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Input and State Observability of Network Systems with a Single Unknown Input

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Abstract: This paper studies network systems affected by a single unknown input, possibly representing an attack or a failure, to be estimated. The main result is a characterization of input and state observability, namely the conditions under which both the whole network state and the unknown input can be reconstructed from some measured local states. This characterization is in terms of observability of a suitably-defined subsystem, which allows the use of known graphical charactizations of observability of network systems, leading to structural results (true for almost all interaction weights) or strong structural results (true for all non-zero interaction weights). We apply our results to an illustrative example, finding a full characterization of input and state observability of a path graph, affected by a single unknown input and with measurement of a small number of local states.

Keywords: Input and state observability, Structural observability, Strong structural observability, Network systems, Cyber-physical security

1. INTRODUCTION

The physical and the technological worlds around us are made of several entities that interact with each other and then give rise to some complex emerging behaviors. They include social networks, chemical networks, robotic networks, to cite a few. This last decade has seen a significant increase in the number of works devoted to network systems with a particular focus on how to control and/or monitor these networks. Precisely, it is of paramount importance to understand how a given target can be reached by controlling a subset of entities and how to infer the state of the whole network from measurements obtained from a subset of entities. These two questions are clearly related to the issues of controllability and observability, well known by system theorists. Here, we focus on the observability issue, and on linear network systems. Two ways of characterization can be considered. In a first approach, the network is fully known: both the topology and the weights of each interaction are known. In such a case, one can resort to classical matrix rank test tools (Kalman observability matrix or PBH test), but algebraic characterization can become numerically infeasible when the network is large, and also gives little insight into the effects of the network topology. Therefore, graph tools have been derived in the recent literature, mainly for consensus networks, see e.g. Ji and Egerstedt (2007); Parlangeli and Notarstefano (2012); O'Cleary et al. (2013); Kibangou and Commault (2014). In a second approach, the network is partially known: only the topology is known. The edge weights being unknown, they are considered as free parameters. In this framework, one can seek for structural results (true for almost all choice of free parameters) or strongly structural results (true for all nonzero free parameters). This line of work is the theory of structured systems (see Shields and Pearson (1976) and Dion et al. (2003)).

Network systems are prone to failures, that can be intentional (from a malicious external entity) or not. Therefore, it is interesting to observe not only the state of the network system, but also a possible unknown input. This problem is known as *input and state observability* (*ISO*). Similarly to observability, ISO can be studied either for a fully known network system, with particular numerical values, or seeking for (strongly) structural results with respect to free parameters for interaction weights. For the first approach, there are well-known algebraic characterizations with matrix rank test similar to Kalman and PBH; in Sect. 2 we will recall the main results which will be used for our developments. For the second approach, a characterization of structural ISO has been derived in Boukhobza et al. (2007), while no result is known about strong structural ISO. In this paper, we focus on the case where a few agents have their state directly observable, and where a single agent is affected by an unknown input (i.e., by the failure or the attack). The first assumption is quite common in the study of structural properties of network systems, see e.g. Trefois and Delvenne (2015) and references therein. The second assumption is introduced e.g. in Teixeira et al. (2010) for detection of cyber attacks. In contrast to the latter work, herein the aim is not to design detection policies but to characterize ISO. Such a characterization can be helpful to select nodes to be monitored, so as to obtain the desired observation without wasting resources in monitoring all nodes.

For this framework with a single unknown input, our main result is Prop. 4, which shows that ISO is equivalent to classical observability of a suitably-defined subsystem.

Using this result and known characterizations of (strong) structural observability, we can characterize both structural ISO and strong structural ISO; the first characterization is simpler than the general one in Boukhobza et al. (2007), and the second one is totally new. Then, we apply our results to an illustrative example: we fully characterize structural and strongly structural ISO of path graphs, depending on the number of nodes and on the number and position of the few nodes whose state is observed.

2. PROBLEM FORMULATION

Consider a network system with N nodes whose interconnections are described by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, 2, \cdots, N\}$ and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ stand for the vertex set and the edge set respectively. We denote by $w_{ij} \in \mathbb{R}$ the weight associated with the edge $(j, i) \in \mathcal{E}$ and by $W = [w_{ij}]$ the weighted adjacency matrix of $\mathcal{G}$, where $w_{ij} = 0$ if $(j, i) \notin \mathcal{E}$.

The network dynamics are assumed to be linear, and affected by a scalar unknown input at a node j:

$$\Sigma : \begin{cases} \mathbf{x}(k+1) = W\mathbf{x}(k) + \mathbf{e}_{j,N} u(k) \\ \mathbf{y}(k) = C\mathbf{x}(k) \end{cases} \tag{1}$$

where $\mathbf{x}(k) \in \mathbb{R}^N$, $u(k) \in \mathbb{R}$ and $\mathbf{y}(k) \in \mathbb{R}^K$ stand for the network state, the unknown input, and the available measurements at time k, respectively, while $\mathbf{e}_{j,N}$ denotes the jth vector of the canonical basis of $\mathbb{R}^N$.

