of synchronization of “natural” trajectories is considered since  $\mathcal{A} \subset \mathcal{W}$ .

In this work we deal with the following synchronization measure:

$$\begin{aligned} \psi &= [\psi_1, \dots, \psi_N]^T, \\ \psi_i &= \begin{cases} (x_{2(i+1)} - x_{2i}), & i = \overline{1, N-1} \\ x_{21} - x_{2N}, & i = N \end{cases}. \end{aligned}$$

From a graph theory point of view, the oscillators are connected through a  $N$ -cycle graph [19] (each oscillator needs only the information of its next neighbor), *i.e.*

$$\psi = M \begin{bmatrix} x_{21} \\ \vdots \\ x_{2N} \end{bmatrix}, \quad M = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \ddots & \\ & & \ddots & \ddots & \ddots \\ & & & & 1 \\ 1 & & & & -1 \end{bmatrix},$$

and any other connection type can be studied similarly. Moreover, the interconnection matrix  $M$  has Metzler form since all off-diagonal elements are positive. Next, let us define the synchronization error among the various states of the oscillators as follows for  $i = \overline{1, N-1}$ :

$$e_{2i-1} = x_{1i} - x_{1(i+1)}, \quad \dot{e}_{2i-1} = x_{2i} - x_{2(i+1)} = e_{2i}$$

and  $e_{2N-1} = x_{1N} - x_{11}$ ,  $\dot{e}_{2N-1} = x_{2N} - x_{21} = e_{2N}$ . Thus,

$$\begin{aligned} \psi_i &= -e_{2i} \quad i = \overline{1, N}, \\ \psi_N &= \sum_{i=1}^{N-1} e_{2i} \end{aligned}$$

and the quantity  $e = [e_1, e_2, \dots, e_{2N}] = 0$  implies that  $\psi = 0$  (the synchronization state is reached). For  $y_i = |x_i|^2 - 1$  the error dynamics can be written in the form:

$$\begin{aligned} \dot{e}_{2i-1} &= e_{2i}, \quad i = \overline{1, N}, \\ \dot{e}_{2i} &= -e_{2i-1} + a_i u_i - a_{i+1} u_{i+1} - b_i x_{2i} y_i \\ &\quad + b_{i+1} x_{2(i+1)} y_{i+1}, \quad i = \overline{1, N-1}, \\ \dot{e}_{2N} &= -e_{2N-1} + a_N u_N - a_1 u_1 - b_N x_{2N} y_N \\ &\quad + b_1 x_{21} y_1. \end{aligned} \quad (16)$$

Since  $e_{2N-j} = \sum_{i=1}^{N-1} e_{2i-j}$  for  $j = 0, 1$ , then formally only  $N - 1$  errors can be considered in (16).

In order to design the controls we will consider in this work the following Lyapunov function

$$V(x) = \sum_{i=1}^N \frac{\alpha_i}{4} y_i^2 + \frac{1}{2} \sum_{i=1}^{2N} e_i^2, \quad (17)$$

where  $\alpha_i \geq 0$  are weighting parameters. Notice that  $V(x) = 0$  for all  $x \in \mathcal{A} \cap \prod_{i=1}^N \mathcal{W}_{2i}$  and positive otherwise. Such a choice of Lyapunov function is very natural for our goal since it has two items: the former one characterizes stability of each oscillator, while the latter item evaluates synchronicity of the network.

## 4.2 Preliminary results

In [35] for  $N = 2$  and

$$u = k\psi, \quad k > 0, \quad (18)$$

e.g.  $\varphi(\psi) = k\psi$  in (15), the following result has been proven using  $V(x)$ :

**Theorem 16.** [35] *The family of Brockett oscillators (14) with  $N = 2$  is synchronized by (18), i.e. the system is globally asymptotically stable with respect to the set  $\mathcal{A}$ .*

It has been observed in numerical experiments that for  $N > 2$  and

$$u = k \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \ddots & \\ & \ddots & \ddots & \ddots & \\ & & & & 1 \\ 1 & & & & -1 \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \\ \vdots \\ x_{2(N-1)} \\ x_{2N} \end{bmatrix} \quad (19)$$

the synchronization persists, but the proof cannot be extended to the case  $N > 2$  since (17) is not a Lyapunov function in such a case.

