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Knapsack problem for automaton groups *

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Abstract

The knapsack problem is a classic optimisation problem that has been recently extended in the setting of groups. Its study reveals to be interesting since it provides many different behaviours, depending on the considered class of groups. In this paper we deal with groups generated by Mealy automata—a class that is often used to study group-theoretical conjectures—and prove that the knapsack problem is undecidable for this class. In a second time, we construct a graph that, if finite, provides a solution to the knapsack problem. We deduce that the knapsack problem is decidable for the so-called bounded automaton groups, a class where the order and conjugacy problems are already known to be decidable.

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1 Introduction

In 2013, Myasnikov, Nikolaev, and Ushakov started to extend and study optimisation problems for general groups in [20]. In particular, they generalized the *knapsack problem*, known to be in **NP** for integers parameters.

Knapsack Problem:

- **Input:** $g_1, \dots, g_i, g \in G$.
- **Output:** Yes if and only if there exists $\epsilon_1, \dots, \epsilon_i \in \mathbb{N}$ such that $g_1^{\epsilon_1} \dots g_i^{\epsilon_i} = g$

Several results on **Knapsack Problem** have been obtained in the seminal paper [20] and later by [7, 15, 16, 19]. Among other things:

- **Knapsack Problem** is in **P** for hyperbolic groups.
- **Knapsack Problem** for the discrete Heisenberg group $H_3(\mathbb{Z})$ is decidable.
- There is a nilpotent group of class 2, formed by the direct product of several copies of the Heisenberg group $H_3(\mathbb{Z})$, for which **Knapsack Problem** is undecidable. This means in particular that the decidability of **Knapsack Problem** is not preserved under direct product.
- **Knapsack Problem** is decidable for all co-context-free groups.

In 1955, Mealy [18] introduced a family of transducers in order to model circuits. Since then, following a suggestion of Gluskov [8], these Mealy automata have been widely used

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in (semi)group theory. They have given numerous interesting groups and semigroups, the most famous being probably the Grigorchuk group, which is an infinite torsion group with intermediate growth, solving both Burnside and Milnor problems [11, 12]. For a more complete introduction to the topic we refer the reader to the survey by Nekrashevych [21] or to the chapter by Bartholdi and Silva [2].

With the underlying automaton structure, these groups are quite tractable for algorithmic problems. For instance **Word Problem** (deciding if a word on the generators represents the trivial element in the group) is always decidable in such a group, whereas it is not for a general (finitely generated) group. Yet not every algorithmic problem becomes trivial: for instance **Conjugacy Problem** is undecidable [24], which shows that automaton groups provide a wide variety of behaviours. We prove that **Knapsack Problem** is undecidable for the whole class of automaton group. The proof is based on a result of undecidability for the direct product of many copies of $H_3(\mathbb{Z})$ by [15].

By requiring additional properties on the automaton, other problems become decidable. If the automaton is 2-state or 2-letter invertible-reversible, then **Finiteness Problem** for the generated group is decidable [13]. Bartholdi studied the Engel property in some restricted class in [1], providing an algorithm that is ensured to terminate under certain conditions. In this paper we are going to focus on the class of bounded automata, a class that contains many groups of great interest (Grigorchuk group, Gupta-Sidki groups, Basilica group, ...). **Order Problem** and **Conjugacy Problem** are decidable in this class [3]. Here, we prove that **Knapsack Problem** is decidable in this class as well, while it is undecidable for a general automaton groups. This is to compare, for instance, to the class of finitely generated nilpotent groups where **Conjugacy Problem** is decidable but **Knapsack Problem** is not ; or the class of finitely generated hyperbolic groups where **Order Problem**, **Conjugacy Problem** and **Knapsack Problem** are decidable [17, 20].

This paper is organised as follows: in Section 2 the basic definitions of Mealy automata and description of the action of a group generated by such an automaton are provided. Then, in Section 3 we prove that **Knapsack Problem** is undecidable for a general automaton group. Finally, in Section 4 we construct a graph, the *Knapgraph*, that allows us to decide **Knapsack Problem** when it is finite, and we prove its finiteness in the case of bounded Mealy automata, whence the decidability of **Knapsack Problem** in this class. Section 5 is devoted to examples.

