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Reduction Model Approach for Linear Time-Varying Systems with Input Delays based on Extensions of Floquet Theory *

Frédéric Mazenc † Michael Malisoff ‡

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Abstract

We solve stabilization problems for linear time-varying systems under input delays. We show how changes of coordinates lead to systems with time invariant drifts, which are covered by the reduction model method and which lead to the problem of stabilizing a time-varying system without delay. For continuous time periodic systems, we can use Floquet theory to find the changes of coordinates. We also prove an analogue for discrete time systems, through a discrete time extension of Floquet theory.

1 Introduction

This note continues our search (begun in [22] and [24]) for extensions of the classical reduction model method that cover time-varying systems with input delays. Input delays are common when controllers are remotely implemented; see [4, 7, 8, 17, 18, 30] for more motivation. The reduction method has its origins in the works [2], [19], and [21] by Artstein and others, who focused on continuous time time-invariant linear systems.

Stabilization problems for linear time-varying systems with delays have been studied in fewer works. In most of them, time-varying Lyapunov functions are needed; see for instance [1] and [27] for the use of strict Lyapunov functions, and [29] for Razumikhin-Lyapunov functions. One useful Lyapunov-based approach to delay systems entails solving the stabilization problem with the input delay set equal to zero, and then using Lyapunov-Krasovskii functionals to look for upper bounds on the input delays that the closed loop system can tolerate without sacrificing the stability performance; see [20, 23]. Linear time-varying systems arise in the context of the local stabilization of a trajectory of a nonlinear system, but are beyond the scope of the classical reduction model method. The main differences between reduction approaches and the Lyapunov-Krasovskii functional approaches such as those in [23] are that (a) under certain delay bounds, methods such as [23] lead to relatively simple controllers that do not require the distributed terms that are used in reduction model methods and (b) reduction model methods usually make it possible to compensate for arbitrarily long input delays, by using the delay value in the dynamic feedback control design.

Our work [24] extends the reduction model method to linear time-varying systems, using two approaches. One approach in [24] leads to a control formula that involves the fundamental matrix for the corresponding uncontrolled system (i.e., the time-varying system obtained from the original system by setting the input equal to zero in the original system), and so may be difficult to apply in practice. The other control design in [24] does not require a formula for the fundamental matrix, but requires that the input delay stay below a suitable constant bound. By contrast, [22] covers time-varying nonlinear systems whose nonlinear parts satisfy certain conditions, and then builds a reduction model control for the linearization of the system, and finally shows local stability of the original nonlinear system in closed loop with the reduction model control.

One natural research direction for addressing the challenges of extending the reduction model method to time-varying linear systems, as well as analogous problems for discrete time systems, is to seek analogues

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of Floquet's theory; see [26, Section 3.5] for the basic results of Floquet theory. Floquet's theory covers systems without controls. One of its basic results is that if a time-varying linear system is periodic, then it can be transformed into a time-invariant system through a periodic change of coordinates. This suggests the possibility of using Floquet theory to transform a time-varying linear control system into a new time-varying linear control system with a time-invariant draft, and then stabilizing the new system using the reduction approach. One of the key observations in this work is that such a transformation can indeed be done under periodicity of the coefficient matrices in the system, and that this simplifies the stabilization problem to one that involves globally asymptotically stabilizing systems with no delay. Our assumptions are novel as well. We also provide analogues for nonperiodic or discrete time systems.

Discrete time systems with delay are important because they can be used to model some engineering devices; see [6], [9], [11], and [14]. However, not many contributions are concerned with time-varying discrete time systems with delay. Our discrete time delayed systems in this work have the form

$$x_{k+1} = A_k x_k + B_k u_{k-r} \tag{1}$$

where $x_k \in \mathbb{R}^n$ is the state, $u_k \in \mathbb{R}^p$ is the control, and $r \in \mathbb{N}$ is the delay. Here and in the sequel, the dimensions and delays are arbitrary. For the case of time-invariant coefficients, the work [9] uses dynamic extensions to transform systems of the form (1) into systems with no delay, in the special case of networked control systems. See also [12] for a prediction based approach for (1) in the time-invariant case. For time-varying continuous time systems with delay in the input, the reduction model approach can be applied under conditions pertaining to the speed of variation of the time-varying matrices; see [24]. However, to the best of our knowledge, no discrete time version of [24] exists. Also, [16] is concerned with time-invariant systems.

We propose a rather general solution to the problem of exponential stabilization of (1) through the reduction model approach, including cases where the coefficient matrices are not necessarily periodic, with an arbitrarily large delay r. It decomposes into two steps. First, under reasonable assumptions, we transform (1) into a system that is autonomous when the control is set equal to zero. Then, we adapt the reduction method to the resulting dynamics, using a novel discrete time analogue of an operator that is used in the predictor based analysis in [16]. Our treatment of (1) also has implications for using reduction model controllers in continuous time systems, because in practice, implementing controllers in continuous time systems uses discretizations, leading to discrete time delay systems of the form (1). We illustrate our theory in two examples, including a discrete time linear system in which the coefficient matrices are not periodic.

