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New Prediction Approach for Stabilizing Time-Varying Systems under Time-Varying Input Delay

Frédéric Mazenc

Michael Malisoff

Abstract—We provide a new sequential predictors approach for the exponential stabilization of linear time-varying systems with pointwise time-varying input delays. Our method circumvents the problem of constructing and estimating distributed terms in the stabilizing control laws, allows arbitrarily large input delay bounds, and covers dynamics with uncertainties. Instead of using distributed terms, our approach to handling longer delays is to increase the number of predictors, and we obtain explicit formulas for lower bounds for the number of required predictors. The lower bounds are functions of the bounds on the delays and their derivatives. We illustrate our method using a pendulum dynamics.

Key Words—Delay, robustness, stability, time-varying

I. INTRODUCTION

This work continues our search (which we began in [17]) for alternative predictive methods for time-varying systems that can be applied without computing Lie derivatives and without computing distributed terms, and which can compensate arbitrarily long input delays. Our work is motivated by the ubiquity of input delays across engineering, coupled with the challenges one may encounter when building delay tolerant feedback controls, if one applies traditional emulation or prediction methods. See, e.g., [5], [23], and [26] for overviews on delay compensating control, [27] for constant electromechanical input delays in muscle response in neuromuscular electrical stimulation (or NMES), and [9] and [22] for extensions to NMES under time-varying input delays.

For constant coefficient linear systems, it often suffices to use linear matrix inequalities (or LMIs) to build delay tolerant controls, but many important linear systems are time-varying. For instance, tracking reference trajectories under input delays leads to time-varying systems, even if the original system is time invariant. See [20], and see [8] for systems with delayed outputs, which naturally lead to input delayed systems when the output for one system is an input for another system.

Traditional input delay compensation methods can roughly be grouped into three categories of approaches. One approach is to solve the control problem with the input delay set to 0, and then look for upper bounds on the input delay that the resulting closed loop system can tolerate without sacrificing

the desired stability properties. Two advantages of this so-called emulation approach are that (a) it makes it possible to use relatively simple controls for undelayed systems (such as Lie derivative feedback design methods and other approaches in [11]) and (b) the strict Lyapunov functions one often obtains from solving the feedback design problem for the corresponding undelayed system can often be transformed into Lyapunov-Krasovskii functionals, which can in turn be used to compute bounds on the input delays that the system can tolerate. See [7] for background on Lyapunov-Krasovskii functionals, and [19] for ways to transform strict Lyapunov functions for undelayed systems into Lyapunov-Krasovskii functionals for the corresponding input delayed systems.

Another approach is the reduction model method, where the control function is expressed implicitly as a solution of an integral equation, which can lead to challenging problems of numerically computing control values, especially when the system is perturbed by uncertainty or the delay is time-varying. Prediction is another useful method, where the state values in the controls are replaced by numerical predictions of the state values. Standard prediction or reduction model methods can compensate for arbitrarily long input delays, and so may have an advantage over emulation methods in the study of communication networks or multi-agent problems that are prone to long delays, but the distributed terms in their controls (where the controls use all values of the input or state on some interval of times) may make them harder to implement than emulation; see [3], [4], [21], and [28].

This note provides a new sequential predictors approach to exponential stabilization of time-varying systems with time-varying pointwise input delays, and therefore builds on recent significant works such as [24] (which used LMI methods for time invariant linear systems to build sequential predictors) and [13] (which extended [24] to linear systems with time-varying delays, and so cover a smaller class of systems than the time-varying systems we consider here). It also extends our work [17], which was confined to constant delays. Since we do not use any distributed terms or Lie derivatives in our control, our work is very different from the classical reduction model or prediction based approaches that have been used by M. Krstic (as in [6], [10], and [12]).

Our sequential predictor approach uses several dynamic extensions, each having the same dimension as the original system, which contrasts with the reduction approach where the integral equation that produces the control has the same dimension as the control. Since we do not use distributed terms, our work also differs from [1] and [2] and other works that use several dynamic extensions and distributed

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terms. Our result gives closed form expressions for the control and explicit lower bounds on the number of required dynamic extensions in terms of the time-varying delays and their derivatives. Our work is mainly a mathematical and methodological development, rather than a specific real-world application. However, we illustrate our work using a pendulum dynamics that was also studied in [20], where we used a predictive approach that led to distributed controls but that did not cover the time-varying delays that we cover here.

We use standard notation and definitions. Throughout the sequel, the dimensions are arbitrary, unless otherwise noted. For simplicity, we omit arguments of functions when they are clear from the context, and we always assume that the initial times t_0 for our solutions of our time-varying systems are $t_0 = 0$, but we can write analogs of our results for general choices of $t_0 \geq 0$. We use $|\cdot|$ to denote the usual Euclidean norm and the induced matrix norm, and $|\phi|_\infty$ (resp., $|\phi|_{\mathcal{I}}$) denotes the essential supremum (resp., supremum over any interval \mathcal{I}) of any bounded measurable function ϕ .

II. MAIN RESULT

We study systems of the form

$$\dot{x}(t) = A(t)x(t) + B(t)u(t - h(t)) + \delta(t), \quad (1)$$

where the state x and the control u are valued in \mathbb{R}^n and \mathbb{R}^m , respectively, h is a known time-varying delay, and $\delta : [0, \infty) \rightarrow \mathbb{R}^n$ is an unknown measurable essentially bounded function that can represent unmodeled features or actuator errors. Throughout this section, we make two assumptions:

Assumption 1: The function $h : \mathbb{R} \rightarrow [0, \infty)$ is C^1 and bounded from above by a constant $c_h > 0$. Also, \dot{h} is bounded from above by some constant $l_h \in (0, 1)$, and \dot{h} admits a global Lipschitz constant $n_h > 0$. \square

Assumption 2: The functions A and B are bounded and continuous, and there is a known bounded continuous function $K : [0, \infty) \rightarrow \mathbb{R}^{m \times n}$ such that

$$\dot{x}(t) = [A(t) + B(t)K(t)]x(t) \quad (2)$$

is exponentially stable to 0. \square

In terms of the known delay $h(t)$ and an integer $m > 1$ that we specify later, we then introduce the functions

$$\Omega_i(t) = t - \frac{i}{m}h(t) \text{ and } \theta_i(t) = \Omega_{m-i+1}^{-1}(\Omega_{m-i}(t)) \quad (3)$$

for all $i \in \{0, \dots, m\}$, and we inductively define the functions

$$R_1(t) = \dot{\theta}_1(t) \text{ and } R_i(t) = \dot{\theta}_i(t)R_{i-1}(\theta_i(t)) \text{ for } i > 1. \quad (4)$$

