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► **To cite this version:**

Edwige Godlewski, Martin Parisot, Jacques Sainte-Marie, Fabien Wahl. Congested shallow water type model: roof modelling in free surface flow. 2016. hal-01368075v1

**HAL Id: hal-01368075**

**<https://inria.hal.science/hal-01368075v1>**

Preprint submitted on 19 Sep 2016 (v1), last revised 11 May 2018 (v3)

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# CONGESTED SHALLOW WATER TYPE MODEL : ROOF MODELLING IN FREE SURFACE FLOW

EDWIGE GODLEWSKI, MARTIN PARISOT, JACQUES SAINTE-MARIE,  
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ABSTRACT. [Abstract here](#)

## 1. INTRODUCTION

[Introduction here](#)

## 2. MATHEMATICAL MODELLING

**2.1. Formal derivation of the shallow water model with roof.** In the current section, a shallow water type model with an additional constraint describing the roof, i.e. an impermeable boundary above the flow, is derived from the Navier-Stokes equations using classical thin layer arguments. For readability, the space variable is split into its horizontal component  $x \in \mathbb{R}^{d-1}$  and its vertical component  $z \in \mathbb{R}$ . In the current work, the time evolution of the bottom and the roof is supposed given and fluid/structure interaction will be considered in a further work.

**2.1.1. Navier-Stokes model.** Let us consider a time depending domain  $\Omega(t) \subset \mathbb{R}^d$ ,  $d \in \{2, 3\}$  confined between two mono-valuated regular enough surfaces  $B(x, t) \in \mathbb{R}$ , from now on called bottom and  $R(x, t) \in \mathbb{R}$ , from now on called roof, such that  $B(x, t) \leq R(x, t)$ , see Figure 1. The bottom (resp. the roof) moves according to its own given velocity with  $u_B(x, t) \in \mathbb{R}^{d-1}$  (resp.  $u_R$ ) the horizontal component and  $w_B(x, t) \in \mathbb{R}$  (resp.  $w_R$ ) the vertical component of the velocity. More precisely for  $\Gamma \in \{B, R\}$ , it satisfies

$$(1) \quad \partial_t \Gamma + u_\Gamma \cdot \nabla_x \Gamma - w_\Gamma = 0.$$

The assumption that at any time, the part of the domain occupied by the fluid is delimited by the bottom and a mono-valuated regular enough surface  $\eta(x, t)$  is made. More precisely the part of the domain occupied by the fluid, called fluid domain in the following is marked by the color function  $\psi(x, z, t) \in \{0, 1\}$  such that  $\psi$  takes the value 1 in the area where the fluid is and 0 elsewhere, given by

$$\psi(x, z, t) = \begin{cases} 1 & \text{if } B(x, t) \leq z \leq \eta(x, t), \\ 0 & \text{otherwise.} \end{cases}$$

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*Date:* September 19, 2016.

*2010 Mathematics Subject Classification.* 35L60, 58J45, 76B07, 35Q35.

*Key words and phrases.* shallow water equations with constraint, fluid-structure interaction, free surface, congestion, pseudo-compressibility approximation.

The fluid satisfies the incompressible Navier-Stokes equations with gravity, i.e. for any  $(x, z, t) \in \Omega \times \mathbb{R}_+$  the velocity of the fluid satisfies

$$(2) \quad \begin{cases} \psi (\nabla_x \cdot u + \partial_z w) = 0, \\ \psi (\partial_t u + (u \cdot \nabla_x) u + w \partial_z u + \nabla_x p - \nabla_x \cdot \sigma_{xx} - \partial_z \sigma_{xz}) = 0, \\ \psi (\partial_t w + u \cdot \nabla_x w + w \partial_z w + \partial_z (p + gz) - \nabla_x \cdot \sigma_{zx} - \partial_z \sigma_{zz}) = 0, \end{cases}$$

with  $u \in \mathbb{R}^{d-1}$  the horizontal component and  $w \in \mathbb{R}$  the vertical component of the velocity,  $p$  the pressure,  $g$  the gravity acceleration and  $\sigma$  the viscosity tensor. The fluid density is supposed constant equal to 1. Considering a Newtonian fluid, the viscosity tensor is given by

$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xz} \\ \sigma_{zx} & \sigma_{zz} \end{pmatrix} \text{ with } \begin{cases} \sigma_{xx} = 2\mu D_x(u), & \sigma_{xz} = \mu (\partial_z u + \nabla_x w), \\ \sigma_{zz} = 2\mu \partial_z w, & \sigma_{zx} = \mu (\partial_z u + \nabla_x w)^\top \end{cases}$$

with  $D_x(u) = \frac{1}{2} (\nabla_x u + (\nabla_x u)^\top)$  the symmetric gradient and  $\mu > 0$  the viscosity coefficient.