2 Preliminary Results in Continuous Time

We use the following notation and definitions. We let $|\cdot|$ denote the usual Euclidean norm of matrices and vectors, and I_n is the $n \times n$ identity matrix. Here and in the sequel, the dimensions of our Euclidean spaces are arbitrary unless other indicated. Given any function $\phi : S \to \mathbb{R}^p$ that is defined on any subset S of a Euclidean space, denote its supremum over any set $J \subseteq S$ by $|\phi|_J$. We set $\mathbb{N} = \{1, 2, \ldots\}$. We often leave out the arguments of functions, when they are clear from the context.

2.1 Fundamental General Result

Consider the system

$$\dot{x} = A(t)x + F(t)u(t-\tau) \tag{2}$$

where the state x and the input u are valued in $\mathbb{R}^n$ and $\mathbb{R}^p$ respectively, the functions $A : \mathbb{R} \to \mathbb{R}^{n\times n}$ and $F : \mathbb{R} \to \mathbb{R}^{n\times p}$ are continuous, and $\tau > 0$ is any positive constant delay. We introduce the following assumption. See below for ways to build the required function P.

Assumption 1. *There exist a constant matrix $A_c \in \mathbb{R}^{n\times n}$, a C^1 function $P : \mathbb{R} \to \mathbb{R}^{n\times n}$ such that $P^{-1}(t)$ is defined for all t, and a constant $p_M > 0$ such that*

$$|P(t)| + |P^{-1}(t)| \le p_M \tag{3}$$

and

$$\dot{P}(t) = A_c P(t) - P(t)A(t). \tag{4}$$

hold for all $t \ge 0$.

We use the following key observations:

Lemma 1. *Assume that the system (2) satisfies Assumption 1. Then the time-varying change of coordinates*

$$z = P(t)x \tag{5}$$

transforms (2) into

$$\dot{z}(t) = A_c z(t) + P(t)F(t)u(t-\tau). \tag{6}$$

Also, the operator

$$Z(t) = z(t) + \int_{t-\tau}^{t} e^{A_c(t-m-\tau)} P(m+\tau)F(m+\tau)u(m)\mathrm{d}m \tag{7}$$

transforms (6) into the system

$$\dot{Z}(t) = A_c Z(t) + e^{-A_c\tau} P(t+\tau)F(t+\tau)u(t) \tag{8}$$

with the state variable Z.

Proof. Our choice (5) of z gives

$$\dot{z}(t) = \dot{P}(t)x(t) + P(t)\dot{x}(t) = \dot{P}(t)x(t) + P(t)[A(t)x(t) + F(t)u(t-\tau)]. \tag{9}$$

Then (6) follows from (4) and our choice of z. Also, the time derivative of (7) along all solutions of (6) is

$$\begin{aligned} \dot{Z}(t) &= A_c z(t) + P(t)F(t)u(t-\tau) + A_c \int_{t-\tau}^{t} e^{A_c(t-m-\tau)} P(m+\tau)F(m+\tau)u(m)\mathrm{d}m \\ &\quad + e^{-A_c\tau} P(t+\tau)F(t+\tau)u(t) - P(t)F(t)u(t-\tau) \\ &= A_c Z(t) + e^{-A_c\tau} P(t+\tau)F(t+\tau)u(t) \end{aligned} \tag{10}$$

which gives the second conclusion. □

The message of Lemma 1 is that if one knows a function P that leads to a time-invariant system when the input is set to zero, then in practice, we can use the reduction model approach to obtain a new system without delays. We study the problem of finding P in the next subsection.

Remark 1. *Before discussing ways to find P, we remark that similar reasoning applies to systems*

$$\dot{x} = A(t)x + \int_{t-\tau}^{t} F(\ell)u(\ell)\mathrm{d}\ell \tag{11}$$

with distributed delay in the input and continuous matrix valued functions A and F. To see how, notice that if Assumption 1 is satisfied, and if we define z by (5) as before and redefine Z to be

$$Z(t) = z(t) + \int_{t-\tau}^{t} e^{A_c(t-m-\tau)} P(m+\tau) \left(\int_{m}^{t} F(\ell)u(\ell)\mathrm{d}\ell \right) \mathrm{d}m, \tag{12}$$

then similar reasoning to the proof of Lemma 1 gives

$$\dot{z}(t) = A_c z(t) + P(t) \int_{t-\tau}^{t} F(\ell)u(\ell)\mathrm{d}\ell \tag{13}$$

and then

$$\dot{Z}(t) = A_c Z(t) + \left(\int_{t-\tau}^{t} e^{A_c(t-\ell-\tau)} P(\ell+\tau)\mathrm{d}\ell \right) F(t)u(t) \tag{14}$$

so we eliminated the distributed delay.

