An averaging technique for transport equations

Philippe Chartier, Nicolas Crouseilles, Mohammed Lemou

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Abstract

In this paper, we develop a new strategy aimed at obtaining high-order asymptotic models for transport equations with highly-oscillatory solutions. The technique relies upon averaging theory for ordinary differential equations, in particular normal form expansions in the vanishing parameter. Noteworthy, the result we state here also allows for the complete recovery of the exact solution from the asymptotic model. This is done by solving a companion transport equation that stems naturally from the change of variables underlying high-order averaging. Eventually, we apply our technique to the Vlasov equation with external electric and magnetic fields. Both constant and non-constant magnetic fields are envisaged, and asymptotic models already documented in the literature and re-derived using our methodology. In addition, it is shown how to obtain new high-order asymptotic models.

1 Introduction

In a large variety of situations, one is confronted to the resolution of a family of transport equations of the form

$$\partial_t f(t,y) + F^\varepsilon(y)\cdot\nabla_y f(t,y) = 0, \quad f(0,y) = f_0(y) \in \mathbb{R}, \quad t \in \mathbb{R}, \quad y \in \mathbb{R}^n, \tag{1.1}$$

indexed by a small positive parameter ε, whose occurrence in *real-life* models often lies at the core of numerous theoretical and numerical difficulties encountered in obtaining a(-n) (approximate-) solution. The nature of the difficulties triggered by the presence of ε may vary according to the form of the vector field $y \mapsto F^\varepsilon(y) \in \mathbb{R}^n$. In this article, we shall address the specific situation where it can be split into two parts

$$F^\varepsilon(y) = \frac{1}{\varepsilon}\omega(y)\, G(y) + K(y) \tag{1.2}$$

where the flow $(t, y_0) \mapsto \Phi_t(y_0)$ associated with the differential equation

$$\dot{y}(t) = G(y(t)), \quad y(0) = y_0, \tag{1.3}$$

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is assumed to be **periodic**, regardless of the specific trajectory (i.e. independently of the initial condition y_0 at time $t = 0$) and where $y \mapsto \omega(y)$ is a scalar function bounded from below by a positive constant. Owing to the $1/\varepsilon$-term in front of the vector field G, the solution of the transport equation evolves in a highly-oscillatory regime as soon as ε becomes small, which is specifically the regime under investigation here. Given their relevance in this context, we shall pay special attention to geometric aspects, such as volume- or energy-preservation, and for this reason, we will also discuss into more details the case where both G and K are divergence-free and/or Hamiltonian. Examples of highly-oscillatory equations of the form (1.1) are numerous [2, 3, 4, 5, 18, 19, 20, 21]. It is obviously out of the scope of this introductory paper to treat all of them: we will rather concentrate on the following model that will constitute hereafter our *target application*, namely the **Vlasov equation with strong magnetic field**:

$$\partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) + \left(E(x) + \frac{1}{\varepsilon} v \times B(x)\right) \cdot \nabla_v f(t,x,v) = 0 \tag{1.4}$$

where $x \in \mathbb{R}^3$ and $v \in \mathbb{R}^3$ denote respectively the position and the velocity of a particle, where $f : \mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3 \mapsto \mathbb{R}$ is the distribution function, i.e. the density of particles at time t, position x and velocity v, and where $E : \mathbb{R}^3 \mapsto \mathbb{R}^3$ and $B : \mathbb{R}^3 \mapsto \mathbb{R}^3$ are respectively the electric and magnetic fields, assumed to be *external* at this stage (i.e. not coupled with f through Maxwell equations for instance). Our first objective is this paper is to derive asymptotic models for equation (1.1) with F^ε satisfying (1.2) and $\omega \equiv 1$. Rather then merely obtain the limit equation where ε tends to zero, we strive after higher-order terms in powers of ε. The methodology we propose relies on the theory of averaging for highly-oscillatory ordinary differential equations [27, 28], and more precisely on normal forms obtained as ε-expansions. Such series have been derived with B-series in [7, 8, 10] or somehow more simply in [24, 25, 26] with word-series. The underlying results we shall lean onto will be presented in Section 3. Prior to that, we shall show in Section 2 how the splitting of the vector field F^ε into two *commuting* vector fields naturally leads to *two independent transport equations*. The first result of this paper (for constant ω) will be stated in Section 4. In Section 5, we will address the much more involved situation of a varying frequency (ω non-constant in (1.2)), which requires to work in an augmented space. In particular, the main result of this paper will be stated there. It allows to rewrite the original transport equation (1.1) as a set of four *non-stiff* equations for a phase-function (S) and a profile-function (h). The two equations for the profile function are the counterpart of the averaged equation obtained elsewhere in the literature. However, solving the equation for the phase-function S allows to recover exactly the complete solution of (1.1). This part is up to our knowledge completely new. Since we use series-expansions, it is possible to write down explicitly and in a systematic way the terms appearing in the four equations for S and h. In Section 6, we shall eventually envisage our target application (1.4) and show how to obtain the terms of these developments. Firstly, in Section 6.1, we will consider the case of a constant magnetic field $B(x) \equiv B$ in two dimensions (1.4), as it appears to be a simple application of the results of Section 4. Secondly, in Section 6.2, we will address the more complicated situation of a varying magnetic field (B non-constant in (1.4)), which requires a preliminary treatment of the transport equation, as exposed in Section 5. At last, we shall treat equation (1.4) in full generality, i.e. in three dimensions and with a non-constant magnetic field, and compare the equations we obtain with our methodology to results previously published in the literature.

2 Decomposition of a transport equation

Let us consider the Liouville equation

$$\partial_t f(t,y) + F(y) \cdot \nabla_y f(t,y) = 0,$$

associated to a split vector field of the form

$$F = F_1 + F_2,$$

and let us make the fundamental assumption that the Lie bracket of F_1 and F_2 vanishes, that is to say that

$$\forall y \in \mathbb{R}^n, \quad [F_1, F_2](y) := (\partial_y F_1)(y)\ F_2(y) - (\partial_y F_2)(y)\ F_1(y) = 0.$$

2.1 A few elementary relations from differential geometry

This *commutation* of vector fields further manifests itself as the commutation of the two flows[1] associated with F_1 and F_2. More precisely, denoting respectively by φ_t, Φ_t, and Ψ_t, the t-flows corresponding to respectively F, F_1 and F_2, as defined by the differential equations

$$\frac{d}{dt}\varphi_t(y) = F \circ \varphi_t(y), \quad \frac{d}{dt}\Phi_t(y) = F_1 \circ \Phi_t(y) \quad \text{and} \quad \frac{d}{dt}\Psi_t(y) = F_2 \circ \Psi_t(y), \tag{2.1}$$

then the following relations hold true

$$\forall (s,t) \in \mathbb{R}^2, \quad \Phi_t \circ \Psi_s = \Psi_s \circ \Phi_t \quad \text{and} \quad \varphi_t = \Phi_t \circ \Psi_t = \Psi_t \circ \Phi_t. \tag{2.2}$$

In other words and equivalently, the pullback of F_1 through Φ_t and Ψ_t is F_1 itself, i.e.

$$\forall t \in \mathbb{R}, \forall y \in \mathbb{R}^n, \quad \partial_y \Phi_t(y)\ F_1(y) = F_1 \circ \Phi_t(y) \ \text{ and } \ \partial_y \Psi_t(y)\ F_1(y) = F_1 \circ \Psi_t(y), \tag{2.3}$$

and the pullback of F_2 through Φ_t and Ψ_t is F_2 itself, i.e.

$$\forall t \in \mathbb{R}, \forall y \in \mathbb{R}^n, \quad \partial_y \Phi_t(y)\ F_2(y) = F_2 \circ \Phi_t(y) \ \text{ and } \ \partial_y \Psi_t(y)\ F_2(y) = F_2 \circ \Psi_t(y). \tag{2.4}$$

These relations can be found in most textbooks on dynamical systems or easily re-derived.

[1] These flows are assumed to be defined for all $t \in \mathbb{R}$ and all $y \in \mathbb{R}^n$ without further notice.

2.2 Characteristics and separation of variables

The method of characteristics immediately gives for any *smooth* solution of (1.1)

$$\forall (t,y) \in \mathbb{R} \times \mathbb{R}^n, \quad \frac{d}{dt} f(t, \varphi_t(y)) = f(y),$$

and given the initial condition at time $t = 0$, the solution of (1.1) may be written as

$$\forall (t,y) \in \mathbb{R} \times \mathbb{R}^n, \quad f(t,y) = f_0(\varphi_{-t}(y)). \tag{2.5}$$

Now, owing to relations (2.2), a somehow *natural* step forward consists in *separating the two times* in Φ and Ψ and defining the new function $\tilde{f}$ with additional variable τ

$$\forall (t,\tau,y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n, \quad \tilde{f}(t,\tau,y) = f_0(\Phi_{-\tau} \circ \Psi_{-t}(y)). \tag{2.6}$$

We are now in position to state a few elementary results:

Lemma 2.1 *The solution $(t,y) \mapsto f(t,y)$ of equation (1.1) may be obtained as the value on the diagonal $\tau = t$ of* **any** *solution $\check{f}$ (it is not unique) of the equation*

$$\partial_t \check{f}(t,\tau,y) + \partial_\tau \check{f}(t,\tau,y) + F(y) \cdot \nabla_y \check{f}(t,t,y) = 0, \quad \check{f}(0,0,y) = f_0(y), \tag{2.7}$$

i.e.

$$\forall (t,y) \in \mathbb{R} \times \mathbb{R}^n, \quad f(t,y) = \check{f}(t,\tau,y)\big|_{\tau=t}.$$

Proof. Let $\check{f}$ be any solution of (2.7). It is straightforward to see that, for any given fixed value $y \in \mathbb{R}^n$, one has

$$(\partial_t \check{f})(t,t,y) + (\partial_\tau \check{f})(t,t,y) = \partial_t(\check{f}(t,t,y))$$

so that $\check{f}(t,t,y)$ satisfies equation (1.1) with the same initial condition as $f(t,y)$. The result follows by uniqueness of the solution of (1.1). ■

This result is of little interest *per se*. More interesting is the following lemma, which shows that, as far as the specific solution $\tilde{f}$ of (2.7) is concerned, it satisfies two **independent** equations.