From the above set-up, it is clear that we restrict our study to the case where the unknown input affects a single node only. Such a scheme can be used to represent an attack on a node, including denial-of-service or deception attacks on the in-coming communications of node j, see Teixeira et al. (2010). We aim at characterizing when it is possible to jointly estimate both the initial conditions of all the nodes of the network and the scalar unknown input, from the direct measurements of the states of a subset $\mathcal{O} \subset \mathcal{V}$ of nodes. More precisely, we are studying input and state observability, defined as follows:

Definition 1. System Σ is *input and state observable (ISO)* if the initial condition $\mathbf{x}(0) \in \mathbb{R}^N$ and the input sequence $u(0), \ldots, u(N-2)$ can be reconstructed from the measurements $\mathbf{y}(0), \ldots, \mathbf{y}(N-1)$.

The following algebraic characterization is immediate:

Lemma 1. Σ is ISO if and only if the matrix $\Psi = [\Theta, \Gamma]$ has full column rank, where $\Theta \in \mathbb{R}^{KN \times N}$ is the Kalman observability matrix of the pair (W, C), i.e.,

$$\Theta = \begin{bmatrix} C \\ CW \\ \cdots \\ CW^{N-1} \end{bmatrix},$$

while $\Gamma \in \mathbb{R}^{KN \times (N-1)}$ is defined by

$$\Gamma = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ C\mathbf{e}_{j,N} & \mathbf{0} & \cdots & \mathbf{0} \\ CW\mathbf{e}_{j,N} & C\mathbf{e}_{j,N} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ CW^{N-2}\mathbf{e}_{j,N} & CW^{N-3}\mathbf{e}_{j,N} & \cdots & C\mathbf{e}_{j,N} \end{bmatrix}.$$

The above characterization is similar to Kalman condition, while the following Lemma 2 is the analogue of PBH test. Notice that ISO is equivalent to strong observability (i.e., the possibility to reconstruct the initial state $x(0)$, in the presence of an unknown input) together with left invertibility with delay 1 (i.e., the possibility to reconstruct all inputs up to $u(N-2)$ when observing outputs up to $y(N-1)$, while knowing $x(0)$). Hence, Thm. 7.17 in Trentelman et al. (2001) together with Thm. 4 in Massey and Sain (1968) give the following characterization.

Lemma 2. Σ is ISO if and only if $Ce_{j,N} \neq 0$ and the matrix $\Phi_z = \begin{bmatrix} zI_N - W & -\mathbf{e}_{j,N} \\ C & 0 \end{bmatrix}$ has full column rank $\forall z \in \mathbb{C}$.

Beyond the study of ISO for a specific W, it is of crucial importance to get results depending on the graph only and not on a particular choice of the edge weights, leading to structural characterization of ISO. Following classical literature on structural controllability and observability, we consider the edge weights (i.e., the entries of W not having beeen fixed to 0) as free parameters, each taking real values, so that each point of $\mathbb{R}^{|\mathcal{E}|}$ is associated with one particular matrix of the system Σ, while all such matrices share a common structure of imposed zeros.

We will be interested in understanding whether a system Σ is ISO for almost all parameters, or cannot be ISO for any choice of the parameters.

Definition 2. Sytem Σ is *structurally ISO* if it is ISO for all choices of edge weights, except those lying in some proper variety of $\mathbb{R}^{|\mathcal{E}|}$.

Inspired by Mayeda and Yamada (1979), we will be also interested in stronger results, which hold true for all choices of non-zero weights, and not only for almost every choice of weights. Indeed, in some cases, it is essential to ensure that system design with some specified pattern of zeros (representing physical absence of links) will never produce a non-observable system.

Definition 3. System Σ is *strongly structurally ISO* if it is ISO for all choices of non-zero edge weights.

Clearly, strong structural ISO implies structural ISO.

3. CHARACTERIZATION OF INPUT AND STATE OBSERVABILITY

By looking at the algebraic characterization of ISO in Lemma 1, one immediately finds the following necessary conditions for input-and-state observability:

i) Ψ has a number of rows greater than or equal to the number of columns;
ii) Θ has full column rank;
iii) Γ has full column rank.

Such conditions can be equivalently re-phrased as follows.

Proposition 3. The following conditions are necessary for input-and-state observability of Σ:

i) C has at least two rows ($K \geq 2$);
ii) the system (W, C) is observable;
iii) $C\mathbf{e}_{j,N} \neq \mathbf{0}$.