2.2 Particular Case

In order to get a function P for which Assumption 1 is satisfied, we impose checkable conditions on the corresponding zero input subsystem

$$\dot{x} = A(t)x \tag{15}$$

which we later show how to verify using Floquet theory when A is periodic. We introduce the following assumption:

Assumption 2. *All solutions of the system (15) are of the form*

$$x(t) = K(t)E(t)x(0), \tag{16}$$

where $K : \mathbb{R} \to \mathbb{R}^{n\times n}$ and $E : \mathbb{R} \to \mathbb{R}^{n\times n}$ are everywhere invertible functions of class C^1, K^{-1} is of class C^1, and E is such that there exists a constant matrix E_c such that

$$\dot{E}(t)E^{-1}(t) = E_c \tag{17}$$

holds for all $t \geq 0$. Moreover, there is a constant $k_c > 0$ such that

$$|K(t)| + |K^{-1}(t)| \leq k_c \tag{18}$$

for all $t \geq 0$.

We can often find functions E satisfying the preceding requirements. For example, they hold if $E(t) = e^{Mt}G$, where $M \in \mathbb{R}^{n\times n}$ and $G \in \mathbb{R}^{n\times n}$ are constant matrices and G is invertible. Indeed, in this case $\dot{E}(t) = Me^{Mt}G$, $E^{-1}(t) = G^{-1}e^{-Mt}$ and therefore also $\dot{E}(t)E^{-1}(t) = M$ are satisfied. See below for cases where Assumption 2 is satisfied. We are ready to state and prove the following result:

Lemma 2. *Let the system (2) be such that (15) satisfies Assumption 2. Then the change of coordinates*

$$z = L(t)x \tag{19}$$

with $L = K^{-1}$ gives the new system

$$\dot{z}(t) = E_c z(t) + L(t)F(t)u(t-\tau) \tag{20}$$

with a time invariant drift term.

Proof. From (15) and (16), it follows that

$$\dot{x}(t) = A(t)K(t)E(t)x(0) \tag{21}$$

and

$$\dot{x}(t) = \dot{K}(t)E(t)x(0) + K(t)\dot{E}(t)x(0) \tag{22}$$

hold for all $x(0) \in \mathbb{R}^n$. This gives

$$A(t)K(t)E(t) = \dot{K}(t)E(t) + K(t)\dot{E}(t). \tag{23}$$

Here and in the sequel, all equalities or inequalities are to be understood to hold for all $t \geq 0$. Since E is invertible everywhere, (23) gives $\dot{K}(t) + K(t)\dot{E}(t)E^{-1}(t) = A(t)K(t)$. From (17), it follows that $\dot{K}(t) + K(t)E_c = A(t)K(t)$. This gives

$$L(t)\dot{K}(t)L(t) + E_c L(t) = L(t)A(t). \tag{24}$$

Since $L(t)K(t) = I_n$ for all t, it follows that $L(t)\dot{K}(t) + \dot{L}(t)K(t) = 0$. Therefore, (24) gives $-\dot{L}(t)K(t)L(t) + E_c L(t) = L(t)A(t)$. We deduce that $\dot{L}(t) = E_c L(t) - L(t)A(t)$. This equality and (18) ensure that Assumption 1 is satisfied with $L = P$ and $E_c = A_c$. Therefore Lemma 1 applies, which proves Lemma 2. □

We next turn to our general ways to find functions K and E that satisfy Assumption 2.

3 Results Based on Floquet's Theory in Continuous Time

3.1 Theoretical result

In the particular case where the function A is periodic, Floquet's theory (e.g., [26, Section 3.5]) guarantees that there are two functions K and E such that Assumption 2 is satisfied. More precisely, thanks to Floquet's theory, we can establish the following result:

Theorem 1. *Consider the system (2) and assume that A is periodic of any period $T > 0$. Then there exist an everywhere invertible C^1 function $H : \mathbb{R} \to \mathbb{R}^{n\times n}$ that is periodic of period T and a constant matrix $R \in \mathbb{R}^{n\times n}$ such that the change of coordinates $z = H(t)x$ transforms (2) into*

$$\dot{z}(t) = Rz(t) + H(t)F(t)u(t-\tau) \tag{25}$$

for all $t \geq 0$.

Proof. Consider the system $\dot{x} = A(t)x$ from (15) associated with the system (2). Let $X : \mathbb{R} \to \mathbb{R}^{n\times n}$ be the fundamental solution of (15) such that $X(0) = I_n$, so $\dot{X}(t) = A(t)X(t)$ and $X(t)$ is invertible for all $t \in \mathbb{R}$. Then Floquet's theory (e.g., from [26, Section 3.5]) ensures that there exist a constant matrix $R \in \mathbb{R}^{n\times n}$, and an everywhere invertible C^1 function $Q : \mathbb{R} \to \mathbb{R}^{n\times n}$ that is periodic of period T, such that

$$X(t) = Q(t)e^{Rt} \tag{26}$$

holds for all $t \in \mathbb{R}$. Then Q^{-1} is C^1, because X is C^1 and invertible. By (26), we deduce that Assumption 2 is satisfied with $K(t) = Q(t)$, $E(t) = e^{Rt}$, and $E_c = R$. It follows from Lemma 2 that the change of coordinates

$$z(t) = Q^{-1}(t)x(t) \tag{27}$$

transforms the system (2) into $\dot{z}(t) = Rz(t) + Q^{-1}(t)F(t)u(t-\tau)$. Since Q^{-1} is periodic of period T, this allows us to conclude by setting $H = Q^{-1}$. □

A challenge in applying Theorem 1 is that determining the function Q is not easy. This fact is a fundamental limitation of Floquet's theory. This remark motivates the next subsection.