2 Mealy Automata

A *Mealy automaton* is a complete deterministic letter-to-letter transducer $\mathcal{A} = (Q, \Sigma, \delta, \rho)$ where Q and Σ are finite sets respectively called the *stateset* and the *alphabet*, and $\delta = (\delta_i : Q \rightarrow Q)_{i \in \Sigma}$, $\rho = (\rho_q : \Sigma \rightarrow \Sigma)_{q \in Q}$ are respectively called the *transition* and *production* functions. These functions can be extended to words as follows: see \mathcal{A} as an automaton with input and output tapes, thus defining mappings from input words over Σ to output words over Σ . Formally, for $q \in Q$, the map $\rho_q : \Sigma^* \rightarrow \Sigma^*$, extending $\rho_q : \Sigma \rightarrow \Sigma$, is defined recursively by:

$$\forall i \in \Sigma, \forall \mathbf{s} \in \Sigma^*, \quad \rho_q(i\mathbf{s}) = \rho_q(i)\rho_{\delta_i(q)}(\mathbf{s}). \quad (1)$$

We can also extend the map ρ to words of states $\mathbf{u} \in Q^*$ by composing the production

functions associated with the letters of \mathbf{u} :

$$\forall i \in \Sigma, \forall q \in Q, \forall \mathbf{u} \in Q^*, \quad \rho_{q\mathbf{u}}(i) = \rho_{\mathbf{u}}(\rho_q(i)). \quad (2)$$

For each automaton transition $q \xrightarrow{a|\rho_q(x)} \delta_x(q)$, we associate the *cross-transition* depicted in the following way:

$$\begin{array}{ccc} & x & \\ & \downarrow & \\ q & \xrightarrow{\quad} & \delta_x(q) \\ & \uparrow & \\ & \rho_q(x) & \end{array} .$$

These diagrams are an useful graphical way to compute transitions and can be composed too:

$$\begin{array}{ccccc} & x_1 & & x_2 \dots x_n & \\ & \downarrow & & \downarrow & \\ q & \xrightarrow{\quad} & \delta_{x_1}(q) & \xrightarrow{\quad} & \delta_{x_2 \dots x_n}(\delta_{x_1}(q)) \\ & \uparrow & & \uparrow & \\ & \rho_q(x_1) & & \rho_{\delta_{x_1}(q)}(x_2 \dots x_n) & \end{array} .$$

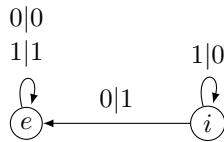
In the same manner they can be composed vertically to express the action of a product of states. A Mealy automaton is said to be *invertible* when ρ_q is a permutation of the alphabet for each $q \in Q$. It is called *reversible* when δ_i is a permutation of the stateset for each $i \in \Sigma$. Moreover an automaton is said to be *bireversible* whenever it is reversible (*i.e.* every input letter induces a permutation of the stateset) and each output letter induces a permutation of the stateset.

Examples of such automata are depicted in Figures 1 and 2.

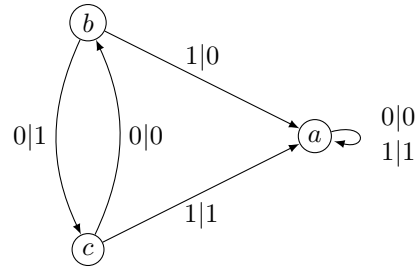
The production functions of an automaton \mathcal{A} generate a semigroup. Whenever \mathcal{A} is invertible, one can define the *group generated by \mathcal{A}* :

$$\langle \mathcal{A} \rangle := \langle \rho_q | q \in Q \rangle = \langle \rho_{\mathbf{u}} | \mathbf{u} \in Q^* \rangle .$$

The action of the group $\langle \mathcal{A} \rangle$ can be seen as an action on Σ^* viewed as a rooted tree. We call $\pi(g)$ the *permutation carried by g* and $g|_x$ the *section of g by x* : if $\mathbf{q} \in Q^*$ is a word that represents g in $\langle \mathcal{A} \rangle$, then $\pi(g) = \rho_{\mathbf{q}|\Sigma}$ and $g|_x$ in $\langle \mathcal{A} \rangle$ is represented by $\delta_x(\mathbf{q}) \in Q^*$. The action



■ **Figure 1** The Adding machine \mathcal{Z} , generating the group \mathbb{Z} .



■ **Figure 2** The Basilica automaton \mathcal{B} .

of an element $g \in \langle \mathcal{A} \rangle$ on $\mathbf{u} = x\mathbf{v} \in \Sigma$ is defined recursively by $g.\mathbf{u} = \pi(g)(x) (g|_x.\mathbf{v})$. For a given element g and a letter x we define the *orbit* of x under g as $\text{Orb}_g(x) = \{g^n.x, n \in \mathbb{N}\}$.