Such functions exist by our upper bound $l_h \in (0, 1)$ on \dot{h} from Assumption 1, which implies that the Ω_i 's are strictly increasing and that their ranges are all of \mathbb{R} . The required inverses in (3) can be computed numerically in practice using standard programs, e.g., the command

$$g = \text{InverseFunction}[\text{Function}[t, t - h[t]/m]] \quad (5)$$

in Mathematica [15] to compute the new function $g(t) = \Omega_1^{-1}(t)$ for a known choice of $h(t)$. In terms of the positive constants

$$u_c = n_h \frac{c_h}{(1-l_h)^2} + \frac{l_h}{1-l_h} \quad (6)$$

and

$$b_1 = \left[1 + \left(1 + \frac{u_c}{m}\right)^m |A|_\infty\right] \left(1 + \frac{u_c}{m}\right)^m |A|_\infty, \quad (7)$$

$$b_2 = \left[1 + \left(1 + \frac{u_c}{m}\right)^m |A|_\infty\right]^2, \text{ and } b_3 = b_2 \left(1 + \frac{u_c}{m}\right),$$

we then prove:

Theorem 1: If Assumptions 1-2 hold and $m \in \mathbb{N}$ satisfies

$$\max \left\{1, 4 \left(\frac{b_1}{\sqrt{2}} + b_3\right) \frac{c_h}{1-l_h}\right\} < m, \quad (8)$$

then we can find positive constants μ_1 and μ_2 such that for all solutions of (1) in closed loop with the control

$$u(t) = K(t + h(t))z_m(t), \quad (9)$$

where z_m is the last n components of the system

$$\begin{aligned} \dot{z}_1(t) &= R_1(t)A(\theta_1(t))z_1(t) + R_1(t)B(\theta_1(t))u(\Omega_{m-1}(t)) \\ &\quad + L_1(t)[z_1(\theta_1^{-1}(t)) - x(t)] \\ \dot{z}_i(t) &= R_i(t)A(G_i(t))z_i(t) + R_i(t)B(G_i(t))u(\Omega_{m-i}(t)) \\ &\quad + L_i(t)[z_i(\theta_i^{-1}(t)) - z_{i-1}(t)], \quad i \in \{2, \dots, m\} \end{aligned} \quad (10)$$

with the choices

$$L_i(t) = -I_n - R_i(t)A(G_i(t)) \text{ and } G_i = \Omega_m^{-1} \circ \Omega_{m-i}, \quad (11)$$

we have

$$|(x(t), \mathcal{E}(t))| \leq \mu_1 (|(x, \mathcal{E})|_{[-c_h, 0]} e^{-\mu_2 t} + |\delta|_{[0, t]}) \quad (12)$$

for all $t \geq 0$, where the vector \mathcal{E} is defined by $\mathcal{E}(t) = (z_1(t) - x(\theta_1(t)), z_2(t) - z_1(\theta_2(t)), \dots, z_m(t) - z_{m-1}(\theta_m(t)))$. \square

Remark 1: The estimate (12) means that the combined (x, \mathcal{E}) dynamics are input-to-state stable (or ISS) with respect to δ . Three key features of the preceding construction are (a) our allowing arbitrarily large delay bounds $c_h > 0$, (b) the fact that our control (9) has no distributed terms and no Lie derivatives, and (c) the fact that only the bottom n components of the dynamic extension (10) are needed to compute the control values (9). This sets our work apart from standard predictive controllers that contain distributed terms, and emulation controllers that limit the allowable upper bounds on the input delay. In general, longer delays and higher dimensional systems of the form (1) lead to larger m values. See our examples below. \square

III. PROOF OF THEOREM 1

We use this lemma, which we prove in the appendix:

Lemma 1: The θ_i 's in (3) and the constant u_c in (6) satisfy

$$|\dot{\theta}_i(t) - 1| \leq \frac{u_c}{m} \text{ and } |\theta_i^{-1}(t) - t| \leq \frac{c_h}{1-l_h} \frac{1}{m} \quad (13)$$

for all $t \geq 0$ and $i \in \{1, 2, \dots, m\}$. \square

By Lemma 1 and our definition of the R_i 's in (4), we get

$$|R_i(t)| \leq \left(1 + \frac{u_c}{m}\right)^m \quad (14)$$

for all i , where throughout the proof, all inequalities and inequalities are for all $t \geq 0$, unless otherwise indicated. In terms of the states z_i of our dynamic controller (10), we define

$$\xi_i(t) = z_{i-1}(\theta_i(t)) \text{ for } i \geq 1, \quad (15)$$

where $z_0 = x$. We divide the rest of the proof into three parts.

Part 1: Error Dynamics. We first derive a useful formula for the dynamics of the error $\mathcal{E} = (\mathcal{E}_1, \dots, \mathcal{E}_m)$, where

$$\mathcal{E}_i(t) = z_i(t) - \xi_i(t) \quad (16)$$

for all $i \in \{1, \dots, m\}$, so $\mathcal{E}_i(t) = z_i(t) - z_{i-1}(\theta_i(t))$ for all $i \in \{2, \dots, m\}$. Our formulas (3) for the Ω_i 's and θ_i 's give

$$\begin{aligned}\dot{\xi}_1(t) &= \dot{\theta}_1(t)A(\theta_1(t))\xi_1(t) + \dot{\theta}_1(t)\delta(\theta_1(t)) \\ &\quad + \dot{\theta}_1(t)B(\theta_1(t))u(\theta_1(t) - h(\theta_1(t))) \\ &= R_1(t)A(\theta_1(t))\xi_1(t) \\ &\quad + R_1(t)B(\theta_1(t))u(\Omega_m(\theta_1(t))) \\ &\quad + \dot{\theta}_1(t)\delta(\theta_1(t)).\end{aligned}\quad (17)$$

Since $\theta_1(t) = \Omega_m^{-1}(\Omega_{m-1}(t))$, we have

$$\Omega_m(\theta_1(t)) = t - \frac{m-1}{m}h(t) = \Omega_{m-1}(t). \quad (18)$$

As a direct consequence, we have

$$\begin{aligned}\dot{\xi}_1(t) &= R_1(t)A(\theta_1(t))\xi_1(t) \\ &\quad + R_1(t)B(\theta_1(t))u(\Omega_{m-1}(t)) + \dot{\theta}_1(t)\delta(\theta_1(t)).\end{aligned}\quad (19)$$

