



**HAL**  
open science

## Transport along stochastic flows in fluid dynamics

Valentin Resseguier, Etienne Mémin, Bertrand Chapron

► **To cite this version:**

Valentin Resseguier, Etienne Mémin, Bertrand Chapron. Transport along stochastic flows in fluid dynamics. Workshop - Statistical methods for dynamical stochastic models - DYNSTOCH 2016, Jun 2016, Rennes, France. hal-01377723

**HAL Id: hal-01377723**

**<https://inria.hal.science/hal-01377723>**

Submitted on 9 Nov 2016

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

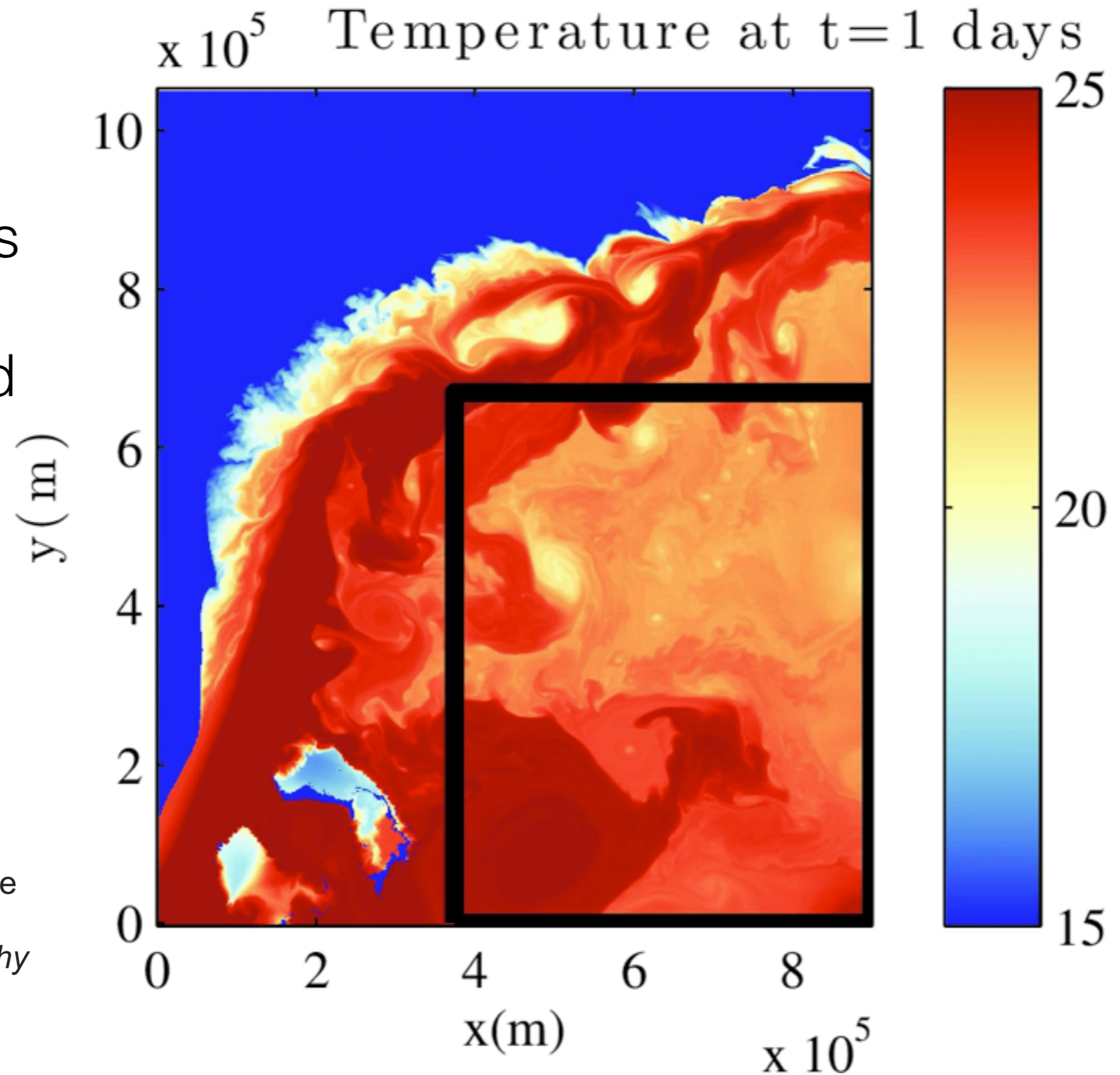
# Transport along stochastic flows in fluid dynamics

Valentin Resseguier,  
Etienne Mémin,  
Bertrand Chapron

Fluids are very complex

with small vortices interacting with large vortices and currents

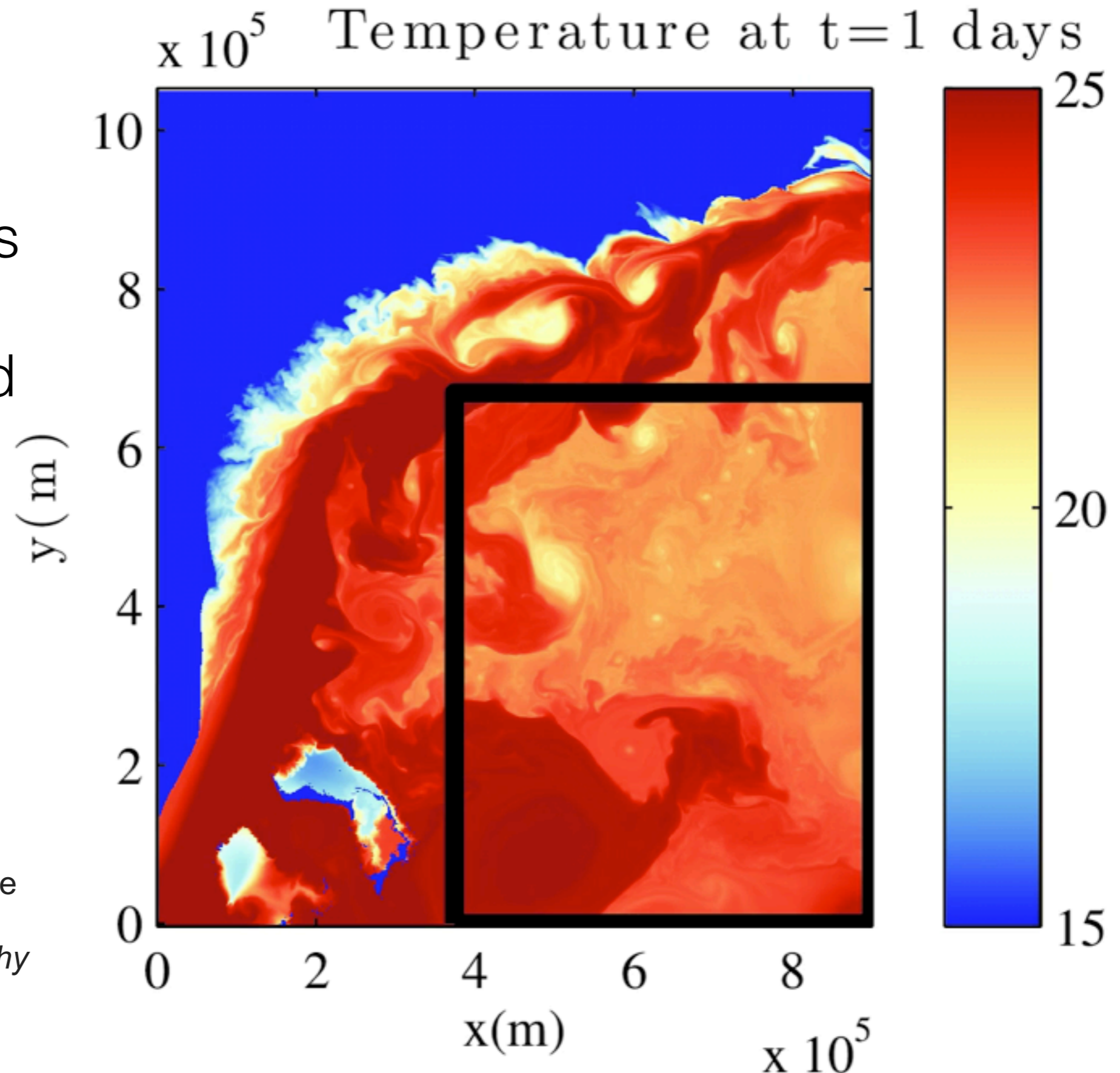
Gula, Jonathan, M. Jeroen Molemaker, and James C. McWilliams  
"Gulf Stream dynamics along the southeastern US seaboard."  
*Journal of Physical Oceanography*  
45.3 (2015): 690-715.



Fluids are very complex

with small vortices interacting with large vortices and currents

Gula, Jonathan, M. Jeroen Molemaker, and James C. McWilliams  
"Gulf Stream dynamics along the southeastern US seaboard."  
*Journal of Physical Oceanography*  
45.3 (2015): 690-715.



