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STABILITY ESTIMATES FOR SCALAR CONSERVATION LAWS WITH MOVING FLUX CONSTRAINTS

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Abstract. We study well-posedness of scalar conservation laws with moving flux constraints. In particular, we show the Lipschitz continuous dependence of BV solutions with respect to the initial data and the constraint trajectory. Applications to traffic flow theory are detailed.

1. **Introduction.** Motivated by the modeling of moving bottlenecks in traffic flow, we consider the Cauchy problem for a scalar conservation law with moving flux constraint

$$\partial_t \rho + \partial_x f(\rho) = 0, \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}, \tag{1a}$$

$$\rho(0,x) = \rho_0(x), \qquad x \in \mathbb{R}, \tag{1b}$$

$$f(\rho(t,y(t))) - \dot{y}(t)\rho(t,y(t)) \leq \widetilde{F}_\alpha(\dot{y}(t)), \qquad t \in \mathbb{R}^+, \tag{1c}$$

where $t \mapsto y(t)$ is a given trajectory, starting from $y(0) = y_0$. Systems of the form (1) arise in the modeling of moving bottlenecks in vehicular traffic [11, 15]: $\rho = \rho(t,x) \in [0,R]$ is the scalar conserved quantity and represents the traffic density, whose maximum attainable value is R. The flux function $f : [0,R] \to \mathbb{R}^+$ is assumed to be strictly concave and such that $f(0) = f(R) = 0$. The time-dependent variable y denotes the constraint position. In the present paper we consider a weakly coupled PDE-ODE system, in the sense that we assume that the constraint trajectory is given, and it does not depend on the solution of (1a).

Let us detail the meaning of inequality (1c). A moving flux constraint located at $x = y(t)$ acts as an obstacle, thus hindering the flow as expressed by the unilateral constraint (1c). There, $\alpha \in]0,1[$ is the dimensionless reduction rate of the road

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capacity (the maximal allowed density) at the bottleneck position. The inequality (1c) is derived by studying the problem in the constraint reference frame, i.e., setting $\tilde{\rho}(t,x) := \rho(t, x + y(t))$ and rewriting the conservation law (1a) as

$$\partial_t \tilde{\rho} + \partial_x F(\tilde{\rho}) = 0, \qquad F(\tilde{\rho}) = f(\tilde{\rho}) - \dot{y}\tilde{\rho}. \tag{2}$$

In fact, let $f_\alpha : [0, \alpha R] \to \mathbb{R}^+$ be the rescaled flux function describing the reduced flow at $x = y(t)$, i.e. $f_\alpha(\rho) = \alpha f(\rho/\alpha)$, and $\rho_\alpha \in]0, \alpha R[$ such that

$$F'_\alpha(\rho_\alpha) = f'_\alpha(\rho_\alpha) - \dot{y}\rho_\alpha = 0 \quad \Leftrightarrow \quad f'_\alpha(\rho_\alpha) = \dot{y}$$

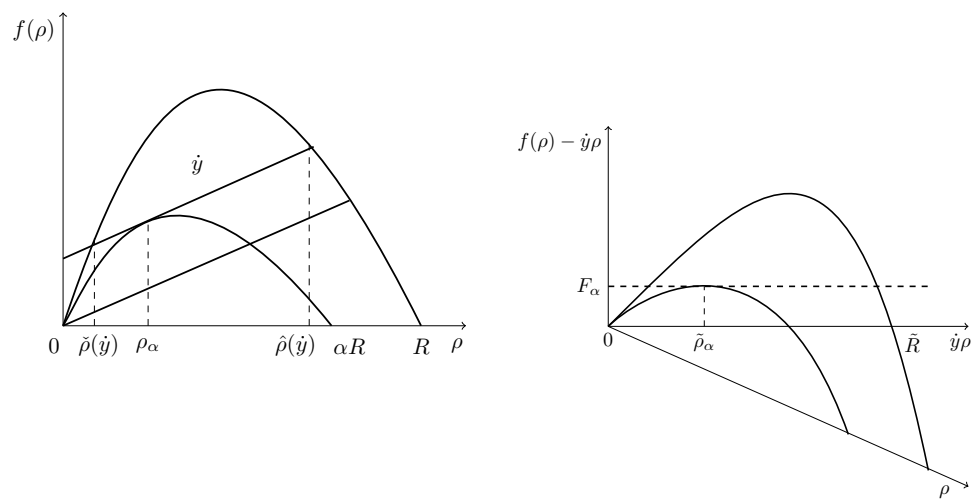
with $F_\alpha(\rho) = f_\alpha(\rho) - \dot{y}\rho$, see Figure 1. Then setting

$$\widetilde{F}_\alpha(\dot{y}) := F_\alpha(\rho_\alpha) = f_\alpha(\rho_\alpha) - \dot{y}\rho_\alpha$$

and imposing that in the obstacle reference frame the flux F is less than the maximum value of the flux of the reduced flow, one gets (1c). Notice that the inequality (1c) is always satisfied if

$$\dot{y}(t) = \frac{f(\rho(t, y(t)\pm))}{\rho(t, y(t)\pm)},$$

since the left-hand side is 0. Moreover, it is well defined even if ρ has a jump at $y(t)$, because of the Rankine-Hugoniot conditions.



(a) Fixed reference frame (b) Obstacle reference frame

FIGURE 1. Graphical representation of the constraint action in the fixed (left) and moving (right) reference frames.

Problem (2), (1b), (1c), can therefore be recast in the framework of conservation laws with fix local constraint, first introduced in [9], then developed in [3, 1] for scalar equations and extended in [12, 2, 13] to systems. Following [11, Definition 4.1] and [5, Definition 1 and 2], solutions of (1) are defined as follows.