The fluid domain is advected by the flow and the color function satisfies the following relation

$$(3) \quad \partial_t \psi + u \cdot \nabla_x \psi + w \partial_z \psi = 0.$$

For each surface  $\Gamma \in \{B, \eta, R\}$ , the outward unit normal, denoted by  $n_\Gamma$ , is defined by

$$n_B = \frac{1}{\sqrt{1 + |\nabla_x B|^2}} \begin{pmatrix} -\nabla_x B \\ 1 \end{pmatrix}, \quad n_\eta = \frac{1}{\sqrt{1 + |\nabla_x \eta|^2}} \begin{pmatrix} -\nabla_x \eta \\ 1 \end{pmatrix}$$

and

$$n_R = \frac{1}{\sqrt{1 + |\nabla_x R|^2}} \begin{pmatrix} \nabla_x R \\ -1 \end{pmatrix}.$$

At the free surface, i.e. where the fluid is not in contact with the roof, which corresponds to  $\eta < R$ , the continuity of the stresses reads

$$(4) \quad (1 - \psi|_{z=R}) (\sigma - pI_d + PI_d) n_\eta = 0$$

with the given atmospheric pressure  $P(x, t)$ .

At the bottom and at the roof, the no-penetration condition such that the flow can detach from the surface is considered, i.e. for  $\Gamma \in \{B, R\}$

$$(5) \quad \psi|_{z=\Gamma} \begin{pmatrix} u|_{z=\Gamma} - u_\Gamma \\ w|_{z=\Gamma} - w_\Gamma \end{pmatrix} \cdot n_\Gamma \leq 0.$$

Note that the no-penetration condition at the bottom becomes an equality as  $\psi(x, B(x, t), t) = 1$  and the color function satisfies (3). In addition, the friction between the fluid and the surface  $\Gamma \in \{B, R\}$  is modeled by the following Navier condition

$$(6) \quad \psi|_{z=\Gamma} \Pi_\Gamma \left( (\sigma - pI_d) n_\Gamma - \kappa_\Gamma \begin{pmatrix} u|_{z=\Gamma} - u_\Gamma \\ w|_{z=\Gamma} - w_\Gamma \end{pmatrix} \right) = 0$$

with  $\kappa_\Gamma > 0$  a friction coefficient and  $\Pi_\Gamma$  the projection on the tangential surface to the normal.

Note that the system (1)-(2)-(3)-(4)-(5)-(6) has to be completed with initial conditions for  $u, w$  and  $p$ .

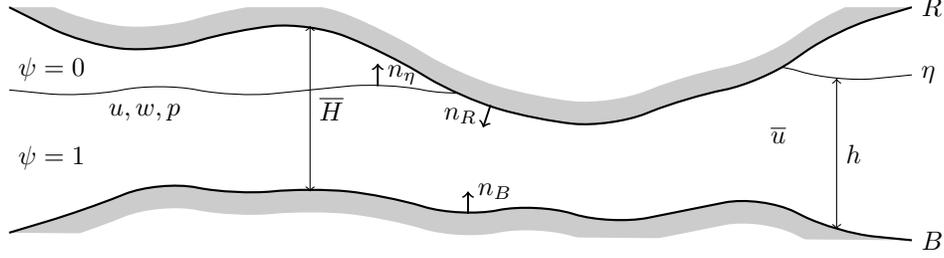


FIGURE 1. Configuration of partially free surface flows

It is well-known that the Navier-Stokes equations (2) completed with the no-penetration condition at the bottom satisfy the non-negativity of the water depth defined by  $h(x, t) = \eta(x, t) - B(x, t)$ . Similarly, we emphasize the fact that the no-penetration condition at the roof (5) ensures that the water stays confined in the opening between the roof and the bottom defined by  $\bar{H}(x, t) = R(x, t) - B(x, t)$ .