Hence, (16) and our choice of the z_1 dynamics in (10) give

$$\dot{\mathcal{E}}_1(t) = R_1(t)A(\theta_1(t))\mathcal{E}_1(t) + L_1(t)\mathcal{E}_1(\theta_1^{-1}(t)) - \dot{\theta}_1(t)\delta(\theta_1(t)). \quad (20)$$

For $i > 1$, we can use the fact that $G_{i-1} \circ \theta_i = G_i$ to get

$$\begin{aligned}\dot{\xi}_i(t) &= \dot{\theta}_i(t) \{ R_{i-1}(\theta_i(t)) [A(G_i(t))z_{i-1}(\theta_i(t)) \\ &\quad + B(G_i(t))u(\Omega_{m-i+1}(\theta_i(t)))] \\ &\quad + L_{i-1}(\theta_i(t)) [z_{i-1}(\theta_i^{-1}(t)) \\ &\quad - z_{i-2}(\theta_i(t))] \} \\ &= R_i(t)B(G_i(t))u(\Omega_{m-i+1}(\theta_i(t))) \\ &\quad + \dot{\theta}_i(t)L_{i-1}(\theta_i(t)) [z_{i-1}(\theta_i^{-1}(t)) \\ &\quad - z_{i-2}(\theta_i(t))] + R_i(t)A(G_i(t))\xi_i(t),\end{aligned}\quad (21)$$

where the last equality used the definition of the R_i 's. Since $\Omega_{m-i+1}(\theta_i(t)) = \Omega_{m-i}(t)$, we deduce that for $i \geq 2$, we get

$$\begin{aligned}\dot{\mathcal{E}}_i(t) &= L_i(t) [z_i(\theta_i^{-1}(t)) - z_{i-1}(t)] \\ &\quad - \dot{\theta}_i(t)L_{i-1}(\theta_i(t)) [z_{i-1}(\theta_i^{-1}(t)) \\ &\quad - z_{i-2}(\theta_i(t))] + R_i(t)A(G_i(t))\mathcal{E}_i(t) \\ &= R_i(t)A(G_i(t))\mathcal{E}_i(t) + L_i(t)\mathcal{E}_i(\theta_i^{-1}(t)) \\ &\quad - \dot{\theta}_i(t)L_{i-1}(\theta_i(t))\mathcal{E}_{i-1}(\theta_i^{-1}(t)).\end{aligned}\quad (22)$$

To summarize, we have the following error dynamics:

$$\begin{cases} \dot{\mathcal{E}}_1(t) = R_1(t)A(G_1(t))\mathcal{E}_1(t) + L_1(t)\mathcal{E}_1(\theta_1^{-1}(t)) \\ \quad - \dot{\theta}_1(t)\delta(\theta_1(t)) \\ \dot{\mathcal{E}}_2(t) = R_2(t)A(G_2(t))\mathcal{E}_2(t) + L_2(t)\mathcal{E}_2(\theta_2^{-1}(t)) \\ \quad - \dot{\theta}_2(t)L_1(\theta_2(t))\mathcal{E}_1(\theta_1^{-1}(\theta_2(t))) \\ \vdots \\ \dot{\mathcal{E}}_m(t) = R_m(t)A(G_m(t))\mathcal{E}_m(t) + L_m(t)\mathcal{E}_m(\theta_m^{-1}(t)) \\ \quad - \dot{\theta}_m(t)L_{m-1}(\theta_m(t))\mathcal{E}_{m-1}(\theta_{m-1}^{-1}(\theta_m(t))) \end{cases}\quad (23)$$

Part 2: ISS of the Error Dynamics (23). We prove that there are positive constants J_1 and J_2 such that

$$|\mathcal{E}(t)| \leq J_1 |\mathcal{E}|_{[-c_h, 0]} e^{-J_2 t} + J_1 |\delta|_{[0, \infty)} \quad (24)$$

holds along all solutions of (23) for all $t \geq 0$, using the triangular structure of (23). Since (23) is triangular, we study the properties of the fundamental system

$$\begin{aligned}\dot{s}_i(t) &= R_i(t)A(G_i(t))s_i(t) + L_i(t)s_i(\theta_i^{-1}(t)) \\ &= -s_i(t) + [L_n + R_i(t)A(G_i(t))] [s_i(t) - s_i(\theta_i^{-1}(t))]\end{aligned}\quad (25)$$

for a fixed i , where the second equality used our choices $L_i(t) = -L_n - R_i(t)A(G_i(t))$. Since $\theta_i^{-1}(t) \geq 0$ for all $t \geq c_h$,

it follows that for all $t \geq c_h$, we have

$$\begin{aligned}\dot{s}_i(t) &= -s_i(t) - L_i(t) \int_{\theta_i^{-1}(t)}^t R_i(l)A(G_i(l))s_i(l)dl \\ &\quad + L_i(t) \int_{\theta_i^{-1}(t)}^t (L_n + R_i(l)A(G_i(l)))s_i(\theta_i^{-1}(l))dl.\end{aligned}\quad (26)$$

We next analyze the stability of (26), using the positive definite quadratic function $Q(s_i) = |s_i|^2/2$.

By our choices (7) of the constants b_1 and b_2 , our second bound in (13) from Lemma 1, and (14), it follows that the time derivative of Q along the trajectories of (26) satisfies

$$\begin{aligned}\dot{Q}(t) &\leq -|s_i(t)|^2 + b_1 |s_i(t)| \int_{t-\frac{c_h}{1-l_h} \frac{1}{m}}^t |s_i(l)| dl \\ &\quad + b_2 |s_i(t)| \int_{t-\frac{c_h}{1-l_h} \frac{1}{m}}^t |s_i(\theta_i^{-1}(l))| dl \\ &\leq -|s_i(t)|^2 + b_1 |s_i(t)| \int_{t-\frac{c_h}{1-l_h} \frac{1}{m}}^t |s_i(l)| dl \\ &\quad + b_2 |s_i(t)| \int_{\theta_i^{-1}(t-\frac{c_h}{1-l_h} \frac{1}{m})}^{\theta_i^{-1}(t)} |s_i(u)| |\dot{\theta}_i(u)| du\end{aligned}\quad (27)$$