*Remark 17.* To overcome this problem, based on the idea presented in [43], the following modification to the control law (19) can be proposed:

$$u_i = k\psi_i + b_i x_{2i} y_i. \quad (20)$$

Since the modified control law (20) compensates the nonlinear part of (14), as a result the closed loop system becomes linear. In this case, it is trivial to show that the closed loop system (14) and (20) is globally asymptotically synchronized.

Theorem 16 guarantees global asymptotic stability of the synchronized behavior, but not the robustness. Note that the controls (19) and (20) are not

bounded, then it is impossible to apply the result of Proposition 14 to prove robust stability of  $\mathcal{W}$ . Moreover, in many application areas, the control is bounded due to actuator limitations [44]. With such a motivation, take a bounded version of (15), then from propositions 10 and 14 convergence of all trajectories in a vicinity of  $\mathcal{W}$  immediately follows. If (15) is sufficiently bounded then any accuracy of approaching  $\mathcal{W}$  can be guaranteed, and the next result summarizes the conditions of synchronization:

**Corollary 18.** *Let the set  $\mathcal{A}$  be compact, it contains all  $\alpha$ - and  $\omega$ -limit sets of (14), (15), and it is decomposable for given bounded  $\varphi_i, i = \overline{1, N}$ , then the interconnection (14), (15) is synchronized, i.e. the system is globally asymptotically stable with respect to the set  $\mathcal{A}$ .*

*Proof.* In the conditions of the corollary Assumption 2 is satisfied for (14), (15). The proof follows from the result of Theorem 11 since Assumption 1 is satisfied due to Proposition 14.  $\square$

If we assume that (15) contains an additional perturbation  $d \in \mathbb{R}^N$ :

$$u_i(t) = \varphi_i [\psi(x(t)) + d_i(t)], \quad i = \overline{1, N},$$

which models the connection errors and coupling imperfections, then ISS property with respect to the set  $\mathcal{A}$  can be proven in the conditions of Corollary 18 (the result of Theorem 11).

### 4.3 Global synchronization control

Consider a variant of synchronization control in the following form:

$$u = k \begin{bmatrix} -2 & 1 & 0 & \cdots & 1 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \cdots & 0 \\ \vdots & & \vdots & \ddots & 0 \\ 1 & \cdots & 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \\ \vdots \\ x_{2(N-1)} \\ x_{2N} \end{bmatrix} \quad (21)$$

$$= kM \begin{bmatrix} e_2 \\ e_4 \\ \vdots \\ e_{2N-2} \\ e_{2N} \end{bmatrix},$$

where  $k > 0$  is the coupling strength and

$$M = \begin{bmatrix} -1 & 0 & 0 & \cdots & 1 \\ 1 & -1 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ \vdots & & \vdots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & -1 \end{bmatrix}.$$

Obviously, the control (21) can be rewritten as (15):

$$u = -kM\psi.$$

With such a control each  $i^{\text{th}}$  oscillator is connected with two its neighbors,  $(i - 1)^{\text{th}}$  and  $(i + 1)^{\text{th}}$  oscillators, and the closed loop network (14), (21) is organized again in the form of  $N$ -cycle graph [19]. Note that for  $N = 2$  the control (21) takes the form of (18).

Let us calculate the derivative of the Lyapunov function  $V(x)$  for (14), (21) (for indexes we will use convention in the calculations below that  $N + 1 = 1$ ):