The class of *bounded automata* has been introduced in [22]. It consists in automata where, from a non-trivial cycle, no non-trivial cycle can be reached by a directed path. For instance the adding machine depicted in Figures 1 and the Basilica automaton drawn on Figure 2 are bounded but the automaton \mathcal{H} in Figure 3 is not: several non-trivial cycles are entangled.

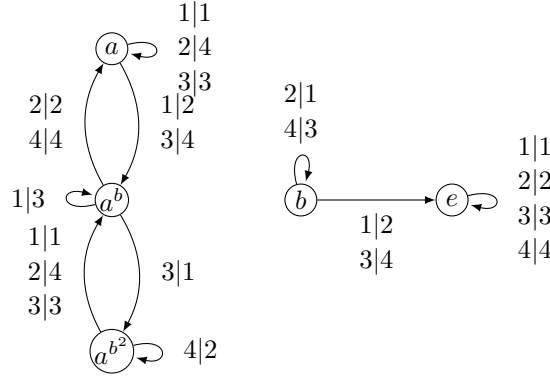
3 Undecidability of Knapsack Problem for automaton groups

We prove that **Knapsack Problem** is undecidable among the class of automaton groups. First recall that the Heisenberg group

$$H_3(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} ; a, b, c \in \mathbb{Z} \right\}$$

is an automaton group:

► **Proposition 1** (Bondarenko-Kavshenko [4]). *The automaton \mathcal{H} , depicted in Figure 3 satisfies $\langle \mathcal{H} \rangle = H_3(\mathbb{Z})$.*



■ **Figure 3** The Mealy automaton \mathcal{H} generating the Heisenberg group $H_3(\mathbb{Z})$.

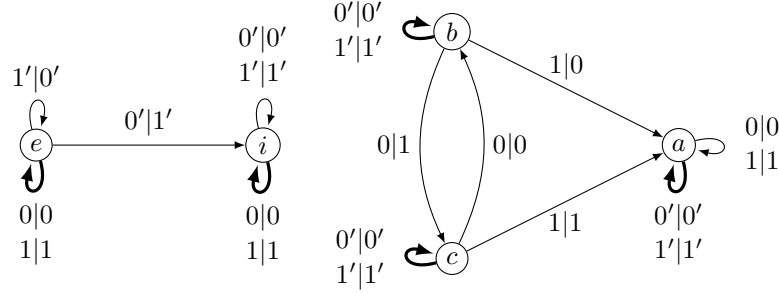
Moreover the class of automaton groups is closed by direct product (in fact several operations on automata can be used to obtain the direct product of two automaton groups). This was proved by Cain in [5] for semigroups and can be extended to groups in several ways. We describe one in what follows.

► **Definition 2.** Let $\mathcal{A}_1 = (Q_1, \Sigma_1, \delta_1, \rho_1)$, $\mathcal{A}_2 = (Q_2, \Sigma_2, \delta_2, \rho_2)$ be two Mealy automata. The *looped direct product* of \mathcal{A}_1 and \mathcal{A}_2 is the automaton $\mathcal{A}_1 \times_{\ell} \mathcal{A}_2 = \mathcal{A} = (Q_1 \sqcup Q_2, \Sigma_1 \sqcup \Sigma_2, \delta, \rho)$, where

- $\delta_{a_1|Q_1} = \delta_{1a_1}, \forall a_1 \in \Sigma_1,$
- $\delta_{a_2|Q_1} = 1|_{Q_1}, \forall a_2 \in \Sigma_2,$
- $\delta_{a_2|Q_2} = \delta_{2a_2}, \forall a_2 \in \Sigma_1,$
- $\delta_{a_1|Q_2} = 1|_{Q_1}, \forall a_1 \in \Sigma_1 ;$
- $\rho_{q_1|\Sigma_1} = \rho_{1q_1}, \forall q_1 \in Q_1,$
- $\rho_{q_1|\Sigma_2} = 1|_{\Sigma_1}, \forall q_1 \in Q_1,$

$$\blacksquare \rho_{q_2|\Sigma_2} = \rho_{2q_2}, \forall q_2 \in Q_2,$$

$$\blacksquare \rho_{q_2|\Sigma_1} = 1_{|\Sigma_2}, \forall q_2 \in Q_2,$$



■ **Figure 4** Looped direct product of the adding machine and the Basilica automaton. New transitions appear in bold.

► **Proposition 3.** Let \mathcal{A}_1 and \mathcal{A}_2 be two Mealy automata. Then $\langle \mathcal{A}_1 \times_{\ell} \mathcal{A}_2 \rangle = \langle \mathcal{A}_1 \rangle \times \langle \mathcal{A}_2 \rangle$.