for all $t \geq \max\{c_h, 2c_h/(m(1-l_h))\}$. Observe that $\theta_i^{-1}(t) \leq t$, because by our choices (3) of the Ω_i 's, this inequality is equivalent to $\Omega_{m-i+1}(t) \leq \Omega_{m-i}(t)$, which is equivalent to $m-i+1 \geq m-i$. Hence the first bound in (13) gives

$$\begin{aligned}\dot{Q}(t) &\leq -|s_i(t)|^2 + b_1 |s_i(t)| \int_{t-\frac{c_h}{1-l_h} \frac{1}{m}}^t |s_i(l)| dl \\ &\quad + b_3 |s_i(t)| \int_{t-\frac{2c_h}{1-l_h} \frac{1}{m}}^t |s_i(u)| du\end{aligned}\quad (28)$$

for all $t \geq \max\{c_h, 2c_h/(m(1-l_h))\}$, where we also used the second bound in (13) to get the new lower bound in the second integral. Since our lower bound (8) on m gives

$$\frac{2c_h}{m(1-l_h)} < \frac{1}{b_1/\sqrt{2}+b_3}, \quad (29)$$

we can use Young's inequality twice (i.e., $s_i(t)r \leq \frac{1}{4}|s_i(t)|^2 + r^2$ for suitable values of $r \geq 0$) and then Jensen's inequality to get

$$\begin{aligned}\dot{Q}(t) &\leq -\frac{1}{2}|s_i(t)|^2 + \frac{(b_1/\sqrt{2})^2}{b_1/\sqrt{2}+b_3} \int_{t-\frac{c_h}{1-l_h} \frac{1}{m}}^t |s_i(l)|^2 dl \\ &\quad + \frac{b_3^2}{b_1/\sqrt{2}+b_3} \int_{t-\frac{2c_h}{1-l_h} \frac{1}{m}}^t |s_i(u)|^2 du \\ &\leq -\frac{1}{2}|s_i(t)|^2 + \left(\frac{b_1}{\sqrt{2}} + b_3 \right) \int_{t-\frac{2c_h}{1-l_h} \frac{1}{m}}^t |s_i(u)|^2 du,\end{aligned}\quad (30)$$

where the first application of Young's and Jensen's inequalities that we used to obtain the first inequality in (30) was

$$\begin{aligned}b_1 |s_i(t)| \int_{t-\frac{c_h}{1-l_h} \frac{1}{m}}^t |s_i(l)| dl \\ \leq \frac{1}{4}|s_i(t)|^2 + \left(b_1 \int_{t-\frac{c_h}{1-l_h} \frac{1}{m}}^t |s_i(l)| dl \right)^2 \\ \leq \frac{1}{4}|s_i(t)|^2 + b_1^2 \frac{c_h}{1-l_h} \frac{1}{m} \int_{t-\frac{c_h}{1-l_h} \frac{1}{m}}^t |s_i(l)|^2 dl \\ \leq \frac{1}{4}|s_i(t)|^2 + \frac{(b_1/\sqrt{2})^2}{b_1/\sqrt{2}+b_3} \int_{t-\frac{c_h}{1-l_h} \frac{1}{m}}^t |s_i(l)|^2 dl,\end{aligned}\quad (31)$$

and the second one was

$$\begin{aligned}
& b_3 |s_i(t)| \int_{t-\frac{2c_h}{1-l_h} \frac{1}{m}}^t |s_i(l)| dl \\
& \leq \frac{1}{4} |s_i(t)|^2 + \left(b_3 \int_{t-\frac{2c_h}{1-l_h} \frac{1}{m}}^t |s_i(l)| dl \right)^2 \\
& \leq \frac{1}{4} |s_i(t)|^2 + b_3^2 \frac{2c_h}{1-l_h} \frac{1}{m} \int_{t-\frac{2c_h}{1-l_h} \frac{1}{m}}^t |s_i(l)|^2 dl \\
& \leq \frac{1}{4} |s_i(t)|^2 + \frac{b_3^2}{b_1/\sqrt{2}+b_3} \int_{t-\frac{2c_h}{1-l_h} \frac{1}{m}}^t |s_i(l)|^2 dl,
\end{aligned} \tag{32}$$

and then we used the fact that $(r^2 + s^2)/(r + s) \leq r + s$ for all $r > 0$ and $s > 0$, with the choices $r = b_1/\sqrt{2}$ and $s = b_3$.

Since our lower bound on m in (8) also allows us to find a constant $\lambda > 1$ such that

$$4 \left(\frac{b_1}{\sqrt{2}} + b_3 \right) \frac{\lambda c_h}{1-l_h} < m, \tag{33}$$

it follows from (30) that we can also find a constant $c_0 > 0$ such that the time derivative of

$$Q^\sharp(s_{i,t}) = Q(s_i(t)) + \lambda \left(\frac{b_1}{\sqrt{2}} + b_3 \right) \int_{t-\frac{2c_h}{1-l_h} \frac{1}{m}}^t \int_w^t |s_i(u)|^2 du dw$$

along all solutions of the fundamental system (25) satisfies

$$\begin{aligned}
\dot{Q}^\sharp & \leq -\frac{1}{2} |s_i(t)|^2 + \left(\frac{b_1}{\sqrt{2}} + b_3 \right) \frac{2c_h}{1-l_h} \frac{\lambda}{m} |s_i(t)|^2 \\
& \quad + (1-\lambda) \left(\frac{b_1}{\sqrt{2}} + b_3 \right) \int_{t-\frac{2c_h}{m(1-l_h)}}^t |s_i(u)|^2 du \\
& \leq -c_0 Q^\sharp(s_{i,t}),
\end{aligned} \tag{34}$$

where the second inequality in (34) used the bound

$$\int_{t-\frac{2c_h}{1-l_h} \frac{1}{m}}^t \int_w^t |s_i(u)|^2 du dw \leq \frac{2c_h}{(1-l_h)m} \int_{t-\frac{2c_h}{1-l_h} \frac{1}{m}}^t |s_i(u)|^2 du$$

and $s_{i,t}(\ell) = s_i(t + \ell)$ for all $\ell \in [-2c_h/((1-l_h)m), 0]$. For instance, we can choose

$$c_0 = \min \left\{ 1 - \frac{4(b_1/\sqrt{2}+b_3)\lambda c_h}{(1-l_h)m}, \frac{(\lambda-1)(1-l_h)m}{2\lambda c_h} \right\}. \tag{35}$$

This concludes our analysis of the fundamental system (25).