$$\begin{aligned} \dot{V} &= \sum_{i=1}^N [\alpha_i (-b_i x_{2i}^2 y_i^2 + a_i x_{2i} y_i u_i) \\ &\quad + e_{2i} (a_i u_i - a_{i+1} u_{i+1} - b_i x_{2i} y_i + b_{i+1} x_{2(i+1)} y_{i+1})] \\ &= \sum_{i=1}^N [a_i (\alpha_i x_{2i} y_i + e_{2i} - e_{2i-2}) u_i \\ &\quad + b_i (e_{2i-2} - e_{2i}) x_{2i} y_i - \alpha_i b_i x_{2i}^2 y_i^2] \\ &= \sum_{i=1}^N [a_i (\alpha_i x_{2i} y_i + e_{2i} - e_{2i-2}) k (e_{2i-2} - e_{2i}) \\ &\quad + b_i (e_{2i-2} - e_{2i}) x_{2i} y_i - \alpha_i b_i x_{2i}^2 y_i^2] \\ &= \sum_{i=1}^N [\{a_i \alpha_i k + b_i\} (e_{2i-2} - e_{2i}) x_{2i} y_i \\ &\quad - k a_i (e_{2i-2} - e_{2i})^2 - \alpha_i b_i x_{2i}^2 y_i^2]. \end{aligned}$$

Select  $\alpha_i = \frac{b_i}{k a_i}$ , then

$$\begin{aligned} \dot{V} &= \sum_{i=1}^N b_i [2(e_{2i-2} - e_{2i}) x_{2i} y_i \\ &\quad - \alpha_i^{-1} (e_{2i-2} - e_{2i})^2 - \alpha_i x_{2i}^2 y_i^2] \\ &= - \sum_{i=1}^N b_i [\alpha_i^{-0.5} (e_{2i-2} - e_{2i}) - \alpha_i^{0.5} x_{2i} y_i]^2 \\ &\leq 0 \end{aligned}$$

Since  $V$  is positive definite with respect to the set  $x \in \mathcal{A} \cap \prod_{i=1}^N \mathcal{W}_{2i}$ , which is compact, then all trajectories in the system are globally bounded. By LaSalle's invariance principle all trajectories of the system converge to the largest invariant set in

$$\begin{aligned} \Omega &= \{x \in M : \dot{V}(x) = 0\} \\ &= \{x \in M : e_{2i-2} - e_{2i} = \alpha_i x_{2i} y_i, i = \overline{1, N}\}. \end{aligned}$$



Note that  $u_i = k(e_{2i-2} - e_{2i}) = k\alpha_i x_{2i} y_i = \frac{b_i}{a_i} x_{2i} y_i$  in the set  $\Omega$ , then on that set the control performs compensation of nonlinearity as (20) and asymptotically the dynamics of synchronization errors take the form for  $i = \overline{1, N}$ :

$$\begin{aligned}\dot{e}_{2i-1} &= e_{2i}, \\ \dot{e}_{2i} &= -e_{2i-1} + a_i u_i - a_{i+1} u_{i+1} - b_i x_{2i} y_i \\ &\quad + b_{i+1} x_{2(i+1)} y_{i+1} \\ &= -e_{2i-1}\end{aligned}$$

and

$$\begin{aligned}\dot{y}_i &= -2b_i x_{2i}^2 y_i + 2a_i x_{2i} u_i = 0, \\ \dot{x}_{1i} &= x_{2i}, \\ \dot{x}_{2i} &= -x_{1i},\end{aligned}$$

*i.e.* the norms  $|x_i|$  and  $|(e_{2i-1}, e_{2i})|$  become constant on  $\Omega$ . Therefore, the following result has been proven:

**Proposition 19.** *For any  $k > 0$  in the system (14), (21) all trajectories are bounded and converge to the largest invariant set in*

$$\begin{aligned}\Omega_\infty &= \{x \in M : |x_i| = \text{const}, e_{2i-1}^2 + e_{2i}^2 = \text{const}, \\ &\quad x_{2(i-1)} + x_{2(i+1)} = (2 + \alpha_i(|x_i|^2 - 1))x_{2i}, \\ &\quad i = \overline{1, N}\}.\end{aligned}$$