# Why a random fluid dynamics?

- Take into account unresolved processes (small scales)
- Physical justification of empirical models
- Predicting possible distinct scenarios, extreme-events, ...
- Quantification of modeling errors for **data assimilation:**  
**ensemble forecasts**

# Contents

- Randomized dynamics
- Simulation of the SQG under Moderate Uncertainty
- Stochastic reduced order model

# Randomized dynamics

# Advection of tracer $\Theta$

- Stochastic flow:

$$d\mathbf{X}_t = \mathbf{w}(\mathbf{X}_t, t)dt + \boldsymbol{\sigma}(\mathbf{X}_t, t)d\mathbf{B}_t$$



# Advection of tracer $\Theta$

- Stochastic flow:

$$d\mathbf{X}_t = \mathbf{w}(\mathbf{X}_t, t)dt + \boldsymbol{\sigma}(\mathbf{X}_t, t)d\mathbf{B}_t$$

with infinite-dimensional Brownian motion

$$\boldsymbol{\sigma}(\bullet, t)d\mathbf{B}_t \triangleq \int_{\Omega} dz \check{\boldsymbol{\sigma}}(\bullet, z, t)d\mathbf{B}_t(z)$$

# Advection of tracer $\Theta$

- Stochastic flow:

$$d\mathbf{X}_t = \mathbf{w}(\mathbf{X}_t, t)dt + \boldsymbol{\sigma}(\mathbf{X}_t, t)d\mathbf{B}_t$$

with infinite-dimensional Brownian motion

$$\boldsymbol{\sigma}(\bullet, t)d\mathbf{B}_t \triangleq \int_{\Omega} dz \check{\boldsymbol{\sigma}}(\bullet, z, t)d\mathbf{B}_t(z)$$

- A tracer is a function conserved along the flow:

$$D_t\Theta(t, \mathbf{X}_t) \triangleq d[\Theta(t, \mathbf{X}_t)] = 0$$

# Ito-Wentzel formula

(Kunita 1997)

If both  $\mathbf{X}$  and  $\Theta$  are semimartingales (w.r.t. time) and  $\Theta$  is twice differentiable w.r.t. space

Then

$$\begin{aligned} d[\Theta(t, \mathbf{X}(t, y))] &= d_t \Theta + (\nabla \Theta)^T d\mathbf{X} + \frac{1}{2} \text{tr}(H_\Theta d \langle \mathbf{X}, \mathbf{X}^T \rangle) \\ &\quad + d_t \langle (\nabla \Theta)^T, \mathbf{X} \rangle \end{aligned}$$

Advection of tracer  $\Theta$

$$D_t \Theta = 0$$

# Advection of tracer $\Theta$

# Advection of tracer $\Theta$

$$d_t \Theta + \boldsymbol{w}^* \cdot \nabla \Theta dt + \boldsymbol{\sigma} d\boldsymbol{B}_t \cdot \nabla \Theta = \nabla \cdot \left( \frac{1}{2} \boldsymbol{a} \nabla \Theta \right) dt$$

# Advection of tracer $\Theta$

Advection

$$d_t \Theta + \mathbf{w}^* \cdot \nabla \Theta dt + \sigma dB_t \cdot \nabla \Theta = \nabla \cdot \left( \frac{1}{2} \mathbf{a} \nabla \Theta \right) dt$$

# Advection of tracer $\Theta$

$$d_t \Theta + \underbrace{w^* \cdot \nabla \Theta dt + \sigma dB_t \cdot \nabla \Theta}_{\text{Advection}} = \underbrace{\nabla \cdot \left( \frac{1}{2} a \nabla \Theta \right)}_{\text{Diffusion}} dt$$



# Advection of tracer $\Theta$

$$d_t \Theta + \underbrace{w^* \cdot \nabla \Theta dt + \sigma dB_t \cdot \nabla \Theta}_{\text{Advection}} = \underbrace{\nabla \cdot \left( \frac{1}{2} a \nabla \Theta \right)}_{\text{Diffusion}} dt$$

$$a = \sigma \sigma^T$$

$$w^* = w - \frac{1}{2} (\nabla \cdot a)^T$$

# Advection of tracer $\Theta$

$$d_t \Theta + \underbrace{w^* \cdot \nabla \Theta dt + \sigma dB_t \cdot \nabla \Theta}_{\text{Advection}} = \underbrace{\nabla \cdot \left( \frac{1}{2} \mathbf{a} \nabla \Theta \right)}_{\text{Diffusion}} dt$$

Drift  
correction

$$\mathbf{a} = \sigma \sigma^T$$

$$w^* = w - \frac{1}{2} (\nabla \cdot \mathbf{a})^T$$

# Advection of tracer $\Theta$

$$d_t \Theta + \underbrace{w^* \cdot \nabla \Theta dt}_{\text{Advection}} + \underbrace{\sigma dB_t \cdot \nabla \Theta}_{\text{Diffusion}} = \nabla \cdot \left( \frac{1}{2} \mathbf{a} \nabla \Theta \right) dt$$

Drift correction

Multiplicative random forcing

$$\mathbf{a} = \sigma \sigma^T$$

$$w^* = w - \frac{1}{2} (\nabla \cdot \mathbf{a})^T$$

# Advection of tracer $\Theta$

$$d_t \Theta + \underbrace{w^* \cdot \nabla \Theta dt}_{\text{Advection}} + \underbrace{\sigma dB_t \cdot \nabla \Theta}_{\text{Diffusion}} = \underbrace{\nabla \cdot \left( \frac{1}{2} a \nabla \Theta \right)}_{\text{Diffusion}} dt$$

Drift correction
Multiplicative random forcing

$$a = \sigma \sigma^T$$

$$w^* = w - \frac{1}{2} (\nabla \cdot a)^T$$

# Advection of tracer $\Theta$

$$d_t \Theta + \underbrace{w^* \cdot \nabla \Theta dt}_{\text{Advection}} + \underbrace{\sigma dB_t \cdot \nabla \Theta}_{\text{Diffusion}} = \nabla \cdot \left( \frac{1}{2} \mathbf{a} \nabla \Theta \right) dt$$

Drift correction
Multiplicative random forcing
Balanced energy exchanges

$$\mathbf{a} = \sigma \sigma^T$$

$$w^* = w - \frac{1}{2} (\nabla \cdot \mathbf{a})^T$$

# $\sigma = ?$

$$\sigma(\bullet, t) d\mathbf{B}_t \triangleq \int_{\Omega} dz \check{\sigma}(\bullet, z, t) d\mathbf{B}_t(z)$$

- Parametric or non-parametric estimation
- Observations can be:
  - Small-scale eulerian velocity  $(\sigma(\mathbf{x}_i, t_j) d\mathbf{B}_{t_j})_{ij}$
  - Small-scale Lagrangian velocity  $(\sigma(\mathbf{X}_{t_j}(\mathbf{x}_i), t_j) d\mathbf{B}_{t_j})_{ij}$
  - **Tracer (solution of the SPDE)**  $(\Theta(\mathbf{x}_i, t_j))_{ij}$

- In the following simulations, very simple model (without estimations)

$$\sigma d\mathbf{B}_t = \int_{\Omega} dz \nabla^{\perp} \check{\psi}(\bullet - z) d\mathbf{B}_t(z) = \nabla^{\perp} \check{\psi} * d\mathbf{B}_t$$

$$\widehat{\check{\psi}}(\mathbf{k}) = A \mathbf{1}_{\{\kappa_1 < \|\mathbf{k}\| < \kappa_2\}} \|\mathbf{k}\|^{-\alpha}$$

# Simulation of the SQG under Moderate Uncertainty

SQG MU

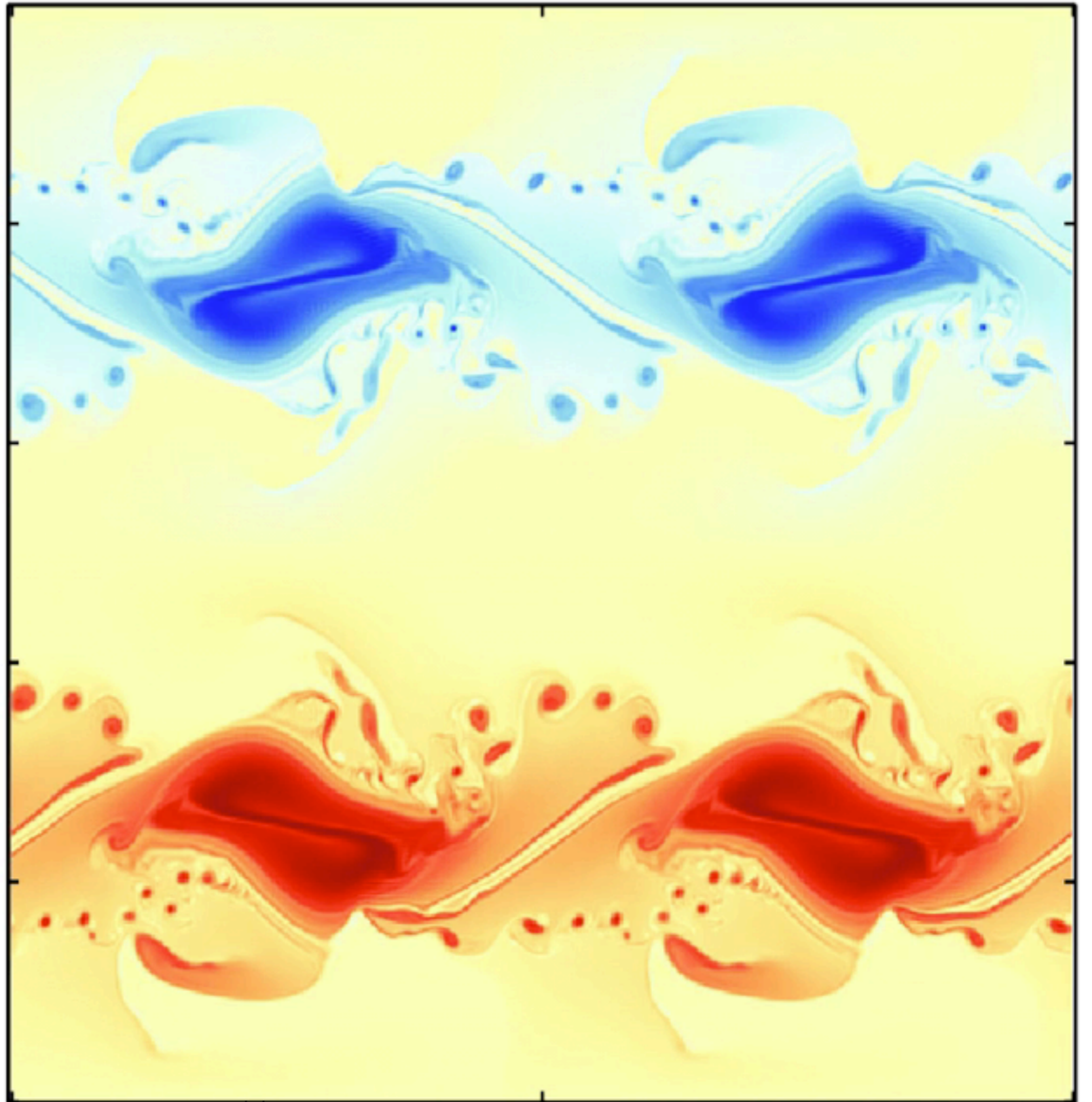
*Code available online*

$t = 17$  days

SQG model:

$$D_t \Theta = \nu \Delta^4 \Theta dt$$

$$\mathbf{w} = \alpha \nabla^\perp (-\Delta)^{-\frac{1}{2}} \Theta$$



Reference flow:

deterministic SQG

512 x 512

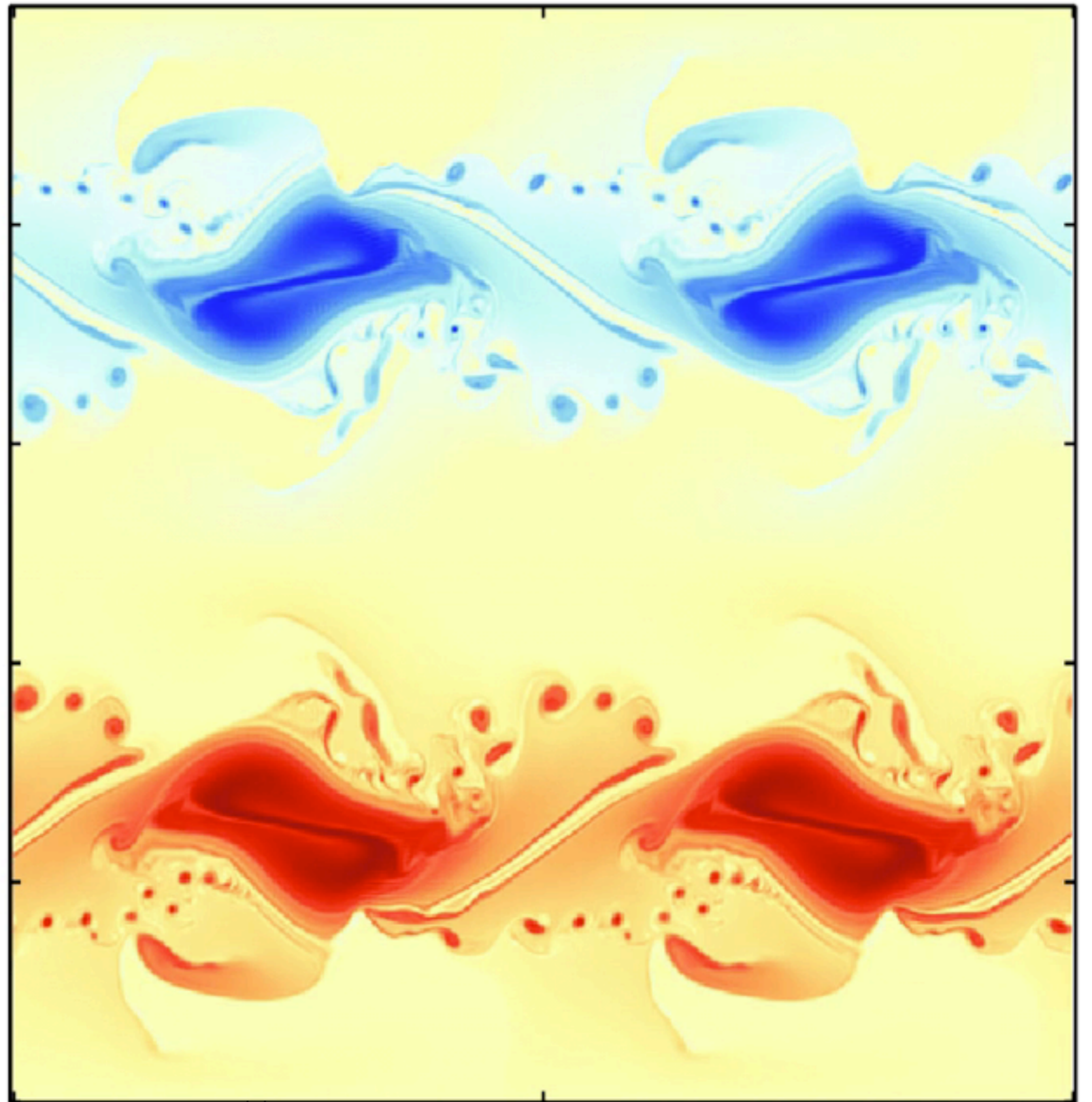


$t = 17$  days

SQG model:

$$D_t \Theta = \nu \Delta^4 \Theta dt$$

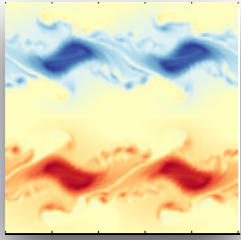
$$\mathbf{w} = \alpha \nabla^\perp (-\Delta)^{-\frac{1}{2}} \Theta$$