Proof. Let $\mathcal{A}_1 = (Q_1, \Sigma_1, \delta_1, \rho_1)$, $\mathcal{A}_2 = (Q_2, \Sigma_2, \delta_2, \rho_2)$ and $\mathcal{A}_1 \times_{\ell} \mathcal{A}_2 = \mathcal{A} = (Q_1 \sqcup Q_2, \Sigma_1 \sqcup \Sigma_2, \delta, \rho)$. Let us prove that $\rho_{u_1} \rho_{u_2} = \rho_{u_2} \rho_{u_1}$, $\forall u_1 \in Q_1^*, u_2 \in Q_2^*$. Let $v_1 \in \Sigma_1^*, v_2 \in \Sigma_2^*$. We have the cross diagrams:

$$\begin{array}{ccc} \begin{array}{c} v_1 \\ \downarrow \\ u_1 \end{array} & \begin{array}{c} \downarrow \\ \rho_{u_1}(v_1) = \rho_{1u_1}(v_1) \\ \downarrow \\ u_2 \end{array} & \begin{array}{c} \delta_{v_1}(u_1) = \delta_{1v_1}(u_1) \\ \downarrow \\ u_2 \end{array} \\ & \begin{array}{c} \downarrow \\ \rho_{1u_1}(v_1) \end{array} & \end{array} \quad \begin{array}{ccc} \begin{array}{c} v_2 \\ \downarrow \\ v_2 \end{array} & \begin{array}{c} \downarrow \\ \rho_{u_2}(v_2) = \rho_{2u_2}(v_2) \\ \downarrow \\ v_2 \end{array} & \begin{array}{c} \delta_{1v_1}(u_1) \\ \downarrow \\ \delta_{v_2}(u_2) = \delta_{2v_2}(u_2) \end{array} \end{array}$$

And, on the other hand:

$$\begin{array}{ccc} \begin{array}{c} v_1 \\ \downarrow \\ u_2 \end{array} & \begin{array}{c} \downarrow \\ \rho_{u_1}(v_1) = \rho_{1u_1}(v_1) \\ \downarrow \\ u_1 \end{array} & \begin{array}{c} \delta_{v_1}(u_1) = \delta_{1v_1}(u_1) \\ \downarrow \\ \delta_{v_2}(u_2) = \delta_{2v_2}(u_2) \end{array} \\ & \begin{array}{c} \downarrow \\ \rho_{2u_2}(v_2) \end{array} & \end{array} \quad \begin{array}{ccc} \begin{array}{c} v_2 \\ \downarrow \\ v_2 \end{array} & \begin{array}{c} \downarrow \\ \rho_{u_2}(v_2) = \rho_{2u_2}(v_2) \\ \downarrow \\ v_2 \end{array} & \begin{array}{c} \delta_{v_2}(u_2) = \delta_{2v_2}(u_2) \\ \downarrow \\ \delta_{1v_1}(u_1) \end{array} \end{array}$$

Hence, by induction, words on Q_1 commute with words over Q_2 . So any word in $(Q_1 \sqcup Q_2)^*$ is equivalent to a word in $Q_1^* \sqcup Q_2^*$. We get that $\langle \mathcal{A}_1 \times_{\ell} \mathcal{A}_2 \rangle \leq \langle \mathcal{A}_1 \rangle \times \langle \mathcal{A}_2 \rangle$.

Moreover any element (a_1, a_2) in $\langle \mathcal{A}_1 \rangle \times \langle \mathcal{A}_2 \rangle$ is represented by an element in $\langle \mathcal{A}_1 \times_{\ell} \mathcal{A}_2 \rangle$: by hypothesis $a_1 = \rho_{1u_1}$ and $a_2 = \rho_{2u_2}$, then $(a_1, a_2) = (\rho_{1u_1}, \rho_{2u_2}) = \rho_{u_1 u_2}$. We can conclude $\langle \mathcal{A}_1 \times_{\ell} \mathcal{A}_2 \rangle = \langle \mathcal{A}_1 \rangle \times \langle \mathcal{A}_2 \rangle$. ◀

► **Theorem 4.** Knapsack Problem is undecidable for the whole class of automaton groups.

Proof. By [15], the problem is undecidable for the direct product of sufficiently many copies of the Heisenberg group. The result follows by Propositions 1 and 3. ◀

However, the automaton \mathcal{H} generating the Heisenberg group does not belong to any of the standard automaton subclasses, such as

- contracting automata [6, 11, 12],
- couded automata [3],
- (bi)reversible automata [9, 10, 14, 23, 25, 26].

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One can hope, by focusing on more restricted classes, to obtain decidability results. This is the aim of the next section.