Proof. The three conditions are equivalent to the above-stated ones. Condition i) is true since Ψ has KN rows and $2N-1$ columns. For condition ii), notice that Θ is the

Kalman observability matrix of the system (W, C). Condition iii) is equivalent to Γ having full column rank. $\square$

This proposition, despite its simplicity, can be effectively used to rule out the possibility of ISO in some cases of interest. In particular, condition i) implies that input-and-state observability from a single node i is impossible. Condition ii) excludes all the systems where the corresponding system (W, C) (without unknown input) is not observable. Finally, condition iii) can be interpreted as the necessity to have a direct influence of the input on node(s) whose state is taken into account in measurements (see Teixeira et al. (2010) for a similar conclusion).

In the sequel, we assume that one can directly measure the states of a set of nodes $\mathcal{O} \subset \mathcal{V}$, $|\mathcal{O}| = K \geq 2$ and that the node j affected by the unknown input belongs to $\mathcal{O}$, say $\mathcal{O} = \{j\} \cup \tilde{\mathcal{O}}$. Hence, rows of C are $\mathbf{e}_{k,N}^T$, $k \in \mathcal{O}$. Without loss of generality we assume that the first row of C is $\mathbf{e}_{j,N}^T$.

Let $S_{j,N}$ denote the $N \times (N-1)$ selection matrix defined by $S_{j,N} = [\, \mathbf{e}_{1,N} \cdots \mathbf{e}_{j-1,N}\ \mathbf{e}_{j+1,N} \cdots \mathbf{e}_{N,N} \,]$. We can therefore define the matrices $\tilde{C} = S_{1,K}^T C S_{j,N}$ and $\tilde{W} = S_{j,N}^T W S_{j,N}$. They are obtained by removing from C the first row and the jth column and from W the jth row and column. The following proposition characterizes ISO in terms of observability of a suitable subsystem:

Proposition 4. Consider a network system Σ described by (1), with state matrix W, where a scalar unknown input affects node j, and a set $\mathcal{O}$ of nodes have their states directly measurable, with $j \in \mathcal{O}$ and $|\mathcal{O}| \geq 2$; this corresponds to an observation matrix C with first row $\mathbf{e}_{j,N}^T$ and other rows $\mathbf{e}_{k,N}^T$, $k \in \mathcal{O} \setminus \{j\}$. This system is ISO if and only if the pair $(\tilde{W}, \tilde{C})$ is observable, where $\tilde{C}$ (resp. $\tilde{W}$) is obtained by removing the first row and the jth column from C (resp. the jth row and jth column from W).

Proof. We define the two following permutation matrices:

$$\Pi_1 = \begin{bmatrix} S_{j,N}^T & 0 \\ 0 & S_{1,K}^T \\ \mathbf{0} & \mathbf{e}_{1,K}^T \\ \mathbf{e}_{j,N}^T & \mathbf{0} \end{bmatrix}, \quad \Pi_2 = \begin{bmatrix} S_{j,N} & \mathbf{e}_{j,N} & \mathbf{0} \\ 0 & 0 & 1 \end{bmatrix}.$$

Applying the corresponding permutations to Φ_z we get:

$$\Pi_1 \Phi_z \Pi_2 = \begin{bmatrix} zI_{N-1} - \tilde{W} & S_{j,N}^T (zI_N - W)\mathbf{e}_{j,N} & \mathbf{0} \\ \tilde{C} & \mathbf{0} & \mathbf{0} \\ 0 & 1 & 0 \\ \mathbf{e}_{j,N}^T (zI_N - W) S_{j,N} & z - w_{jj} & -1 \end{bmatrix}.$$

Since the last row of this matrix is linearly independent of all other rows, $\text{rank}(\Pi_1 \Phi_z \Pi_2) = \text{rank}(\tilde{\Phi}_z) + 1$, with $\tilde{\Phi}_z = \begin{bmatrix} zI_{N-1} - \tilde{W} & S_{j,N}^T (zI_N - W)\mathbf{e}_{j,N} \\ \tilde{C} & \mathbf{0} \\ 0 & 1 \end{bmatrix}$. Now, the last column of $\tilde{\Phi}_z$ is linearly independent from all other columns, so that: $\text{rank}(\tilde{\Phi}_z) = \text{rank}\left(\begin{bmatrix} zI_{N-1} - \tilde{W} \\ \tilde{C} \end{bmatrix} \right) + 1$.