As we can conclude, the set  $\Omega_\infty$  includes the dynamics of interest with synchronization at the unit circle (when  $|x_i| = 1$  for all  $i = \overline{1, N}$ ) or on a circle (when  $|x_i| \neq 0$  for all  $i = \overline{1, N}$ ). Indeed, the relations

$$\beta_i x_{2i} = x_{2(i-1)} + x_{2(i+1)} \quad (22)$$

with constant  $\beta_i = (2 + \alpha_i(|x_i|^2 - 1))$ , which satisfy in the set  $\Omega_\infty$  for all  $i = \overline{1, N}$ , can be interpreted as a kind of synchronization, with another synchronization measure (the previously introduced  $\psi(x(t))$  may be non zero in general case). Note that different, phase or anti-phase, patterns can be obtained in (14), (21) depending on values of parameters. The case when  $|x_i| = 0$  for all  $i = \overline{1, N}$  corresponds also to synchronization, but it is not interesting from application point of view since there is no periodic solutions in this case.

**Theorem 20.** *For any  $k > 0$ , if there is an index  $1 \leq i \leq N$  such that  $2a_i k < b_i$ , then in the system (14), (21) all trajectories are bounded and almost all of them converge to the largest invariant set in*

$$\begin{aligned}\Omega'_\infty &= \{x \in M : |x_i| = \text{const} \neq 0, e_{2i-1}^2 + e_{2i}^2 = \text{const}, \\ &\quad x_{2(i-1)} + x_{2(i+1)} = (2 + \alpha_i(|x_i|^2 - 1))x_{2i}, \\ &\quad i = \overline{1, N}\}.\end{aligned}$$

*Proof.* Since all conditions of Proposition 19 are satisfied, then all trajectories converge to the set  $\Omega_\infty$ . By substitution of the control (21) in the equations of (14) we obtain:

$$\begin{aligned}
\dot{x}_{1i} &= x_{2i}, \\
\dot{x}_{2i} &= a_i u_i - x_{1i} - b_i x_{2i} (|x_i|^2 - 1) \\
&= a_i k (x_{2(i-1)} - 2x_{2i} + x_{2(i+1)}) - x_{1i} \\
&\quad - b_i x_{2i} (|x_i|^2 - 1) \\
&= -x_{1i} - (2a_i k - b_i) x_{2i} + a_i k (x_{2(i-1)} \\
&\quad + x_{2(i+1)}) - b_i x_{2i} |x_i|^2.
\end{aligned}$$

Linearizing this system around the origin ( $|x_i| = 0$  for all  $i = \overline{1, N}$ ) we conclude that this equilibrium is unstable if there exists at least one index  $1 \leq i \leq N$  with  $2a_i k < b_i$ . Thus, for almost all initial conditions trajectories converge to a subset of  $\Omega_\infty$  where  $|x_i| \neq 0$ , *i.e.* to the set  $\Omega'_\infty$ .  $\square$

On the set  $\Omega'_\infty$  we have  $x_{2i} = r_i \sin(t + \phi_i)$  for all  $i = \overline{1, N}$ , where  $r_i = |x_i|$  and  $\phi_i \in [0, 2\pi)$  are some constants depending on the system parameters and initial conditions, then from (22)

$$\begin{aligned}
(2 + \alpha_i (r_i^2 - 1)) r_i \sin(t + \phi_i) &= r_{i-1} \sin(t + \phi_{i-1}) \\
&\quad + r_{i+1} \sin(t + \phi_{i+1})
\end{aligned}$$

and

$$\begin{aligned}
(2 + \alpha_i (r_i^2 - 1)) r_i &= \sqrt{r_{i-1}^2 + r_{i+1}^2 + 2r_{i-1} r_{i+1} \cos(\phi_{i+1} - \phi_{i-1})}, \\
\phi_i &= \phi_{i-1} + \arctan \left( \frac{r_{i+1} \sin(\phi_{i+1} - \phi_{i-1})}{r_{i-1} + r_{i+1} \cos(\phi_{i+1} - \phi_{i-1})} \right)
\end{aligned}$$

providing expressions of possible relations of phases and amplitudes of oscillation for neighboring agents in (14), (21).