4 Knapsack Problem is decidable for bounded automata

Let $\mathcal{A} = (Q, \Sigma, \delta, \rho)$ and consider **Knapsack Problem** in $\langle \mathcal{A} \rangle$.

We are going to use the structure of the action of $\langle \mathcal{A} \rangle$ on the rooted tree Σ^* to tackle **Knapsack Problem**. If two elements are equal in the group, then their actions must coincide on every vertex of the tree Σ^* . In particular, if $\prod_{j=1}^i g_j^{n_j} = g$ holds, then

$$\pi\left(\left(\prod_j g_j^{n_j}\right)_{|x}\right) = \pi(g_{|x})$$

must hold for any $x \in \Sigma^*$. We are going to use the automaton structure to construct a branching procedure that follows this principle. Examples are displayed in Section 5.

A high-level description of our procedure to solve **Knapsack Problem** is as follows.

If n_1, \dots, n_i is a solution for the inputs $\{g_1, \dots, g_i, g\}$, then the actions of $\prod_j g_j^{n_j}$ and g must coincide on the first level. Hence each n_j is of the form $\alpha_j + p_j \mathbb{Z}$, where $p_j = \text{order}(\pi(g_j))$, $\alpha_j \in \{0, \dots, p_j - 1\}$ and

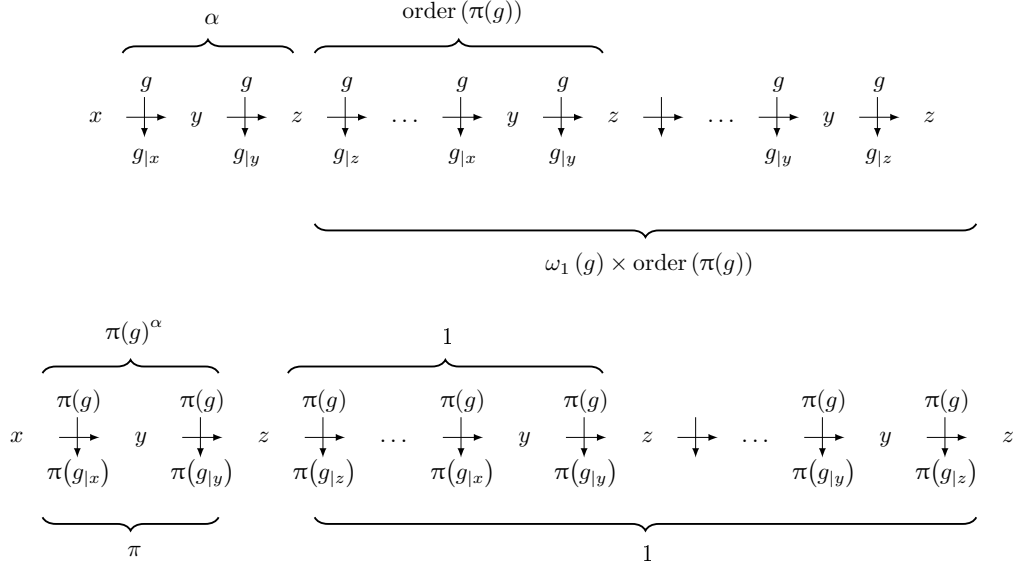
$$\pi\left(\prod_j g_j^{\alpha_j}\right) = \pi(g).$$

Then one should make the second level coincide. And from $g_j^{\alpha_j + p_j \mathbb{Z}}$ there are only finitely many permutations $\pi\left(\left(\prod_j g_j^{\alpha_j + p_j \mathbb{Z}}\right)_{|x}\right)$ that are reachable. Then we consider those that make the second level coincide, and iterate the process from $\pi\left(\left(\prod_j g_j^{\alpha_j + k_j p_j + q_j \mathbb{Z}}\right)_{|x}\right)$.

► **Lemma 5.** *Let \mathcal{A} be an invertible Mealy automaton, $g \in \langle \mathcal{A} \rangle$. Then there exists an integer p such that, for all letters $x \in \Sigma$ and integers i, j , $\pi\left(g^{i+jp}_{|x}\right) = \pi\left(g^i_{|x}\right)$*

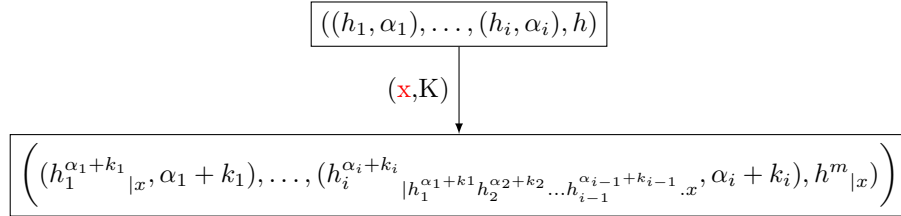
Proof. Since \mathcal{A} is invertible, the action of g on Σ is a permutation $\pi(g)$. Moreover the set $\{g_{|x}, x \in \Sigma\}$ is finite, hence the result. ◀

Henceforth we write $\omega_1(g) \times \text{order}(\pi(g)) = \text{lcm}_x(\text{order}\left(\pi\left(g^{\text{order}(\pi(g))}_{|x}\right)\right))$ the smallest integer fulfilling the condition of Lemma 5.