By reapplying Young's inequality, it follows that along all trajectories of the \mathcal{E}_1 subsystem in (23), we have

$$\begin{aligned}
\dot{Q}^\sharp & \leq -c_0 Q^\sharp(\mathcal{E}_{1,t}) + |\mathcal{E}_1(t)| |\dot{\theta}_1(t)| |\delta|_\infty \\
& \leq -\frac{c_0}{2} Q^\sharp(\mathcal{E}_{1,t}) + \frac{1}{c_0} |\dot{\theta}_1(t)|^2 |\delta|_\infty^2.
\end{aligned} \tag{36}$$

By Lemma 1, $\dot{\theta}_1$ is bounded, so we can multiply (36) through by $e^{c_0 t/2}$ and integrate both sides and use the quadratic lower bound on Q^\sharp and the relation $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for suitable $a \geq 0$ and $b \geq 0$ to find positive constants c_1 and c_2 such that

$$|\mathcal{E}_1(t)| \leq c_1 (e^{-c_2 t} |\mathcal{E}_1|_{[-c_h, 0]} + |\delta|_\infty) \tag{37}$$

holds along all trajectories of the \mathcal{E}_1 dynamics. Similar reasoning implies that along all solutions of the \mathcal{E}_2 dynamics, we have

$$|\mathcal{E}_2(t)| \leq c_1 (e^{-c_2 t/2} |\mathcal{E}_2|_{[t/2-c_h, t/2]} + |\mathcal{E}_1(\theta_1 \circ \theta_2^{-1})|_{[t/2, t]}) \tag{38}$$

for all $t \geq 0$, which allows us to build constants $\hat{c}_1 > 0$ and $\hat{c}_2 > 0$ such that along all trajectories of the $(\mathcal{E}_1, \mathcal{E}_2)$ dynamics in (23), we have

$$|(\mathcal{E}_1, \mathcal{E}_2)(t)| \leq \hat{c}_1 (|(\mathcal{E}_1, \mathcal{E}_2)|_{[-c_h, 0]} e^{-\hat{c}_2 t} + |\delta|_\infty) \tag{39}$$

for all $t \geq 0$. Then using the fact that the error dynamics (23) are triangular and repeating the preceding construction,

it follows that we can find positive constants J_1 and J_2 satisfying (24) along all solutions of (23).

Part 3: ISS of (1) in Closed Loop with (9). By our error variable formulas (16) and $\mathcal{E}_i(t) = z_i(t) - z_{i-1}(\theta_i(t))$ for all $i \in \{2, \dots, m\}$, we have

$$\begin{aligned}
z_m(t) & = \xi_m(t) + \mathcal{E}_m(t) = z_{m-1}(\theta_m(t)) + \mathcal{E}_m(t) \\
& = z_{m-2}(\theta_{m-1}(\theta_m(t))) + \mathcal{E}_{m-1}(\theta_m(t)) + \mathcal{E}_m(t),
\end{aligned} \tag{40}$$

and therefore also

$$\begin{aligned}
z_m(t) & = x(\theta_1(\dots\theta_{m-1}(\theta_m(t))\dots)) \\
& \quad + \mathcal{E}_1(\theta_2(\dots\theta_m(t)\dots)) + \dots + \mathcal{E}_m(t),
\end{aligned} \tag{41}$$

by arguing inductively. Recalling the formulas $\theta_i(t) = \Omega_{m-i+1}^{-1}(\Omega_{m-i}(t))$ then gives

$$\theta_{m-1}(\theta_m(t)) = \Omega_2^{-1}(\Omega_1(\Omega_1^{-1}(\Omega_0(t)))) = \Omega_2^{-1}(t). \tag{42}$$

We deduce that $\theta_1(\dots\theta_{m-1}(\theta_m(t))\dots) = \Omega_m^{-1}(t)$, and so also

$$z_m(t) = x(\Omega_m^{-1}(t)) + \mathcal{E}_1(\theta_2(\dots\theta_m(t)\dots)) + \dots + \mathcal{E}_m(t). \tag{43}$$

Hence, our control (9) can be written as

$$\begin{aligned}
u(t-h(t)) & = K(t)[x(\Omega_m^{-1}(t-h(t)))] \\
& \quad + \mathcal{E}_1(\theta_2(\dots\theta_m(t-h(t)\dots))) + \dots + \mathcal{E}_m(t-h(t)).
\end{aligned} \tag{44}$$

Since $\Omega_m(t) = t - h(t)$, the closed loop system is then

$$\begin{aligned}
\dot{x}(t) & = (A(t) + B(t)K(t))x(t) \\
& \quad + \{B(t)[K(t)\mathcal{E}_1(\theta_2(\dots\theta_m(t-h(t)\dots)))] \\
& \quad + \dots + K(t)\mathcal{E}_m(t-h(t))\} + \delta(t).
\end{aligned} \tag{45}$$

Also, Assumption 2 implies that (2) admits a quadratic Lyapunov function $V(t, x) = x^\top P(t)x$ for some C^1 bounded function P such that $\dot{V} \leq -|x(t)|^2$ holds along all solutions of (2) (using [11, Theorem 4.14]). Since the triangle inequality gives $\dot{V} \leq -\frac{1}{2}|x(t)|^2 + 2|P|_\infty^2 |\delta^\sharp|_\infty^2 \leq -(1/(2|P|_\infty))V(x(t)) + 2|P|_\infty^2 |\delta^\sharp|_\infty^2$ along all solutions of

$$\dot{x}(t) = [A(t) + B(t)K(t)]x(t) + \delta^\sharp(t), \tag{46}$$

for all measurable essentially bounded functions δ^\sharp , we can integrate as we did to get (36) to find positive constants \tilde{c}_1 and \tilde{c}_2 such that

$$|x(t)| \leq \tilde{c}_1 (e^{-\tilde{c}_2 t} |x(0)| + |\delta^\sharp|_{[0, t]}) \tag{47}$$

holds along all solutions of (46). Hence, our theorem now follows by choosing δ^\sharp to be the quantity in curly braces in (45) and then combining (24) and (47), by reasoning as we did to convert the ISS estimate for \mathcal{E}_1 into an ISS estimate for \mathcal{E} . \square