Definition 1.1. Let $y \in \mathbf{W^{1,1}}(\mathbb{R}^+; \mathbb{R})$ and $\rho_0 \in \mathbf{L^1} \cap \mathbf{L^\infty}(\mathbb{R}; [0, R])$ be given. A function $\rho \in \mathcal{C}^0\left(\mathbb{R}^+; \mathbf{L^1} \cap \mathbf{L^\infty}(\mathbb{R}; [0, R])\right)$ is a solution to (1) if

1. ρ satisfies Kružhkov entropy conditions [14] on $(\mathbb{R}^+ \times \mathbb{R}) \setminus \{(t, y(t)) \colon t \in \mathbb{R}^+\}$, i.e. for every $k \in [0,1]$ and for all $\varphi \in \mathcal{C}^1_c(\mathbb{R}^2; \mathbb{R}^+)$ and $\varphi(t, y(t)) = 0$, $t > 0$,
$$\int_{\mathbb{R}^+} \int_{\mathbb{R}} \left(|\rho - k| \partial_t \varphi + \operatorname{sgn}(\rho - k) \left(f(\rho) - f(k)\right) \partial_x \varphi\right) dx \; dt + \int_{\mathbb{R}} |\rho_o - k| \varphi(0, x) \; dx \geq 0 \; ; \tag{3a}$$
2. for a. e. $t > 0$ the left and right traces of ρ at $x = y(t)$ satisfy
$$(\rho(t, y(t)-), \rho(t, y(t)+)) \in \mathcal{G}_\alpha(\dot{y}(t)). \tag{3b}$$

The set $\mathcal{G}_\alpha(\dot{y})$ in (13b) is defined as follows, see [4, 3, 5].

Definition 1.2. The *admissibility germ* $\mathcal{G}_\alpha(\dot{y}) \subset [0, R]^2$ for (1a), (1c) is the union $\mathcal{G}_\alpha(\dot{y}) := \mathcal{G}_1(\dot{y}) \cup \mathcal{G}_2(\dot{y}) \cup \mathcal{G}_3(\dot{y})$, where
$$\mathcal{G}_1(\dot{y}) := \left\{(c_L, c_R) \in [0, R]^2 \colon c_L > c_R, \; f(c_L) - \dot{y} c_L = f(c_R) - \dot{y} c_R = F_\alpha(\dot{y})\right\},$$
$$\mathcal{G}_2(\dot{y}) := \left\{(c, c) \in [0, R]^2 \colon f(c) - \dot{y} c \leq F_\alpha(\dot{y})\right\},$$
$$\mathcal{G}_3(\dot{y}) := \left\{(c_L, c_R) \in [0, R]^2 \colon c_L < c_R, \; f(c_L) - \dot{y} c_L = f(c_R) - \dot{y} c_R \leq F_\alpha(\dot{y})\right\}.$$

The equivalence between Definition 1.1 and [11, Definition 4.1] can be proved as in [3, Proposition 2.6].

Systems of the type (1) arise in the modeling of moving bottlenecks in traffic flows, see [11, 15] and Section 3 below, where they are coupled with an ODE depending on the downstream traffic velocity and describing the trajectory of a slow moving vehicle (a bus or a truck) acting as a bottleneck.

This paper is a first step towards establishing well-posedness for the strongly coupled models [11, 15]. Section 2 presents the main result, stating the $\mathbf{L^1}$ Lipschitz continuous dependence of solutions of (1) from the initial data and the constraint trajectory. Section 3 describes in details the related traffic flow model with moving bottleneck. Technical proof details are deferred to Appendix A.

2. Lipschitz continuous dependence of BV solutions with respect to $\dot{y}$.

Let us fix $y \in W^{1,1}(\mathbb{R}^+)$ and $\rho_0 \in \mathrm{BV}(\mathbb{R}; [0, R])$, and let $\rho \in \mathbf{L}^\infty(\mathbb{R}^+ \times \mathbb{R}; [0, R])$ be a solution of (1) in the sense of Definition 1.1. Applying the coordinate transformation $\tilde{\rho}(t, x) := \rho(t, x + y(t))$, $\tilde{\rho}$ is a weak entropy solution of the problem
$$\partial_t \tilde{\rho} + \partial_x F(\tilde{\rho}) = 0, \qquad (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \tag{4a}$$
$$\tilde{\rho}(0, x) = \rho_0(x + y_0), \qquad x \in \mathbb{R}, \tag{4b}$$
$$F(\tilde{\rho}(t, 0)) \leq \widetilde{F}_\alpha(\dot{y}(t)), \qquad t \in \mathbb{R}^+, \tag{4c}$$
in the sense of [3, Proposition 2.6(A)] where we have set $F(\tilde{\rho}) := f(\tilde{\rho}) - \dot{y}\, \tilde{\rho}$. Existence and uniqueness for (4) are proved in [3, 9]. In particular, we have that for every $k \in [0,1]$ and for all $\varphi \in \mathcal{C}^1_c(\mathbb{R}^2; \mathbb{R}^+)$ such that $\varphi(t, 0) = 0$, $t > 0$,
$$\int_{\mathbb{R}^+} \int_{\mathbb{R}} \left(|\tilde{\rho} - k| \partial_t \varphi + \Phi(\dot{y}; \tilde{\rho}, k) \partial_x \varphi\right) dx \; dt + \int_{\mathbb{R}} |\rho_o(x + y_0) - k| \varphi(0, x) \; dx \geq 0 \; ,$$
where we have set
$$\Phi(\dot{y}; a, b) := \operatorname{sgn}(a - b) \left(f(a) - f(b)\right) - \dot{y} |a - b|, \qquad \text{for } a, b \in \mathbb{R},$$
and
$$(\tilde{\rho}(t, 0-), \tilde{\rho}(t, 0+)) \in \mathcal{G}_\alpha(\dot{y}(t)) \quad \text{for a. e. } t > 0,$$

see [3, Proposition 2.6]. Moreover, we note that, since $\alpha < 1$ and f is strictly concave, we have

$$\widetilde{F}_\alpha(\dot y) = f_\alpha(\rho_\alpha) - \dot y \rho_\alpha < f(\rho_\alpha) - \dot y \rho_\alpha = \max_{\rho\in[0,R]} F(\rho). \tag{5}$$

Remark 1. Solution to (4) are not in BV in general, since $y \in W^{1,1}(\mathbb{R}^+)$ and thus $\dot y \notin \mathrm{BV}(\mathbb{R}^+)$. Yet, if $\dot y \in \mathrm{BV}([0,t];\mathbb{R})$, $\tilde\rho(t,\cdot) \in \mathrm{BV}(\mathbb{R};[0,R])$ due to (5).
In any case, the left and right traces at $x = 0$ do exist, see [3, Section 2].

We compare solutions of (4) corresponding to different constraint trajectories y and z.