■ **Figure 5** Example of transition with $\alpha = 2$. On top the group elements and the associated permutations on the bottom. For graphical convenience, elements of $\langle \mathcal{A} \rangle$ are displayed horizontally.

Let $\Gamma = \{g_1, \dots, g_i, g \in \langle \mathcal{A} \rangle\}$ be the inputs of **Knapsack Problem**. We construct the (rooted, directed) graph $\kappa_{\mathcal{A}}(\Gamma)$, having vertex in $(\langle \mathcal{A} \rangle \times \mathbb{N})^i \times \langle \mathcal{A} \rangle$ and (labelled) edges as follows:



■ **Figure 6** An edge of the Knapgraph.

if and only if

$$\left(\prod_{j=1}^i \pi \left(h_j^{\alpha_j+k_j} |_{h_1^{\alpha_1+k_1} h_2^{\alpha_2+k_2} \dots h_{j-1}^{\alpha_{j-1}+k_{j-1}} .x} \right) \right)^m = \pi(h^m|_x)$$

and

$$K = (k_1, \dots, k_i) \in K_1 \times \dots \times K_i,$$

where $K_j = \{0, \text{order}(\pi(h_j)), \dots, \omega_1(h_j) \text{order}(\pi(h_j))\}$ and $m = \#\text{Orb}_h(x)$.

One can notice that if two vertices $((h_i, \alpha_i), h)$, $((f_i, \beta_i), f)$ are linked by an edge with label (x, K) , it means that the product of the permutations induced by $h_i^{\alpha_i}$ (resp. $h_i^{\alpha_i+k_i}$,

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$f_i^{\alpha_i+k_i}$ is equal to $\pi(h)$ (*resp.* $\pi(h), \pi(f)$) and that

$$\left(\prod_{j=1}^i h_j^{\alpha_j+k_j} \right)_{|x} = \prod_{j=1}^i f_j^{\alpha_j+k_j} = \prod_{j=1}^i f_j^{\beta_j}$$

(*resp.* $h^m_{|x} = f$).

If, from a vertex v and for some K , there is no outgoing edge for some letter $x \in \Sigma$ then we delete every edge going out of v labeled by K . Moreover if some vertex v has no outgoing edge labelled with K , then we delete every edge labelled by K on edges at the same distance from the root as v . Finally, if some edge has no outgoing edge, we delete it.

We construct the *Knapgraph* $\kappa_{\mathcal{A}}(\Gamma)$, following this procedure from the initial vertices

$$((g_1^{\pm 1, \alpha_1}, \alpha_1), \dots, (g_i^{\pm 1, \alpha_i}, \alpha_i), g)$$

satisfying

$$\left(\prod_{j=1}^i \pi(g_j^{\pm 1, \alpha_j}) \right) = \pi(g).$$

Clearly, if the Knapgraph is empty then the answer to **Knapsack Problem** is NO. Otherwise assume that the graph is finite. If the answer to **Knapsack Problem** is YES then every loop admits $K = (0, \dots, 0)$ as a label. Indeed it means that the powers on the g_i stabilise at some point, and the reciprocal implies that we cannot find finite exponents to solve **Knapsack Problem**. Moreover the suitable exponent can be read on the graph. However due to the over-approximation one can get "false-positive" (see the case $i^{-4} \approx i^4$ in Figure 7). Still, since the graph is finite and the word problem decidable we are still able to give the proper answer.

Now, we do not know if the Knapgraph is finite. However, if we project each vertex on its last coordinate and the edge label on the letter, we obtain a subgraph of the *Orbit Graph*. The Orbit Graph has been introduced in [3]. It is a graph which captures the dynamics of the action of an element, and whose vertices are elements of $\langle \mathcal{A} \rangle$ and edges are defined by

$$(h) \xrightarrow{x, m} (h^m_{|x})$$

with $m = \#\text{Orb}_h(x)$.

The Orbit Graph of a given element g consists in those vertices which are accessible from the initial vertex g .