Remark 2: The fact that $\Omega_m(t) = t - h(t)$ made it possible to write our control as

$$\begin{aligned}
u(t-h(t)) & = K(t)x(t) + \\
& \quad K(t)[\mathcal{E}_1(\theta_2(\dots\theta_m(t-h(t)\dots))) + \dots + \mathcal{E}_m(t-h(t))],
\end{aligned} \tag{48}$$

which may at first seem to contradict our assumption that the current value of the state $x(t)$ is not available to use in the control. However, our formulas (3) for the θ_i 's imply that $\theta_2 \circ \dots \circ \theta_m(t-h(t)) = \theta_1^{-1}(t)$ for all $t \geq 0$. This gives

$$\begin{aligned}
\mathcal{E}_1(\theta_2(\dots\theta_m(t-h(t)\dots))) & = z_1(\theta_1^{-1}(t)) - \xi_1(\theta_1^{-1}(t)) \\
& = z_1(\theta_1^{-1}(t)) - x(t),
\end{aligned} \tag{49}$$

so the two terms in (48) containing $x(t)$ cancel each other. This proves that our control does not require current values $x(t)$ of the state. \square

IV. EXAMPLES

Our Assumptions 1-2 on A , B , and h hold for broad classes of systems of the form $\dot{x}(t) = A(t)x(t) + B(t)u(t - h(t))$, including all systems with constant controllable pairs (A, B) . We first illustrate Theorem 1 in a scalar case. Then we revisit the linearized pendulum model that we covered in [20]. Our work [20] covered the linearized pendulum under constant delays, using a predictive controller that has distributed terms. Our work [17] covered the original nonlinear pendulum dynamics using sequential predictors without distributed terms but required the delays to be constant. Hence, this section improves on [17] and [20] in the case of the pendulum because we allow nonconstant delays and because our control has no distributed terms.

A. Scalar Example

Consider the scalar case

$$\dot{x}(t) = rx(t) + u(t - h(t)) + \delta(t) \quad (50)$$

for any constant $r > 0$ with the oscillating delay

$$h(t) = 1 + \alpha \sin(t) \quad (51)$$

for any constant $\alpha \in (0, 1)$. Then Assumptions 1-2 are satisfied using $n = 1$, $A = r$, $B = 1$, $K = -2r$, $c_h = 1 + \alpha$, and $n_h = l_h = \alpha$, so our ISS conclusion from Theorem 1 is satisfied using the control $u(t) = -2rz_m(t)$, where z_m is the m th component of the dynamic extension from Theorem 1, if the integer $m > 1$ satisfies the lower bound (8). To compute the lower bound on the allowable values of m , notice that in terms of our earlier notation (6)-(7), we have

$$\begin{aligned} u_c &= \frac{2\alpha}{(1-\alpha)^2}, \quad b_1 = r \left(1 + r \left(1 + \frac{u_c}{m}\right)^m\right) \left(1 + \frac{u_c}{m}\right)^m, \\ b_2 &= \left(1 + r \left(1 + \frac{u_c}{m}\right)^m\right)^2, \\ \text{and } b_3 &= \left(1 + r \left(1 + \frac{u_c}{m}\right)^m\right)^2 \left(1 + \frac{u_c}{m}\right). \end{aligned} \quad (52)$$

Hence, our requirement on the integer $m \geq 2$ from (8) is

$$\begin{aligned} m - 4 \frac{1+\alpha}{1-\alpha} \left[\frac{r}{\sqrt{2}} \left(1 + r \left(1 + \frac{u_c}{m}\right)^m\right) \left(1 + \frac{u_c}{m}\right)^m \right. \\ \left. + \left(1 + r \left(1 + \frac{u_c}{m}\right)^m\right)^2 \left(1 + \frac{u_c}{m}\right) \right] > 0. \end{aligned} \quad (53)$$

For suitable constants $\alpha \in (0, 1)$ and $r > 0$, we can compute the left side of (53) as a function $\phi_{\alpha,r}(m)$ of m and then choose m such that $\phi_{\alpha,r}(m) > 0$. For instance, if $\alpha = 1/7$ and $r = 1/2$, then $\phi_{1/7,1/2}(22) = 0.847755$, so our theorem applies with $m = 22$.

B. Pendulum

Consider the model

$$\begin{cases} \dot{r}_1(t) &= r_2(t) \\ \dot{r}_2(t) &= -\frac{g}{l} \sin(r_1(t)) + \frac{1}{\mathcal{M}l^2} v(t - h(t)) \end{cases} \quad (54)$$

of the simple pendulum with a time-varying delay $h(t)$, where $g = 9.8$ m/s is the gravity constant, l is the pendulum length in meters, \mathcal{M} is the pendulum mass, and v is the input. As in [17] and [20], our control objective is to track a given C^1 reference trajectory $(r_{1,s}(t), r_{2,s}(t))$ such that $\dot{r}_{1,s}(t) = r_{2,s}(t)$. Using the error variables $\tilde{r}_i = r_i - r_{i,s}(t)$ for $i = 1, 2$ and the change of feedback

$$u(t - h(t)) = \frac{1}{\mathcal{M}l^2} v(t - h(t)) - \dot{r}_{2,s}(t) - \frac{g}{l} \sin(r_{1,s}(t)), \quad (55)$$

we obtain the tracking system

$$\begin{cases} \dot{\tilde{r}}_1(t) &= \tilde{r}_2(t) \\ \dot{\tilde{r}}_2(t) &= \frac{g}{l} [\sin(r_{1,s}(t)) - \sin(\tilde{r}_1(t) + r_{1,s}(t))] \\ &\quad + u(t - h(t)). \end{cases} \quad (56)$$

Our joint work [20] with Niculescu showed that when $r_{1,s}(t) = \omega t$ and $\omega > 0$ is a large enough constant and the delay is the constant $h = 1$, the linearization

$$\begin{cases} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -\frac{g}{l} \cos(\omega t) x_1(t) + u(t - h), \end{cases} \quad (57)$$

of (56) at 0 has the globally exponentially stabilizing distributed control u defined by

$$\begin{aligned} u(t) &= -0.6x_1(t) - 0.4x_2(t) \\ &\quad - \int_{t-1}^t (0.6(t-r-1) + 0.4) u(r) dr. \end{aligned} \quad (58)$$

Our work [17] designed a globally asymptotically stabilizing sequential predictor control for the original nonlinear system (56) for cases where h is constant. However, it is not obvious how to extend the analysis from [17] to time-varying delays. Therefore, we apply Theorem 1 to its linearization (57) using the time-varying delay (51), where $\alpha \in (0, 1)$ is a known constant, but analogous reasoning applies for any delay $h(t)$ satisfying Assumption 1 above.