Theorem 2.1. *Assume $y, z \in W^{1,1}(\mathbb{R}^+)$ with $0 \le \dot y, \dot z < f'(0)$, $\tilde\rho_0, \tilde\sigma_0 \in \mathbf{L}^\infty(\mathbb{R},[0,R])$ and $\tilde\rho_0 - \tilde\sigma_0 \in \mathbf{L^1}(\mathbb{R})$. Let $\tilde\rho, \tilde\sigma \in \mathbf{L}^\infty(\mathbb{R}^+\times\mathbb{R};[0,R])$ be the solutions of (4) corresponding respectively to $y, \tilde\rho_0$ and $z, \tilde\sigma_0$. Moreover we assume that $TV(\tilde\rho(t,\cdot)) \le C$ for all $t > 0$. Then we have*

$$\|\tilde\rho(t,\cdot) - \tilde\sigma(t,\cdot)\|_{\mathbf{L^1}(\mathbb{R})} \le \|\tilde\rho_0 - \tilde\sigma_0\|_{\mathbf{L^1}(\mathbb{R})} + (C+2R)\|\dot y - \dot z\|_{\mathbf{L^1}([0,t])}. \tag{6}$$

Proof. The two solutions $\tilde\rho$, $\tilde\sigma$ satisfy

$$\partial_t|\tilde\rho - k| + \partial_x\Phi(\dot y;\tilde\rho,k) \le 0, \tag{7}$$
$$\partial_t|\tilde\sigma - k| + \partial_x\Phi(\dot z;\tilde\sigma,k) \le 0, \tag{8}$$

in $\mathcal{D}'(\mathbb{R}^+\times\mathbb{R}^*)$ (where we have noted $\mathbb{R}^* = \mathbb{R}\setminus\{0\}$). Following the proofs of [5, Lemma 15] and [6, Theorem 3.1] we observe that

$$\partial_t|\tilde\rho - k| + \partial_x\Phi(\dot y;\tilde\rho,k) = \partial_t|\tilde\rho - k| + \partial_x\Phi(\dot z;\tilde\rho,k) + \partial_x\Phi(\dot y;\tilde\rho,k) - \partial_x\Phi(\dot z;\tilde\rho,k)$$

therefore

$$\begin{aligned}\partial_t|\tilde\rho - k| + \partial_x\Phi(\dot z;\tilde\rho,k) &\le \partial_x\Phi(\dot z;\tilde\rho,k) - \partial_x\Phi(\dot y;\tilde\rho,k)\\ &\le |\dot y - \dot z||\partial_x\tilde\rho|.\end{aligned} \tag{9}$$

Applying the classical Kružkov doubling of variables technique [14], with a test function $\psi \in \mathcal{C}^1_c(\mathbb{R}^2;\mathbb{R}^+)$ such that $\psi(t,0) = 0$, we deduce the following Kato inequality

$$\begin{aligned}&\int_{\mathbb{R}^+}\int_{\mathbb{R}} (|\tilde\rho - \tilde\sigma|\partial_t\psi + \Phi(\dot z;\tilde\rho,\tilde\sigma)\partial_x\psi)\,dx\,dt\\ &\qquad + \int_{\mathbb{R}} |\tilde\rho_0 - \tilde\sigma_0|\psi(0,x)\,dx + C\|\psi\|_\infty \int_{\mathbb{R}^+} |\dot y - \dot z|dt \ge 0\ ,\end{aligned}$$

We now choose the test function $\psi(t,x) = \theta_\varepsilon(x)\xi(t,x)$, where $\xi \in \mathcal{C}^1_c(\mathbb{R}^2;\mathbb{R}^+)$ is an approximation of the characteristic function of the trapezoid

$$\{(s,x)\colon |x| \le M + L(t-s),\ 0 \le s \le t\}$$

(where $L \ge |V - \dot z|$)and θ_ε a smooth approximation of $x \to \min\{|x|/\varepsilon, 1\}$. Following the proof of [9, Proposition 4.4] and letting $\varepsilon \searrow 0$ we get

$$\begin{aligned}\int_{-M}^{M} |\tilde\rho(t,x) - \tilde\sigma(t,x)|dx &\le \int_{-M-Lt}^{M+Lt} |\tilde\rho_0(x) - \tilde\sigma_0(x)|dx + C\int_0^t |\dot y(s) - \dot z(s)|ds\\ &\quad + \int_0^t (\Phi(\dot z;\tilde\rho(t,0+),\tilde\sigma(t,0+)) - \Phi(\dot z;\tilde\rho(t,0-),\tilde\sigma(t,0-)))\,ds.\end{aligned}$$

By Lemma A.1, the last integrand can be bounded by

$$\int_0^t \left(\Phi(\dot{z};\tilde{\rho}(t,0+),\tilde{\sigma}(t,0+)) - \Phi(\dot{z};\tilde{\rho}(t,0-),\tilde{\sigma}(t,0-))\right) ds \leq 2R \int_0^t |\dot{y}(s) - \dot{z}(s)| ds. \tag{10}$$

Letting $M \to \infty$, we recover (6). □

3. **Application to traffic modeling.** Setting

$$f(\rho) := \rho v(\rho),$$

where $v(\rho) = V(1 - \rho/R)$ is the mean traffic speed, V being the maximal velocity allowed on the road, problem (1) can be used to describe the situation of a moving bottleneck along a road, see [11]. In this case, we get $f_\alpha(\rho) = V\rho\left(1 - \frac{\rho}{\alpha R}\right)$ and $\rho_\alpha = \frac{\alpha R}{2}\left(1 - \frac{\dot{y}}{V}\right)$, so that

$$\widetilde{F}_\alpha(\dot{y}) = \frac{\alpha R}{4V}(V - \dot{y})^2.$$

Let us suppose that a slow and large vehicle, like for example a bus or a truck moves on the road. The slow vehicle, that in the following we will refer as bus, reduces the road capacity and moves with a trajectory given by the following ODE:

$$\begin{cases} \dot{y}(t) = \omega(\rho(t, y(t)+)), & t \in \mathbb{R}^+, \\ y(0) = y_0, \end{cases} \tag{11}$$

where the velocity of the bus is given by the following traffic density dependent function (see Figure 2)

$$\omega(\rho) = \begin{cases} V_b & \text{if } \rho \leq \rho^* \doteq R(1 - V_b/V), \\ v(\rho) & \text{otherwise.} \end{cases} \tag{12}$$

This means that if the traffic is not too congested, the bus moves at its own maximal speed $V_b < V$. When the surrounding traffic density becomes too high, the bus adapts its velocity accordingly. In particular, it is not possible for the bus to overtake the cars.