As a consequence,

$$\text{rank}(\Phi_z) = \text{rank}(\Pi_1 \Phi_z \Pi_2) = \text{rank}\left(\begin{bmatrix} zI_{N-1} - \tilde{W} \\ \tilde{C} \end{bmatrix} \right) + 2.$$

From Lemma 2, we know that the system Σ is ISO if and only if $\text{rank}(\Phi_z) = N + 1$, which in turn is equivalent to $\text{rank}\left(\begin{bmatrix} zI_{N-1} - \tilde{W} \\ \tilde{C} \end{bmatrix} \right) = N - 1$. The latter corresponds to the PBH observability test applied to the system $\tilde{\Sigma}$ with state matrix $\tilde{W}$ and observation matrix $\tilde{C}$. $\square$

Prop. 4 can be interpreted as follows. System Σ can be decomposed in two coupled subsystems:

$$\begin{cases} \tilde{\mathbf{x}}(k+1) = \tilde{W}\tilde{\mathbf{x}}(k) + S_{j,N}^T W \mathbf{e}_{j,N} x_j(k) \\ \tilde{\mathbf{y}}(k) = \tilde{C}\tilde{\mathbf{x}}(k) \end{cases}$$

$$\begin{cases} x_j(k+1) = w_{jj} x_j(k) + \mathbf{e}_{j,N}^T W S_{j,N} \tilde{\mathbf{x}}(k) + u(k) \\ y_j(k) = x_j(k). \end{cases}$$

Since x_j is directly measured, the first system has a known input, so its state can be reconstructed when the pair $\tilde{W}, \tilde{C}$ is observable. Having reconstructed $\tilde{\mathbf{x}}(k-1)$ and measured $x_j(k)$, one can find $u(k)$ from the second subsystem.

4. STRUCTURAL AND STRONGLY STRUCTURAL INPUT AND STATE OBSERVABILITY

The richness of Proposition 4 lays in the fact that it turns the more complex problem of ISO into a problem of observability of a related system. In particular, this means that (strong) structural ISO can be assessed by exploiting the rich literature characterizing (strong) structural observability of network systems, and also the results on the dual problem of reachability (or controllability in the continuous-time case). In this section, we will briefly mention some relevant results.

Throughout this section, we consider the system $\tilde{\Sigma}$ with state $\tilde{\mathbf{x}}(k) \in \mathbb{R}^n$, state matrix $\tilde{W}$, and observation matrix $\tilde{C}$. To this system, we associate a graph $\tilde{\mathcal{G}} = (\tilde{\mathcal{V}}, \tilde{\mathcal{E}})$ which is an induced subgraph of $\mathcal{G}$. Observability of $(\tilde{W}, \tilde{C})$ for a given choice of the parameters is characterized by classical results such as Kalman observability matrix having full column rank and PBH test, and is equivalent to reachability of $(\tilde{W}^T, \tilde{C}^T)$ (or its controllability, in the continuous-time case).

Structural observability is defined as the system being observable for at least one choice of the free parameters, or, equivalently, as the system being observable for almost all choices of parameters (i.e., all except those lying in a proper variety of $\mathbb{R}^{|\tilde{\mathcal{E}}|}$), since observability is known to be a generic property (see Shields and Pearson (1976) and Dion et al. (2003)). The survey paper (Dion et al., 2003) revises some characterizations of structural controllability and observability. Here we will consider a characterization involving a matching in a bipartite graph.

To $\tilde{\mathcal{G}}$, one can associate a bipartite graph $\mathcal{B} = (\mathcal{U}, \mathcal{W}, \mathcal{E}_{\mathcal{B}})$, with $\mathcal{U} = \tilde{\mathcal{V}} \setminus \tilde{\mathcal{O}}$ and $\mathcal{W} = \tilde{\mathcal{V}}$, constructed as follows: the two vertices in $\mathcal{U}$ and $\mathcal{W}$ corresponding to a same element $v \in \tilde{\mathcal{V}}$ are denoted u_v and w_v, respectively, and $\mathcal{E}_{\mathcal{B}} = \{(u_i, w_j) \in \mathcal{U} \times \mathcal{W} \text{ s.t. } (i,j) \in \tilde{\mathcal{E}}\}$. With this notation, structural observability of $(\tilde{W}, \tilde{C})$ can be characterized as follows:

Proposition 5. The system $(\tilde{W}, \tilde{C})$ is structurally observable if and only if the following two conditions are satisfied:

- for all $v \in \tilde{\mathcal{V}}$, in $\tilde{\mathcal{G}}$ there exists a path from v to w for some $w \in \tilde{\mathcal{O}}$ ($\tilde{\mathcal{G}}$ is output-connected);

- there exists a matching in $\mathcal{B}$ of size $n - |\tilde{\mathcal{O}}|$.

Prop. 5 is the same as Thm. 1 in Chapman and Mesbahi (2013), just phrased here for observability instead of controllability. This result can be traced back to Commault et al. (2002) and Commault et al. (2008).