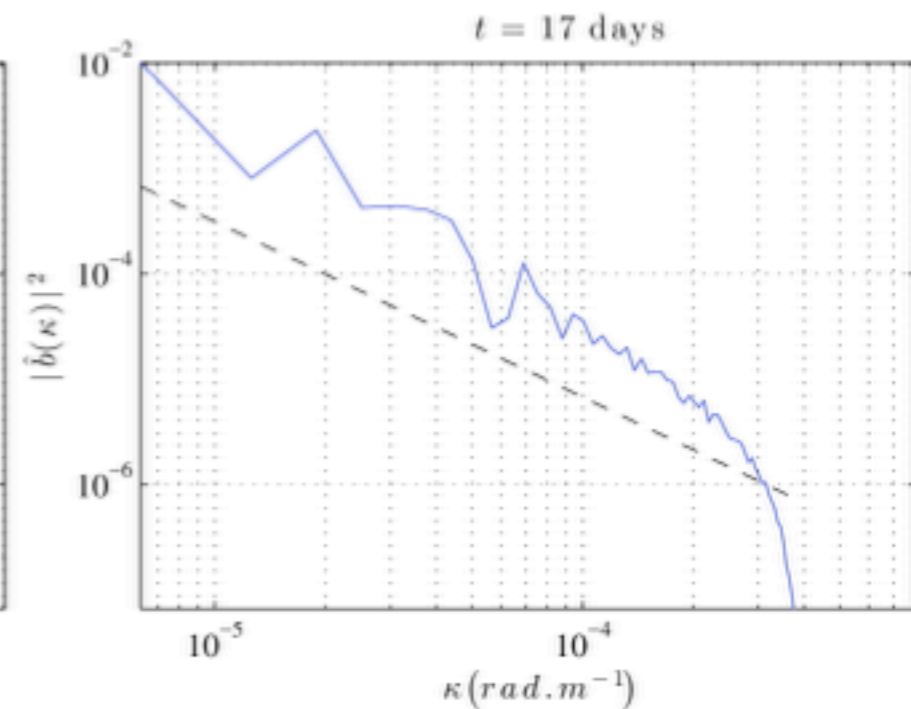
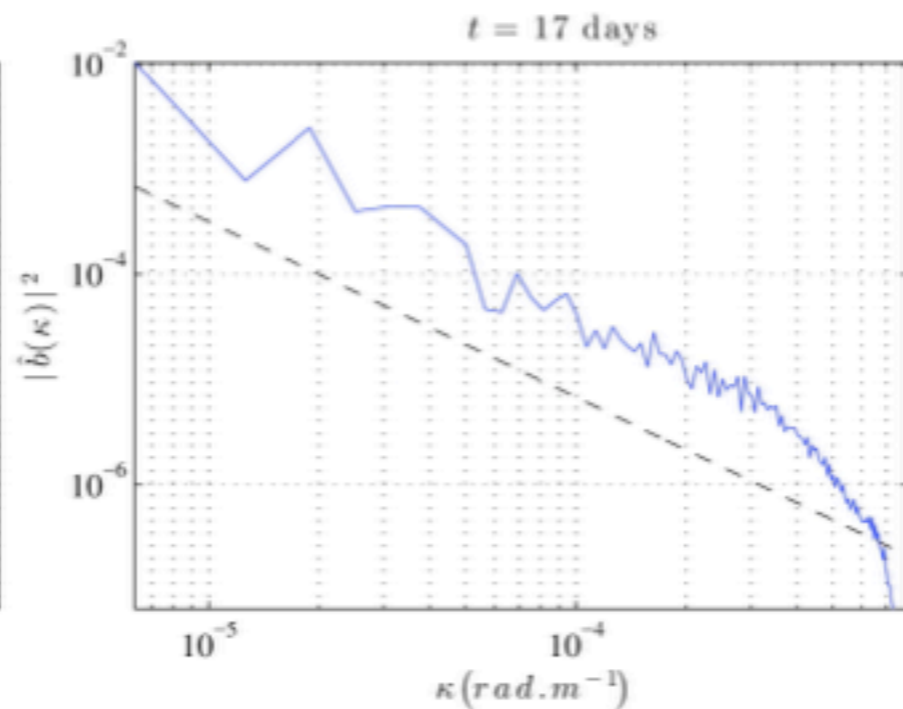
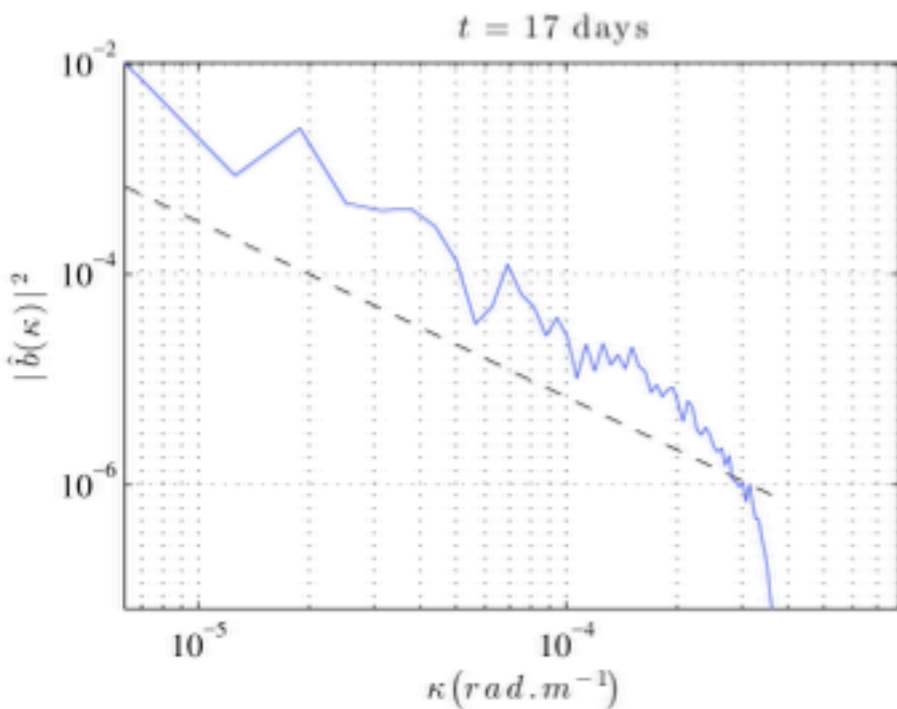
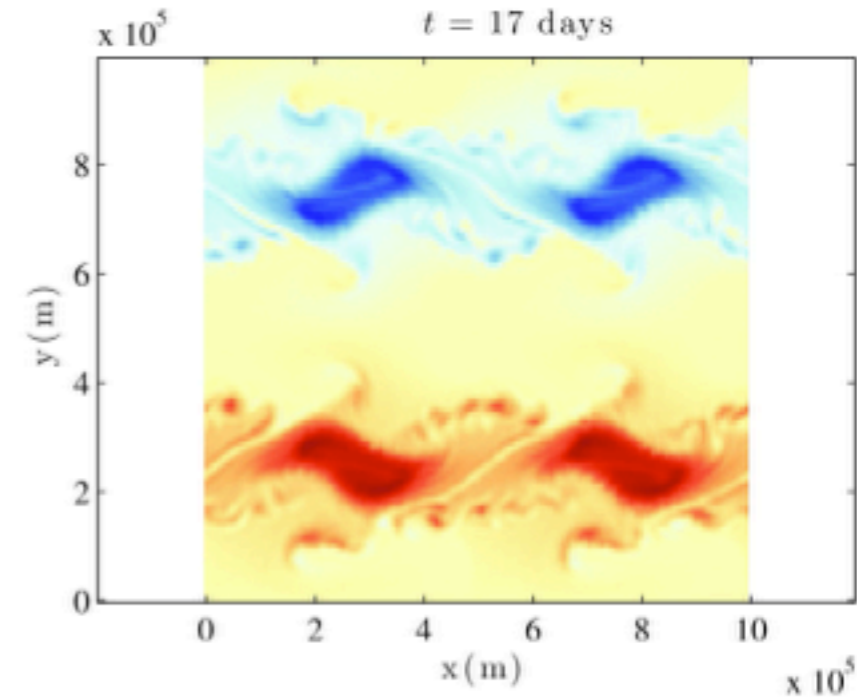
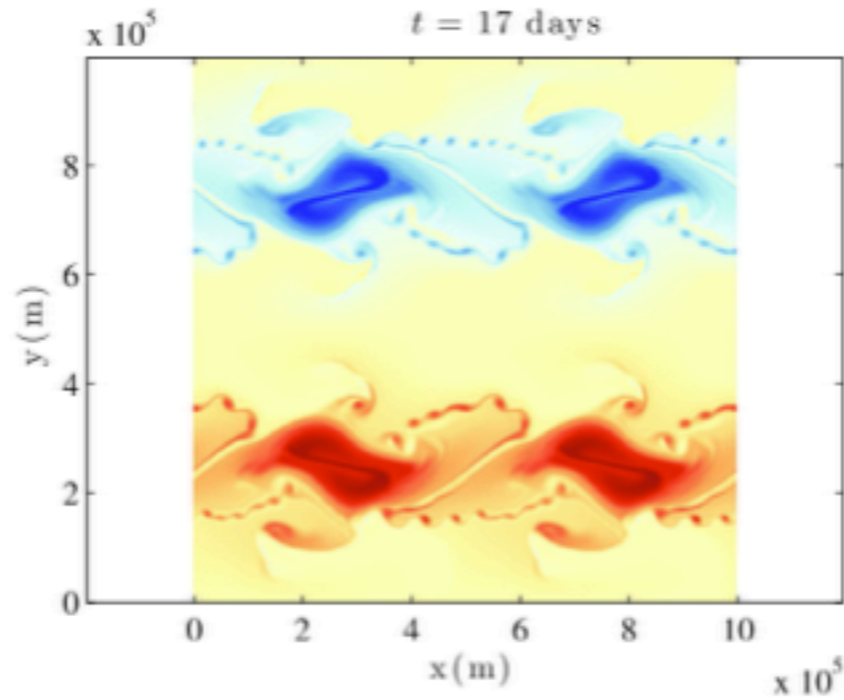
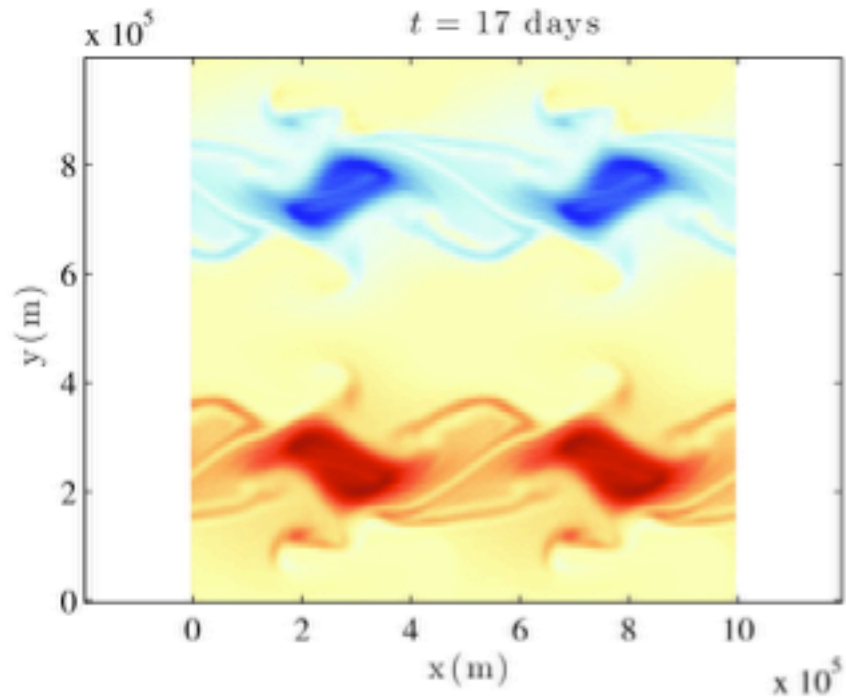
Reference flow:

deterministic SQG

512 x 512



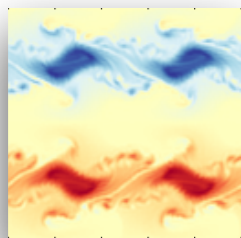
# One realization



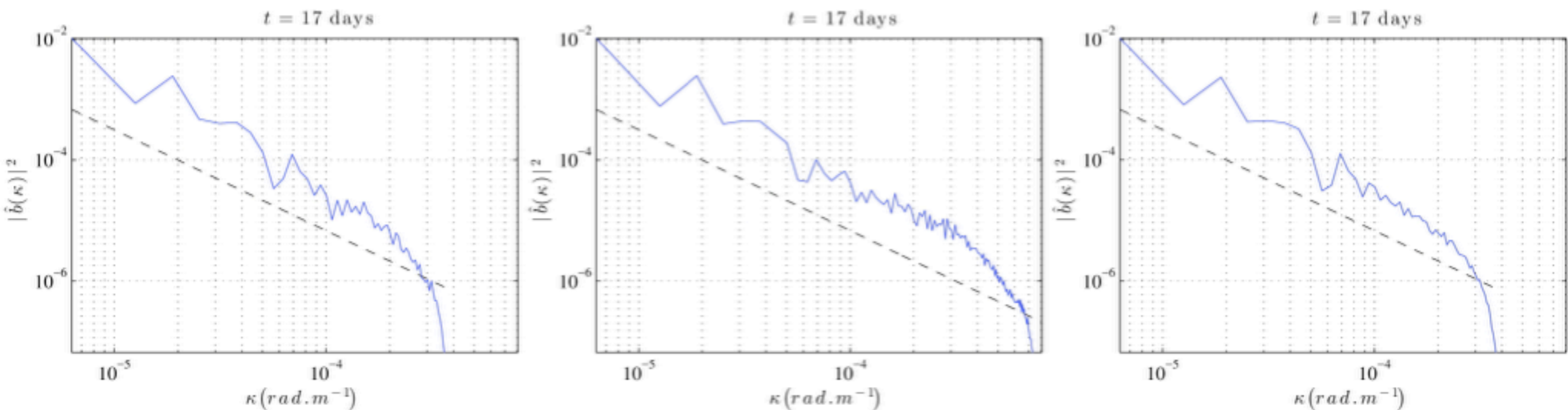
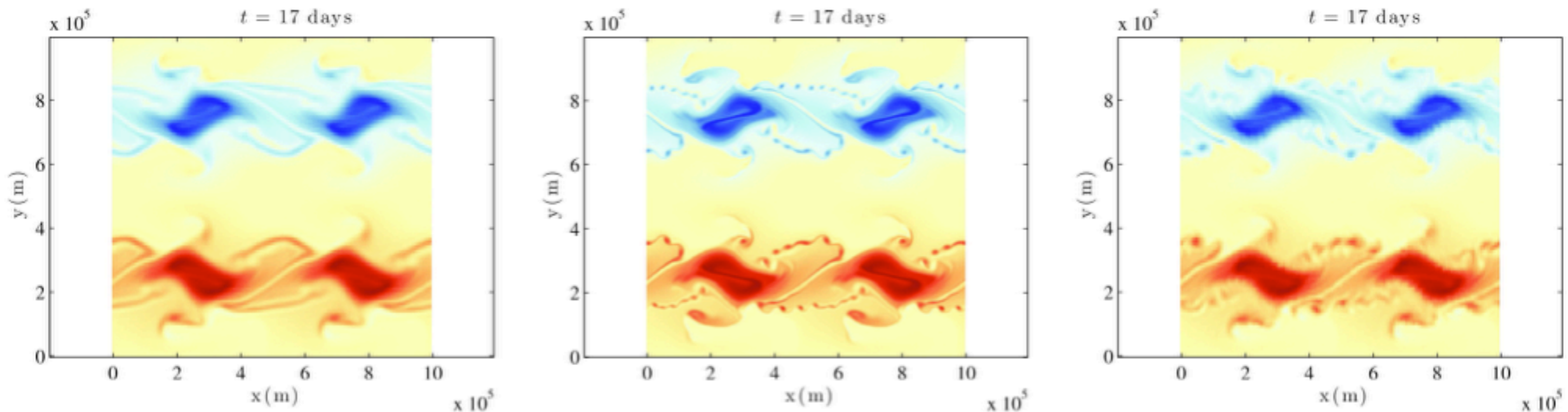
**Deterministic 128x128**