Lemma 2.2 *The function $\tilde{f}$ defined by (2.6) satisfies the following two transport equations:*

$$\forall (t,\tau,y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n, \quad \partial_\tau \tilde{f}(t,\tau,y) + F_1(y) \cdot \nabla_y \tilde{f}(t,\tau,y) = 0, \tag{2.8}$$

and

$$\forall (t,\tau,y) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n, \quad \partial_t \tilde{f}(t,\tau,y) + F_2(y) \cdot \nabla_y \tilde{f}(t,\tau,y) = 0, \tag{2.9}$$

with initial condition $\tilde{f}(0,0,y) = f_0(y)$. The solution of (2.8,2.9) with this initial condition is unique.

Proof. We have on the one hand

$$\begin{aligned}
\partial_\tau \tilde{f}(t,\tau,y) &= \partial_y f_0 \circ \Phi_{-\tau} \circ \Psi_{-t}(y)\ \ \partial_\tau \Phi_{-\tau} \circ \Psi_{-t}(y) \\
&= -\partial_y f_0 \circ \Phi_{-\tau} \circ \Psi_{-t}(y)\ \ F_1 \circ \Phi_{-\tau} \circ \Psi_{-t}(y) \\
&= -\left(\partial_y f_0\ F_1\right) \circ \Phi_{-\tau} \circ \Psi_{-t}(y)
\end{aligned}$$

where we have used the second relation of (2.1) with τ replaced by $-\tau$ and y by $\Psi_{-t}(y)$, and on the other hand

$$\begin{aligned}
\partial_y \tilde{f}(t,\tau,y)\ F_1(y) &= \partial_y f_0 \circ \Phi_{-\tau} \circ \Psi_{-t}(y)\ \ \partial_y \Phi_{-\tau} \circ \Psi_{-t}(y)\ \ \partial_y \Psi_{-t}(y)\ \ F_1(y) \\
&= \partial_y f_0 \circ \Phi_{-\tau} \circ \Psi_{-t}(y)\ \ \partial_y \Phi_{-\tau} \circ \Psi_{-t}(y)\ \ F_1 \circ \Psi_{-t}(y) \\
&= \partial_y f_0 \circ \Phi_{-\tau} \circ \Psi_{-t}(y)\ \ F_1 \circ \Phi_{-\tau} \circ \Psi_{-t}(y), \\
&= \left(\partial_y f_0\ F_1\right) \circ \Phi_{-\tau} \circ \Psi_{-t}(y)
\end{aligned}$$

where we have used the second relation of (2.3). Adding up the two terms, we get equation (2.8). Now, repeating similar arguments, using the third relation of (2.1) with t replaced by $-t$ and the first relation of (2.4), we get

$$\begin{aligned}
\partial_t \tilde{f}(t,\tau,y) &= \partial_y f_0 \circ \Phi_{-\tau} \circ \Psi_{-t}(y)\ \ \partial_y \Phi_{-\tau} \circ \Psi_{-t}(y)\ \ \partial_t \Psi_{-t}(y) \\
&= -\partial_y f_0 \circ \Phi_{-\tau} \circ \Psi_{-t}(y)\ \ \partial_y \Phi_{-\tau} \circ \Psi_{-t}(y)\ \ F_2 \circ \Psi_{-t}(y) \\
&= -\left(\partial_y f_0\ F_2\right) \circ \Phi_{-\tau} \circ \Psi_{-t}(y)
\end{aligned}$$

and on the other hand, using the second relation of (2.4)

$$\begin{aligned}
\partial_y \tilde{f}(t,\tau,y)\ \ F_2(y) &= \partial_y f_0 \circ \Phi_{-\tau} \circ \Psi_{-t}(y)\ \ \partial_y \Phi_{-\tau} \circ \Psi_{-t}(y)\ \ \partial_y \Psi_{-t}(y)\ \ F_2(y) \\
&= \partial_y f_0 \circ \Phi_{-\tau} \circ \Psi_{-t}(y)\ \ \partial_y \Phi_{-\tau} \circ \Psi_{-t}(y)\ \ F_2 \circ \Psi_{-t}(y) \\
&= \partial_y f_0 \circ \Phi_{\tau} \circ \Psi_{-t}(y)\ \ F_2 \circ \Phi_{-\tau} \circ \Psi_{t}(y) \\
&= \left(\partial_y f_0\ F_2\right) \circ \Phi_{-\tau} \circ \Psi_{-t}(y),
\end{aligned}$$

and again, upon adding up these two terms, we obtain equation (2.9). ■

It is worth emphasizing that, as stated in previous lemma, the initial condition

$$\tilde{f}(0,0,y) = f_0(y)$$

is enough to determine a unique solution: in order to get $\tilde{f}$ at (t,τ), the two equations (2.8) and (2.9) have to be solved *successively*. Nevertheless, the order in which they are solved does not play any role.

3 Averaging of ordinary differential equations in a nutshell

Since our approach for averaging the transport equation (1.1) consists in averaging first the characteristics and then rewrite the corresponding Liouville equations, we hereafter recall the main results upon which we shall lean. In this expository paper, we content ourselves with formal expansions, thus neglecting at this stage the occurence of error terms. This is justified by the fact that these errors actually become exponentially small in terms of ε (they are bounded by $Ce^{-C/\varepsilon}$ for some positive constant C). A completely rigorous treatment of this part can be found for in [8] and the consequences in our situation will be analysed in a forthcoming paper [11], together with the numerical counterpart of the technique presented here.

3.1 A normal form theorem

Consider the highly-oscillatory differential equation

$$\dot{y} = F^{\varepsilon}(y) := \frac{1}{\varepsilon} G(y) + K(y) \tag{3.1}$$

i.e. equation (1.2) with $\omega \equiv 1$, where both vector fields G and K are assumed to be smooth[2]. As already alluded to in the Introduction section, the fundamental assumption (**H**) required to go any further is that:

(H) G generates a **periodic flow**, regardless of the specific trajectory (i.e. with a period which remains independent of the initial value). By convention, we will suppose here that this period is 2π.

Now, in accordance with previous notations, let us write φ_t^{ε} the t-flow of the differential equation

$$\frac{d}{dt}\varphi_t^{\varepsilon}(y) = (F^{\varepsilon} \circ \varphi_t^{\varepsilon})(y)$$

and Φ_τ and Ψ_t the τ- and t-flows of the two differential equations

$$\frac{d}{dt}\Phi_t(y) = (G \circ \Phi_t)(y) \qquad \text{and} \qquad \frac{d}{dt}\Psi_t(y) = (K \circ \Psi_t)(y).$$

Since the Lie-bracket of G and K has here no reason to vanish, the two flows Φ_τ and Ψ_t do not commute and we can not reproduce right away the analysis conducted in previous section. It is precisely the aim of averaging to rewrite F^{ε} as the sum of two commuting fields[3]. As already emphasized, this is in general possible only up to small error terms, so that the theorem stated below is to be understood in a formal sense.

Theorem 3.1 *Suppose that the vector fields F^{ε} can be split according to equation (3.1) and that G satisfies assumption* (**H**). *Then there exist two vector fields G^{ε} and K^{ε} such that*

(i) the Lie-bracket of G^{ε} and K^{ε} vanish, i.e. $[G^{\varepsilon}, K^{\varepsilon}](y) = 0$ for all $y \in \mathbb{R}^n$;

(ii) the vector field G^{ε} generates a flow Φ_t^{ε} which is 2π-periodic, regardless of the specific trajectory, i.e.

$$\forall (t, y) \in \mathbb{R} \times \mathbb{R}^n, \quad \Phi_{t+2\pi}^{\varepsilon}(y) = \Phi_t^{\varepsilon}(y).$$

[2] Either of class C^k or analytic. The precise smoothness assumption determines the type of error bounds, either polynomial or exponential in ε and is thus not essential here (see [11]).

[3] At least, this is one way to envisage averaging for ordinary differential equations and this is the point of view adopted both in [10] and in the recent series of papers by Murua and Sanz-Serna [24, 25, 26].

Remark 3.2 *As a consequence of the first point of Theorem 3.1, we have*

$$\forall t, \quad \varphi_t^\varepsilon = \Phi_{t/\varepsilon}^\varepsilon \circ \Psi_t^\varepsilon = \Psi_t^\varepsilon \circ \Phi_{t/\varepsilon}^\varepsilon,$$

where Ψ_t^ε denotes the t-flow associated with vector field K^ε.

This result brings us back to Section 2 and indeed allows to *split* equation (1.1) into two equations of the form (2.8, 2.9); details will be given in Section 4. We conclude this subsection with a few additional statements related to the conservation of geometric properties by stroboscopic averaging.