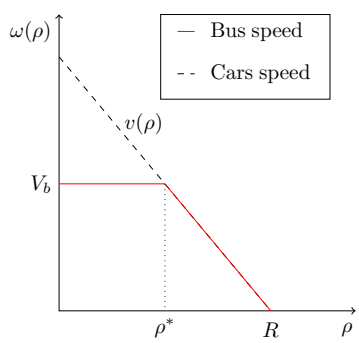


FIGURE 2. Bus and cars speed.

Solutions of the coupled system (1), (11) for general initial data are defined as follows.

Definition 3.1. A couple $(\rho, y) \in \mathcal{C}^0\left(\mathbb{R}^+; \mathbf{L^1} \cap \mathrm{BV}(\mathbb{R}; [0,1])\right) \times \mathbf{W^{1,1}}(\mathbb{R}^+; \mathbb{R})$ is a solution to (1) if

1. ρ satisfies Kružhkov entropy conditions [14] on $(\mathbb{R}^+ \times \mathbb{R}) \setminus \{(t, y(t)) \colon t \in \mathbb{R}^+\}$, i.e. for every $k \in [0,1]$ and for all $\varphi \in \mathcal{C}^1_c(\mathbb{R}^2; \mathbb{R}^+)$ and $\varphi(t, y(t)) = 0$, $t > 0$,

$$\int_{\mathbb{R}^+} \int_{\mathbb{R}} \left(|\rho - k| \partial_t \varphi + \operatorname{sgn}(\rho - k) \left(f(\rho) - f(k) \right) \partial_x \varphi \right) dx \; dt + \int_{\mathbb{R}} |\rho_o - k| \varphi(0, x) \; dx \geq 0 \; ; \tag{13a}$$

2. for a. e. $t > 0$ the one-sided traces of ρ at $x = y(t)$ satisfy

$$(\rho(t, y(t)-), \rho(t, y(t)+)) \in \mathcal{G}_\alpha(\dot{y}(t)) \, ; \tag{13b}$$

3. y is a Carathéodory solution of (11), i.e. for a.e. $t \in \mathbb{R}^+$

$$y(t) = y_o + \int_0^t \omega(\rho(s, y(s)+)) \; ds \; . \tag{13c}$$

The proof of existence of solutions for the general Cauchy problem (1) strongly coupled with the bus trajectory (11) with BV initial data can be found in [11]. For completeness, we recall the definition the solution of a Riemann problem, as given in [11]. Let us consider a Riemann type initial datum

$$\rho_0(x) = \begin{cases} \rho_L & \text{if } x < 0, \\ \rho_R & \text{if } x > 0, \end{cases} \qquad y_0 = 0. \tag{14}$$

Denote by $\mathcal{R}$ the standard (i.e., without the constraint (1c)) Riemann solver for (1a)-(1b)-(14), i.e., the (right continuous) map $(t, x) \mapsto \mathcal{R}(\rho_L, \rho_R)(x/t)$ given by the standard weak entropy solution, see for instance [7, Chapter 5]. Moreover, assume that $\dot{y}$ is constant and let $\check{\rho} = \check{\rho}(\dot{y})$ and $\hat{\rho} = \hat{\rho}(\dot{y})$, with $\check{\rho} \leq \hat{\rho}$, be the intersection points of the flux function $f(\rho)$ with the line $f_\alpha(\rho_\alpha) + \dot{y}(\rho - \rho_\alpha)$ (see Figure 1(a)):

$$\begin{aligned} \check{\rho}(\dot{y}) &= \frac{R}{2V} \left(1 - \sqrt{1 - \alpha}\right) (V - \dot{y}), \\ \hat{\rho}(\dot{y}) &= \frac{R}{2V} \left(1 + \sqrt{1 - \alpha}\right) (V - \dot{y}). \end{aligned} \tag{15}$$

Definition 3.2. The constrained Riemann solver $\mathcal{R}^\alpha : [0, R]^2 \to \mathbf{L^1_{loc}}(\mathbb{R}; [0, R])$ corresponding to (1), (11), (14) is defined as follows.

1. If $f(\mathcal{R}(\rho_L, \rho_R)(V_b)) > F_\alpha + V_b \mathcal{R}(\rho_L, \rho_R)(V_b)$, then

$$\mathcal{R}^\alpha(\rho_L, \rho_R)(x/t) = \begin{cases} \mathcal{R}(\rho_L, \hat{\rho}(V_b))(x/t) & \text{if } x < V_b t, \\ \mathcal{R}(\check{\rho}(V_b), \rho_R)(x/t) & \text{if } x \geq V_b t, \end{cases} \qquad \text{and} \quad y(t) = V_b t.$$

2. If $V_b \mathcal{R}(\rho_L, \rho_R)(V_b) \leq f(\mathcal{R}(\rho_L, \rho_R)(V_b)) \leq F_\alpha + V_b \mathcal{R}(\rho_L, \rho_R)(V_b)$, then

$$\mathcal{R}^\alpha(\rho_L, \rho_R) = \mathcal{R}(\rho_L, \rho_R) \quad \text{and} \quad y(t) = V_b t.$$

3. If $f(\mathcal{R}(\rho_L, \rho_R)(V_b)) < V_b \mathcal{R}(\rho_L, \rho_R)(V_b)$, i.e., $v(\mathcal{R}(\rho_L, \rho_R)(V_b)) < V_b$ then

$$\mathcal{R}^\alpha(\rho_L, \rho_R) = \mathcal{R}(\rho_L, \rho_R) \quad \text{and} \quad y(t) = v(\rho_R) t.$$

Note that, when the constraint is enforced (point 1. in the above definition), a non-classical shock arises between $\hat{\rho}(V_b)$ and $\check{\rho}(V_b)$, which satisfies the Rankine-Hugoniot condition but violates the Lax entropy condition, see Figure 3 for an example.

Remark 2. Unfortunately, no result about the Lipschitz continuous dependence of the solution $y = y(t)$ of (11) from the solution $\rho = \rho(t,x)$ of (1) is known at present. Related results [8, 10] concerning uniqueness and continuous dependence for ODEs of the form (11) hold under hypothesis on the speed function ω that are not satisfied by (12).

Appendix A. Proof of (10).