3.2 More Applied Results Based on Floquet's Theory

We again consider the system (15) in the case where A is periodic of some period T. Then the decomposition (26) of the fundamental matrix gives $I_n = X(0) = Q(0) = Q(T)$, so $Q(T) = I_n$ and $X(T) = e^{RT}$. Through numerical simulations, we can determine $X(T)$, which is invertible. Recalling from [10] that an invertible real matrix has a real matrix logarithm (denoted by log) if and only if each Jordan block in its Jordan decomposition corresponding to a negative eigenvalue occurs an even number of times, it follows that if $X(T)$ has no eigenvalues that are contained in $(-\infty, 0)$, then we can set $R = \frac{1}{T}\log(X(T))$; see [15, Section 1.6] for the definition of and formulas for matrix logarithms of invertible matrices.

Next, we return to (2). Let us show how we can determine a function Q such that $X(t) = Q(t)e^{Rt}$, which makes it possible to apply Theorem 1 in practice. Since $X(t) = Q(t)e^{Rt}$ is a fundamental solution of $\dot{x} = A(t)x$, we get $\dot{X}(t) = \dot{Q}(t)e^{Rt} + Q(t)Re^{Rt} = A(t)Q(t)e^{Rt}$, so cancelling e^{Rt} from both sides gives $Q^{-1}(t)\dot{Q}(t)Q^{-1}(t) + RQ^{-1}(t) = Q^{-1}(t)A(t)$. This gives

$$RH(t) - H(t)A(t) = -Q^{-1}\dot{Q}(t)Q^{-1}(t) = \dot{H}(t), \tag{28}$$

where $H = Q^{-1}$, and where the last equality follows by computing the derivative of $Q(t)Q^{-1}(t)$ using the product rule. Therefore, we consider the system (2) with the dynamic extension $\dot{\varpi}(t) = R\varpi(t) - \varpi(t)A(t)$. Then, since $Q(0) = I_n$, it follows that when $\varpi(0) = I_n$, we have $H = \varpi$. Hence, through a simple dynamic extension with the identity as the initial condition, we have H.

Then Theorem 1 implies that the change of coordinates $z(t) = H(t)x(t)$ gives

$$\dot{z}(t) = Rz(t) + \varpi(t)F(t)u(t-\tau). \tag{29}$$

Therefore, when we apply the reduction model approach, we consider the operator Z from (7), namely,

$$Z(t) = z(t) + \int_{t-\tau}^{t} e^{R(t-m-\tau)}\varpi(m+\tau)F(m+\tau)u(m)\mathrm{d}m. \tag{30}$$

Although this involves future values of ϖ, the function ϖ is periodic of period T. Therefore, we can replace $\varpi(m+\tau)$ by $\varpi(m+\tau-kT)$, where k is a sufficiently large integer. After such a substitution, (30) is only valid for sufficiently large t. Then the reasoning that gave the delay free system (8) gives

$$\dot{Z}(t) = RZ(t) + e^{-R\tau}\varpi(t+\tau-kT)F(t+\tau)u(t). \tag{31}$$

This means that if we stabilize this system using a state feedback, it will depend on the past ϖ values. Alternatively, we can consider the system

$$\left\{\begin{array}{rcl} \dot{x}(t) & = & A(t)x(t) + F(t)u(t-\tau) \\ \dot{\varpi}(t) & = & R\varpi(t) - \varpi(t)A(t) \\ \varpi(lT) & = & I_n \ , \ l \in \{0,1,2,\ldots\} \end{array}\right. \tag{32}$$

to obtain the closed loop system.