**Deterministic 512x512**

**Stochastic 128x128**



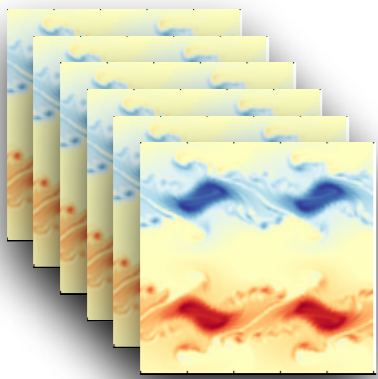
# One realization



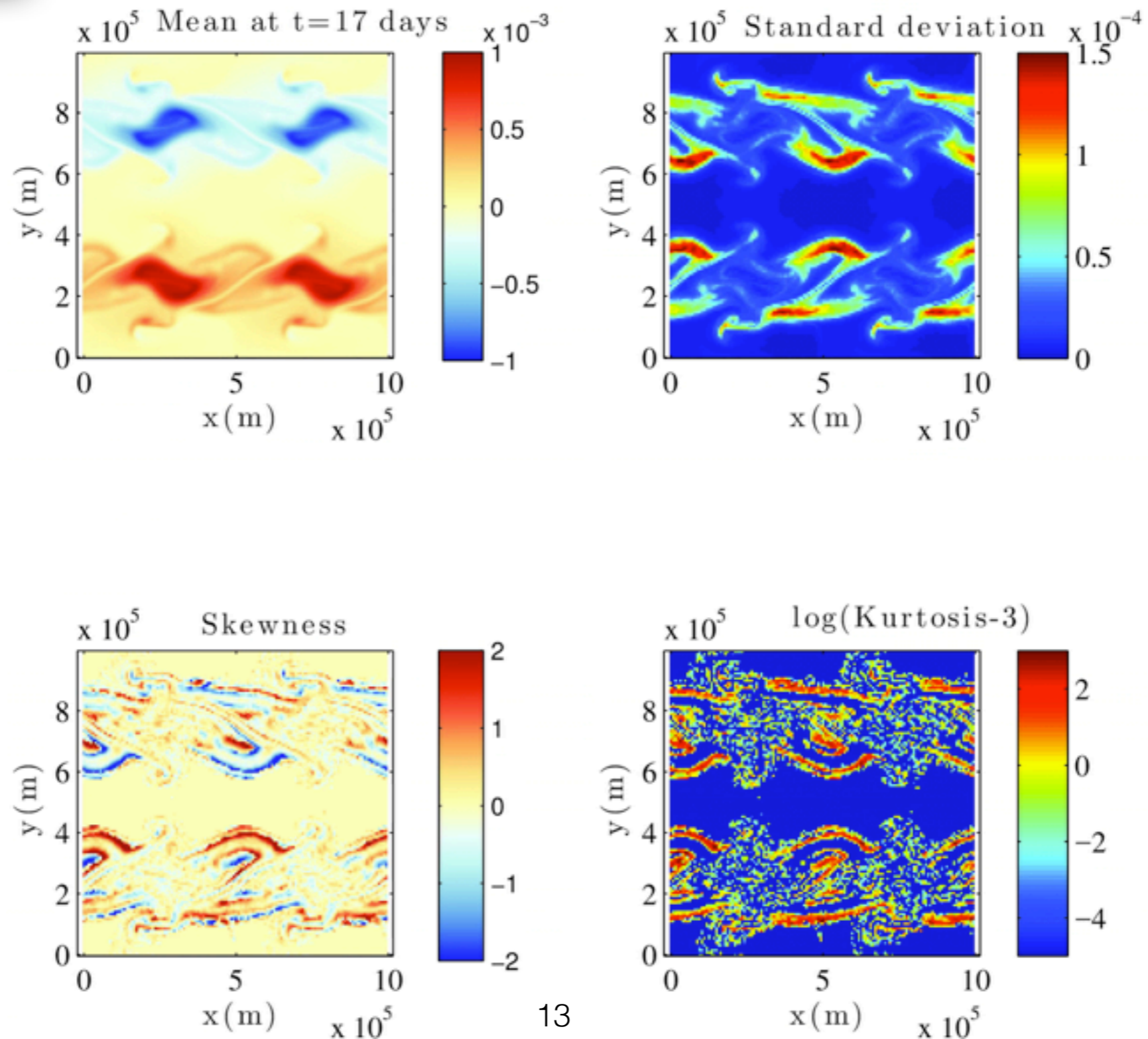
**Deterministic 128x128**

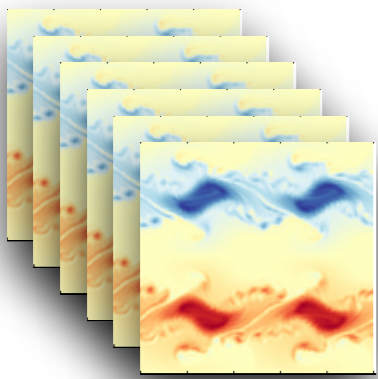
**Deterministic 512x512**

**Stochastic 128x128**

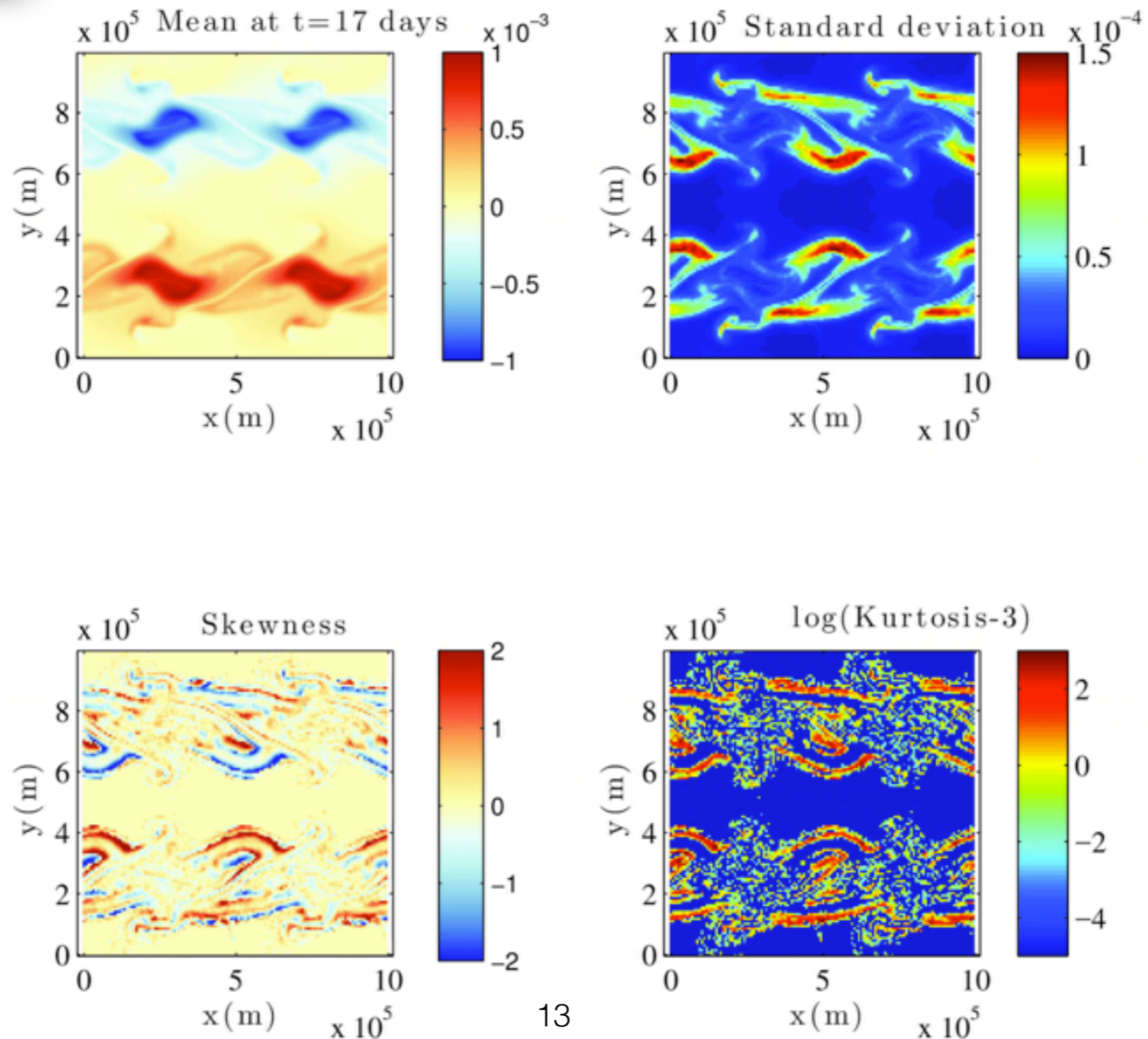


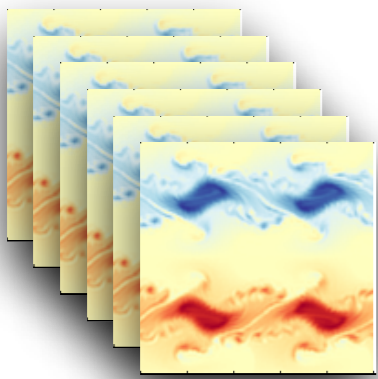
# Ensemble



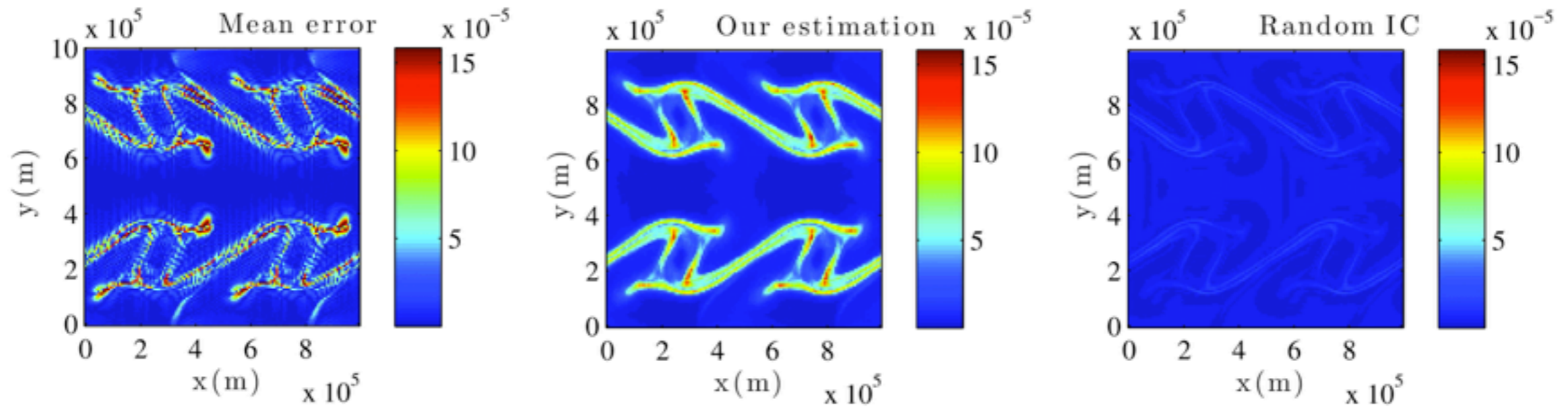


# Ensemble

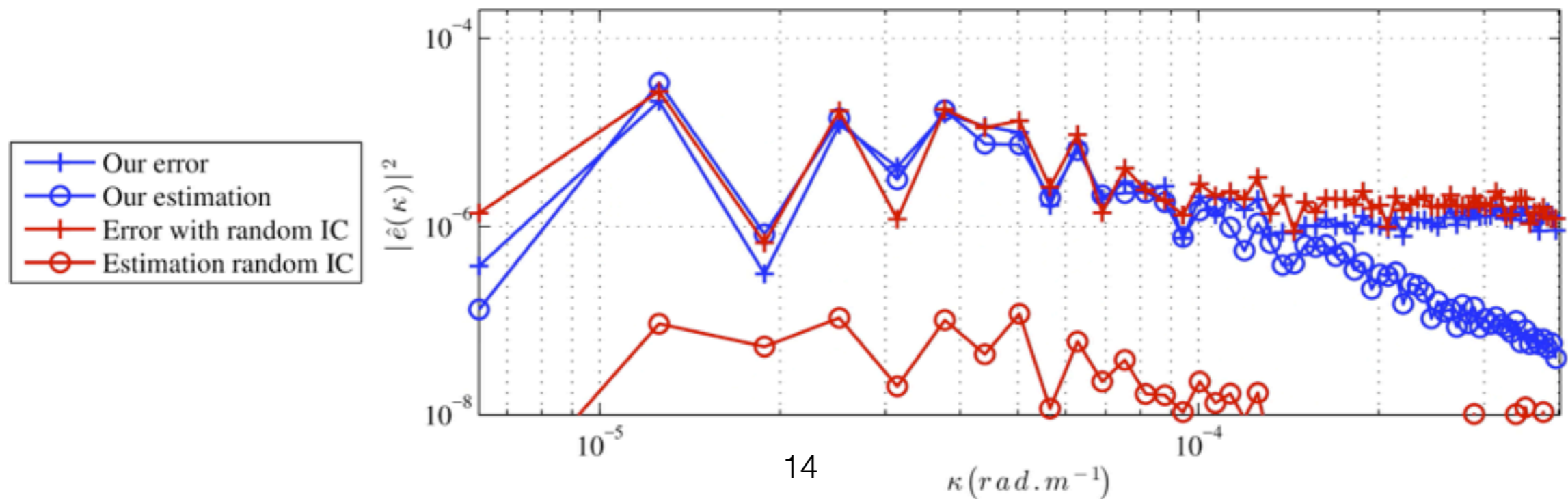


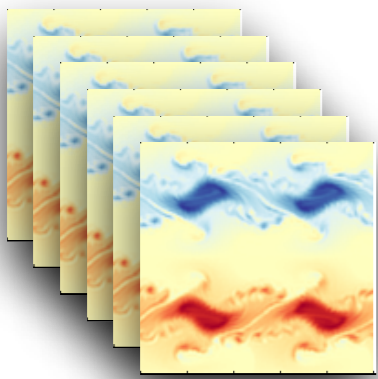


# Ensemble

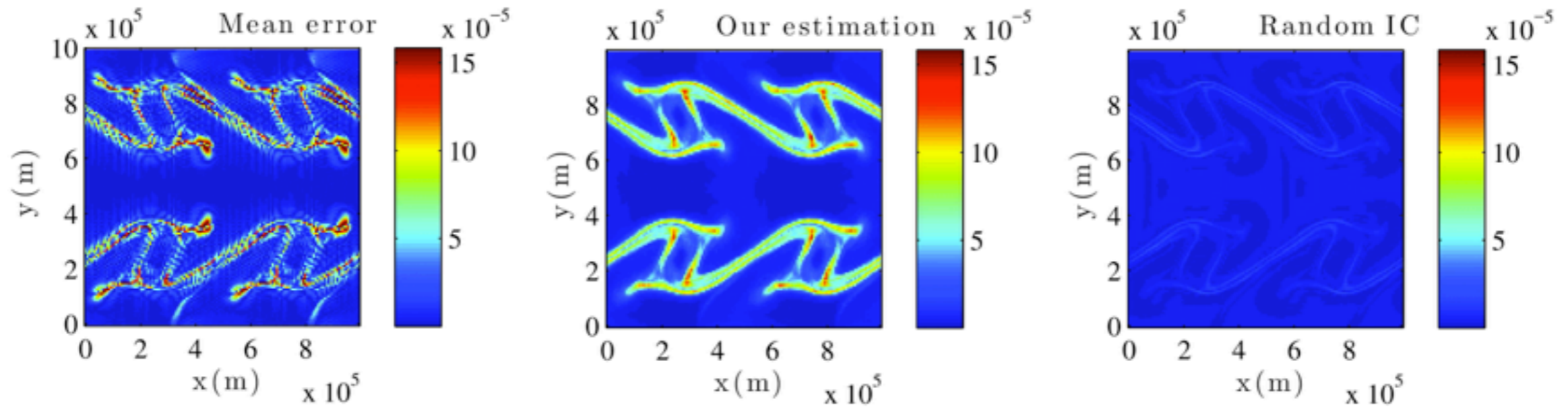


Spectrum of the errors and its estimation at  $t=12$  days

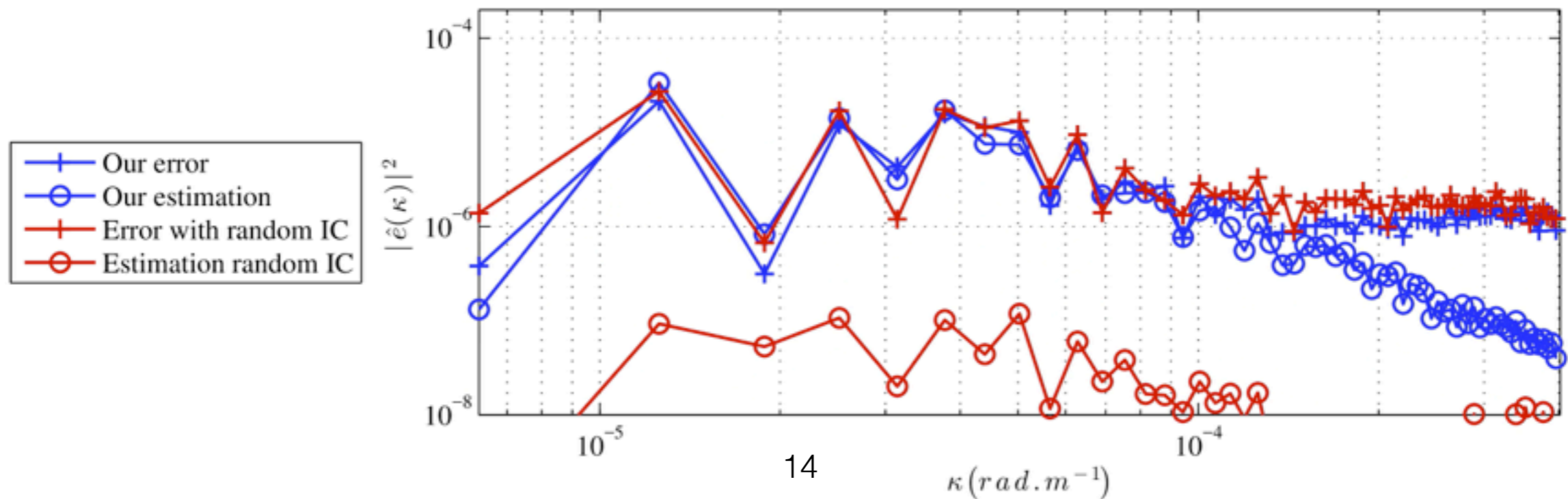




# Ensemble



Spectrum of the errors and its estimation at  $t=12$  days



# Random reduced order model from stochastic Navier-Stokes equation



# Why a reduced order model?

- Very fast simulation of very complex system (e.g. for industrial application)

# What is a reduced order model?

- Physical model (PDE) simplified using observations

# PCA on data to reduce the state-space dimension

$$\mathbf{v}(\mathbf{x}, t) = \bar{\mathbf{v}}(\mathbf{x}) + \sum_{i=0}^n b_i(t) \phi_i(\mathbf{x}) + \sum_{i=n+1}^N b_i(t) \phi_i(\mathbf{x})$$

# PCA on data to reduce the state-space dimension

**Resolved modes**

$$v(\boldsymbol{x}, t) = \bar{v}(\boldsymbol{x}) + \sum_{i=0}^n b_i(t) \phi_i(\boldsymbol{x}) + \sum_{i=n+1}^N b_i(t) \phi_i(\boldsymbol{x})$$

$w$



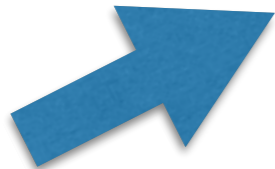
# PCA on data to reduce the state-space dimension

**Resolved modes**

**Unresolved modes**

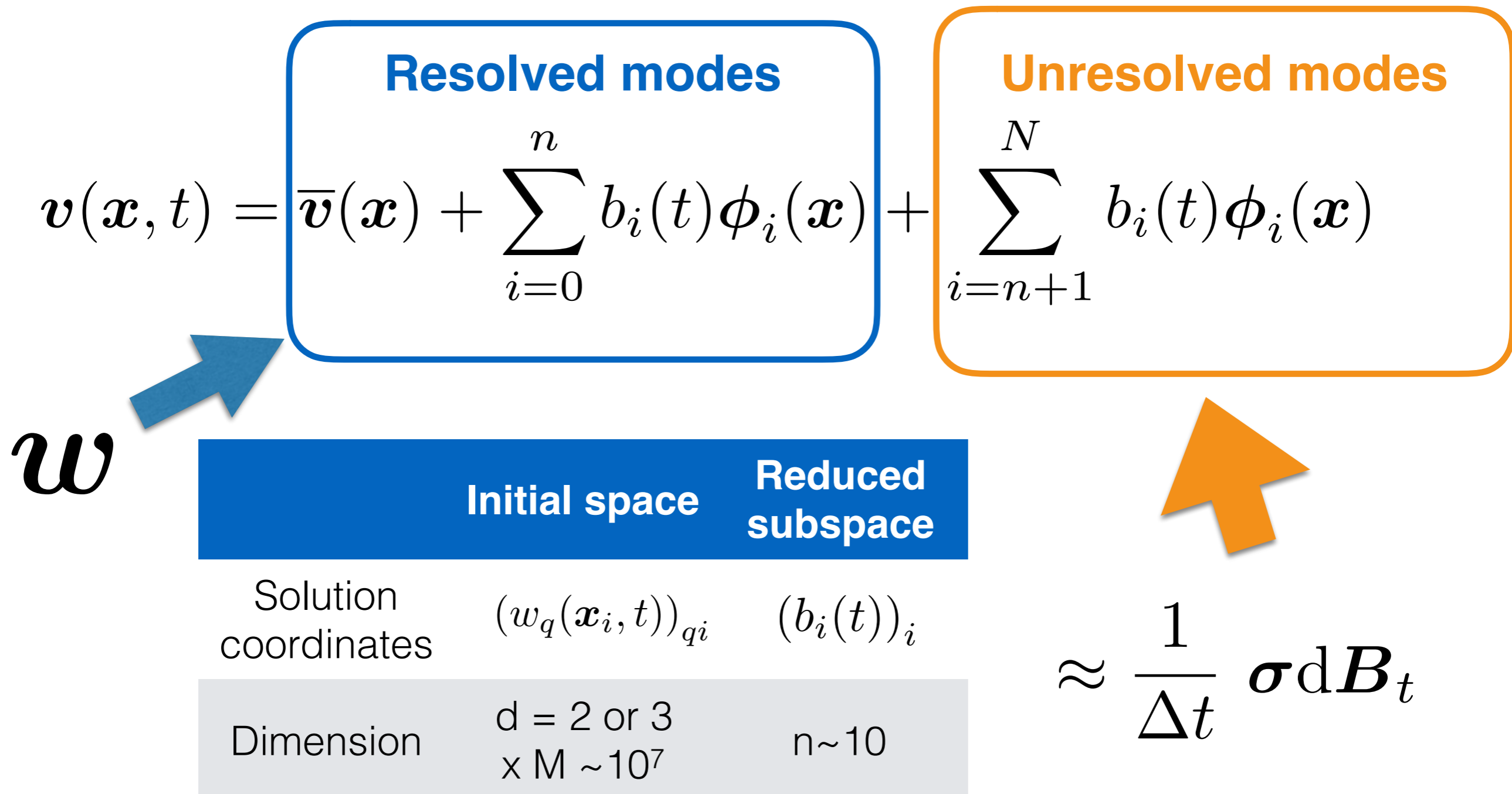
$$v(\boldsymbol{x}, t) = \bar{v}(\boldsymbol{x}) + \sum_{i=0}^n b_i(t) \phi_i(\boldsymbol{x}) + \sum_{i=n+1}^N b_i(t) \phi_i(\boldsymbol{x})$$