Theorem 3.3 *Suppose that the vector fields F^ε can be split according to equation (3.1) and that G satisfies assumption* (**H**)*. Then the two vector fields G^ε and K^ε of Theorem 3.1 have the following properties:*

(i) if both G and K are divergence-free vector fields, then so are G^ε and K^ε;

(ii) if both G and K are Hamiltonian vector fields, then so are G^ε and K^ε;

(iii) if Λ^ε is an invariant *of the flow φ_t^ε, i.e. if*

$$\forall (t,y), \quad \Lambda^\varepsilon(\varphi_t^\varepsilon(y)) = \Lambda^\varepsilon(y)$$

then it is also an invariant of both Φ_t^ε and Ψ_t^ε where Ψ_t^ε is the t-flow of K^ε, i.e.

$$\forall (t,y), \quad \Lambda^\varepsilon(\Phi_t^\varepsilon(y)) = \Lambda^\varepsilon(\Psi_t^\varepsilon(y)) = \Lambda^\varepsilon(y).$$

Remark 3.4 *The properties of Theorem 3.3 are intimately linked to the choice of stroboscopic averaging (see [10, 14]), which is the only averaging procedure preserving geometric properties of the initial vector field F^ε.*

3.2 Expansions in powers of ε of the vector fields G^ε and K^ε

Since we wish in particular to identify the asymptotic behaviour of (1.1) in the limit when ε tends to zero as well as higher-order terms in ε, it is essential to consider ε-expansions of the various functions appearing in Theorem 3.1. Since this was precisely the point of view adopted in [10, 14], where B-series were used to this aim, we shall again quote the following result[4]:

Theorem 3.5 *Consider the Fourier series of*

$$K_\tau(y) = \left(\frac{\partial \Phi_\tau}{\partial y}(y)\right)^{-1} (K \circ \Phi_\tau)(y) = \sum_{k\in\mathbb{Z}} e^{ik\tau} \hat{K}_k(y). \tag{3.2}$$

The averaged *vector field K^ε admits the following formal ε-expansion*

$$K^\varepsilon = \sum_{r=1}^{+\infty} \varepsilon^{r-1} K^{[r]} = \sum_{r=1}^{+\infty} \frac{\varepsilon^{r-1}}{r} \sum_{(i_1,\ldots,i_r)\in\mathbb{Z}^r} \bar{\beta}_{i_1\cdots i_r} \left[\ldots[\hat{K}_{i_1}, \hat{K}_{i_2}], \hat{K}_{i_3}], \ldots, \hat{K}_{i_r}\right] \tag{3.3}$$

[4]Note that an alternative proof, somewhat simpler, of this result may also be found in [24] and [25].

where the coefficients $\bar\beta$ are universal (problem-independent). Similarly, the vector field G^ε admits the following formal ε-expansion

$$G^\varepsilon \quad = \quad \varepsilon(F^\varepsilon - K^\varepsilon). \tag{3.4}$$

Remark 3.6 *The fact that geometric properties of G^ε and K^ε are inherited from F^ε may also be seen as a direct consequence of the form of previous expansions, which are linear combinations of embedded Lie-brackets of the $\hat K_k$'s. As a matter of fact, the process of changing the variables in K_τ and forming Lie-brackets of the $\hat K_k$'s commute. For instance, if both G and K are Hamiltonian, then K_τ is of the form*

$$K_\tau(y) = J^{-1}\nabla_y H_\tau(y) = \sum_{k\in\mathbb{Z}} e^{i\tau}\hat H_k(y)$$

and all Fourier coefficients $\hat K_k(y) = J^{-1}\nabla_y \hat H_k(y)$ are also Hamiltonian. Since

$$\forall (k,l)\in\mathbb{Z}^2, \quad [\hat K_k, \hat K_l] = J^{-1}\nabla_y\{\hat H_k, \hat H_l\}$$

where $\{\cdot,\cdot\}$ denotes the Poisson bracket operation, it is then immediate to see that both G^ε and K^ε are Hamiltonian with Hamiltonians given by formulas (3.3) and (3.4) where Lie brackets are replaced by Poisson brackets and the $\hat K_k$'s by the $\hat H_k$'s. Similarly, if $\mathrm{div}\,(G) = \mathrm{div}\,(K) = 0$, then $\mathrm{div}\,(\hat K_k) = 0$ for all $k\in\mathbb{Z}$ and a standard computation shows that

$$\forall (k,l)\in\mathbb{Z}^2, \quad \mathrm{div}\,\left([\hat K_k, \hat K_l]\right) = 0$$

so that again both G^ε and K^ε are divergence-free.

In order to be able to derive the expansions of G^ε and K^ε, it still remains to give the value of the coefficients $\bar\beta$ appearing in formula (3.3). This is the purpose of next proposition.

Proposition 3.7 *The coefficients $\bar\beta$ can be computed recursively from the following formulas, which hold for all values of $j\in\mathbb{Z}^*$, $r,s\in\mathbb{N}^*$ and $(l_1,\ldots,l_s)\in\mathbb{Z}^s$:*

$$\begin{array}{llllll}
\bar\beta_0 & = & 1, & \bar\beta_j & = & 0,\\
\bar\beta_{0^{r+1}} & = & 0, & \bar\beta_{0^r j} & = & \frac{i}{j}\left(\bar\beta_{0^{r-1}j} - \bar\beta_{0^r}\right),\\
\bar\beta_{jl_1\cdots l_s} & = & \frac{i}{j}\left(\bar\beta_{l_1\cdots l_s} - \bar\beta_{(j+l_1)l_2\cdots l_s}\right), & \bar\beta_{0^r jl_1\cdots l_s} & = & \frac{i}{j}\left(\bar\beta_{0^{r-1}jl_1\cdots l_s} - \bar\beta_{0^r(j+l_1)l_2\cdots l_s}\right).
\end{array}$$

For the sake of illustration and later use, we now give the first terms of $K^\varepsilon = K_1 + \varepsilon K_2 + \varepsilon^2 K_3 + \mathcal{O}(\varepsilon^3)$, as stated in [7]:

$$
\begin{aligned}
K^{[1]} &= \hat{K}_0, \\
K^{[2]} &= \sum_{k>0} \frac{i}{k}\left([\hat{K}_k, \hat{K}_{-k}] + [\hat{K}_0, \hat{K}_k - \hat{K}_{-k}]\right), \\
K^{[3]} &= \sum_{k\neq 0} \frac{1}{k^2}\left([[\hat{K}_k, \hat{K}_0], \hat{K}_0] + [[\hat{K}_{-k}, \hat{K}_k], \hat{K}_k] - \frac{1}{2}[[\hat{K}_{-2k}, \hat{K}_k], \hat{K}_k] + [[\hat{K}_0, \hat{K}_k], \hat{K}_{-k}]\right) \\
&\quad - \sum_{0\neq m\neq -l\neq 0} \frac{1}{l(m+l)}[[\hat{K}_0, \hat{K}_l], \hat{K}_m] + \sum_{k<-|l|} \frac{1}{lk}[[\hat{K}_k, \hat{K}_l], \hat{K}_{-l}] \\
&\quad - \sum_{0>k<m, m+k\neq 0} \frac{1}{km}[[\hat{K}_k, \hat{K}_{-k}], \hat{K}_m] \\
&\quad - \sum_{0\neq m\neq \pm l\neq 0, m>-m-l<l} \frac{1}{m(m+l)}[[\hat{K}_{-m-l}, \hat{K}_l], \hat{K}_m]. \qquad (3.5)
\end{aligned}
$$

Remark 3.8 *The following expressions of the first three terms of the averaged equation have also been derived in various places and do not use Fourier coefficients:*

$$
\begin{aligned}
K^{[1]}(y) &= \frac{1}{2\pi}\int_0^{2\pi} K_\tau(y)d\tau, \quad K^{[2]}(y) = \frac{-1}{4\pi}\int_0^{2\pi}\int_0^{\tau} [K_s(y), K_\tau(y)]dsd\tau, \\
K^{[3]}(y) &= \frac{1}{8\pi}\int_0^{2\pi}\int_0^{\tau}\int_0^{s} [[K_r(y), K_s(y)], K_\tau(y)]drdsd\tau \\
&\qquad + \frac{1}{24\pi}\int_0^{2\pi}\int_0^{\tau}\int_0^{\tau} [K_r(y), [K_s(y), K_\tau(y)]]drdsd\tau.
\end{aligned}
$$

Further terms can be formally obtained by using a non-linear Magnus expansion [1]. Each of these is a linear combination of iterated integrals of iterated brackets of K_τ.

4 Averaging for highly-oscillatory kinetic equations

Compiling the arguments of the two previous sections, it is now straightforward to obtain the following corollary, which establishes in particular the *formal* existence of an averaged transport equation for problems of the form (1.1, 1.2).