Remark 2. *If we consider the system $\dot{x}(t) = f(x(t))$ with x valued in $\mathbb{R}^2$ with a periodic solution $r(t)$ of some period T and with $\dot{r}(0) \neq 0$ and evaluate the linear approximation around this solution, then for the corresponding linearized system $\dot{x} = A(t)x$ (where $A(t)$ is the Jacobian matrix $A(t) = Df(r(t))$), we know that 1 is one of the eigenvalues of the corresponding matrix e^{RT}. This follows because $\dot{r}(t)$ is a solution of $\dot{x} = A(t)x$, so $\dot{r}(0) = \dot{r}(T) = Q(T)e^{RT}\dot{r}(0) = e^{RT}\dot{r}(0)$. Then, using the relation*

$$\rho_1\rho_2 = \exp\left(\int_0^T \mathrm{trace}(A(\ell))\mathrm{d}\ell\right) \tag{33}$$

between the eigenvalues ρ_1 and ρ_2 of e^{RT} that is provided by Floquet's theory (e.g., [3, Proposition 2.39]), we get $\rho_1 = 1$ and

$$\rho_2 = \exp\left(\int_0^T \mathrm{trace}(A(\ell))\mathrm{d}\ell\right). \tag{34}$$

Therefore one does not need to use a computer to get the two eigenvalues of e^{RT} to compute the matrix logarithm for the Floquet theory decomposition of the fundamental matrix.

Before illustrating our theory in the continuous time case in Section 6, we provide the following discrete time analogues of the preceding results.

4 Preliminary Results in Discrete Time

We propose a more general version of Floquet's theory for discrete time systems than what was developed in [13] and [28]. We wish to design delay compensating stabilizing controllers for systems of the form

$$x_{k+1} = A_k x_k + B_k u_{k-r} \tag{35}$$

where $x_k \in \mathbb{R}^n$ is the state, $u_k \in \mathbb{R}^p$ is the control, and $r \in \mathbb{N}$ is the delay. We first consider the system

$$x_{k+1} = A_k x_k \tag{36}$$

with $k \in \mathbb{N}$ and x_k valued in $\mathbb{R}^n$. Since our extension does not require A_k to be periodic, we believe that it is new. Moreover, our changes of coordinates are explicit in many interesting cases. The work [28] shows that in the periodic context, matrices similar to the R_k's defined below can be found, but in [13], the change of coordinates is not explicit, and [13, 28] do not cover reduction model controls. We introduce two assumptions.

Assumption 3. *For all $k \geq 0$, the matrix A_k is invertible.*

Assumption 3 allows us to introduce a sequence of matrices defined by

$$P_k = \prod_{i=0}^{k-1} A_i^{-1} \tag{37}$$

for all $k \geq 1$ and $P_0 = I_n$. Then P_k is invertible for all $k \in \mathbb{N}$. Our second assumption is:

Assumption 4. *There are an invertible matrix $A_* \in \mathbb{R}^{n\times n}$ and a real number $c > 0$ such that the sequence*

$$R_k = A_*^k P_k \tag{38}$$

satisfies

$$|R_k^{-1}| + |R_k| \leq c \tag{39}$$

for all $k \in \mathbb{N}$.

We prove the following:

Lemma 3. *Let Assumptions 3-4 hold. Then for each solution sequence x_k of (36), the sequence*

$$z_k = R_k x_k \tag{40}$$

satisfies

$$z_{k+1} = A_* z_k \tag{41}$$

for all k.

Proof. We have $z_{k+1} = R_{k+1} A_k x_k = R_{k+1} A_k R_k^{-1} z_k$. Here and in the sequel, all equalities or inequalities should be understood to hold for all integers $k \geq 0$, unless otherwise indicated. Our definitions give

$$R_{k+1} A_k R_k^{-1} \;=\; A_*^{k+1} \left(\prod_{i=0}^{k} A_i^{-1} \right) A_k P_k^{-1} (A_*^k)^{-1} \;=\; A_*^{k+1} \left(\prod_{i=0}^{k-1} A_i^{-1} \right) P_k^{-1} (A_*^k)^{-1} \;=\; A_* \,. \tag{42}$$

By right multiplying through (42) by z_k, it follows that z_k is solution of (41), which proves the lemma. □

If A_* is known, then R_k is given by an explicit formula. Finding A_* can be difficult. However, if A_k is periodic with some period p, and if $A_{p-1} A_{p-2} \ldots A_0$ has no eigenvalues that lie in the interval $(-\infty, 0]$, then [5] and [15, Theorem 7.2] provide a matrix $A_* \in \mathbb{R}^{n \times n}$ such that $A_*^p = A_{p-1} A_{p-2} \ldots A_0$, as well as algorithms for computing this pth root matrix. Since (42) gives $R_{k+1} = A_* R_k A_k^{-1}$ and $R_1 = A_* A_0^{-1}$, we get $R_k = A_*^k A_0^{-1} A_1^{-1} \ldots A_{k-1}^{-1}$ for all $k \geq 1$, which gives $R_p = I_n$ and periodicity of R_k. Hence, this A_* satisfies Assumption 4. To check that this pth root matrix is invertible, notice that if $v \in \mathbb{R}^n \setminus \{0\}$ were in the null space of A_*, then $0 = A_*^p v = A_{p-1}(A_{p-2} \ldots A_0 v)$, which contradicts the invertibility of the A_k's. See Section 6 for a nonperiodic example where we find A_*.