Lemma A.1. *For any $(\rho_-,\rho_+) \in \mathcal{G}_\alpha(\dot y)$ and $(\sigma_-,\sigma_+) \in \mathcal{G}_\alpha(\dot z)$, it holds*

$$\Phi(\dot z;\rho_+,\sigma_+) - \Phi(\dot z;\rho_-,\sigma_-) \le 2R|\dot y - \dot z|.$$

Proof. For simplicity, we consider the case $f(\rho) = V\rho(1-\rho/R)$. First of all, let us remark that

$$\begin{aligned}
&\Phi(\dot z;\rho_+,\sigma_+) - \Phi(\dot z;\rho_-,\sigma_-) && (16)\\
&= \operatorname{sgn}(\rho_+ - \sigma_+)\left(f(\rho_+) - \dot z\rho_+ - f(\sigma_+) + \dot z\sigma_+\right)\\
&\quad - \operatorname{sgn}(\rho_- - \sigma_-)\left(f(\rho_-) - \dot z\rho_- - f(\sigma_-) + \dot z\sigma_-\right)\\
&= (\lambda(\rho_+,\sigma_+) - \dot z)|\rho_+ - \sigma_+| - (\lambda(\rho_-,\sigma_-) - \dot z)|\rho_- - \sigma_-|, && (17)
\end{aligned}$$

where

$$\lambda(\rho,\sigma) = \frac{f(\rho) - f(\sigma)}{\rho - \sigma} = V\left(1 - \frac{\rho+\sigma}{R}\right).$$

Without loss of generality, we can assume $\dot y < \dot z$. Therefore we get

$$F_\alpha(\dot y) - F_\alpha(\dot z) = \frac{\alpha R}{4V}(2V - \dot z - \dot y)(\dot z - \dot y) > 0. \tag{18}$$

We distinguish the following cases:

1. $(\rho_-,\rho_+) \in \mathcal{G}_1(\dot y)$: we observe that

$$\rho_- = \hat\rho,\ \rho_+ = \check\rho \quad \text{and} \quad f(\rho_-) - \dot y\rho_- = f(\rho_+) - \dot y\rho_+ = F_\alpha(\dot y).$$

Depending on the values of σ_-, σ_+, we different situations can occur as shown in Figure 4:

1.1 $(\sigma_-,\sigma_+) \in \mathcal{G}_1(\dot z)$: in this case $\sigma_- = \hat\sigma$, $\sigma_+ = \check\sigma$, as shown in Figure 4a and $f(\sigma_-) - \dot z\sigma_- = f(\sigma_+) - \dot z\sigma_+ = F_\alpha(\dot z)$, therefore

$$\begin{aligned}
(16) &= (F_\alpha(\dot y) + \dot y\rho_+ - \dot z\rho_+ - F_\alpha(\dot z)) - (F_\alpha(\dot y) + \dot y\rho_- - \dot z\rho_- - F_\alpha(\dot z))\\
&= (\rho_- - \rho_+)(\dot z - \dot y)\\
&\le R|\dot y - \dot z|.
\end{aligned}$$

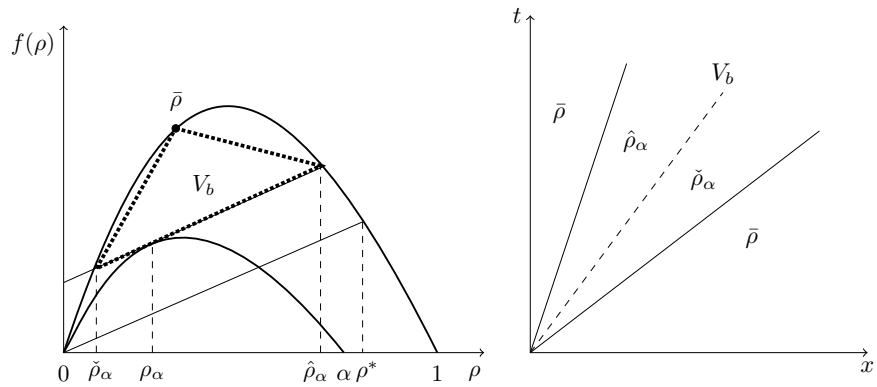
1.2 $(\sigma_-,\sigma_+) \in \mathcal{G}_2(\dot z)$: we set

$$\sigma := \sigma_- = \sigma_+ \quad \text{and} \quad f(\sigma) - \dot z\sigma =: F(\sigma) \le F_\alpha(\dot z).$$

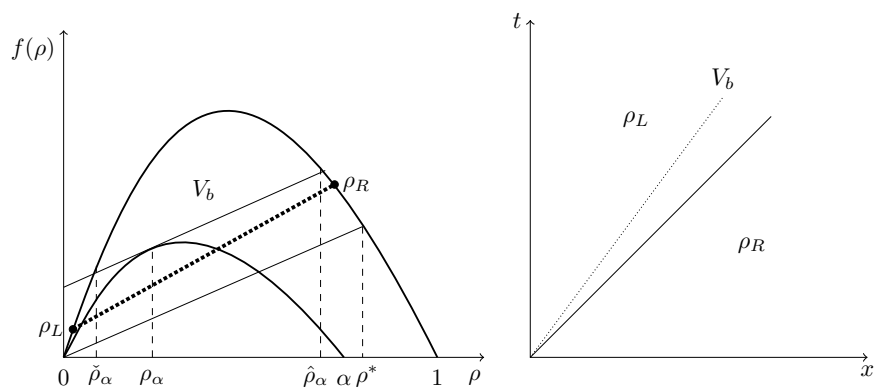
The following cases can occur:

- $0 \le \sigma \le \check\sigma$:

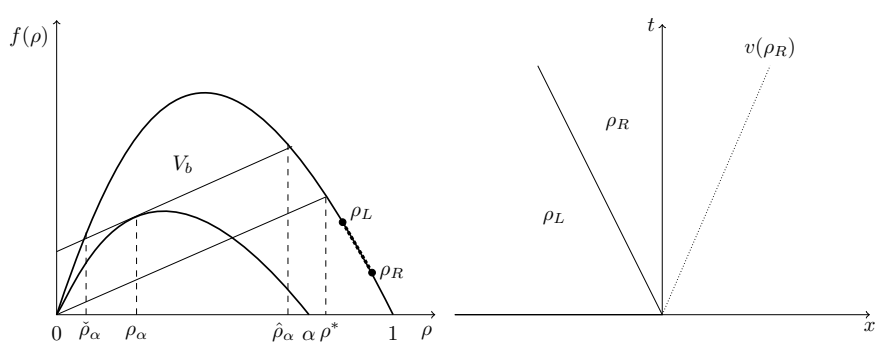
$$\begin{aligned}
(16) &= (F_\alpha(\dot y) + \dot y\rho_+ - \dot z\rho_+ - F(\sigma)) - (F_\alpha(\dot y) + \dot y\rho_- - \dot z\rho_- - F(\sigma))\\
&= (\rho_- - \rho_+)(\dot z - \dot y)\\
&\le R|\dot y - \dot z|.
\end{aligned}$$



(a) Example of solution corresponding to case 1. of Definition 3.2 for $\rho_L = \rho_R = \bar\rho$, $\check\rho_\alpha < \bar\rho < \hat\rho_\alpha$.