Corollary 4.1 *Let $F^\varepsilon = \frac{1}{\varepsilon}G^\varepsilon + K^\varepsilon$ be the normal form splitting of a highly-oscillating vector field $F^\varepsilon = \frac{1}{\varepsilon}G + K$ satisfying* (**H**). *The solution of the transport equation*

$$\partial_t f(t,y) + F^\varepsilon(y) \cdot \nabla_y f(t,y) = 0$$

may be obtained as the diagonal value (i.e. for the value $\tau = t/\varepsilon$) of the two-scale function $\tilde{f}(t,\tau,y)$, 2π-periodic in τ, satisfying the following set of two equations

$$
\begin{aligned}
(i) &\quad \forall (t,\tau,y), \quad \partial_\tau \tilde{f}(t,\tau,y) + G^\varepsilon(y)\cdot\nabla_y \tilde{f}(t,\tau,y) = 0, \\
(ii) &\quad \forall (t,\tau,y), \quad \partial_t \tilde{f}(t,\tau,y) + K^\varepsilon(y)\cdot\nabla_y \tilde{f}(t,\tau,y) = 0.
\end{aligned}
$$

Equation (ii) is the so-called averaged kinetic equation. *Moreover, the ε-expansions of G^ε and K^ε are given by formulas (3.3, 3.4) of Theorem 3.5. If in addition G and K are both divergence-free, then so are G^ε and K^ε, and similarly, if G and K are both Hamiltonian, then so are G^ε and K^ε, with Hamiltonians that can be obtained again from formulas (3.3, 3.4) by replacing Lie brackets by Poisson brackets.*

Proof. The statements follow immediately by taking $F_1 = \frac{1}{\varepsilon}G^\varepsilon$ and $F_2 = K^\varepsilon$ in Lemma 2.2. ■

As a first illustration of this corollary, we consider the very simplified case of a set of particles evolving in a constant electric field (independent of time and phase-space variables) and submitted to a constant magnetic field. The corresponding equation

$$\partial_t f + v \cdot \nabla_x f + \left(\frac{1}{\varepsilon} Jv + E\right) \cdot \nabla_v f = 0, \tag{4.1}$$

-where f depends on time $t \in \mathbb{R}$, position $x \in \mathbb{R}^2$ and velocity $v \in \mathbb{R}^2$- is obviously of the form (1.1) with $y = (x_1, x_2, v_1, v_2)^T \in \mathbb{R}^4$. Here, J denotes here the 2×2-matrix

$$\left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right).$$

Due to the extreme simplicity of the vector field F^ε, the solution $f(t,x,v)$ can be written explicitly as $\tilde{f}(t,\tau,x,v)$ for $\tau = t/\varepsilon$ and

$$\tilde{f}(t,\tau,x,v) = f_0\left(x - \varepsilon J(e^{\tau J} - I)v - \varepsilon^2(e^{\tau J}E - E) + \varepsilon t JE, e^{\tau J}v - \varepsilon J(e^{\tau J} - I)E\right). \tag{4.2}$$

In order to apply the result of this section, we write

$$F^\varepsilon(y) = \left(\begin{array}{c} v \\ \frac{1}{\varepsilon}Jv + E \end{array}\right) = \frac{1}{\varepsilon}G + K \quad \text{with} \quad G(y) = \left(\begin{array}{c} 0 \\ Jv \end{array}\right) \quad \text{and} \quad K(y) = \left(\begin{array}{c} v \\ E \end{array}\right).$$

The corresponding flow Φ_τ (associated with G) simply reads

$$\Phi_\tau(y) = \left(\begin{array}{c} x \\ e^{\tau J}v \end{array}\right)$$

and

$$K_\tau(y) = \left(\begin{array}{c} e^{\tau J}v \\ e^{-\tau J}E \end{array}\right) = e^{i\tau}\hat{K}_1(y) + e^{-i\tau}\hat{K}_{-1}(y)$$

with

$$\hat{K}_1(y) = \frac{1}{2}\left(\begin{array}{c} v - iJv \\ E + iJE \end{array}\right) \quad \text{and} \quad \hat{K}_{-1}(y) = \frac{1}{2}\left(\begin{array}{c} v + iJv \\ E - iJE \end{array}\right),$$

where we have used the relation $e^{\theta J} = (\cos\theta)I + (\sin\theta)J$ and have written $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$. Formula (3.3) then gives

$$\begin{aligned} K^{[1]} &= \hat{K}_0 = 0, \\ K^{[2]} &= i[\hat{K}_1, \hat{K}_{-1}] = -2\Im\left((\partial_y \hat{K}_1)\hat{K}_{-1}\right) = \left(\begin{array}{c} JE \\ 0 \end{array}\right) \end{aligned}$$

and all other $K^{[r]}$ for $r \geq 3$ vanish as can be checked by easy calculations. Equation (i) of Corollary 4.1 for $\tilde{f}(t,\tau,x,v)$ thus has the following form

$$\partial_\tau \tilde{f} + (\varepsilon v - \varepsilon^2 JE) \cdot \nabla_x \tilde{f} + (Jv + \varepsilon E) \cdot \nabla_v \tilde{f} = 0$$

while equation (ii) is simply

$$\partial_t \tilde{f} + \varepsilon JE \cdot \nabla_x \tilde{f} = 0.$$

By direct differentiation of formula (4.2) w.r.t. t and then τ, it can be seen that $\tilde{f}$ satisfies both equations (i) and (ii), as expected.

5 High-oscillations with varying frequency

In this section, we again consider the transport equation

$$\partial_t f(t,y) + F^\varepsilon(y) \cdot \nabla_y f(t,y) = 0 \tag{5.1}$$

where the vector field F^ε is now of the form

$$F^\varepsilon(y) = \frac{1}{\varepsilon}\omega(y)G(y) + K(y) \tag{5.2}$$

with G still generating a 2π-periodic flow Φ_τ, independently of the initial condition. In this form, Theorem 4.1 does not directly apply, owing to the non-existence of a common frequency for all trajectories (if ω varies). In order to rewrite (5.1) in a more amenable form, we thus divide it by ω

$$\frac{1}{\omega(y)}\partial_t f(t,y) + \frac{1}{\omega(y)}F^\varepsilon(y) \cdot \nabla_y f(t,y) = 0. \tag{5.3}$$

Upon denoting $Y = (t,y)$, previous equation may then be rewritten as

$$D_{\check{F}^\varepsilon}[f](Y) = 0 \tag{5.4}$$

where $D_{\check{F}^\varepsilon}[f]$ is the Lie derivative of f in the direction of the **augmented** vector field

$$\begin{aligned} \check{F}^\varepsilon(Y) &= \begin{pmatrix} \frac{1}{\omega(y)} \\ \frac{1}{\omega(y)}F^\varepsilon(y) \end{pmatrix} = \frac{1}{\varepsilon}\begin{pmatrix} 0 \\ G(y) \end{pmatrix} + \begin{pmatrix} \frac{1}{\omega(y)} \\ \frac{1}{\omega(y)}K(y) \end{pmatrix} \\ &:= \frac{1}{\varepsilon}\check{G}(Y) + \check{K}(Y), \end{aligned} \tag{5.5}$$

where $\check{G}$ still generates a 2π-periodic flow. Denoting $\check{\varphi}_s^\varepsilon$ the s-flow associated with the vector field $\check{F}^\varepsilon$, we may write from (5.4)

$$\frac{d}{ds} f(\check{\varphi}_s(Y)) = D_{\check{F}^\varepsilon}[f](\check{\varphi}_s(Y)) = 0 \tag{5.6}$$

so that

$$\forall s \in \mathbb{R}, \quad f \circ \check{\varphi}_{-s}^\varepsilon(t,y) = f(t,y).$$

5.1 Immersion as the stationary solution of an extended equation

The *extended* function $g(s, Y) = f(\check{\varphi}_s^\varepsilon(Y))$ can be regarded as the (stationary) solution of the transport equation

$$\partial_s g(s, Y) + \check{F}^\varepsilon(Y) \cdot \nabla_Y g(s, Y) = 0, \quad g(0, Y) = f(Y) = f(t, y).$$

Proceeding as in Section 2, we then get two equations for

$$\tilde{g}(s, \tau, Y) = g(0, \check{\Phi}^\varepsilon_{-\tau} \circ \check{\Psi}^\varepsilon_{-s}(Y))$$

of the form

$$(i) \quad \partial_s \tilde{g}(s, \tau, Y) + \check{K}^\varepsilon(Y) \cdot \nabla_Y \tilde{g}(s, \tau, Y) = 0 \tag{5.7}$$

$$(ii) \quad \partial_\tau \tilde{g}(s, \tau, Y) + \check{G}^\varepsilon(Y) \cdot \nabla_Y \tilde{g}(s, \tau, Y) = 0 \tag{5.8}$$

which in principle can be solved one after another in any order. Note the usual relation $\tilde{g}(s, s/\varepsilon, Y) = g(s, Y) = f(Y)$. However, there is here no initial condition at $s = \tau = 0$, since $\tilde{g}(0, 0, Y) = g(0, Y) = f(t, y)$, i.e. precisely the unknown of the original problem.

5.2 Eliminating the extra-variable s

Our objective in this subsection is to transform the two equations (5.7,5.8) into new equations which do not involve the variable s and are provided with a proper initial condition. We will then show how to recover the original solution $f(t, y)$ using only these new equations. To this aim, we thus introduce a phase-function $(t, \tau, y) \mapsto S(t, \tau, y)$ in the spirit of [22], which will be defined later on as the solution of a transport equation, and a profile-function $(t, \tau, y) \mapsto h(t, \tau, y)$ defined by

$$h(t, \tau, y) = \tilde{g}(S(t, \tau, y), \tau, t, y), \tag{5.9}$$

which will satisfy a companion transport equation. The following relations

$$\begin{aligned}
\partial_t h &= (\partial_s \tilde{g}(S, \tau, t, y))\, \partial_t S + \partial_t \tilde{g}(S, \tau, t, y), \\
\partial_y h &= (\partial_s \tilde{g}(S, \tau, t, y))\, \partial_y S + \partial_y \tilde{g}(S, \tau, t, y), \\
\partial_\tau h &= (\partial_s \tilde{g}(S(t, \tau, y), \tau, t, y))\, \partial_\tau S + \partial_\tau \tilde{g}(S, \tau, t, y),
\end{aligned}$$

where we have omitted the obvious arguments of functions h and S, may be straightforwardly obtained. Together with equations (5.7) and (5.8), they lead immediately to

$$\begin{aligned}
&\check{K}_1^\varepsilon(y) \partial_t h(t, \tau, y) + \check{K}_2^\varepsilon(y) \cdot \nabla_y h(t, \tau, y) \\
&= (\partial_s \tilde{g}(S, \tau, t, y)) \left(\check{K}_1^\varepsilon(y) \partial_t S(t, \tau, y) + \check{K}_2^\varepsilon(y) \partial_y S(t, \tau, y) - 1 \right),
\end{aligned}$$