Strong structural controllability has been studied starting from Mayeda and Yamada (1979); an interesting discussion of both new and old results can be found in Jarczyk et al. (2011). More recently, for the case where columns of the input matrix are of the form $\mathbf{e}_{u,N}$, two equivalent characterizations have been found. A first one is graphical, based on the existence of constrained matchings in two suitably-defined bipartite graphs, see Chapman and Mesbahi (2013). A second one involves the so-called zero-forcing sets, and is described in Trefois and Delvenne (2015). Here we will describe the first one, suitably adapted to deal with observability instead of controllability. We will use the above-described bipartite graph $\mathcal{B}$. We will denote by $\mathcal{E}_{\text{loop}} \subset \mathcal{E}_{\mathcal{B}}$ the edges of the form $\{u_i, w_i\}$, if any exist; they correspond to self-loops in the graph $\tilde{\mathcal{G}}$. We will also need a second bipartite graph $\mathcal{B}_\times$, obtained from $\mathcal{B}$ by adding to $\mathcal{E}_{\mathcal{B}}$ all edges $\{u_i, w_i\}$, $i \in \tilde{\mathcal{V}}$, also for those vertices i not having a self-loop in $\tilde{\mathcal{G}}$. We recall that, given a bipartite graph with vertex sets $\mathcal{X}, \mathcal{Y}$ and edge set $\mathcal{F}$, a matching $\mathcal{M} \subseteq \mathcal{F}$ is *constrained* if there is no other matching involving the same vertex set. In bipartite graphs, the following characteriztion of constrained matching is given in Golumbic et al. (2001), Thm. 3.1: a matching $\mathcal{M}$ is constrained if and only if there exists a reordering of vertices $\mathcal{X} = \{a_1, \dots, a_{|\mathcal{X}|}\}$ and $\mathcal{Y} = \{b_1, \dots, b_{|\mathcal{Y}|}\}$ such that $\mathcal{M} = \{\{a_1, b_1\}, \dots, \{a_{|\mathcal{M}|}, b_{|\mathcal{M}|}\}\}$ and moreover for all $1 \leq j < i \leq |\mathcal{M}|$, $\{a_i, b_j\} \notin \mathcal{F}$. With this notation and definitions, we can state the following characterization of strong structural observability.

Proposition 6. The system $(\tilde{W}, \tilde{C})$ is strongly structurally observable if and only if the following two conditions are satisfied:

- there exists a constrained matching $\mathcal{M} \subseteq \mathcal{E}_{\mathcal{B}}$ of size $n - |\tilde{\mathcal{O}}|$;
- there exists a constrained matching $\mathcal{M}_\times \subseteq \mathcal{E}_{\mathcal{B}_\times}$ of size $n - |\tilde{\mathcal{O}}|$ and such that $\mathcal{M}_\times \cap \mathcal{E}_{\text{loop}} = \emptyset$.

Prop. 6 is a rephrasing for observability of the characterization of strong structural controllability in Chapman and Mesbahi (2013), Thm. 5.

5. STRUCTURAL INPUT AND STATE OBSERVABILITY OF PATH GRAPHS

Fig. 1. A path graph with 7 nodes.

In this section, we consider an illustrative example: the path graph, namely a graph with $\mathcal{V} = \{1, \dots, N\}$ and $(i, j) \in \mathcal{E}$ if and only if $|i - j| = 1$. Fig. 1 shows for example a path graph with 7 nodes. We are considering the undirected path graph, in the sense that all edges appear in both orientations, but we are not enforcing any symmetry in the edge weights. Moreover, we consider that no self-loop is present.

For the path graph, in Prop. 9 we give a full characterization of (strong) structural ISO from a small number of nodes (1, 2 and 3 nodes), depending on the position of the observing nodes and on the size N. To obtain this result, we will first characterize (strong) structural observability of a path graph from one and two nodes (Propositions 7 and 8), and then reduce the study of ISO to the observability problem over subgraphs.

Observability being a generic property, 'the system is not structurally observable' means that there exists no choice of the free parameters such that the system is observable, while 'the system is structurally observable' means that it is observable for almost all choices of parameters. The same is true for ISO. Also recall that strong structural observability implies structural observability.

In a path graph, the first and the last nodes will be called *terminal* nodes, while their neighbors (the second, and the second-last nodes) will be called *near-terminal*.

Proposition 7. Consider a path graph with n nodes, observed by a single node i. The associated network system

1) is not structurally observable, if i is even and n is odd;
2) is strongly structurally observable in two cases:
   a) i is terminal;
   b) i is near-terminal and n is even;
3) is structurally observable, but not strongly structurally observable, in all other cases, namely, if i is neither terminal nor near-terminal and one of the following holds:
   a) i is odd;
   b) i is even and n is even.