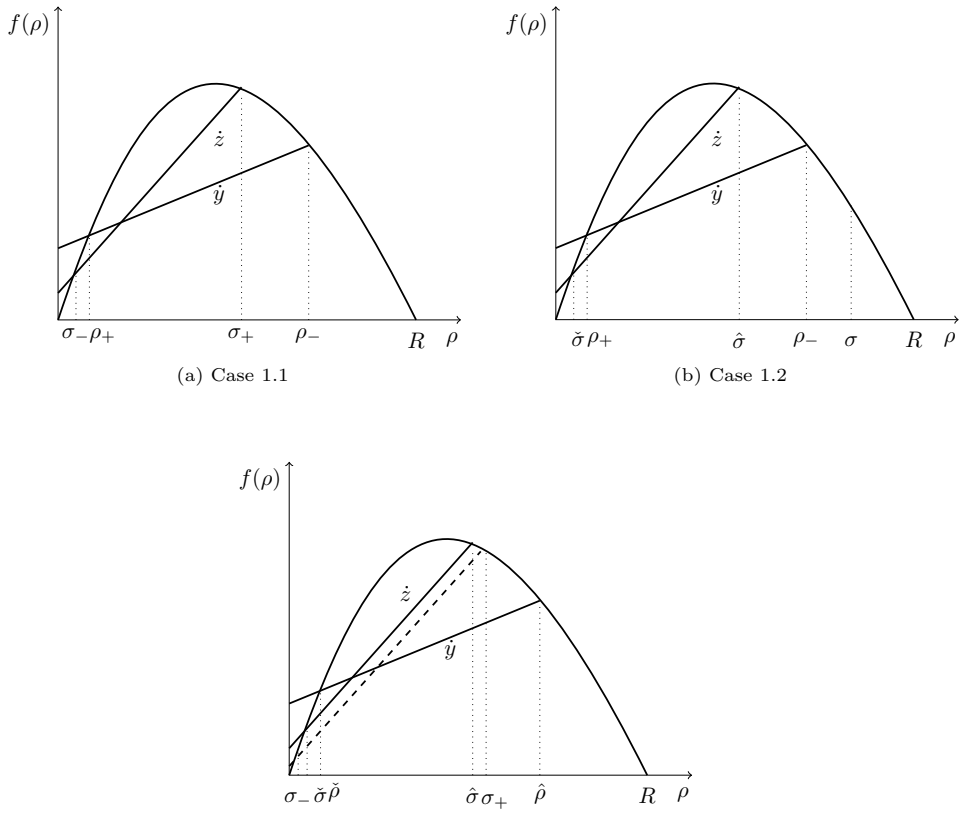


(b) Example of solution corresponding to case 2. of Definition 3.2 for $0 < \rho_L < \check\rho_\alpha$ and $\hat\rho_\alpha < \rho_R < \rho^*$.



(c) Example of solution corresponding to case 3. of Definition 3.2 for $\rho_L, \rho_R > \rho^*$.

FIGURE 3. Different solutions of the Riemann problem (14). Each subfigure illustrates a point of the Definition 3.2: fundamental diagram representation (left) and space-time diagram (right).



(a) Case 1.1

(b) Case 1.2

(c) Case 1.3

FIGURE 4. Case 1

- $\hat{\sigma} \le \sigma \le \hat{\rho}$, see Figure 4b:

$$
\begin{aligned}
(16) &= -\left(F_\alpha(\dot{y}) + \dot{y}\rho_+ - \dot{z}\rho_+ - F(\sigma)\right) - \left(F_\alpha(\dot{y}) + \dot{y}\rho_- - \dot{z}\rho_- - F(\sigma)\right) \\
&= 2F(\sigma) - 2F_\alpha(\dot{y}) + (\rho_+ + \rho_-)(\dot{z} - \dot{y}) \\
&\le 2F_\alpha(\dot{z}) - 2F_\alpha(\dot{y}) + 2R(\dot{z} - \dot{y}) \\
&= -\frac{\alpha R}{2V}(2V - \dot{z} - \dot{y})(\dot{z} - \dot{y}) + 2R(\dot{z} - \dot{y}) \\
&\le 2R|\dot{y} - \dot{z}|.
\end{aligned}
$$

- $\hat{\rho} \le \sigma \le R$:

$$
\begin{aligned}
(16) &= -\left(F_\alpha(\dot{y}) + \dot{y}\rho_+ - \dot{z}\rho_+ - F(\sigma)\right) + \left(F_\alpha(\dot{y}) + \dot{y}\rho_- - \dot{z}\rho_- - F(\sigma)\right) \\
&= (\rho_+ - \rho_-)(\dot{z} - \dot{y}) \le 0.
\end{aligned}
$$

1.3 $(\sigma_-, \sigma_+) \in \mathcal{G}_3(\dot{z})$: we set

$$f(\sigma_-) - \dot{z}\sigma_- = f(\sigma_+) - \dot{z}\sigma_+ =: F(\sigma) \le F_\alpha(\dot{z}).$$