The last coordinate of the Knapgraph $\kappa_{\mathcal{A}}(g_1, \dots, g_i, g)$ mimics the Orbit Graph of g . Since the other coordinates belong to finite sets, we get that an automaton with finite Orbit Graph has finite Knapgraph, for any input of the problem, hence has a decidable **Knapsack Problem**.

Consider the class of *bounded automata* introduced in [22]. This class is known to be quite tractable to decision problems. In particular **Order Problem** and **Conjugacy Problem** are decidable in this class. This comes from the finiteness of the *Orbit Graph* [3]. Using this structural property, we obtain:

► **Theorem 6.** *Knapsack Problem is decidable in the class of bounded automata.*

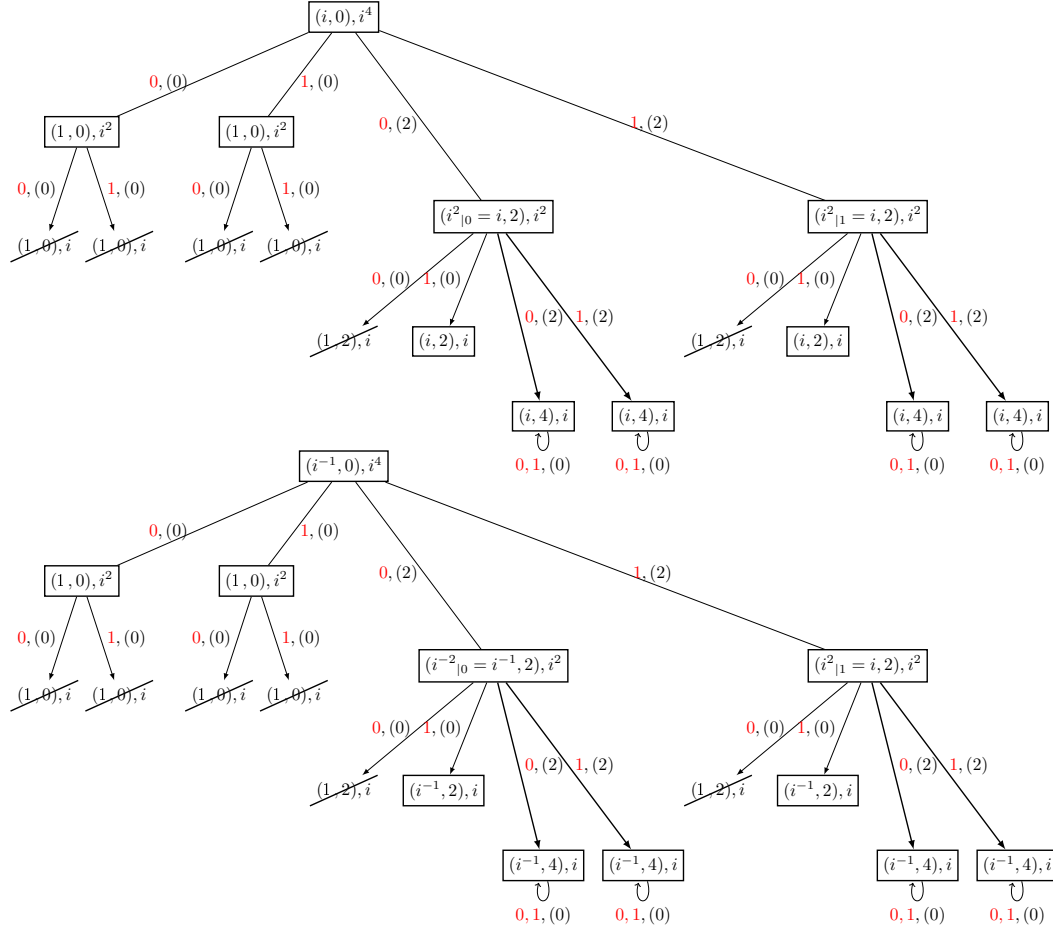
This class happens to contain many interesting groups, including the Grigorchuk group, the Basilica group (Figure 2), the Gupta-Sidki groups, or the adding machine (Figure 1). In the next section we develop some instances of **Knapsack Problem**.

5 Examples

First let us describe an easy example on the adding machine \mathcal{Z} :

- **Input:** $i, i^4 \in \langle \mathcal{Z} \rangle$.
- Does there exist n such that $i^n = i^4$?

Here the answer is obviously **Yes**, and the Knapgraph is displayed on Figure 7.