$w$



$$\approx \frac{1}{\Delta t} \boldsymbol{\sigma} d\boldsymbol{B}_t$$

# PCA on data to reduce the state-space dimension




Assumptions of finite variations for  $\boldsymbol{w}$  and Galerkin projection gives **ODEs** for resolved modes:

$$\int_{\Omega} \phi_i \cdot (\text{stochastic Navier-Stokes})$$

$$db_i = F_i(\mathbf{b})dt$$

Assumptions of finite variations for  $\mathbf{w}$  and Galerkin projection gives **ODEs** for resolved modes:

$$\int_{\Omega} \phi_i \cdot (\text{stochastic Navier-Stokes})$$

$$db_i = \boxed{F_i(\mathbf{b})} dt$$


2<sup>nd</sup> order polynomial:  
coefficients given by physics,

$$(\phi_j)_j \text{ and } \mathbf{a}(\mathbf{x}, \mathbf{x}, t) = \frac{1}{t} \langle (\boldsymbol{\sigma}(\mathbf{x}, t) \mathbf{B})_{obs}, (\boldsymbol{\sigma}(\mathbf{x}, t) \mathbf{B})_{obs}^T \rangle$$

Assumptions of finite variations for  $\mathbf{w}$  and Galerkin projection gives **ODEs** for resolved modes:

$$\int_{\Omega} \phi_i \cdot (\text{stochastic Navier-Stokes})$$

$$db_i = \boxed{F_i(\mathbf{b})} dt$$



2<sup>nd</sup> order polynomial:  
coefficients given by physics,

$$(\phi_j)_j \text{ and } \mathbf{a}(\mathbf{x}, \mathbf{x}, t) = \frac{1}{t} \langle (\boldsymbol{\sigma}(\mathbf{x}, t) \mathbf{B})_{obs}, (\boldsymbol{\sigma}(\mathbf{x}, t) \mathbf{B})_{obs}^T \rangle$$

**Model and estimation**  
(inspired by Genon-Catalot, Laredo, Picard 1992)

$$\mathbf{a}(\mathbf{x}, \mathbf{x}, t) = \mathbf{z}_0(\mathbf{x}) + \sum_{i=0}^n b_i(t) \mathbf{z}_i(\mathbf{x})$$

$$\mathbf{z}_i(\mathbf{x}) = \frac{1}{t} \int_0^t b_i(t) d \langle \boldsymbol{\sigma}(\mathbf{x}, t) \mathbf{B}, \boldsymbol{\sigma}(\mathbf{x}, t) \mathbf{B} \rangle$$