and

$$\begin{aligned}
&\partial_\tau h(t, \tau, y) + \check{G}_1^\varepsilon(y) \partial_t h(t, \tau, y) + \check{G}_2^\varepsilon(y) \cdot \nabla_y h(t, \tau, y) \\
&= (\partial_s \tilde{g}(S(t, \tau, y), \tau, t, y)) \left(\check{G}_1^\varepsilon(y) \partial_t S(t, \tau, y) + \check{G}_2^\varepsilon(y) \partial_y S(t, \tau, y) + \partial_\tau S(t, \tau, y) \right),
\end{aligned}$$

where the index 1 in $\check{K}_1^\varepsilon$ and $\check{G}_1^\varepsilon$ refers to the first component of $\check{K}^\varepsilon$ and $\check{G}^\varepsilon$, while the index 2 in $\check{K}_2^\varepsilon$ and $\check{G}_2^\varepsilon$ refers to *all remaining components* of $\check{K}^\varepsilon$ and $\check{G}^\varepsilon$. Now, in order to eliminate the dependence on s from the previous two equations, one has to choose S and h such that

$$\begin{aligned}&\check{K}_1^\varepsilon(y)\partial_t S(t,\tau,y) + \check{K}_2^\varepsilon(y)\cdot\nabla_y S(t,\tau,y) = 1,\\ &\partial_\tau S(t,\tau,y) + \check{G}_1^\varepsilon(y)\partial_t S(t,\tau,y) + \check{G}_2^\varepsilon(y)\cdot\nabla_y S(t,\tau,y) = 0,\end{aligned} \tag{5.10}$$

and

$$\begin{aligned}&\check{K}_1^\varepsilon(y)\partial_t h(t,\tau,y) + \check{K}_2^\varepsilon(y)\cdot\nabla_y h(t,\tau,y) = 0,\\ &\partial_\tau h(t,\tau,y) + \check{G}_1^\varepsilon(y)\partial_t h(t,\tau,y) + \check{G}_2^\varepsilon(y)\cdot\nabla_y h(t,\tau,y) = 0,\end{aligned} \tag{5.11}$$

with initial conditions

$$S(0,0,y) = 0, \qquad h(0,0,y) = f_0(y). \tag{5.12}$$

From these functions S and h, one can recover the distribution function $f(t,y)$ as follows. For any given (t,y) we define $\tau(t,y)$ as a solution of

$$\tau(t,y) = \frac{S(t,\tau(t,y),y)}{\varepsilon}.$$

Then we have

$$\begin{aligned}h(t,\tau(t,y),y) &= \tilde{g}\left(S(t,\tau(t,y),y), \frac{S(t,\tau(t,y),y)}{\varepsilon}, t, y\right),\\ &= g(S(t,\tau(t,y),y),t,y)\\ &= f(t,y).\end{aligned}$$

Theorem 5.1 *Consider the functions S and h given by the following two separate Cauchy problems*

$$\check{K}_1^\varepsilon(y)\partial_t S(t,\tau,y) + \check{K}_2^\varepsilon(y)\cdot\nabla_y S(t,\tau,y) = 1, \tag{5.13}$$

$$\check{K}_1^\varepsilon(y)\partial_\tau S(t,\tau,y) + \Big(\check{K}_1^\varepsilon(y)\check{G}_2^\varepsilon(y) - \check{G}_1^\varepsilon(y)\check{K}_2^\varepsilon(y)\Big)\cdot\nabla_y S(t,\tau,y) = -\check{G}_1^\varepsilon(y), \tag{5.14}$$

$$S(0,0,y) = 0, \tag{5.15}$$

and

$$\check{K}_1^\varepsilon(y)\partial_t h(t,\tau,y) + \check{K}_2^\varepsilon(y)\cdot\nabla_y h(t,\tau,y) = 0, \tag{5.16}$$

$$\check{K}_1^\varepsilon(y)\partial_\tau h(t,\tau,y) + \Big(\check{K}_1^\varepsilon(y)\check{G}_2^\varepsilon(y) - \check{G}_1^\varepsilon(y)\check{K}_2^\varepsilon(y)\Big)\cdot\nabla_y h(t,\tau,y) = 0, \tag{5.17}$$

$$h(0,0,y) = f_0(y). \tag{5.18}$$

Then the exact solution $f(t,y)$ of equation (5.1) with (5.2) can be recovered from h through the formula

$$f(t,y) = h(t,\tau(t,y),y),$$

where $(t,y)\mapsto\tau(t,y)\in\mathbb{R}$ is implicitly defined (locally) from S by the relation

$$\varepsilon\tau(t,y) = S(t,\tau(t,y),y).$$

Proof. Assuming that $\check{K}_1^\varepsilon$ does not vanish, equations (5.13, 5.14) and (5.16, 5.17) are straightforwardly derived from (5.10) and (5.11). ∎

Remark 5.2 *Several remarks are in order:*

- *The result in previous theorem is at this stage only* formal. *It will be the subject of a forthcoming paper to prove error estimates for the defects in (5.10) and (5.11).*
- *It can be easily seen from the two second equations of (5.10) and (5.11) that both S and h are periodic in τ. This indeed follows from Theorem 3.1 and the method of characteristics.*
- *The two equations (5.13, 5.14) and (5.16, 5.17) can be solved in arbitrary order. As a matter of fact, recalling that the fields $\check{K}^\varepsilon(Y)$ and $\check{G}^\varepsilon(Y)$, used in the averaged models* (5.7) *and* (5.8)*, have a vanishing Lie-bracket, the following relations hold true:*

$$\partial_y \check{K}_1^\varepsilon(y)\ \check{G}_2^\varepsilon(y) - \partial_y \check{G}_1^\varepsilon(y)\ \check{K}_2^\varepsilon(y) = 0, \quad \partial_y \check{K}_2^\varepsilon(y)\ \check{G}_2^\varepsilon(y) - \partial_y \check{G}_2^\varepsilon(y)\ \check{K}_2^\varepsilon(y) = 0. \quad (5.19)$$

The commutation of the flows of (5.16) *and* (5.17) *now follows provided that*

$$\partial_y \left(\frac{\check{K}_2^\varepsilon}{\check{K}_1^\varepsilon}\right)\left(\check{G}_2^\varepsilon - \check{G}_1^\varepsilon \frac{\check{K}_2^\varepsilon}{\check{K}_1^\varepsilon}\right) - \partial_y \left(\check{G}_2^\varepsilon - \check{G}_1^\varepsilon \frac{\check{K}_2^\varepsilon}{\check{K}_1^\varepsilon}\right)\frac{\check{K}_2^\varepsilon}{\check{K}_1^\varepsilon} = 0,$$

which an easy consequence of relations (5.19) and of the following equation, which holds for any scalar function α, any vector field $\mathcal{F}$ and any vector $u \in \mathbb{R}^n$:

$$\partial_y \left(\alpha(y)\mathcal{F}(y)\right) u = (\nabla_y \alpha(y) \cdot u)\mathcal{F}(y) + \alpha(y)(\partial_y \mathcal{F}(y))\, u.$$

5.3 An illustrative elementary example

Our aim here is to illustrate the result of Section 5 on an elementary example for which exact solutions can be easily obtained. Consider the following transport equation

$$\partial_t f + \left(\frac{1}{\varepsilon}\omega(y)Jy + y\right) \cdot \nabla_y f = 0, \qquad f(0,y) = f_0(y), \tag{5.20}$$

where $y \in \mathbb{R}^2$, and where

$$\omega(y) = 1 + |y|^2 = 1 + y_1^2 + y_2^2 \quad \text{and} \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

This equation can be solved as follows: let $\varphi_t^\varepsilon(y)$ be the flow of the characteristics equation

$$\dot{y} = \frac{1}{\varepsilon}\omega(y)Jy + y.$$

By taking its inner product by y, we have immediately $|\varphi_t^\varepsilon(y)| = \exp(t)|y|$, so that

$$\varphi_t^\varepsilon(y) = \exp(t)\exp\left(\frac{1}{\varepsilon}\left(t + (e^{2t} - 1)\frac{|y|^2}{2}\right)J\right) y.$$

As a consequence, the explicit solution of (5.20) reads

$$f(t,y) = f_0\left(\exp\left(-t - \frac{1}{\varepsilon}\left(t + (1-e^{-2t})\frac{|y|^2}{2}\right)J\right)y\right). \tag{5.21}$$

Now, we observe that the two fields $\omega(y)Jy$ and $K(y) = y$ do not commute, and in order to transform the problem into a highly-oscillatory problem with y-independent frequency, one has to divide equation (5.20) by ω and immerse the equation on f into an augmented one for the unknown $g(s,t,y)$

$$\partial_s g + \frac{1}{\omega(y)}\partial_t g + \left(\frac{1}{\varepsilon}Jy + \frac{y}{\omega(y)}\right)\cdot\nabla_y g = 0, \qquad g(0,t,y) = f(t,y). \tag{5.22}$$

Unlike the fields $\omega(y)Jy$ and K, we now observe that the two augmented fields $\check{G}(y) = (0, Jy)^T$ and $\check{K}(y) = \left(\frac{1}{\omega(y)}, \frac{y}{\omega(y)}\right)^T$ do commute. This means that equation (5.22) is already written in a normal form and therefore the averaged fields in this case are simply

$$\check{G}^\varepsilon = (0, Jy)^T, \quad \check{K}^\varepsilon = (\check{K}_1^\varepsilon, \check{K}_2^\varepsilon)^T, \quad \text{with} \quad \check{K}_1^\varepsilon = \frac{1}{\omega(y)}, \quad \check{K}_2^\varepsilon = \frac{y}{\omega(y)}.$$