To this end, first note that for any choice of the constant $\omega > 0$, Assumption 2 is satisfied using

$$A(t) = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} \cos(\omega t) & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad (59)$$

$$\text{and } K(t) = \begin{bmatrix} \frac{g}{l} \cos(\omega t) - 1 & -1 \end{bmatrix}$$

and then $|A|_\infty = \max\{g/l, 1\}$. Therefore, our requirement (8) on $m \geq 2$ from Theorem 1 holds if

$$\begin{aligned} \frac{m}{4} - \frac{1+\alpha}{1-\alpha} \left[\frac{\max\{g/l, 1\}}{\sqrt{2}} \left(1 + \max\left\{\frac{g}{l}, 1\right\} \left(1 + \frac{u_c}{m}\right)^m\right) \left(1 + \frac{u_c}{m}\right)^m \right. \\ \left. + \left(1 + \max\left\{\frac{g}{l}, 1\right\} \left(1 + \frac{u_c}{m}\right)^m\right)^2 \left(1 + \frac{u_c}{m}\right) \right] > 0. \end{aligned} \quad (60)$$

Condition (60) can easily be checked for suitable choices of α , and l . For instance, if we pick $\alpha = 1/7$, any $\omega > 0$, any $l > g$, and the choice $m = 47$, then we get the value 0.376057 for the left side of (60), so Theorem 1 applies with $m = 47$.

V. CONCLUSIONS AND FUTURE WORK

We designed new exponentially stabilizing sequential predictor controllers for a large class of time-varying linear systems under general conditions on the time-varying input delays. Our closed loop systems enjoy input-to-state stability (or ISS) with respect to additive uncertainty. By allowing arbitrarily large bounds on the input delays, we overcame a possible challenge in applying emulation methods to systems that can have long input delays.

We hope to extend our work to adaptive systems, where additional dynamic extensions would make it possible to identify unknown parameters. We also hope to cover nonlinear systems, by building sequential predictors for the linear parts and proving that the controls ensure local stabilization of the original nonlinear systems, which would be a sequential predictor analog of our work [16] on local stabilization

of nonlinear systems, which was confined to reduction model controllers. Finally, we plan to merge our methods with the methods in [14] for state constrained problems, by proving robust forward invariance under time-varying delays for curve tracking under sequential predictors.

APPENDIX: PROOF OF LEMMA 1

Fix any $i \in \{1, 2, \dots, m\}$ and $t \geq 0$. Our formula $\theta_i(t) = \Omega_{m-i+1}^{-1}(\Omega_{m-i}(t))$ from (3) gives

$$\theta_i(t) = t + \Omega_{m-i+1}^{-1}(\Omega_{m-i+1}(t) + \frac{1}{m}h(t)) - \Omega_{m-i+1}^{-1}(\Omega_{m-i+1}(t)). \quad (\text{A.1})$$

Hence, the Mean Value Theorem provides a $w \in \mathbb{R}$ (depending on t and i in general) such that

$$\theta_i(t) = t + \frac{1}{\Omega_{m-i+1}(w)} \frac{1}{m}h(t). \quad (\text{A.2})$$

Therefore, Assumption 1 gives

$$|\theta_i(t) - t| \leq \frac{1}{1-l_h} \frac{1}{m}h(t) \leq \frac{c_h}{1-l_h} \frac{1}{m}. \quad (\text{A.3})$$

Hence, the second inequality in (13) follows by replacing t by $\theta_i^{-1}(t)$ in (A.3).

To check the first inequality in (13), first note that

$$\begin{aligned} \dot{\theta}_i(t) &= \frac{1 - \frac{m-i}{m} \dot{h}(t)}{1 - \frac{m-i+1}{m} \dot{h}(\Omega_{m-i+1}^{-1}(\Omega_{m-i}(t)))} \\ &= \frac{1 - \frac{m-i+1}{m} \dot{h}(t)}{1 - \frac{m-i+1}{m} \dot{h}(\Omega_{m-i+1}^{-1}(\Omega_{m-i}(t)))} \\ &\quad + \frac{\frac{1}{m} \dot{h}(t)}{1 - \frac{m-i+1}{m} \dot{h}(\Omega_{m-i+1}^{-1}(\Omega_{m-i}(t)))}. \end{aligned} \quad (\text{A.4})$$

Since $\Omega_{m-i}(t) = \Omega_{m-i+1}(t) + \frac{1}{m}h(t)$, the function

$$\mathcal{G}_i(t) = \Omega_{m-i+1}^{-1}(\Omega_{m-i+1}(t) + \frac{1}{m}h(t)) \quad (\text{A.5})$$

is such that

$$\begin{aligned} \dot{\theta}_i(t) &= \frac{1 - \frac{m-i+1}{m} \dot{h}(t)}{1 - \frac{m-i+1}{m} \dot{h}(\mathcal{G}_i(t))} + \frac{\frac{1}{m} \dot{h}(t)}{1 - \frac{m-i+1}{m} \dot{h}(\Omega_{m-i+1}^{-1}(\Omega_{m-i}(t)))} \\ &= \frac{m-i+1}{m} \frac{\dot{h}(t) + \mathcal{G}_i(t) - \Omega_{m-i+1}^{-1}(\Omega_{m-i+1}(t)) - \dot{h}(t)}{1 - \frac{m-i+1}{m} \dot{h}(\mathcal{G}_i(t))} \\ &\quad + \frac{1}{m} \frac{\dot{h}(t)}{1 - \frac{m-i+1}{m} \dot{h}(\Omega_{m-i+1}^{-1}(\Omega_{m-i}(t)))} + 1, \end{aligned} \quad (\text{A.6})$$

by writing $\dot{h}(t)$ in the first numerator in (A.6) as $\dot{h}(t) = (\dot{h}(t) - \dot{h}(\mathcal{G}_i(t))) + \dot{h}(\mathcal{G}_i(t))$. It follows that

$$|\dot{\theta}_i(t) - 1| \leq \frac{m-i+1}{m} n_h \frac{|\mathcal{G}_i(t) - \Omega_{m-i+1}^{-1}(\Omega_{m-i+1}(t))|}{1 - \frac{m-i+1}{m} l_h} + \frac{1}{m} \frac{l_h}{1 - \frac{m-i+1}{m} l_h}, \quad (\text{A.7})$$

using the Lipschitz constant n_h for \dot{h} . Hence, our choice of \mathcal{G}_i and the Mean Value Theorem applied to Ω_{m-i+1}^{-1} gives

$$\begin{aligned} |\dot{\theta}_i(t) - 1| &\leq \frac{m-i+1}{m} n_h \frac{|\Omega_{m-i+1}(t) + \frac{1}{m}h(t) - \Omega_{m-i+1}(t)|}{(1 - \frac{m-i+1}{m} l_h)^2} \\ &\quad + \frac{1}{m} \frac{l_h}{1 - \frac{m-i+1}{m} l_h} \\ &\leq \frac{1}{m} \frac{m-i+1}{m} n_h \frac{c_h}{(1 - \frac{m-i+1}{m} l_h)^2} + \frac{1}{m} \frac{l_h}{1 - \frac{m-i+1}{m} l_h}. \end{aligned}$$

The lemma now follows from our formula (6) for u_c .