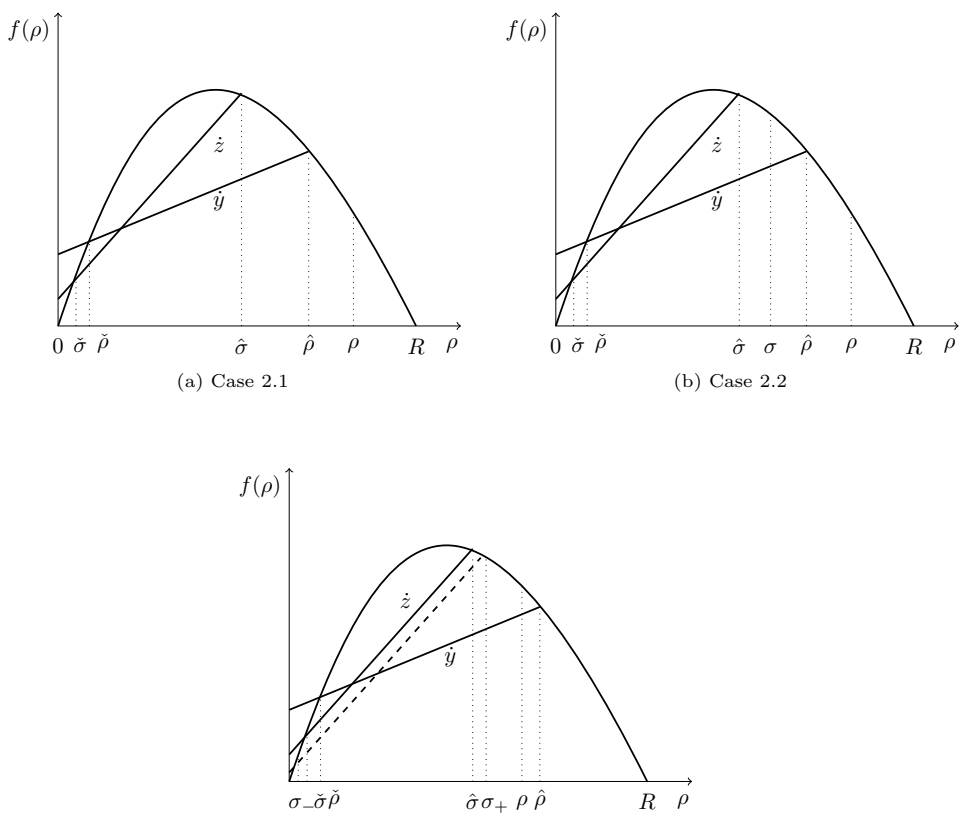
We observe that $\operatorname{sgn}(\rho_+ - \sigma_+) < 0$ and $\operatorname{sgn}(\rho_- - \sigma_-) > 0$, see Figure 4c. Therefore

$$
\begin{aligned}
(16) &= -\left(F_\alpha(\dot y) + \dot y\rho_+ - \dot z\rho_+ - F(\sigma)\right) - \left(F_\alpha(\dot y) + \dot y\rho_- - \dot z\rho_- - F(\sigma)\right)\\
&= 2F(\sigma) - 2F_\alpha(\dot y) + (\rho_+ + \rho_-)(\dot z - \dot y)\\
&\le 2F_\alpha(\dot z) - 2F_\alpha(\dot y) + 2R(\dot z - \dot y)\\
&= -\frac{\alpha R}{2V}(2V - \dot z - \dot y)(\dot z - \dot y) + 2R(\dot z - \dot y)\\
&\le 2R|\dot y - \dot z|.
\end{aligned}
$$

2. $(\rho_-, \rho_+) \in \mathcal{G}_2(\dot y)$: we set

$$\rho := \rho_- = \rho_+ \quad \text{and} \quad f(\rho) - \dot y\rho =: F(\rho) \le F_\alpha(\dot y).$$

as illustrated in Figure 5.



(a) Case 2.1 (b) Case 2.2

(c) Case 2.3

FIGURE 5. Case 2

2.1 $(\sigma_-, \sigma_+) \in \mathcal{G}_1(\dot z)$:

- $0 \le \rho \le \check\sigma$:

$$(16) = -\left(F(\rho) + \dot y\rho - \dot z\rho - F_\alpha(\dot z)\right) + \left(F(\rho) + \dot y\rho - \dot z\rho - F_\alpha(\dot z)\right) = 0.$$

- $\check{\sigma} \leq \rho \leq \check{\rho}$, this case is displayed in Figure 5a:

$$\begin{aligned}(16) &= (F(\rho) + \dot{y}\rho - \dot{z}\rho - F_\alpha(\dot{z})) + (F(\rho) + \dot{y}\rho - \dot{z}\rho - F_\alpha(\dot{z})) \\ &= 2F(\rho) - 2F_\alpha(\dot{z}) + 2\rho(\dot{y} - \dot{z}) \\ &\leq 2\left(F_\alpha(\dot{y}) - F_\alpha(\dot{z})\right) \\ &\leq \alpha R|\dot{y} - \dot{z}|.\end{aligned}$$

- $\hat{\rho} \leq \rho \leq R$:

$$(16) = (F(\rho) + \dot{y}\rho - \dot{z}\rho - F_\alpha(\dot{z})) - (F(\rho) + \dot{y}\rho - \dot{z}\rho - F_\alpha(\dot{z})) = 0.$$

2.2 $(\sigma_-, \sigma_+) \in \mathcal{G}_2(\dot{z})$, illustrated in Figure 5b:
We observe that $\operatorname{sgn}(\rho_+ - \sigma_+) = \operatorname{sgn}(\rho_- - \sigma_-) = \operatorname{sgn}(\rho - \sigma)$, therefore

$$(16) = \operatorname{sgn}(\rho - \sigma)\left[(F(\rho) + \dot{y}\rho - \dot{z}\rho - F(\sigma)) - (F(\rho) + \dot{y}\rho - \dot{z}\rho - F(\sigma))\right] = 0.$$

2.3 $(\sigma_-, \sigma_+) \in \mathcal{G}_3(\dot{z})$ shown in Figure 5c: If $\operatorname{sgn}(\rho - \sigma_+) = \operatorname{sgn}(\rho - \sigma_-)$, we get

$$(16) = \operatorname{sgn}(\rho - \sigma_+)\left[(F(\rho) + \dot{y}\rho - \dot{z}\rho - F(\sigma)) - (F(\rho) + \dot{y}\rho - \dot{z}\rho - F(\sigma))\right] = 0.$$

Otherwise, we have that $\sigma_- \leq \rho \leq \check{\rho}$ or $\hat{\rho} \leq \rho \leq \sigma_+$. In this case

$$\begin{aligned}\lambda(\rho_+, \sigma_+) - \dot{z} &= V\left(1 - \frac{\rho + \sigma_+}{R}\right) - V\left(1 - \frac{\sigma_+ + \sigma_-}{R}\right) = \frac{V}{R}(\sigma_- - \rho) \leq 0, \\ \lambda(\rho_-, \sigma_-) - \dot{z} &= V\left(1 - \frac{\rho + \sigma_-}{R}\right) - V\left(1 - \frac{\sigma_+ + \sigma_-}{R}\right) = \frac{V}{R}(\sigma_+ - \rho) \geq 0.\end{aligned}$$

Therefore, (16) ≤ 0 by (17).

3. $(\rho_-, \rho_+) \in \mathcal{G}_3(\dot{y})$: we set

$$f(\rho_-) - \dot{y}\rho_- = f(\rho_+) - \dot{y}\rho_+ =: F(\rho) \leq F_\alpha(\dot{y}).$$

See Figure 6, for a graphical representation.