We now apply Theorem 5.1 in this particular case. The solution $h = h(t,0,y)$ to

$$\partial_t h + y\cdot\nabla_y h = 0, \qquad h(0,0,y) = f_0(y),$$

is $h(t,0,y) = f_0(e^{-t}y)$. As a consequence, the solution $h = h(t,\tau,y)$ to

$$\partial_\tau h + Jy\cdot\nabla_y h = 0, \qquad h(t,0,y) = f_0(e^{-t}y),$$

is

$$h(t,\tau,y) = f_0(e^{-t}e^{-\tau J}y). \tag{5.23}$$

The solution $S = S(t,0,y)$ to

$$\partial_t S + y\cdot\nabla_y S = \omega(y), \qquad S(0,0,y) = 0,$$

is simply $S(t,0,y) = t + (1-e^{-2t})\frac{|y|^2}{2}$, so that the solution $S = S(t,\tau,y)$ to

$$\partial_\tau S + Jy\cdot\nabla_y S = 0, \qquad S(t,0,y) = t + (1-e^{-2t})\frac{|y|^2}{2},$$

is constant w.r.t. τ, given that $|e^{\tau J}y|^2 = |y|^2$, i.e.

$$S(t,\tau,y) = t + (1-e^{-2t})\frac{|y|^2}{2}. \tag{5.24}$$

Theorem 5.1 asserts that $f(t,y) = h(t,\tau(t,y),y)$ where $\tau(t,y)$ is given by $\varepsilon\tau(t,y) = t + (1-e^{-2t})\frac{|y|^2}{2}$, an assertion which can be easily checked on our explicit example.

6 The Vlasov equation with a strong magnetic field

In this section, we consider the case of particles submitted to a strong magnetic field and evolving in an electric field $E(x)$ depending on the position. We recall hereinafter the corresponding equation (1.4)

$$\partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) + \left(E(x) + v \times \frac{B(x)}{\varepsilon} \right) \cdot \nabla_v f(t,x,v) = 0 \qquad (6.1)$$

which is closely related to the illustrative example of Section 4, though with the additional difficulty that E and B may vary. We further assume here that E derives from a potential U, i.e. that $E(x) = -\nabla_x U(x)$.

6.1 Constant magnetic field

Over a first phase, we assume that the magnetic field is constant. This means that, up to constant rotation, we have $B(x) = (0,0,b(x))^T$ and $b(x) \equiv b$. Upon rescaling the time $t \to t/b$ in f, i.e. considering the equation for $f(t/b,x,v)$ instead of $f(t,x,v)$ we may even assume that $b = 1$. This hypothesis allows to assert that equation $n = 4$ and

$$F^\varepsilon(y) = \begin{pmatrix} v \\ \frac{1}{\varepsilon} Jv + E(x) \end{pmatrix} = \frac{1}{\varepsilon} G + K \text{ with } G(y) = \begin{pmatrix} 0 \\ Jv \end{pmatrix} \text{ and } K(y) = \begin{pmatrix} v \\ E(x) \end{pmatrix}.$$

We now repeat the steps followed for the example of Section 4, starting first with the flow Φ_τ (associated with G)

$$\Phi_\tau(y) = \begin{pmatrix} x \\ e^{\tau J} v \end{pmatrix}.$$

The time-dependent vector field K_τ then writes

$$K_\tau(y) = \begin{pmatrix} e^{\tau J} v \\ e^{-\tau J} E(x) \end{pmatrix} = e^{i\tau} \hat{K}_1(y) + e^{-i\tau} \hat{K}_{-1}(y)$$

with

$$\hat{K}_1(y) = \frac{1}{2} \begin{pmatrix} v - iJv \\ E(x) + iJE(x) \end{pmatrix} \quad \text{and} \quad \hat{K}_{-1}(y) = \frac{1}{2} \begin{pmatrix} v + iJv \\ E(x) - iJE(x) \end{pmatrix}.$$

Formula (3.3) then gives

$$\begin{aligned} K^{[1]} &= \hat{K}_0 = 0, \\ K^{[2]} &= i[\hat{K}_1, \hat{K}_{-1}] = -2\Im\left((\partial_y \hat{K}_1) \hat{K}_{-1} \right) = \begin{pmatrix} JE(x) \\ \frac{1}{2}(\Delta U(x)) Jv \end{pmatrix} \end{aligned}$$

where we used computed successively

$$\frac{\partial \hat{K}_1}{\partial y} = \frac{1}{2} \begin{pmatrix} 0 & (I - iJ) \\ -(I + iJ)\nabla_x^2 U(x) & 0 \end{pmatrix}$$

and[5]

$$\begin{aligned}
\frac{\partial \hat{K}_1}{\partial y}\,\hat{K}_{-1} &= \frac{1}{4}\begin{pmatrix} (I-iJ)^2E(x) \\ -(I+iJ)\nabla_x^2U(x)(I+iJ)v \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} 2(I-iJ)E(x) \\ -\big(\nabla_x^2U(x)-J\nabla_x^2U(x)J+i(J\nabla_x^2U(x)+\nabla_x^2U(x)J)\big)\,v \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} 2(I-iJ)E(x) \\ -\big(\nabla_x^2U(x)+\det(\nabla_x^2U(x))I+i\Delta U(x)J\big)\,v \end{pmatrix}.
\end{aligned}$$

At second order, equation (i) of Corollary 4.1 for $\tilde{f}(t,\tau,x,v)$ thus has the following form

$$\partial_\tau \tilde{f} + \varepsilon(v-\varepsilon JE(x))\cdot\nabla_x\tilde{f} + ((1-\varepsilon^2\Delta U(x))Jv+\varepsilon E(x))\cdot\nabla_v\tilde{f} = 0$$

while equation (ii) is simply

$$\partial_t\tilde{f} + \varepsilon JE(x)\cdot\nabla_x\tilde{f} + \frac{\varepsilon}{2}(\Delta U(x))Jv\cdot\nabla_v\tilde{f} = 0.$$

6.2 Magnetic field with varying intensity and constant direction

Over this second phase, we address the case of a magnetic field with varying intensity $b(x)$ and constant direction $B(x) = (0,0,b(x))^T$. In order to handle this new difficulty, one has to re-parametrize the time as in Section 5. Equivalently, this amounts to dividing equation (1.4) by $b(x)$. In order to do so, $b(x)$ should not vanish for any x in $\mathbb{R}^2$ and we will make this assumption for the remaining of this section. The "new" equation writes

$$\frac{1}{b(x)}\partial_t f(t,x,v) + \frac{1}{b(x)}v\cdot\nabla_x f(t,x,v) + \left(\frac{1}{\varepsilon}Jv - \frac{1}{b(x)}\nabla_x U(x)\right)\cdot\nabla_v f(t,x,v) = 0. \quad (6.2)$$

Denoting $Y = (t,x_1,x_2,v_1,v_2)\in\mathbb{R}^5$ the now **extended** phase-space variable, we may also write previous equation as the Lie-derivative of f in the direction of an **extended** vector field $\check{F}^\varepsilon$, i.e. $D_{\check{F}^\varepsilon}f = 0$ with

$$\check{F}^\varepsilon(Y) = \begin{pmatrix} \frac{1}{b(x)} \\ \frac{1}{b(x)}v \\ \frac{1}{\varepsilon}Jv - \frac{1}{b(x)}\nabla U(x) \end{pmatrix}.$$

or equivalently as the stationary solution of

$$\partial_s g(s,Y) + \check{F}^\varepsilon(Y)\cdot\nabla_Y g(s,Y) = 0$$

[5]Note that if S is a 2×2 symmetric matrix then

$$JS+SJ = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \alpha & \gamma \\ \gamma & \beta \end{pmatrix} + \begin{pmatrix} \alpha & \gamma \\ \gamma & \beta \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = (\alpha+\beta)J$$

so that $J\nabla^2U + \nabla^2UJ = (\Delta U)J$ and

$$JSJ = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \alpha & \gamma \\ \gamma & \beta \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\det(S)I.$$

where g then denotes a function of the 2 variables (s, Y). We may now resume the derivation of the equations (i) and (ii) of Theorem 4.1, by first splitting $\check{F}^\varepsilon$ into $\check{F}^\varepsilon = \frac{1}{\varepsilon}\check{G} + \check{K}$ with

$$\check{G}(Y) = \begin{pmatrix} 0 \\ 0 \\ Jv \end{pmatrix} \quad \text{and} \quad \check{K}(Y) = \frac{1}{b(x)} \begin{pmatrix} 1 \\ v \\ -\nabla U(x) \end{pmatrix}.$$

It is clear that $\check{G}$ now generates a 2π-periodic flow

$$\check{\Phi}_\tau(Y) = \check{\Phi}_\tau \begin{pmatrix} t \\ x \\ v \end{pmatrix} = \begin{pmatrix} t \\ x \\ e^{\tau J} v \end{pmatrix}$$

whose period is independent of the trajectory. The function $\check{K}_\tau$ becomes

$$\check{K}_\tau(Y) = \frac{1}{b(x)} \begin{pmatrix} 1 \\ e^{\tau J} v \\ e^{-\tau J} E(x) \end{pmatrix}$$

and the corresponding Fourier modes are all vanishing except the modes 1, -1 and 0 (the additional one w.r.t. the case of a constant field):

$$\hat{K}_0(Y) = \begin{pmatrix} \frac{1}{b(x)} \\ 0 \\ 0 \end{pmatrix}, \hat{K}_1(Y) = \frac{1}{2b(x)} \begin{pmatrix} 0 \\ (I - iJ)v \\ (I + iJ)E(x) \end{pmatrix}, \hat{K}_{-1}(Y) = \frac{1}{2b(x)} \begin{pmatrix} 0 \\ (I + iJ)v \\ (I - iJ)E(x) \end{pmatrix}.$$