## 5 Examples and simulations

To illustrate the theoretical results, we will consider both identical and non identical Brockett oscillators. First, let us choose 4 non-identical Brockett oscillators (14) with parameters  $k = 0.5$ ,  $a_i = b_i = i$ ,  $i = \overline{1, 3}$ ,  $a_4 = 3$ ,  $b_4 = 4$ . For the case of identical oscillators, let us choose the parameters  $k = 0.5$ ,  $a_i = 0.5$ ,  $b_i = 1$ ,  $i = \overline{1, 4}$ . The chosen parameters respect the condition of Theorem 20. For both cases, let us consider the control (21). Then the system (14) synchronizes (in/anti-phase) and converges to the set  $\Omega'_\infty$  (which includes unit circle) according to Theorem 20.

The simulation result for non-identical oscillators can be seen in Fig. 1, while Fig. 2 shows the simulation result for the case of identical oscillators. From the simulation result, in-phase synchronization is present for non-identical

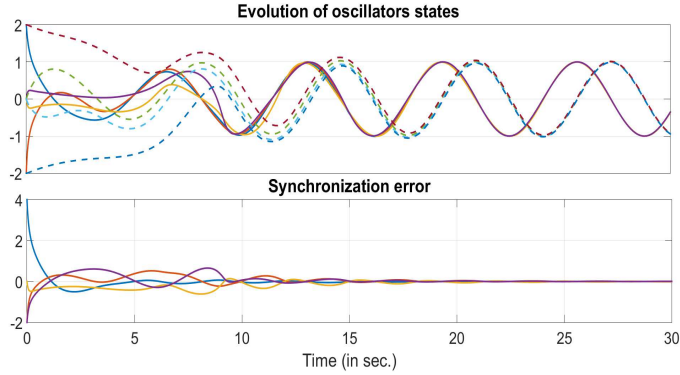


Figure 1: Synchronization result with control (21) for the case of non-identical oscillators. In the top figure, Solid line -  $x_2$ , dashed line -  $x_1$

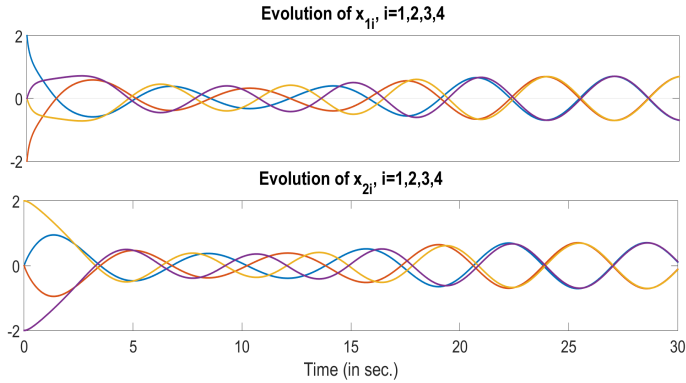


Figure 2: Synchronization result with control (21) for the case of identical oscillators

oscillators while anti-phase synchronization is present for identical oscillators. In the case of in-phase synchronization, oscillators converge to the unit circle. However, for anti-phase synchronization, the oscillators converge to a circle which is in  $\Omega'_\infty$ . In this case, the radius of the circle depends on the system parameters and initial conditions.

Moreover, adding a small perturbation in the case of identical oscillators leads to in-phase synchronization (difference in parameters for non-identical oscillators can be considered as a perturbation also). Thus, in this example synchronization is a phenomenon, which follows the agents imperfections rather than similarity.

## 6 Conclusions

This technical report studied the problem of global robust synchronization of non-identical Brockett oscillators. To this end, global stability and ISS analysis were done for an individual oscillator (with respect to the set  $\mathcal{W}$  composed by the equilibrium at the origin and the limit cycle at the unit sphere). These results make Brockett oscillator a promising benchmark model for the investigation of synchronization and consensus phenomena. Next, two synchronization control strategies were proposed. The first one imposes restriction on the synchronization control amplitude and uses generic ISS arguments. The second synchronization control design is based on a special Lyapunov function proposed in this work, and it allows the kind of synchronous motions to be evaluated. Numerical simulations demonstrated the effectiveness of our method by applying it to a network of non-identical Brockett oscillators.

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