**Proposition 1.** *Assume that at the initial time, the water depth is non-negative and bounded by the opening, i.e.  $0 \leq h(x, 0) \leq \bar{H}(x, 0)$ . Then for any time  $t > 0$ , the water depth is non-negative and bounded by the opening, i.e.*

$$0 \leq h(x, t) \leq \bar{H}(x, t).$$

*Proof.* By integration of (3) in the vertical and horizontal space direction as well as in time and using the no-penetration condition at the roof (5),

$$\int_{\mathbb{R}^d} \int_R^{+\infty} \psi(x, z, t) \, dz \, dx \leq \int_{\mathbb{R}^d} \int_R^{+\infty} \psi(x, z, 0) \, dz \, dx.$$

The assumption on the initial condition allows to conclude.  $\square$

2.1.2. *Rescaled model and closure.* Following the strategy proposed in [4], we introduce a ratio aspect  $\varepsilon = \frac{H}{L}$  where  $H$  and  $L$  are two characteristic dimensions along the vertical and the horizontal dimension. In addition, the vertical-averaged quantity is denoted by

$$(7) \quad \bar{u} = \frac{1}{h(x, t)} \int_B^{\eta} u(x, z, t) \, dz.$$

**Proposition 2.** *For  $\varepsilon$  small enough, assume that*

$$\mu = \varepsilon \mu^0 = O(\varepsilon), \quad \kappa_B = \varepsilon \kappa_B^0 = O(\varepsilon), \quad \kappa_R = \varepsilon \kappa_R^0 = O(\varepsilon)$$

and

$$\nabla_x B = O(\varepsilon), \quad \nabla_x R = O(\varepsilon),$$

then as long as  $\nabla_x \eta = O(\varepsilon)$ , the shallow water type model

$$(8) \quad \begin{cases} \partial_t h + \nabla_x \cdot (h \bar{u}) & = 0, \\ \partial_t (h \bar{u}) + \nabla_x \cdot (h \bar{u} \otimes \bar{u}) & = -h \nabla_x \phi + D(\bar{u}) - \kappa(\bar{u} - U) \end{cases}$$

with  $\kappa = \widetilde{\kappa}_B + \mathbb{1}_{h \geq \overline{H}} \widetilde{\kappa}_R$ ,  $U = \frac{\widetilde{\kappa}_B u_B + \mathbb{1}_{h \geq \overline{H}} \widetilde{\kappa}_R u_R}{\kappa}$ ,  $\phi = g(h + B) + \widetilde{p}_\eta$  and  $D(\bar{u}) = 2\varepsilon\mu^0 (\nabla_x \cdot (hD_x(\bar{u})) + \nabla_x (h\nabla_x \cdot \bar{u}))$ ,

$$\widetilde{\kappa}_R = \begin{cases} \frac{\kappa_R^0}{1 + \frac{\varepsilon h}{3\mu^0} \left( \mathbb{1}_{h \geq \overline{H}} \kappa_R^0 - \frac{1}{2} \kappa_B^0 \right)}, & \text{if } \kappa_B^0 \leq 2\kappa_R^0 \\ \kappa_R^0 \left( 1 - \frac{\varepsilon h}{3\mu^0} \left( \mathbb{1}_{h \geq \overline{H}} \kappa_R^0 - \frac{1}{2} \kappa_B^0 \right) \right), & \text{otherwise} \end{cases}$$

and

$$\widetilde{\kappa}_B = \begin{cases} \frac{\kappa_B^0}{1 + \frac{\varepsilon h}{3\mu^0} \left( \kappa_B^0 - \mathbb{1}_{h \geq \overline{H}} \frac{1}{2} \kappa_R^0 \right)}, & \text{if } \kappa_B^0 \leq 2\kappa_R^0 \\ \kappa_B^0 \left( 1 - \frac{\varepsilon h}{3\mu^0} \left( \kappa_B^0 - \mathbb{1}_{h \geq \overline{H}} \frac{1}{2} \kappa_R^0 \right) \right), & \text{otherwise} \end{cases}$$

and completed with the constraints

$$(9) \quad h \leq \overline{H} \quad \text{and} \quad (h - \overline{H})(\widetilde{p}_\eta - P) = 0$$

is formally an approximation in  $O(\varepsilon^2)$  of the Navier-Stokes system with roof (1)-(2)-(3)-(4)-(5)-(6).

*Proof.* Note that thanks to the constraint from Proposition 1, the color function at the roof can be formulated as an indicator function, i.e.  $\psi|_{z=R}(x, t) = \mathbb{1}_{h \geq \overline{H}}(x, t)$ .