3.1 $(\sigma_-, \sigma_+) \in \mathcal{G}_1(\dot{z})$, see Figure 6a.

$$\begin{aligned}(16) &= (F(\rho) + \dot{y}\rho_+ - \dot{z}\rho_+ - F_\alpha(\dot{z})) + (F(\rho) + \dot{y}\rho_- - \dot{z}\rho_- - F_\alpha(\dot{z})) \\ &= 2F(\rho) - 2F_\alpha(\dot{z}) + (\rho_+ + \rho_-)(\dot{y} - \dot{z}) \\ &\leq 2\left(F_\alpha(\dot{y}) - F_\alpha(\dot{z})\right) \\ &\leq \alpha R|\dot{y} - \dot{z}|.\end{aligned}$$

3.2 $(\sigma_-, \sigma_+) \in \mathcal{G}_2(\dot{z})$: If $\operatorname{sgn}(\rho_+ - \sigma) = \operatorname{sgn}(\rho_- - \sigma)$, we get

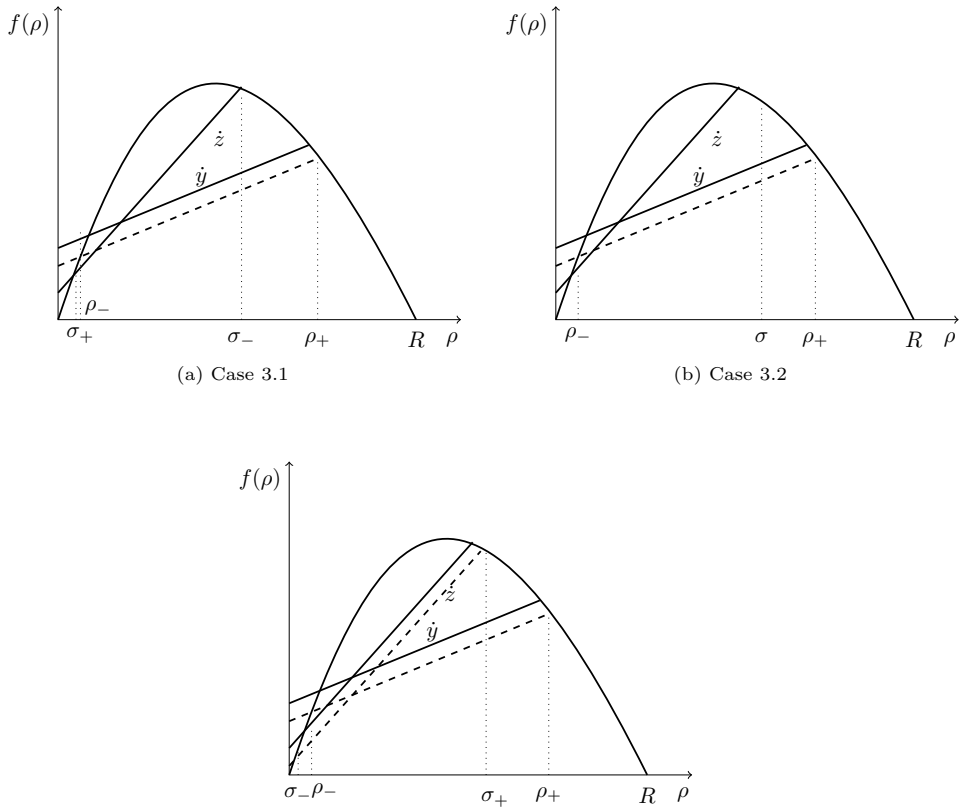
$$\begin{aligned}(16) &= \operatorname{sgn}(\rho_+ - \sigma)\left[(F(\rho) + \dot{y}\rho_+ - \dot{z}\rho_+ - F(\sigma)) - (F(\rho) + \dot{y}\rho_- - \dot{z}\rho_- - F(\sigma))\right] \\ &\leq (\rho_+ - \rho_-)(\dot{z} - \dot{y}) \\ &\leq R|\dot{y} - \dot{z}|.\end{aligned}$$

Otherwise as shown in Figure 6b, we have that $\rho_- \leq \sigma \leq \check{\sigma}$ or $\hat{\sigma} \leq \sigma \leq \rho_+$. In this case

$$\begin{aligned}\lambda(\rho_+, \sigma_+) - \dot{z} &\leq \lambda(\rho_+, \sigma_+) - \dot{y} \\ &= V\left(1 - \frac{\rho_+ + \sigma}{R}\right) - V\left(1 - \frac{\rho_+ + \rho_-}{R}\right) = \frac{V}{R}(\rho_- - \sigma) \leq 0.\end{aligned}$$

Moreover, we observe that $\lambda(\rho_-, \sigma_-) > \dot{y}$. Indeed

$$\lambda(\rho_-, \sigma_-) - \dot{y} = V\left(1 - \frac{\rho_- + \sigma}{R}\right) - V\left(1 - \frac{\rho_+ + \rho_-}{R}\right) = \frac{V}{R}(\rho_+ - \sigma) \geq 0.$$



(a) Case 3.1

(b) Case 3.2

(c) Case 3.3

FIGURE 6. Case 3

Therefore, by (17)

$$(16) \leq R|\dot{y} - \dot{z}|.$$

3.3 $(\sigma_-, \sigma_+) \in \mathcal{G}_3(\dot{z})$:
We observe that one of the following relations must hold

$$\rho_- \leq \sigma_- < \sigma_+ \leq \rho_+, \quad \sigma_- \leq \rho_- < \sigma_+ \leq \rho_+, \quad \sigma_- \leq \rho_- < \rho_+ \leq \sigma_+.$$

For an example see Figure 6c. Therefore

$$\lambda(\rho_+, \sigma_+) - \dot{z} = V\left(1 - \frac{\rho_+ + \sigma_+}{R}\right) - V\left(1 - \frac{\sigma_+ + \sigma_-}{R}\right) = \frac{V}{R}(\sigma_- - \rho_+) \leq 0,$$

$$\lambda(\rho_-, \sigma_-) - \dot{z} = V\left(1 - \frac{\rho_- + \sigma_-}{R}\right) - V\left(1 - \frac{\sigma_+ + \sigma_-}{R}\right) = \frac{V}{R}(\sigma_+ - \rho_-) \geq 0,$$

hence (16) ≤ 0 by (17).

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□

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