5 Model Reduction in Discrete Time

5.1 Main Result

In this section, we use Lemma 3 to extend the reduction model approach to a general family of systems of the type (35). We introduce the following assumption, which is a natural analogue of the usual requirement of controllable pairs of coefficient matrices, generalized to time-varying systems with delays:

Assumption 5. *The system (35) is such that the sequence A_k satisfies Assumptions 3-4, and the sequence B_k is bounded. Also, there is a sequence K_k such that the sequence $K_k R_k^{-1}$ is bounded and such that the system*

$$\zeta_{k+1} = H_k \zeta_k, \quad \text{where} \quad H_k = A_k + R_{k+1}^{-1} R_{k+r+1} B_{k+r} K_k \tag{43}$$

is uniformly exponentially stable, where R_k is the sequence defined in (38).

We prove the following:

Theorem 2. *Assume that the system (35) satisfies Assumption 5. Then we can find positive constants $\bar{c}_1$ and $\bar{c}_2$ such that all trajectories of (35), in closed loop with*

$$u_k = K_k R_k^{-1} \left(A_*^r R_k x_k + \sum_{i=1}^{r} A_*^{i-1} R_{k+r-i+1} B_{k+r-i} u_{k-i} \right), \tag{44}$$

satisfy $|x_k| \leq \bar{c}_1 e^{\bar{c}_2 (k_0 - k)} (|x_{k_0}| + |u|_{[k_0 - r, k_0]})$ for all integers $k_0 \geq 1$ and $k \geq k_0$.

Proof. We can use (42) to show that the change of coordinates $z_k = R_k x_k$ applied to (35) gives

$$z_{k+1} = A_* z_k + G_k u_{k-r}, \tag{45}$$

where $G_k = R_{k+1} B_k$. Since A_* is time-invariant, we can adapt ideas from [12] to the problem of stabilizing the system (45), as follows.

We use the operator

$$\Gamma_k = A_*^r z_k + \sum_{i=1}^{r} A_*^{i-1} G_{k+r-i} u_{k-i}. \tag{46}$$

Then,

$$\begin{aligned}
\Gamma_{k+1} &= A_*^r (A_* z_k + G_k u_{k-r}) + \sum_{i=1}^{r} A_*^{i-1} G_{k+1+r-i} u_{k+1-i} \\
&= A_*^{r+1} z_k + A_*^r G_k u_{k-r} + \sum_{i=2}^{r} A_*^{i-1} G_{k+1+r-i} u_{k+1-i} + G_{k+r} u_k \\
&= A_*^{r+1} z_k + \sum_{i=2}^{r+1} A_*^{i-1} G_{k+1+r-i} u_{k+1-i} + G_{k+r} u_k.
\end{aligned} \tag{47}$$

Using the definition of Γ_k, we get

$$\begin{aligned}
\Gamma_{k+1} &= A_* \left(\Gamma_k - \sum_{i=1}^{r} A_*^{i-1} G_{k+r-i} u_{k-i} \right) + \sum_{i=2}^{r+1} A_*^{i-1} G_{k+1+r-i} u_{k+1-i} + G_{k+r} u_k \\
&= A_* \Gamma_k - \sum_{i=1}^{r} A_*^{i} G_{k+r-i} u_{k-i} + \sum_{i=1}^{r} A_*^{i} G_{k+r-i} u_{k-i} + G_{k+r} u_k \;=\; A_* \Gamma_k + G_{k+r} u_k.
\end{aligned} \tag{48}$$

Then we can choose $u_k = K_k R_k^{-1} \Gamma_k$. This feedback is equivalent to the feedback in (44). Then (42) gives

$$\begin{aligned}
\Gamma_{k+1} &= \left(A_* + G_{k+r} K_k R_k^{-1} \right) \Gamma_k \\
&= \left(R_{k+1} A_k R_k^{-1} + G_{k+r} K_k R_k^{-1} \right) \Gamma_k \\
&= R_{k+1} \left(A_k + R_{k+1}^{-1} R_{k+r+1} B_{k+r} K_k \right) R_k^{-1} \Gamma_k \;=\; R_{k+1} H_k R_k^{-1} \Gamma_k.
\end{aligned} \tag{49}$$

From Assumption 5, it follows that $R_k^{-1}\Gamma_k$ is solution of a uniformly exponentially stable system. Hence, there are constants $c_1 > 0$ and $c_2 > 0$ such that for all $k_0 \geq 1$ and $k \geq k_0$, the inequality

$$|\Gamma_k| \leq c_1 e^{c_2 (k_0 - k)} |\Gamma_{k_0}| \tag{50}$$

is satisfied. Recalling that the sequence $K_k R_k^{-1}$ is bounded, we conclude that there is a constant $c_3 > 0$ such that for all $k_0 \geq 1$ and $k \geq k_0$, the inequality

$$|u_k| \leq c_3 e^{c_2 (k_0 - k)} |\Gamma_{k_0}| \tag{51}$$

is satisfied, namely, $c_3 = c_1 \sup_k |K_k R_k^{-1}|$. Next, observe that (46) gives

$$x_k = R_k^{-1} \left((A_*^r)^{-1} \Gamma_k - (A_*^r)^{-1} \sum_{i=1}^{r} A_*^{i-1} G_{k+r-i} u_{k-i} \right). \tag{52}$$