Considering the main term of the horizontal momentum equation and the Navier conditions at the bottom and the roof (6), leads to

$$\left\{ \begin{array}{l} \partial_{zz} u = -\frac{\varepsilon}{\mu^0 h} \left( \kappa_B^0 (\bar{u} - u_B) + \mathbb{1}_{h \geq \overline{H}} \kappa_R^0 (\bar{u} - u_R) \right) + O(\varepsilon^2), \\ \partial_z u|_{z=B} = \frac{\varepsilon \kappa_B^0}{\mu^0} (\bar{u} - u_B) + O(\varepsilon^2), \\ \mathbb{1}_{h \geq \overline{H}} \partial_z u|_{z=R} = -\mathbb{1}_{h \geq \overline{H}} \frac{\varepsilon \kappa_R^0}{\mu^0} (\bar{u} - u_R) + O(\varepsilon^2). \end{array} \right.$$

The reader can check that this system together with (7) has the following solution  $u = \frac{\alpha_1}{2} (z - B)^2 + \alpha_2 (z - B) + \alpha_3 + O(\varepsilon^2)$  with

$$\begin{aligned} \alpha_1 &= -\frac{\varepsilon}{\mu^0 h} \left( \kappa_B^0 (\bar{u} - u_B) + \mathbb{1}_{h \geq \overline{H}} \kappa_R^0 (\bar{u} - u_R) \right), \\ \alpha_2 &= \frac{\varepsilon}{\mu^0} \kappa_B^0 (\bar{u} - u_B), \\ \text{and } \alpha_3 &= \bar{u} + \frac{\varepsilon h}{\mu^0} \left( \mathbb{1}_{h \geq \overline{H}} \frac{\kappa_R^0}{6} (\bar{u} - u_R) - \frac{\kappa_B^0}{3} (\bar{u} - u_B) \right). \end{aligned}$$

The main term of the vertical momentum equation leads to  $p = \widetilde{p}_\eta + g(\eta - z) - 2\varepsilon\mu^0 \nabla_x \cdot \bar{u} + O(\varepsilon^2)$  where  $\widetilde{p}_\eta = p_\eta - 2\varepsilon\mu^0 \partial_z w|_{z=\eta}$ . The water volume conservation and the momentum equation can be established as done in [4].

It remains to establish the constraints (9). The first constraint is a direct implication of Proposition 1. Introducing the first equation of the continuity of stresses at the surface (4) into the second gives

$$\left( 1 - \mathbb{1}_{h \geq \overline{H}} \right) (\widetilde{p}_\eta - P) = O(\varepsilon^2)$$

which leads to the second constraint after multiplication by the water depth.  $\square$

**Remark 1.** Note that the two formulations of the friction parameters  $\widetilde{\kappa}_B$  and  $\widetilde{\kappa}_R$  are similar in  $O(\varepsilon^2)$ . The definition is chosen in order to get positive parameters whatever the water depth is.

**Remark 2.** Roof modelling in free surface flow can be realized using water wave or Boussinesq equations (see [reference]).

2.1.3. Preliminary Analysis.

**Proposition 3** (Steady state at rest). For any  $\eta_0 \in \mathbb{R}_+$ , the partially free surface model (8)-(9) admits a steady state at rest, i.e.  $h\bar{u} = 0$ ,  $\partial_t h = 0$  and  $\partial_t(h\bar{u}) = 0$ , given by  $\bar{u} = 0$ ,  $h = \min(\max(0, \eta_0 - B), \bar{H})$  and  $\tilde{p}_\eta = P - \mathbb{1}_{\eta_0 \geq B + \bar{H}} g(\bar{H} + B - \eta_0)$ .

*Proof.* The steady state at rest in (8) gives  $h = 0$  or  $\phi$  constant. If  $h \neq 0$ , it reads  $g\eta_0 + P = g\eta + \tilde{p}_\eta$ . Considering the two following cases gives the value for  $h$  and  $p$

- if  $\eta_0 < B + \bar{H}$ ,  $\tilde{p}_\eta = P$  and thus  $\eta = \eta_0$
- if  $\eta_0 \geq B + \bar{H}$ ,  $h = \bar{H}$  and thus  $\tilde{p}_\eta = P - g(\bar{H} + B - \eta_0)$

□

We define the mechanical energy  $\mathcal{E} = \mathcal{K} + \mathcal{P}$  with  $\mathcal{K} = \frac{1}{2}h\|\bar{u}\|^2$  the kinetic energy and  $\mathcal{P} = gh\left(B + \frac{h}{2}\right) + hP$  the potential energy.