Proof. We use the characterizations from Propositions 5 and 6. First, the graph is output-connected, since it is strongly connected.

Then, we look at the graph $\mathcal{B}$. We notice that it has two disjoint subgraphs $\mathcal{B}_1$, $\mathcal{B}_2$: the first one has vertex sets $\mathcal{U}_{\text{odd}} = \{u_h \in \mathcal{U} \text{ s.t. } h \text{ odd}\}$ and $\mathcal{W}_{\text{even}} = \{w_h \in \mathcal{W} \text{ s.t. } h \text{ even}\}$; vice-versa the second one has vertex sets $\mathcal{U}_{\text{even}}$ and $\mathcal{W}_{\text{odd}}$ defined accordingly. When i is even and n is odd, it is impossible to find a matching of $\mathcal{B}$ of size $n-1$. Indeed, $\mathcal{B}_1$ has $|\mathcal{U}_{\text{odd}}| = (n+1)/2 > |\mathcal{W}_{\text{even}}| = (n-1)/2$, and hence no matching of this subgraph can cover all nodes of $\mathcal{U}_{\text{odd}}$, also implying that no matching of $\mathcal{B}$ can cover all nodes of $\mathcal{U}$. As an example, Fig. 2 illustrates the case with $n = 7$ and $i = 2$.

In the cases 2) and 3) we can find a matching of $\mathcal{B}$ of size $n - 1$, which is also constrained. When n is even (regardless i being even or odd), we have the following constrained matching: $\mathcal{M} = \{\{u_h, w_{h+1}\} \forall u_h \in \mathcal{U}_{\text{odd}}\} \cup \{\{u_h, w_{h-1}\} \forall u_h \in \mathcal{U}_{\text{even}}\}$, as it is illustrated in Fig. 3.

When n is odd and i is odd, we have the constrained matching $\mathcal{M} = \{\{u_h, w_{h+1}\} \forall u_h \in \mathcal{U}_{\text{odd}}, h < i\} \cup \{\{u_h, w_{h-1}\} \forall u_h \in \mathcal{U}_{\text{odd}}, h > i\} \cup \{\{u_h, w_{h-1}\} \forall u_h \in \mathcal{U}_{\text{even}}\}$.

By Prop. 5 we can conclude that in cases 2) and 3) the system is structurally observable, since $\tilde{\mathcal{G}}$ is output-connected and there exists a matching in $\mathcal{B}$ of size $n - 1$.

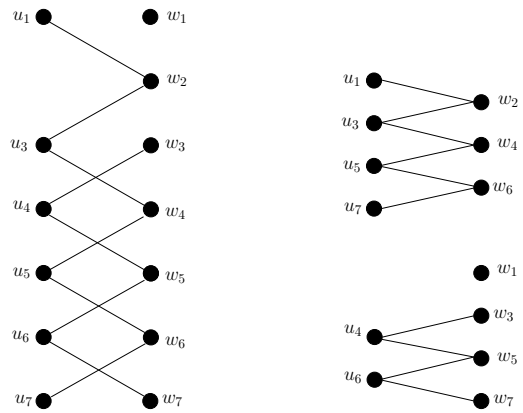

Fig. 2. Left: the bipartite graph $\mathcal{B}$ for a path graph with $n = 7$ nodes and $i = 2$. Right: a re-drawing of the same graph $\mathcal{B}$, highlighting the two disjoint subgraphs $\mathcal{B}_1$ with vertex sets $\mathcal{U}_{\text{odd}}$, $\mathcal{W}_{\text{even}}$ (above) and $\mathcal{B}_2$ with vertex sets $\mathcal{U}_{\text{even}}$, $\mathcal{W}_{\text{odd}}$ (below). Since n is odd and i is even, $\mathcal{B}_1$ has no matching covering $\mathcal{U}_1$.

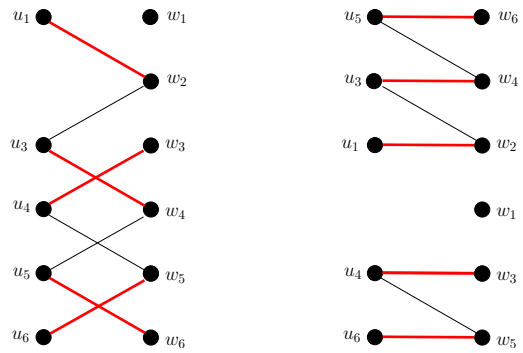

Fig. 3. Left: the bipartite graph $\mathcal{B}$ for a path graph with $n = 6$ nodes and $i = 2$; in red thick lines a matching covering $\mathcal{U}$. Right: the same graph and matching, with nodes re-ordered to show that the matching is constrained.