Also, the boundedness of the sequences B_k and R_k gives boundedness of the sequence G_k. From (50), (51), (52) and the properties of the matrices R_k, we can conclude. □

Remark 3. *If we know a strict quadratic Lyapunov function for (43), then, by adapting ideas of [25], we can build a function of Lyapunov-Krasovskii functional type for (35) in closed loop with (44).*

5.2 Important Remark on Discretizations

Consider a continuous time time-varying system

$$\dot{x}(t) = M(t)x(t) + N(t)u(t - \tau) \tag{53}$$

where the delay $\tau > 0$ is constant. Assume that M is periodic, and that M and N are continuous. Since we do not assume that a fundamental solution of $\dot{\xi}(t) = M(t)\xi(t)$ is known, no standard stabilization technique available in the literature applies.

It is natural to approximate the system (53) by

$$\dot{x}(t) = M(t_k)x(t) + N(t_k)u(t_{k-r}), \tag{54}$$

where $t_{k+1} - t_k = \nu$, $\nu > 0$ is a sufficiently small constant, and $r > 0$ is an integer. By integrating (54), we obtain

$$x(t_{k+1}) = e^{\nu M(t_k)}x(t_k) + \left(\int_{t_k}^{t_{k+1}} e^{M(t_k)(t_k+\nu-\ell)}\mathrm{d}\ell\right) N(t_k)u(t_{k-r}). \tag{55}$$

That way, we can handle cases which are not covered by the results of our work [24], e.g., because we do not need an upper bound on the allowable delays, nor do we need a formula for the fundamental matrix for $\dot{x} = M(t)x$ to use in the reduction model control. Notice that, to simplify, we can approximate (55) by

$$x_{k+1} = [I_n + \nu M(t_k)]x_k + \nu N(t_k)u_{t_{k-r}}. \tag{56}$$

If ν is sufficiently small, then all of the matrices $A_k = I_n + \nu M(t_k)$ are invertible. Moreover, we can select the sequence t_k so that the sequence A_k is periodic.

6 Examples

6.1 Continuous Time Example

Consider the system

$$\dot{x}(t) = \left(\sin(t) + \tfrac{1}{2}\right) x(t) + u(t-\tau), \tag{57}$$

where x and u are scalar valued. It is unstable when the input is set to zero, but Theorem 1 applies to (57). In fact, the fundamental matrix of $\dot{x} = \left(\sin(t) + \frac{1}{2}\right) x$ is

$$X(t) = e^{\frac{1}{2}t+1}e^{-\cos(t)}. \tag{58}$$

Therefore, Theorem 1 leads us to consider the change of coordinates (27), which in our case leads to our choice $z = e^{\cos(t)}x$. It yields

$$\dot{z} = \tfrac{1}{2}z + e^{\cos(t)}u(t-\tau). \tag{59}$$

Applying the reduction model approach from Lemma 1, we obtain the undelayed system

$$\dot{Z}(t) = \tfrac{1}{2}Z(t) + e^{\cos(t+\tau)-\frac{\tau}{2}}u(t), \tag{60}$$

where

$$Z(t) = e^{\cos(t)}x(t) + \textstyle\int_{t-\tau}^{t} e^{\frac{1}{2}(t-m-\tau)}e^{\cos(m+\tau)}u(m)\mathrm{d}m. \tag{61}$$

This system is asymptotically stabilized by

$$u(t) = -e^{-\cos(t+\tau)+\frac{\tau}{2}}Z(t) = -e^{-\cos(t+\tau)+\frac{\tau}{2}}\left[e^{\cos(t)}x(t) + \textstyle\int_{t-\tau}^{t} e^{\frac{1}{2}(t-m-\tau)}e^{\cos(m+\tau)}u(m)\mathrm{d}m\right], \tag{62}$$

which gives the closed loop system $\dot{Z}(t) = -\frac{1}{2}Z(t)$.

6.2 Discrete Time Example

Consider the system (35) with $B_k = 1$ for all k and with the sequence of real numbers A_k defined as follows:

$$A_{2l} = 2 \;\text{ and }\; A_{2l+1} = \frac{2l^2+1}{l^2+1} \;\text{ for all }\; l \geq 0 \tag{63}$$

Observe that this sequence is not periodic. With the notation of Section 4, we have $P_0 = 1$, $P_1 = \frac{1}{2}$, and the following for all $l \geq 1$:

$$P_{2l+1} = \prod_{i=0}^{2l} A_i^{-1} = \frac{1}{2^{l+1}}\prod_{i=0}^{l-1}\frac{i^2+1}{2i^2+1} \quad\text{and}\quad P_{2l} = \prod_{i=0}^{2l-1} A_i^{-1} = P_{2l+1}A_{2l} = \frac{1}{2^l}\prod_{i=0}^{l-1}\frac{i^2+1}{2i^2+1} \tag{64}$$