**Proposition 4** (Energy balance). For smooth enough solutions, the mechanical energy  $\mathcal{E}$  of the partially free surface model (8)-(9) satisfies

$$\begin{aligned} \partial_t \mathcal{E} + \nabla_x \cdot ((\mathcal{K} + h(\phi - 2\mu(\nabla \cdot \bar{u} + D_x(\bar{u})))) \bar{u}) &= -(\tilde{p}_\eta - P) \partial_t \bar{H} + h \partial_t (P + gB) \\ &\quad - \frac{\kappa}{2} (\|\bar{u}\|^2 - \|U\|^2 + \|\bar{u} - U\|^2) \\ &\quad - 2\mu h (\|D_x(\bar{u})\|^2 + |\nabla_x \cdot \bar{u}|^2). \end{aligned}$$

*Proof.* Multiplying the momentum balance of (8) by  $u$ , the kinetic energy balance classically writes

$$\begin{aligned} \partial_t \mathcal{K} + \nabla_x \cdot ((\mathcal{K} - 2\mu h(\nabla \cdot \bar{u} + D_x(\bar{u}))) \bar{u}) &= -h\bar{u} \cdot \nabla_x \phi \\ &\quad - 2\mu h (\|D_x(\bar{u})\|^2 + |\nabla_x \cdot \bar{u}|^2) \\ &\quad - \frac{\kappa}{2} (\|\bar{u}\|^2 - \|U\|^2 + \|\bar{u} - U\|^2). \end{aligned}$$

Furthermore the potential energy satisfies

$$\begin{aligned} \partial_t \mathcal{P} &= (g(h + B) + P) \partial_t h + gh \partial_t B + h \partial_t P \\ &= -\phi \nabla \cdot (hu) - (\tilde{p}_\eta - P) \partial_t (h - \bar{H}) - (\tilde{p}_\eta - P) \partial_t \bar{H} + gh \partial_t B + h \partial_t P. \end{aligned}$$

Considering the constraint (9),

$$0 = \partial_t ((\tilde{p}_\eta - P)(h - \bar{H})) = (\tilde{p}_\eta - P) \partial_t (h - \bar{H}) + (h - \bar{H}) \partial_t (\tilde{p}_\eta - P)$$

which implies that  $(\tilde{p}_\eta - P) \partial_t (h - \bar{H}) = 0$  as either  $\tilde{p}_\eta - P = 0$  or  $h - \bar{H} = 0$ .

It remains to sum the two equations. □

**2.2. Relaxed model.** The constraint makes the model (8)-(9) difficult to handle analytically as well as numerically. For this reason a relaxed model is introduced in this section.

2.2.1. *Pseudo-compressibility approximation.* To get a constraint under the form of a vanishing divergence, the mass conservation equation of (8) is multiplied by the indicator function for  $h \geq \bar{H}$  and  $h$  is replaced by  $\bar{H}$  in the partial time derivative to define

$$(10) \quad \overline{\text{DIV}} := \mathbf{1}_{h \geq \bar{H}} (\partial_t \bar{H} + \nabla_x \cdot (h\bar{u})) = 0.$$

The constraint (9) can be replaced by (10), which is easier to manipulate. Following the pseudo-compressibility method for the incompressible Euler equations [13], the constraint (10) is relaxed and for  $\lambda > 0$  the relaxed model reads

$$(11) \quad \begin{cases} \partial_t h_\lambda + \nabla_x \cdot (h_\lambda \bar{u}_\lambda) & = 0, \\ \partial_t (h_\lambda \bar{u}_\lambda) + \nabla_x \cdot (h_\lambda \bar{u}_\lambda \otimes \bar{u}_\lambda) & = -h_\lambda \nabla_x \phi_\lambda + D(\bar{u}_\lambda) - \kappa(\bar{u}_\lambda - U), \\ \lambda^2 \mathcal{L}(\tilde{p}_\lambda - P) + g \overline{\text{DIV}}_\lambda & = 0, \end{cases}$$

where  $\mathcal{L} : \mathbb{R} \rightarrow \mathbb{R}$  is an operator,  $\overline{\text{DIV}}_\lambda := \mathbf{1}_{h_\lambda \geq \bar{H}} (\partial_t \bar{H} + \nabla_x \cdot (h_\lambda \bar{u}_\lambda))$  and  $\phi_\lambda = g(h_\lambda + B) + \tilde{p}_\lambda$ .