The difference between the two cases is about $\mathcal{B}_\times$: a constrained matching exists only in case 2), not in case 3). When i is terminal, say $i = 1$ (the case $i = n$ is obtained symmetrically), we have the constrained matching $\mathcal{M}_\times = \{\{u_h, w_{h-1}\} \forall u_h \in \mathcal{U}\}$. When i is near-terminal and n is even, say $n = 2$ (again, the case $i = n - 1$ is obtained symmetrically), we have the constrained matching $\mathcal{M}_\times = \{\{u_1, w_1\}\} \cup \{\{u_h, w_{h-1}\} \forall u_h \in \mathcal{U}, h > 2\}$. Since $\mathcal{E}_{\text{loop}} = \emptyset$, we trivially have $\mathcal{M}_\times \cap \mathcal{E}_{\text{loop}} = \emptyset$, and so we can conclude by Prop. 6 that the system is strongly structurally observable, in cases 2a) and 2b). On the contrary, in case 3), we can show that there exists no constrained matching of $\mathcal{B}_\times$ covering $\mathcal{U}$. We notice that removing w_i disconnects $\mathcal{B}_\times$ in two disconnected subgraphs: $\mathcal{B}_<$ with vertices $\mathcal{U}_< = \{u_j, j < i\}$ and $\mathcal{W}_< = \{w_j, j < i\}$, and $\mathcal{B}_>$ with vertices $\mathcal{U}_> = \{u_j, j > i\}$ and $\mathcal{W}_> = \{w_j, j > i\}$. Since i is neither terminal nor near-terminal, $|\mathcal{U}_<| = |\mathcal{V}_<| \geq 2$ and $|\mathcal{U}_>| = |\mathcal{V}_>| \geq 2$. Any matching of $\mathcal{B}_\times$ covering $\mathcal{U}$ necessarily contains a perfect matching of one of the two subgraphs $\mathcal{B}_<$, $\mathcal{B}_>$. However, such a perfect matching is not unique, there are at least two different matchings on the same set of vertices, since in particular there are vertices $u_h, u_{h+1}, w_h, w_{h+1}$ which could be matched either by edges $\{u_h, w_h\}$, $\{u_{h+1}, w_{h+1}\}$ or by edges $\{u_h, w_{h+1}\}$, $\{u_{h+1}, w_h\}$. As a consequence, no matching of $\mathcal{B}_\times$ covering $\mathcal{U}$ can be constrained. By Prop. 6, we conclude that, when i is neither terminal nor near-terminal, the system is not strongly structurally observable. □

Proposition 8. Consider a path graph with n nodes, observed by two distinct nodes i, i'. The associated network system

1) is not structurally observable, if n is odd and both i and i' are even;
2) is strongly structurally observable if at least one of the following happens
   a) i or i' is terminal (or both);
   b) n is even and i or i' is near-terminal (or both);
   c) i and i' are neighbors, i.e., $|i - i'| = 1$;
   d) $|i - i'| = 2$ and moreover either i, i' are odd, or, in case they are even, n is also even.
3) is structurally observable, but not strongly structurally observable, in all other cases, namely, if at least one of i, i' is odd, or, in case both are even, n is also even, and moreover:
   * neither i nor i' are terminal nor near-terminal;
   * $|i - i'| \geq 3$.

Proof. The proof for case 1) is the same as the proof of case 1) in Prop. 7: having two even observer nodes i, i' instead of only one does not change $\mathcal{B}_1$, and hence it is still impossible to find a matching of $\mathcal{B}$ covering U.

Cases 2a) and 2b) are immediate from the corresponding cases in Prop. 7, since observability is preserved when adding a second row to $\tilde{C}$ when the system is already observable from a first row. More precisely, adding a second observation node i' means removing $u_{i'}$ from $\mathcal{U}$ in the graphs $\mathcal{B}$ and $\mathcal{B}_\times$ of the system with one observation node i. If any (constrained) matching covering $\mathcal{U}$ was already present before the removal, then a (constrained) matching exists also after the removal. Hence, from the proof of Prop. 7, we know that a constrained matching of $\mathcal{B}$ covering $\mathcal{U}$ exists if at least one of i, i' is odd, or if both i and i' are even and n is even. This, together with $\tilde{\mathcal{G}}$ being output-connected, implies that the system is structurally observable in cases 2) and 3) (notice that 3c) implies that one of i and i' is odd). Also, this helps us understanding strong observability, which will then depend only on the existence of a constrained matching of $\mathcal{B}_\times$, having already found a constrained matching for $\mathcal{B}$. Having already proved 2a) and 2b), we will exhibit a constrained matching of $\mathcal{B}_\times$ for the cases 2c) and 2d), and prove that a constrained matching of size $n - 2$ cannot exist in case 3). For the case 2c), say $i' = i + 1$, we have the constrained matching $\mathcal{M}_\times = \{\{u_h, w_{h+1} \forall h < i\}\} \cup \{\{u_h, w_{h-1}\} \forall h > i'\}$. For the case 2d), say $i' = i + 2$, we have the constrained matching $\mathcal{M}_\times = \{\{u_h, w_{h+1} \forall h < i\}\} \cup \{\{u_{i+1}, w_{i+1}\}\} \cup \{\{u_h, w_{h-1}\} \forall h > i'\}$ (assumptions mentioning even or odd do not matter for finding $\mathcal{M}_\times$, but they were needed to have a constrained matching $\mathcal{M}$ of size $n - 2$ in $\mathcal{B}$).