According to Theorem 4.1, we thus have

$$K^{[1]}(Y) = \hat{K}_0(Y)$$

and

$$K^{[2]} = i\left([\hat{K}_1, \hat{K}_{-1}] + [\hat{K}_0, \hat{K}_1 - \hat{K}_{-1}]\right) = -2\Im([\hat{K}_0, \hat{K}_1]) - 2\Im\left((\partial_Y \hat{K}_1)\hat{K}_{-1}\right).$$

Omitting the argument x in E, U and b, and denoting simply ∇ for ∇_x, we have

$$\frac{\partial \hat{K}_0}{\partial Y} = \frac{-1}{b^2} \begin{pmatrix} 0 & \nabla^T b & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and

$$\frac{\partial \hat{K}_1}{\partial Y} = \frac{1}{2b^2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -(I - iJ)v\,\nabla^T b & b(I - iJ) \\ 0 & -b(I + iJ)\nabla^2 U + (I + iJ)\nabla U\,\nabla^T b & 0 \end{pmatrix}$$

so that

$$
\begin{aligned}
(\partial_Y \hat{K}_1)\hat{K}_{-1} &= \frac{1}{4b^3}\begin{pmatrix} 0 & 0 & 0 \\ 0 & -(I-iJ)v\,\nabla^T b & b(I-iJ) \\ 0 & -b(I+iJ)\nabla^2 U + (I+iJ)\nabla U\,\nabla^T b & 0 \end{pmatrix}\begin{pmatrix} 0 \\ (I+iJ)v \\ (I-iJ)E \end{pmatrix} \\
&= \frac{1}{4b^3}\begin{pmatrix} 0 \\ -(I-iJ)v\,\nabla^T b\,(I+iJ)v + b(I-iJ)^2 E \\ -b(I+iJ)\nabla^2 U\,(I+iJ)v + (I+iJ)\nabla U\,\nabla^T b\,(I+iJ)v \end{pmatrix} \\
&= \frac{1}{4b^3}\begin{pmatrix} 0 \\ -(\nabla b\cdot v + i\nabla b\cdot Jv)v - (\nabla b\cdot Jv - i\nabla b\cdot v)Jv + 2b(I-iJ)E \\ -b\nabla^2 U\,v - ib(\Delta U)Jv + b\det(\nabla^2 U)v \end{pmatrix}
\end{aligned}
$$

and finally

$$
-2\Im\left((\partial_Y \hat{K}_1)\hat{K}_{-1}\right) = \frac{1}{2b^3}\begin{pmatrix} 0 \\ (\nabla b\cdot Jv)v - (\nabla b\cdot v)Jv + 2JE \\ b(\Delta U)Jv \end{pmatrix}.
$$

Besides, we have

$$
\begin{aligned}
(\partial_Y \hat{K}_0)\hat{K}_1 &= \frac{-1}{2b^3}\begin{pmatrix} 0 & \nabla^T b & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ (I-iJ)v \\ (I+iJ)E(x) \end{pmatrix} \\
&= \frac{-1}{2b^3}\begin{pmatrix} \nabla b\cdot v - i\nabla b\cdot Jv \\ 0 \\ 0 \end{pmatrix}
\end{aligned}
$$

and

$$
(\partial_Y \hat{K}_1)\hat{K}_0 = \frac{1}{2b^3}\begin{pmatrix} 0 & 0 & 0 \\ 0 & -(I-iJ)v\,\nabla^T b & b(I-iJ) \\ 0 & -b(I+iJ)\nabla^2 U + (I+iJ)\nabla U\,\nabla^T b & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0
$$

so that

$$
-2\Im([\hat{K}_0, \hat{K}_1]) = \frac{-1}{b^3}\begin{pmatrix} \nabla b\cdot Jv \\ 0 \\ 0 \end{pmatrix}.
$$

Finally, at first order in ε, we have

$$
K^\varepsilon = K^{[1]} + \varepsilon K^{[2]} = \frac{1}{b}\begin{pmatrix} 1 - \varepsilon\,\frac{\nabla b\cdot Jv}{b^2} \\ -\varepsilon\,\frac{(\nabla b\cdot v)}{2b^2}Jv + \varepsilon\,\frac{(\nabla b\cdot Jv)}{2b^2}v - \varepsilon\,\frac{1}{b}J\nabla U \\ \frac{\varepsilon(\nabla b\cdot v)}{2b^2}J\nabla U + \frac{\varepsilon(\nabla b\cdot Jv)}{2b^2}\nabla U - \frac{\varepsilon\Delta U}{2b}Jv \end{pmatrix} = \begin{pmatrix} K_1^\varepsilon \\ K_2^\varepsilon \end{pmatrix},
$$

and

$$
G^\varepsilon = \varepsilon(F^\varepsilon - K^\varepsilon) = \frac{1}{b}\begin{pmatrix} 0 \\ \varepsilon v \\ \frac{1}{b(x)}Jv - \varepsilon\nabla U(x) \end{pmatrix}.
$$

Therefore the transport equations on h are at first order in ε:

$$\partial_t h + \frac{\varepsilon}{2b}\left(\frac{\nabla b \cdot v}{b} Jv - \frac{\nabla b \cdot Jv}{b} v + 2J\nabla U\right) \cdot \nabla_x h$$
$$+ \frac{\varepsilon}{2b}\left(\frac{\nabla b \cdot v}{b} J\nabla U + \frac{\nabla b \cdot Jv}{b} \nabla U - (\Delta U) Jv\right) \cdot \nabla_v h = 0, \tag{6.3}$$

and

$$\partial_\tau h + \frac{\varepsilon}{b} v \cdot \nabla_x h + \frac{\varepsilon}{b^2}(Jv) \cdot \nabla_v h = 0, \tag{6.4}$$

with the initial condition $h(0, 0, y) = f_0(y)$. Similarly the transport equations on S are

$$\partial_t S + \frac{\varepsilon}{2b}\left(\frac{\nabla b \cdot v}{b} Jv - \frac{\nabla b \cdot Jv}{b} v + 2J\nabla U\right) \cdot \nabla_x S$$
$$+ \frac{\varepsilon}{2b}\left(\frac{\nabla b \cdot v}{b} J\nabla U + \frac{\nabla b \cdot Jv}{b} \nabla U - (\Delta U) Jv\right) \cdot \nabla_v S = b(x)\left(1 + \varepsilon\, \frac{\nabla b \cdot Jv}{b^2}\right), \tag{6.5}$$

and

$$\partial_\tau S + \frac{\varepsilon}{b} v \cdot \nabla_x S + \frac{\varepsilon}{b^2}(Jv) \cdot \nabla_v S = 0, \tag{6.6}$$

with the initial condition $S(0, 0, y) = 0$.

Now we make an important comment on these transport equations. First, the transport equation (6.3) coincides with the gyro-kinetic model derived by Frénod & Sonnendrücker in [19]. However, our approach provides more information through the phase S and the dependence in τ. These informations are necessary to correctly reconstruct the full original distribution function f (and not only the averaged model) at first order in ε. This reconstruction may be performed through the relation $f(t, x, v) = h(t, \tau(t, x, v), x, v)$ where $\tau(t, x, v)$ is a solution to $\varepsilon\tau = S(t, \tau, x, v)$. Up to our knowledge, no such construction can be found in the literature.

6.3 Magnetic field with varying intensity and varying direction

We now consider the transport kinetic equation in its general form (1.4). This means that now we allow variations of B in both amplitude and direction. Our aim in this part is to extend our previous approach to this more general case.

We first immerse the model (1.4) in an augmented problem in the unkown $g(s, t, x, v)$, as follows

$$\partial_s g + \frac{1}{|B(x)|}\partial_t g + \frac{v}{|B(x)|} \cdot \nabla_x g + \left(\frac{E(x)}{|B(x)|} + \frac{1}{\varepsilon} v \times \frac{B(x)}{|B(x)|}\right) \cdot \nabla_v g = 0 \tag{6.7}$$

with the initial condition $g(0, t, x, v) = f(t, x, v)$. The main interest of this form is that the oscillatory part in the variable s is now driven by the vector field $\frac{B(x)}{|B(x)|} \times v$, which, as we shall see, generates a periodic flow with a constant period 2π. In particular the period does not depend on the trajectory, although the unitary vector $\frac{B(x)}{|B(x)|}$ depends on this trajectory . Indeed let e_0 a unit vector and let (e_0, e_1, e_2) an orthonormal basis such that $e_0 \times e_1 = e_2$ and $e_1 \times e_2 = e_0$. The matrix representing the skew-symmetric linear map $\mathcal{L}_{e_0} : v \mapsto v \times e_0$ in the basis (e_0, e_1, e_2), is simply $\mathcal{J} = \begin{pmatrix} 0 & 0 \\ 0 & J \end{pmatrix}$. Since $\exp(t\mathcal{J})$ is 2π-periodic, the flow $\exp(t\mathcal{L}_{e_0})$ is 2π-periodic. We now apply our methodology on model (6.7). Here the vector field $F^\varepsilon = \frac{1}{\varepsilon}G + K$ is given by

$$K(t,x,v) = \frac{1}{|B(x)|}\begin{pmatrix} 1 \\ v \\ E(x) \end{pmatrix}, \qquad G(t,x,v) = \begin{pmatrix} 0 \\ 0 \\ v \times \frac{B(x)}{|B(x)|} \end{pmatrix}.$$

We introduce the following notations

$$e(x) = \frac{B(x)}{|B(x)|}, \quad \mathcal{L}_e v = v \times e, \quad \mathcal{P}_e v = (e \cdot v)e, \quad \forall e \in \mathbb{S}^2, v \in \mathbb{R}^3, x \in \mathbb{R}^3. \tag{6.8}$$