**Remark 3.** *One can prove that in a similar context the model (11) converges to the initial model (8)-(9) when  $\lambda$  goes to 0 (see [8, 7]).*

It remains to discuss the choice of  $\mathcal{L}$ . Classically (see [13]) the operator  $\mathcal{L}$  is chosen between  $I_d$ ,  $-\Delta$ ,  $\partial_t$  or  $-\Delta(\partial_t)$ . This list is not exhaustive but common. The choice of  $\mathcal{L}$  should recover the steady state solution (Proposition 3) as well as the energy conservation property (Proposition 4).

**Property 2.1.** *To recover the steady state at rest, the operator  $\mathcal{L}$  has to be under the form  $\partial_t(\tilde{\mathcal{L}})$  with  $\tilde{\mathcal{L}} : \mathbb{R} \rightarrow \mathbb{R}$ . In particular the operators  $I_d$  and  $-\Delta$  do not preserve the steady state at rest.*

*Proof.* Let us introduce the mass conservation in the last equation of (11)

$$\lambda^2 \mathcal{L}(\tilde{p}_\lambda - P) = g \mathbf{1}_{h_\lambda \geq \bar{H}} \partial_t (h_\lambda - \bar{H}).$$

At the steady state, the equality is equivalent to  $\tilde{p}_\lambda - P \in \ker \mathcal{L}$ . Yet the operators  $\mathcal{L}$  under the form  $\partial_t(\tilde{\mathcal{L}})$  with  $\tilde{\mathcal{L}} : \mathbb{R} \rightarrow \mathbb{R}$  verify this condition.  $\square$

The second order projection method, introduced in [12], which consists in taking  $\mathcal{L} = -\Delta(\partial_t)$ , results in  $\tilde{p}_\lambda$  being a linear function. It is obvious that this is not the case in a general framework.

**Remark 4.** *A solution could be to take a linear combination of the two operators  $-\Delta(\partial_t)$  and  $\partial_t$  under the form  $-\alpha \Delta(\tilde{p}_\lambda - P) + (1 - \alpha)(\tilde{p}_\lambda - P) = \mathbf{1}_{h_\lambda \geq \bar{H}} \frac{g(h_\lambda - \bar{H})}{\lambda^2}$  with  $\alpha \in [0, 1]$ . This might be discussed in a further work.*

In the sequel only the so called artificial compressibility method [2, 14] is analyzed.

2.2.2. *Artificial compressibility method.* The artificial compressibility method introduced in [2, 14] consists in choosing  $\mathcal{L} = \partial_t$ . In this case the relaxed model

writes for  $\lambda > 0$

$$(12) \quad \begin{cases} \partial_t h_\lambda + \nabla_x \cdot (h_\lambda \bar{u}_\lambda) & = 0, \\ \partial_t (h_\lambda \bar{u}_\lambda) + \nabla_x \cdot (h_\lambda \bar{u}_\lambda \otimes \bar{u}_\lambda) & = -h_\lambda \nabla_x \phi_\lambda + D(\bar{u}_\lambda) - \kappa(\bar{u}_\lambda - U), \\ \widetilde{p}_\lambda & = P + g \frac{(h_\lambda - \bar{H})_+}{\lambda^2} \end{cases}$$

with  $(\psi)_\pm = \frac{|\psi| \pm \psi}{2}$  for any  $\psi$  the positive and negative part function.

**Proposition 5** (Steady state at rest). *For any  $\eta_\lambda^0 \in \mathbb{R}_+$  and  $\lambda > 0$ , the relaxed model (12) admits a steady state at rest, i.e.  $h_\lambda \bar{u}_\lambda = 0$ ,  $\partial_t h_\lambda = 0$  and  $\partial_t (h_\lambda \bar{u}_\lambda) = 0$ , given by  $\bar{u}_\lambda = 0$ ,  $h_\lambda = \max\left(0, \frac{\lambda^2 \eta_\lambda^0 + \mathbf{1}_{\eta_\lambda^0 \geq B + \bar{H}}(B + \bar{H})}{\lambda^2 + \mathbf{1}_{\eta_\lambda^0 \geq B + \bar{H}}} - B\right)$  and  $\widetilde{p}_\lambda = P + g \frac{(h_\lambda - \bar{H})_+}{\lambda^2}$ .*

*Proof.* The steady state at rest in the momentum equation of (12) gives  $h_\lambda = 0$  or  $\phi_\lambda$  constant. If  $h_\lambda \neq 0$  and by taking into account the expression for the pressure, it reads  $g\eta_\lambda^0 + P = g\eta_\lambda + P + \frac{g}{\lambda^2}(h_\lambda - \bar{H})_+$ .