Let us choose $A_* = 2$. Then, for all $l \geq 1$, the functions $R_k = A_*^k P_k$ from Assumption 4 are

$$R_{2l+1} \;=\; 2^{2l+1}\frac{1}{2^{l+1}}\prod_{i=0}^{l-1}\frac{i^2+1}{2i^2+1} \;=\; 2^l\prod_{i=0}^{l-1}\frac{i^2+1}{2i^2+1} \;=\; R_{2l}. \tag{65}$$

Consequently, for all $l \geq 1$, we have

$$\begin{aligned} 0 \;<\; R_{2l} \;=\; R_{2l+1} \;&\leq\; 2^l\prod_{i=0}^{l-1}\frac{i^2+1}{2i^2+1} \;=\; \exp\left(\sum_{i=0}^{l-1}\ln\left(\frac{2i^2+2}{2i^2+1}\right)\right) \;=\; \exp\left(\sum_{i=0}^{l-1}\ln\left(1+\frac{1}{2i^2+1}\right)\right) \\ &\leq\; \exp\left(\sum_{i=0}^{l-1}\frac{1}{2i^2+1}\right) \;\leq\; \exp\left(1+\frac{1}{2}\sum_{i=1}^{\infty}\frac{1}{i^2}\right) \;<\; \infty \end{aligned} \tag{66}$$

and

$$0 \;<\; R_{2l}^{-1} = R_{2l+1}^{-1} \;\leq\; \frac{1}{2^l}\prod_{i=0}^{l-1}\frac{2i^2+1}{i^2+1} \;\leq\; 1. \tag{67}$$

Hence, we can satisfy the bound requirement (39) from Assumption 4 with the bound

$$c = \exp\left(1+\frac{1}{2}\sum_{i=1}^{\infty}\frac{1}{i^2}\right). \tag{68}$$

We now apply Theorem 2 to the corresponding discrete time system (35), with the preceding choices of A_k, $B_k = 1$ for all k, and $r = 4$. With the choices

$$K_k = -\frac{A_k}{R_{k+1}^{-1}R_{k+5}}, \tag{69}$$

the matrices H_k from Assumption 5 become

$$H_k = A_k + R_{k+1}^{-1}R_{k+5}K_k = 0, \tag{70}$$

so Assumption 5 is satisfied. Then Theorem 2 provides us with the stabilizing control law

$$u_k = -16\frac{A_kR_{k+1}}{R_{k+5}}x_k - \sum_{i=1}^{4}\frac{A_kR_{k+1}R_{k+5-i}}{R_kR_{k+5}}2^{i-1}u_{k-i}. \tag{71}$$

Using our formulas (65) for the R_k's and (71), we obtain the formulas

$$\begin{aligned} u_{2l} \;&=\; -32\frac{R_{2l}}{R_{2l+4}}x_{2l} - 2\sum_{i=1}^{4}\frac{R_{2l+5-i}}{R_{2l+4}}2^{i-1}u_{2l-i} \\ &=\; 2\left(\frac{2(l+1)^2+1}{(l+1)^2+1}\left\{-2\frac{2l^2+1}{l^2+1}(2x_{2l}+u_{2l-4}) - u_{2l-2} - 2u_{2l-3}\right\} - u_{2l-1}\right) \quad \text{and} \\ u_{2l+1} \;&=\; -\frac{2l^2+1}{l^2+1}\left(16\frac{R_{2l+2}}{R_{2l+6}}x_{2l+1} + \sum_{i=1}^{4}\frac{R_{2l+2}R_{2l+6-i}}{R_{2l+1}R_{2l+6}}2^{i-1}u_{2l+1-i}\right) \\ &=\; -\frac{2l^2+1}{l^2+1}\left\{4\left(\prod_{i=l+1}^{l+2}\frac{2i^2+1}{i^2+1}\right)x_{2l+1} + \frac{l^2+1}{2l^2+1}\frac{2(l+2)^2+1}{(l+2)^2+1}(u_{2l}+2u_{2l-1})\right. \\ &\qquad \left. + 2\frac{l^2+1}{2l^2+1}\prod_{i=l+1}^{l+2}\frac{2i^2+1}{i^2+1}(u_{2l-2}+2u_{2l-3})\right\} \end{aligned} \tag{72}$$

for all l, which ensure the desired exponential stability property from Theorem 2.

7 Conclusions

We provided new reduction model method input delay compensating controllers for time-varying continuous and discrete time systems, using Floquet's theory and its extensions. Compared with other reduction model methods for time-varying continuous time linear systems (such as our recent work [24]), two potential advantages of our approach are that (a) we do not require closed form expressions for the fundamental matrix for the corresponding uncontrolled drift systems and (b) our new results allow arbitrarily long input delays. In future works, we will investigate stability problems for systems with several and time-varying delays.

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