Using Theorem 3.5, the vector field K_τ can be easily computed from

$$\Phi_\tau(t,x,v) = \begin{pmatrix} t \\ x \\ \exp\left(\tau \mathcal{L}_{e(x)}\right) v \end{pmatrix}.$$

The following elementary identities

$$\mathcal{L}_e^2 = -I + \mathcal{P}_e, \qquad \mathcal{L}_e\mathcal{P}_e = \mathcal{P}_e\mathcal{L}_e = 0$$

imply that

$$\begin{aligned}\Phi_\tau(t,x,v) &= \begin{pmatrix} t \\ x \\ (\cos\tau)v + (1-\cos\tau)\mathcal{P}_{e(x)}v + (\sin\tau)\mathcal{L}_{e(x)}v \end{pmatrix} \\ &= \begin{pmatrix} t \\ x \\ (\cos\tau)v + (1-\cos\tau)(e(x)\cdot v)e(x) + (\sin\tau)v \times e(x) \end{pmatrix}.\end{aligned} \tag{6.9}$$

We then deduce the expression of the Jacobian matrix $\partial_{t,x,v}\Phi_\tau = (\partial_t\Phi_\tau, \partial_x\Phi_\tau, \partial_v\Phi_\tau)$:

$$\partial_{t,x,v}\Phi_\tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & I & 0 \\ 0 & R & Q \end{pmatrix},$$

where

$$\begin{array}{ll} R & = (1-\cos\tau)\partial_x\left(\mathcal{P}_{e(x)}v\right) + (\sin\tau)\partial_x\left(\mathcal{L}_{e(x)}v\right) \\ & = \alpha_0 + \alpha e^{i\tau} + \overline{\alpha}e^{-i\tau}, \\ Q_\tau & = (\cos\tau)I + (1-\cos\tau)\mathcal{P}_{e(x)} + (\sin\tau)\mathcal{L}_{e(x)} \\ & = a_0 + ae^{i\tau} + \overline{a}e^{-i\tau}, \end{array},$$

and

$$\begin{array}{l} a_0 = \mathcal{P}_{e(x)}, \qquad \alpha_0 = \partial_x\left(\mathcal{P}_{e(x)}v\right), \\ 2a = I - \mathcal{P}_{e(x)} - i\mathcal{L}_{e(x)}, \\ 2\alpha = -\partial_x\left(\mathcal{P}_{e(x)}v + i\mathcal{L}_{e(x)}v\right). \end{array}$$

In order to compute the inverse of the matrix $\partial_{t,x,v}\Phi_\tau$, we observe that

$$(\nabla_{t,x,v}\Phi_\tau)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & I & 0 \\ 0 & -Q_\tau^{-1}R & Q_\tau^{-1} \end{pmatrix},$$

which means that we only need to compute Q_τ^{-1}. Using again the identity $\mathcal{L}_e^2 = -I + \mathcal{P}_e$, one may check

$$\begin{aligned} Q_\tau^{-1} &= (\cos\tau)I + (1-\cos\tau)\mathcal{P}_{e(x)} - (\sin\tau)\mathcal{L}_{e(x)}, \\ &= a_0 + \overline{a}e^{i\tau} + ae^{-i\tau} = Q_{-\tau}. \end{aligned}$$

Now we also have

$$K \circ \Phi_\tau(t,x,v) = \frac{1}{|B(x)|}\begin{pmatrix} 1 \\ Q_\tau v \\ E(x) \end{pmatrix}.$$

Therefore

$$K_\tau(t,x,v) = \frac{1}{|B(x)|}\begin{pmatrix} 1 \\ Q_\tau v \\ -Q_{-\tau}RQ_\tau v + Q_{-\tau}E(x) \end{pmatrix}.$$

One can easily see that the Fourier expansion of K_τ (in the periodic variable τ) only contains modes $k \in \mathbb{Z}$ with $|k| \le 3$. Note that we can recover the previous case ($B(x)$ with constant direction and $(x,v) \in \mathbb{R}^2 \times \mathbb{R}^2$) by taking $\mathcal{P}_{e(x)}v = 0$, $\mathcal{L}_{e(x)} \equiv J$ and $\alpha = 0$, which means that $R = 0$ and $Q_\tau = e^{\tau J}$.

Although all the Fourier coefficients of K_τ can be derived from this expression, we just give for simplicity the zero-th mode:

$$\hat{K}_0(x,v) = \frac{1}{|B(x)|}\begin{pmatrix} 1 \\ \mathcal{P}_{e(x)}v = (e(x)\cdot v)e(x) \\ (\hat{K}_0)_3 \end{pmatrix} = K^{[1]},$$

with

$$\begin{aligned} (\hat{K}_0)_3(x,v) &= a_0E(x) - (a_0b_0a_0 + a_0b\overline{a} + a_0\overline{b}a + \overline{a}b_0\overline{a} + \overline{a}\overline{b}a_0 + ab_0a + aba_0)v \\ &= \mathcal{P}_{e(x)}E(x) - \Big[4\mathcal{P}_{e(x)}\partial_x\left(\mathcal{P}_{e(x)}v\right)\mathcal{P}_{e(x)} + \frac{1}{2}\mathcal{P}_{e(x)}\partial_x\left(\mathcal{L}_{e(x)}v\right)\mathcal{L}_{e(x)} \\ &\quad -\frac{1}{2}\mathcal{L}_{e(x)}\partial_x\left(\mathcal{P}_{e(x)}v\right)\mathcal{L}_{e(x)} - \frac{1}{2}\mathcal{L}_{e(x)}\partial_x\left(\mathcal{L}_{e(x)}v\right)\mathcal{P}_{e(x)} \\ &\quad -\mathcal{P}_{e(x)}\partial_x\left(\mathcal{P}_{e(x)}v\right) - \partial_x\left(\mathcal{P}_{e(x)}v\right)\mathcal{P}_{e(x)} + \frac{1}{2}\partial_x\left(\mathcal{P}_{e(x)}v\right)\Big]v. \end{aligned}$$

We then deduce the vector field G^ε at the 0^{th} order in ε:

$$\begin{aligned} G^{[1]} = \varepsilon(F^\varepsilon - K^{[1]}) + \mathcal{O}(\varepsilon) &= \tfrac{1}{|B(x)|}\begin{pmatrix} 0 \\ \varepsilon\left(v - \mathcal{P}_{e(x)}v\right) \\ |B(x)|\mathcal{L}_{e(x)}v + \varepsilon\Big(E(x) - (\hat{K}_0)_3(x,v)\Big) \end{pmatrix} + \mathcal{O}(\varepsilon) \\ &= \begin{pmatrix} 0 \\ 0 \\ \mathcal{L}_{e(x)}v \end{pmatrix} + \mathcal{O}(\varepsilon). \end{aligned}$$

The averaged model at the 0^{th} order in ε can now be written in terms of $h(t,\tau,x,v)$ and $S(t,\tau,x,v)$. We have

$$\begin{aligned}
&\partial_t h + \left(\tfrac{B(x)}{|B(x)|}\cdot v\right)\tfrac{B(x)}{|B(x)|}\cdot\nabla_x h + (\hat{K}_0)_3(x,v)\cdot\nabla_v h = 0,\\
&\partial_\tau h + \left(v\times\tfrac{B(x)}{|B(x)|}\right)\cdot\nabla_v h = 0,
\end{aligned}$$

and

$$\begin{aligned}
&\partial_t S + \left(\tfrac{B(x)}{|B(x)|}\cdot v\right)\tfrac{B(x)}{|B(x)|}\cdot\nabla_x S + (\hat{K}_0)_3(x,v)\cdot\nabla_v S = |B(x)|,\\
&\partial_\tau S + \left(v\times\tfrac{B(x)}{|B(x)|}\right)\cdot\nabla_v S = 0,
\end{aligned}$$

with the initial conditions: $h(0,0,x,v) = f_0(x,v)$ and $S(0,0,x,v) = 0$. Note that in the particular case where $B(x)$ has a constant direction $B(x) = (0,0,b(x))^T$, we get

$$\begin{aligned}
&\partial_t h + \ v_\parallel \partial_{x_\parallel} h + E_\parallel \partial_{v_\parallel} h = 0,\\
&\partial_\tau h + v_\perp\cdot\nabla_{v_\perp} h = 0.
\end{aligned}$$

and

$$\begin{aligned}
&\partial_t S + \ v_\parallel \partial_{x_\parallel} S + E_\parallel \partial_{v_\parallel} S = b(x),\\
&\partial_\tau S + v_\perp\cdot\nabla_{v_\perp} S = 0.
\end{aligned}$$

The averaged equations at the first order in ε can also be derived in the case of a magnetic field $B(x)$ with constant direction $B(x) = (0,0,b(x))^T$, with $b(x) > 0$. In this case we have $R = 0$, $e(x)$ is the constant unit-vector e_z, $|B(x)| = b(x)$, and therefore

$$K_\tau(t,x,v) = \frac{1}{b(x)}\begin{pmatrix} 1 \\ Q_\tau v \\ Q_{-\tau}E(x)\end{pmatrix}.$$

which implies

$$\hat{K}_1 = \frac{1}{b(x)}\begin{pmatrix} 0 \\ av \\ \overline{a}E(x)\end{pmatrix}, \qquad \hat{K}_{-1} = \frac{1}{b(x)}\begin{pmatrix} 0 \\ \overline{a}v \\ aE(x)\end{pmatrix}$$

The computation of K^ε at first order in ε can then be derived from Theorem 3.5 and provides, up to a rescaling in time, an asymptotic model which is similar to the one recently derived by Degond & Filbet in [17].

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