Considering the two following cases gives the value for  $h_\lambda$

- if  $\eta_\lambda^0 < B + \bar{H}$ ,  $\eta_\lambda = \eta_\lambda^0$
- if  $\eta_\lambda^0 \geq B + \bar{H}$ ,  $\eta_\lambda^0 + \frac{B + \bar{H}}{\lambda^2} = \left(1 + \frac{1}{\lambda^2}\right)\eta_\lambda \Rightarrow \eta_\lambda = \frac{\eta_\lambda^0 + \frac{B + \bar{H}}{\lambda^2}}{1 + \frac{1}{\lambda^2}}$

□

We define the mechanical energy  $\mathcal{E}_\lambda = \mathcal{K}_\lambda + \mathcal{P}_\lambda$  with  $\mathcal{K}_\lambda = \frac{1}{2}h_\lambda \|\bar{u}_\lambda\|^2$  the kinetic energy and  $\mathcal{P}_\lambda = gh_\lambda \left(B + \frac{h_\lambda}{2}\right) + h_\lambda P + \frac{g}{2} \frac{(h_\lambda - \bar{H})_+^2}{\lambda^2}$  the potential energy.

**Proposition 6** (Energy balance). *For smooth enough solutions and any  $\lambda > 0$ , the mechanical energy  $\mathcal{E}_\lambda$  of the relaxed model (12) satisfies*

$$(13) \quad \begin{aligned} \partial_t \mathcal{E}_\lambda + \nabla_x \cdot ((\mathcal{K}_\lambda + h_\lambda(\phi_\lambda - 2\mu(\nabla \cdot \bar{u}_\lambda + D_x(\bar{u}_\lambda))))\bar{u}_\lambda) = & \\ & - (\widetilde{p}_\lambda - P) \partial_t \bar{H} + h_\lambda \partial_t (P + gB) \\ & - \frac{\kappa}{2} (\|\bar{u}_\lambda\|^2 - \|U\|^2 + \|\bar{u}_\lambda - U\|^2) \\ & - 2\mu h_\lambda (\|D_x(\bar{u}_\lambda)\|^2 + |\nabla_x \cdot \bar{u}_\lambda|^2). \end{aligned}$$

*Proof.* As done in the proof of Proposition 4, the kinetic energy balance writes

$$\begin{aligned} \partial_t \mathcal{K}_\lambda + \nabla_x \cdot ((\mathcal{K}_\lambda - 2\mu h_\lambda (\nabla \cdot \bar{u}_\lambda + D_x(\bar{u}_\lambda)))\bar{u}_\lambda) = & -h_\lambda \bar{u}_\lambda \cdot \nabla_x \phi_\lambda \\ & - 2\mu h_\lambda (\|D_x(\bar{u}_\lambda)\|^2 + |\nabla_x \cdot \bar{u}_\lambda|^2) \\ & - \frac{\kappa}{2} (\|\bar{u}_\lambda\|^2 - \|U\|^2 + \|\bar{u}_\lambda - U\|^2). \end{aligned}$$

For the potential energy balance, since  $\partial_{h_\lambda} \mathcal{P}_\lambda = \phi_\lambda$ ,

$$\partial_t \mathcal{P}_\lambda = \phi_\lambda \partial_t h_\lambda + gh_\lambda \partial_t B + h_\lambda \partial_t P - (\widetilde{p}_\lambda - P) \partial_t \bar{H}$$

and it remains to sum the two equations as done in the proof of Proposition 4. □

**Proposition 7** (Hyperbolicity). *For any strictly positive  $\lambda$  and  $h_\lambda$ , the homogeneous relaxed model (12) is strictly hyperbolic.*

In particular, denoting  $\bar{u}_\lambda = (u_\lambda, v_\lambda)^\top$ , for a given unit vector  $n = \begin{pmatrix} n_x \\ n_y \end{pmatrix}$  the eigenvalues of the Jacobian matrix in direction  $n$  are  $l_1 = u_n - c < l_2 = u_n < l_3 = u_n + c$  with  $u_n = u_\lambda n_x + v_\lambda n_y$  and  $c = \sqrt{\left(1 + \frac{\mathbb{1}_{h_\lambda \geq \bar{H}}}{\lambda^2}\right)} gh_\lambda$ . A possible choice for the as-

sociated eigenvectors are  $r_1 = \begin{pmatrix} 1 \\ u_\lambda - cn_x \\ v_\lambda - cn_y \end{pmatrix}$ ,  $r_2 = \begin{pmatrix} 0 \\ -n_y \\ n_x \end{pmatrix}$  and  $r_3 = \begin{pmatrix} 1 \\ u_\lambda + cn_x \\ v_\lambda + cn_y \end{pmatrix}$ .