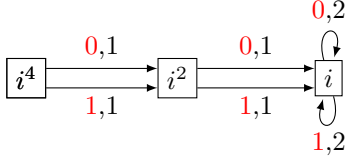


■ **Figure 7** An example of Knapgraph in the adding machine \mathcal{Z} .

By examination of the graph we get two candidates: $n = 4$ and $n = -4$. Applying the algorithm to solve **Word Problem** in $\langle \mathcal{Z} \rangle$ (which is based on composition and minimisation of the automaton \mathcal{Z}), we obtain a positive answer to **Knapsack Problem**.

We recall the *Orbit Graph* $OG(i^4)$:

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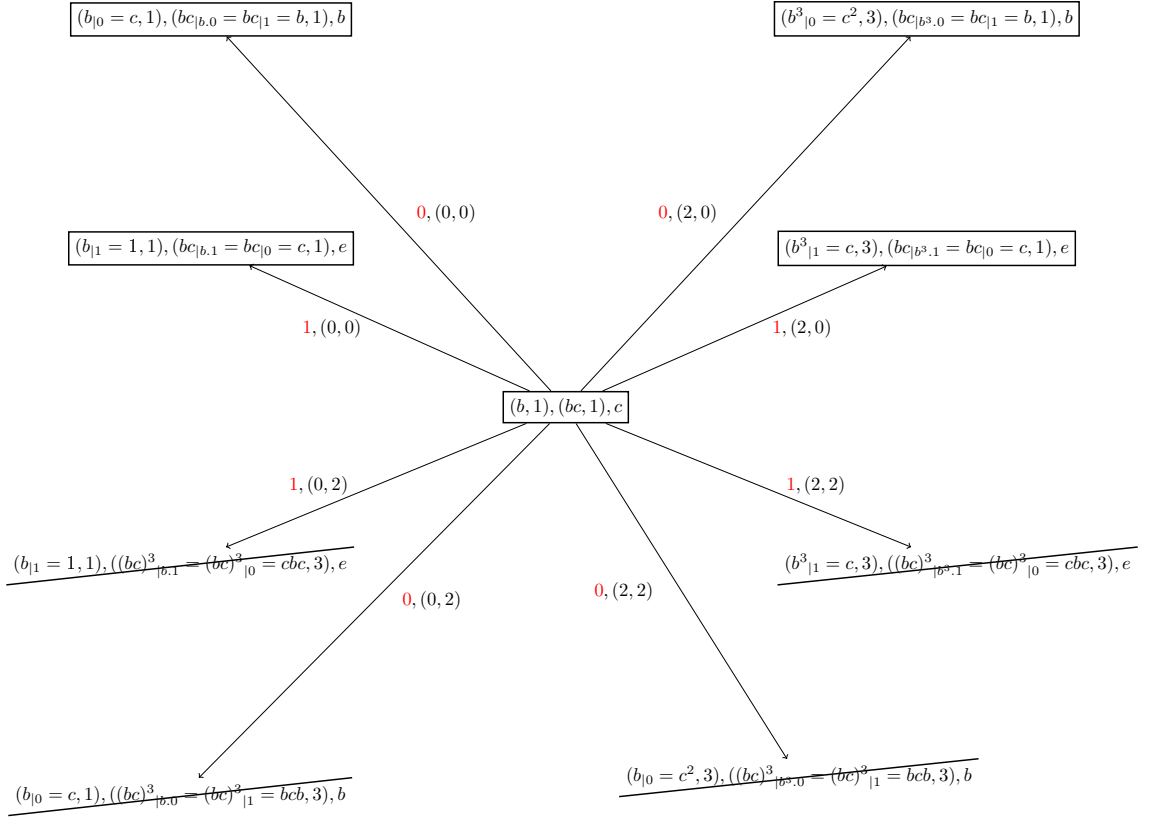


■ **Figure 8** The *Orbit Graph* $OG(i^4)$.

Let us now consider the following question for the Basilica automaton \mathcal{B} (described in Figure 2).

- **Input:** $b, bc, c \in \langle \mathcal{B} \rangle$.
- Do there exist n_1, n_2 such that $b^{n_1}(bc)^{n_2} = c$?

Here the answer is **Yes**, for $n_1 = -1$ and $n_2 = 1$. The fragment of radius 1, centred on initial vertex $((b, 1), (bc, 1), c)$, of Knapgraph $\kappa_{\mathcal{B}}(b, bc, c)$ is drawn on Figure 9.



■ **Figure 9** The fragment of radius 1, centred on initial vertex $((b, 1), (bc, 1), c)$, of Knapgraph $\kappa_{\mathcal{B}}(b, bc, c)$ in the Basilica \mathcal{B} .

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