REFERENCES

[1] T. Ahmed-Ali, I. Karafyllis, M. Krstic, and F. Lamnabhi-Lagarriague. Robust stabilization of nonlinear globally Lipschitz delay systems. volume 4 of *Advances in Delays and Dynamics*, pages 43–60. Springer, New York, 2016.

[2] T. Ahmed-Ali, I. Karafyllis, and F. Lamnabhi-Lagarriague. Global exponential sampled-data observers for nonlinear systems with delayed measurements. *Systems and Control Letters*, 67(7):539–549, 2013.

[3] Z. Artstein. Linear systems with delayed controls: a reduction. *IEEE Trans. Automat. Control*, 27(4):869–879, 1982.

[4] N. Bekiaris-Liberis and M. Krstic. Compensation of time-varying input and state delays for nonlinear systems. *Journal of Dynamic Systems, Measurement and Control*, 134(1):1–14, 2012.

[5] N. Bekiaris-Liberis and M. Krstic. *Nonlinear Control Under Non-constant Delays*. Society for Industrial and Applied Mathematics, Philadelphia, PA, 2013.

[6] D. Bresch-Pietri and N. Petit. Robust compensation of a chattering time-varying input delay. In *Proceedings of the IEEE Conference on Decision and Control*, pages 457–462, Los Angeles, LA, 2014.

[7] E. Fridman and S-I. Niculescu. On complete Lyapunov-Krasovskii functional techniques for uncertain systems with fast-varying delays. *International Journal of Robust and Nonlinear Control*, 18(3):364–374, 2008.

[8] A. Germani, C. Manes, and P. Pepe. A new approach to state observation of nonlinear systems with delayed output. *IEEE Trans. Automat. Control*, 47(1):96–101, 2002.

[9] R. Kamalapurkar, N. Fischer, S. Obuz, and W. Dixon. Time-varying input and state delay compensation for uncertain nonlinear systems. *IEEE Trans. Automat. Control*, 2016, to appear.

[10] I. Karafyllis and M. Krstic. Stabilization of nonlinear delay systems using approximate predictors and high-gain observers. *Automatica*, 49(12):3623–3631, 2013.

[11] H. Khalil. *Nonlinear Systems, Third Edition*. Prentice Hall, Upper Saddle River, NJ, 2002.

[12] M. Krstic. *Delay Compensation for Nonlinear, Adaptive, and PDE Systems*. Birkhauser, Boston, 2009.

[13] V. Léchappé. *Predictive Control and Estimation of Uncertain Systems with an Input Delay*. PhD Thesis, Ecole Centrale de Nantes, Nantes, France, 2015.

[14] M. Malisoff and F. Zhang. Robustness of adaptive control under time delays for three-dimensional curve tracking. *SIAM Journal on Control and Optimization*, 53(4):2203–2236, 2015.

[15] Mathematica. The world's definitive system for modern technical computing, 2015. <http://www.wolfram.com/mathematica/>.

[16] F. Mazenc and M. Malisoff. Local stabilization of nonlinear systems through the reduction model approach. *IEEE Trans. Automat. Control*, 59(11):3033–3039, 2014.

[17] F. Mazenc and M. Malisoff. Stabilization of nonlinear time-varying systems through a new prediction based approach. *IEEE Trans. Automat. Control*, submitted.

[18] F. Mazenc, M. Malisoff, and T. Dinh. Robustness of nonlinear systems with respect to delay and sampling of the controls. *Automatica*, 49(6):1925–1931, 2013.

[19] F. Mazenc, M. Malisoff, and Z. Lin. Further results on input-to-state stability for nonlinear systems with delayed feedbacks. *Automatica*, 44(9):2415–2421, 2008.

[20] F. Mazenc, M. Malisoff, and S-I. Niculescu. Reduction model approach for linear time-varying systems with delays. *IEEE Trans. Automat. Control*, 59(8):2068–2082, 2014.

[21] F. Mazenc, S-I. Niculescu, and M. Krstic. Lyapunov-Krasovskii functionals and application to input delay compensation for linear time-invariant systems. *Automatica*, 48(7):1317–1323, 2012.

[22] M. Merad, R. Downey, S. Obuz, and W. Dixon. Isometric torque control for neuromuscular electrical stimulation with time-varying input delay. *IEEE Trans. Control Systems Technology*, 2016, to appear.

[23] W. Michiels and S-I. Niculescu. *Stability and Stabilization of Time-Delay Systems*. Society for Industrial and Applied Mathematics, Philadelphia, PA, 2007.

[24] M. Najafi, S. Hosseinnia, F. Sheikholeslam, and M. Karimadin. Closed-loop control of dead time systems via sequential sub-predictors. *International Journal of Control*, 86(4):599–609, 2013.

[25] H. Omran, L. Hetel, J-P. Richard, and F. Lamnabhi-Lagarriague. Stability analysis of bilinear systems under aperiodic sampled-data control. *Automatica*, 50(4):1288–1295, 2014.

[26] J-P. Richard. Time-delay systems: an overview of some recent advances and open problems. *Automatica*, 39(10):1667–1694, 2003.

[27] N. Sharma, C. Gregory, and W. Dixon. Predictor-based compensation for electromechanical delay during neuromuscular electrical stimulation. *IEEE Trans. Neural Systems and Rehabilitation Engineering*, 19(6):601–611, 2011.

[28] E. Witrant, C. Canudas de Wit, D. Georges, and M. Alami. Remote stabilization via communication networks with a distributed control law. *IEEE Trans. Automat. Control*, 52(8):1480–1485, 2007.