In case 3), say $i < i'$. There are two vertices w_i, $w_{i'}$ whose removal separates $\mathcal{B}_\times$ in three disjoint graphs: $\mathcal{B}_<$, $\mathcal{B}_{\text{middle}}$, and $B_>$, whose vertices have index $h < i$, $i < h < i'$, and $h > i'$, respectively. The assumptions i, i' neither terminal nor near-terminal and $i' - i \geq 3$ imply that each of the three subgraphs has at least four nodes, in

particular it contains a subgraph having nodes u_k, u_{k+1} and w_k, w_{k+1} for some k. Since w_i and $w_{i'}$ can be matched to at most two nodes in $\mathcal{U}$, any matching of $\mathcal{B}_\times$ covering U contains a perfect matching of one of the three subgraphs $\mathcal{B}_<$, $\mathcal{B}_{\text{middle}}$, and $\mathcal{B}_>$, but this means that the matching cannot be constrained, since such a perfect matching is not unique. □

Proposition 9. Consider a path graph (without self-loops) with N nodes and a single input in a node j and an observation matrix C with rows $\mathbf{e}_{u,N}^T$ for $u \in \mathcal{O}$.

1) If $j \notin \mathcal{O}$, the system is not structurally ISO.
2) If $|\mathcal{O}| = 1$, the system is not structurally ISO.
3) If $|\mathcal{O}| = 2$ and $j \in \mathcal{O}$:
   a) if j is not terminal, then the system is not structurally ISO;
   b) if j is terminal, then the system is (strongly) structurally ISO if and only if the system $(\tilde{W}, \tilde{C})$ obtained by removing j is (strongly) observable; the system $(\tilde{W}, \tilde{C})$ corresponds to a path graph observed by a single node, whose structural observability is characterized in Prop 7.
4) If $|\mathcal{O}| = 3$ and $j \in \mathcal{O}$:
   a) if j is not terminal, then the system $\tilde{W}, \tilde{C}$ has a corresponding graph $\tilde{\mathcal{G}}$ consisting of two smaller paths. If both nodes of $\mathcal{O} \setminus \{j\}$ belong to a same path, then the system is not structurally ISO. If they belong one to each path, then the system is (strongly) structurally ISO if and only if both paths are (strongly) structurally observable; the structural observability of each path can be assessed using Prop 7.
   b) if j is terminal, the system is (strongly) structurally ISO if and only if the system $(\tilde{W}, \tilde{C})$ is (strongly) structurally observable; $(\tilde{W}, \tilde{C})$ is a single path graph, observed from two nodes, and its observability is assessed uing Prop 8.

Proof. Items 1) and 2) are from Prop. 3. For the other cases, the main result Prop. 4 reduces ISO to observability of the subsystem $(\tilde{W}, \tilde{C})$ obtained by removing j. Then, cases where the graph $\tilde{\mathcal{G}}$ corresponding to the system $(\tilde{W}, \tilde{C})$ is not output-connected are excluded, using Prop 5: case 3a), and 4a) with both elements of $\mathcal{O} \setminus \{j\}$ being in a same path (thus leaving the second path not output-connected). In the remaining cases, (strong) structural ISO boils down to (strong) structural observability of one or two smaller path graph(s), each observed from one or two of its nodes, whose characterization is given in Propositions 7 and 8. □

6. CONCLUSION

We have studied linear network systems affected by a scalar unknown input, directly influencing a single local state (i.e., one node). We have studied input and state observability, namely the conditions under which both the whole network state and the unknown input can be reconstructed from the measurements of a few local states. We have fully characterized input and state observability in terms of observability of a suitable subsystem, and then used known results for structural and strongly structural observability to infer corresponding structural results for our problem, in the illustrative example of a path graph.

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