Galerkin projection gives **SDEs** for resolved modes:

$$\int_{\Omega} \phi_i \cdot (\text{stochastic Navier-Stokes})$$

$$db_i = F_i(\mathbf{b})dt + (\boldsymbol{\alpha}_{\bullet i} d\mathbf{B}_t)^T \mathbf{b} + (\boldsymbol{\theta}_i d\mathbf{B}_t)$$

Galerkin projection gives **SDEs** for resolved modes:

$$\int_{\Omega} \phi_i \cdot (\text{stochastic Navier-Stokes})$$

$$n \times M \quad M \times 1 \quad n \times 1 \quad 1 \times M \quad M \times 1$$

$$db_i = F_i(\mathbf{b})dt + (\alpha_{\bullet i} d\mathbf{B}_t)^T \mathbf{b} + (\theta_i d\mathbf{B}_t)$$

Galerkin projection gives **SDEs** for resolved modes:

$$\int_{\Omega} \phi_i \cdot (\text{stochastic Navier-Stokes})$$

$$db_i = F_i(\mathbf{b})dt + \underbrace{\left( \overset{n \times M}{\alpha_{\bullet i}} \overset{M \times 1}{d\mathbf{B}_t} \right)^T \overset{n \times 1}{\mathbf{b}}}_{\text{multiplicative noise}} + \underbrace{\left( \overset{1 \times M}{\theta_i} \overset{M \times 1}{d\mathbf{B}_t} \right)}_{\text{additive noise}}$$

Galerkin projection gives **SDEs** for resolved modes:

$$\int_{\Omega} \phi_i \cdot (\text{stochastic Navier-Stokes})$$

$$db_i = \boxed{F_i(\mathbf{b})} dt + \boxed{(\alpha_{\bullet i} d\mathbf{B}_t)^T \mathbf{b}} + \boxed{(\theta_i d\mathbf{B}_t)}$$

$n \times M$     $M \times 1$     $n \times 1$     $1 \times M$     $M \times 1$

multiplicative noise   additive noise



2<sup>nd</sup> order polynomial:  
coefficients given by physics,

$$(\phi_j)_j \quad \text{and} \quad \mathbf{a}(\mathbf{x}, \mathbf{x}) = \frac{1}{t} \langle (\boldsymbol{\sigma}(\mathbf{x})\mathbf{B})_{obs}, (\boldsymbol{\sigma}(\mathbf{x})\mathbf{B})_{obs}^T \rangle_t$$

Galerkin projection gives **SDEs** for resolved modes:

$$\int_{\Omega} \phi_i \cdot (\text{stochastic Navier-Stokes})$$

$$db_i = \boxed{F_i(\mathbf{b})} dt + \boxed{(\alpha_{\bullet i} d\mathbf{B}_t)^T \mathbf{b}} + \boxed{(\theta_i d\mathbf{B}_t)}$$

$n \times M$     $M \times 1$     $n \times 1$     $1 \times M$     $M \times 1$   
multiplicative noise   additive noise

**Correlations to estimate**

2<sup>nd</sup> order polynomial:  
coefficients given by physics,

$$(\phi_j)_j \quad \text{and} \quad \mathbf{a}(\mathbf{x}, \mathbf{x}) = \frac{1}{t} \langle (\boldsymbol{\sigma}(\mathbf{x})\mathbf{B})_{obs}, (\boldsymbol{\sigma}(\mathbf{x})\mathbf{B})_{obs}^T \rangle_t$$

# Estimate of the correlation $\alpha_{pi} \alpha_{qj}^T$

SDE

$$db_i = F_i(\mathbf{b})dt + (\alpha_{\bullet i} d\mathbf{B}_t)^T \mathbf{b} + (\theta_i d\mathbf{B}_t)$$

multiplicative  
noise

Classical estimate

$$\alpha_{pi} \alpha_{qj}^T = \frac{1}{t} \langle \alpha_{pi} \mathbf{B}, \alpha_{qj} \mathbf{B} \rangle_t$$

Problem:

$$\begin{matrix} \bullet & \underbrace{(\alpha_{pi} d\mathbf{B}_t)}_{1 \times M} \underbrace{)_{obs}}_{M \times 1} = \int_{\Omega} \underbrace{G_{pi}}_{\text{Known but complex linear operator}} [ \underbrace{(\sigma d\mathbf{B}_t)}_{d \times M} ]_{obs} \end{matrix}$$

(as a function of space)



Too complex to be computed for each time step

•

# Estimate of the correlation $\alpha_{pi} \alpha_{qj}^T$

SDE

$$db_i = F_i(\mathbf{b})dt + \underbrace{(\alpha_{\bullet i} d\mathbf{B}_t)^T \mathbf{b}}_{\text{multiplicative noise}} + (\theta_i d\mathbf{B}_t)$$

Classical estimate

$$\alpha_{pi} \alpha_{qj}^T = \frac{1}{t} \langle \alpha_{pi} \mathbf{B}, \alpha_{qj} \mathbf{B} \rangle_t$$

Problem:

- $$\underbrace{(\alpha_{pi} d\mathbf{B}_t)}_{1 \times M} \underbrace{d\mathbf{B}_t}_{M \times 1} \underbrace{obs}_{obs} = \int_{\Omega} \underbrace{\mathbf{G}_{pi}}_{\text{Known but complex linear operator}} [ \underbrace{(\sigma d\mathbf{B}_t)}_{d \times M} \underbrace{obs}_{obs} ]$$

➔ Too complex to be computed for each time step
- $$\underbrace{\mathbf{a}(x, y)}_{d \times d \times M \times M} = \frac{1}{t} \langle (\sigma(x) \mathbf{B})_{obs}, (\sigma(y) \mathbf{B})_{obs}^T \rangle_t$$

➔ Cannot be even memorized

# Estimate of the correlation $\alpha_{pi} \alpha_{qj}^T$

SDE

$$db_i = F_i(\mathbf{b})dt + (\alpha_{\bullet i} d\mathbf{B}_t)^T \mathbf{b} + (\theta_i d\mathbf{B}_t)$$

multiplicative  
noise

Solution:

$$\begin{aligned} \bullet \frac{1}{t} \left\langle b_i, \int_0^t b_p (\alpha_{qj} d\mathbf{B}_t) \right\rangle_t &= \sum_k \alpha_{ki} \alpha_{qj}^T \underbrace{\frac{1}{t} \int_0^t b_k b_p}_{=\lambda_p \delta_{kp}} + \theta_i \alpha_{qj}^T \underbrace{\frac{1}{t} \int_0^t b_p}_{=0} \\ &= \lambda_p \alpha_{pi} \alpha_{qj}^T \end{aligned}$$

(from the PCA)

•



# Estimate of the correlation $\alpha_{pi} \alpha_{qj}^T$

SDE

$$db_i = F_i(\mathbf{b})dt + (\alpha_{\bullet i} d\mathbf{B}_t)^T \mathbf{b} + (\theta_i d\mathbf{B}_t)$$

multiplicative  
noise

Solution:

$$\begin{aligned} \bullet \frac{1}{t} \left\langle b_i, \int_0^t b_p (\alpha_{qj} d\mathbf{B}_t) \right\rangle_t &= \sum_k \alpha_{ki} \alpha_{qj}^T \underbrace{\frac{1}{t} \int_0^t b_k b_p}_{=\lambda_p \delta_{kp}} + \theta_i \alpha_{qj}^T \underbrace{\frac{1}{t} \int_0^t b_p}_{=0} \\ &= \lambda_p \alpha_{pi} \alpha_{qj}^T \end{aligned}$$

(from the PCA)

$$\bullet \frac{1}{t} \left\langle b_i, \int_0^t b_p (\alpha_{qj} d\mathbf{B}_{t'}) \right\rangle_t = \int_{\Omega} \mathbf{G}_{qj} \left[ \frac{1}{t} \left\langle (b_i)_{obs}, \int_0^t (b_p \sigma d\mathbf{B}_{t'})_{obs} \right\rangle_t \right]$$

By linearity of  $\mathbf{G}_{qj}$

# Estimate of the correlation $\alpha_{pi} \alpha_{qj}^T$

SDE

$$db_i = F_i(\mathbf{b})dt + (\alpha_{\bullet i} d\mathbf{B}_t)^T \mathbf{b} + (\theta_i d\mathbf{B}_t)$$

multiplicative  
noise

Solution:

$$\bullet \frac{1}{t} \left\langle b_i, \int_0^t b_p (\alpha_{qj} d\mathbf{B}_t) \right\rangle_t = \sum_k \alpha_{ki} \alpha_{qj}^T \underbrace{\frac{1}{t} \int_0^t b_k b_p}_{=\lambda_p \delta_{kp}} + \theta_i \alpha_{qj}^T \underbrace{\frac{1}{t} \int_0^t b_p}_{=0}$$

(from the PCA)

$$= \lambda_p \alpha_{pi} \alpha_{qj}^T$$

d x M

$$\bullet \frac{1}{t} \left\langle b_i, \int_0^t b_p (\alpha_{qj} d\mathbf{B}_{t'}) \right\rangle_t = \int_{\Omega} \mathbf{G}_{qj} \left[ \frac{1}{t} \left\langle (b_i)_{obs}, \int_0^t (b_p \sigma d\mathbf{B}_{t'})_{obs} \right\rangle_t \right]$$

By linearity of  $\mathbf{G}_{qj}$

Independent of time t !

# Conclusion

# Conclusion

- Random transport applicable to any fluid dynamics models
- Better small scales
- Estimate position and amplitude of errors, extreme events, likely scenarios
- Possible applications for your previous and future estimation methods (e.g. MLE with  $(\Theta(\mathbf{x}_i, t_j))_{ij}$  )

Thank you for your attention

Code SQG MU:  
link from Fluminance website - V. Resseguier