The Riemann invariants are  $w_1 = u_n - 2c$ ,  $w_2 = u_n$  and  $w_3 = u_n + 2c$ .

*Proof.* The proof relies on simple computations and is not detailed here.  $\square$

### 3. NUMERICAL SCHEME

In the current section, a numerical scheme is described to solve the relaxed model (12) and its main properties are proved. Splitting into a hyperbolic step with a conservative source term and a second step taking into account the dissipative sources

$$\begin{cases} W^{n+\frac{1}{2}} = W^n - \Delta t \nabla \cdot (F(W^n)) + \Delta t S_\phi(W^n) \\ W^{n+1} = W^{n+\frac{1}{2}} + \Delta t S_{f,\mu}(W^{n+\frac{1}{2}}) \end{cases}$$

with  $W^n = (h_\lambda^n, h_\lambda^n \bar{u}_\lambda^n)^\top$ ,  $F(W^n) = (h_\lambda^n \bar{u}_\lambda^n, h_\lambda^n \bar{u}_\lambda^n \otimes \bar{u}_\lambda^n)^\top$ , a time step  $\Delta t$ , so that the numerical unknowns are estimated at time  $t^{n+1} = t^n + \Delta t$ ,  $S_\phi(W^n) = (0, -h_\lambda^n \nabla_x \phi_\lambda^n)^\top$  and  $S_{f,\mu}(W^n) = (0, D(\bar{u}_\lambda^n) - \kappa(\bar{u}_\lambda^n - U^n))^\top$ . In the following only the first step is considered. The second step, taking into account the friction and viscosity terms, is not detailed here.

It is well-known that classical Riemann solvers require a very restrictive CFL condition and are too much dissipative for a stiff potential force, i.e. a small  $\lambda$  in our case (see [6]). Thus, in the next section, the Centered Potential Regularization (CPR) scheme is described. This scheme has the advantage of a CFL condition not restrictive for small  $\lambda$ . Moreover the first order accuracy is recovered.

Let  $\mathbb{T}$  be a tessellation of  $\Omega$  composed of star-shaped control volumes. We denote  $k \in \mathbb{T}$  a control volume,  $\mathbb{F}_k$  the set of its faces,  $|k|$  its volume and  $|\partial k|$  its surface area. Furthermore for a face  $f$ , its area is denoted  $|f|$  and  $k_f$  is the neighbor control volume of  $k$  such that  $k \cap k_f = f$ . Thus the space step  $l_k = \frac{|k|}{|\partial k|}$  and the normalized face size  $\mu_f^k = \frac{|f|}{|\partial k|}$ . The unit normal to  $f$  outward to  $k$  is denoted as  $N_f^k$ . Eventually for any  $\psi$  the following notations are introduced  $2[\psi]_k^{k_f} = \psi_{k_f}^n - \psi_k^n$  and  $2(\psi)_f = \psi_k + \psi_{k_f}$ .

**3.1. Description of the CPR scheme.** The CPR scheme introduced in [10] is an adaptation of an Advection Upstream Splitting Method, i.e. separating the flux into the convective flux and the pressure flux (see [9]). Furthermore the numerical flux is regularized by the gradient of the potential of the conservative forces (see [5]) and a centered estimation is used for the non-conservative terms (see [3]). The scheme is based on a cell-centered finite volume method, i.e. numerical unknowns are the approximated averaged value of the unknown in each control volume.

For  $\gamma \geq 0$ , the numerical scheme for the volume conservation equation writes

$$(14) \quad h_k^{n+1} = h_k^n - \frac{\Delta t}{l_k} \sum_{f \in \mathbb{F}_k} \mathcal{F}_f^{n+1} \cdot N_f^k \mu_f^k$$

and for the momentum balance it writes

$$(15) \quad \begin{aligned} h_k^{n+1} u_k^{n+1} &= h_k^n u_k^n - \frac{\Delta t}{l_k} h_k^{n+1} \sum_{f \in \mathbb{F}_k} [\phi^{n+1}]_f^{k_f} N_f^k \mu_f^k \\ &\quad - \frac{\Delta t}{l_k} \sum_{f \in \mathbb{F}_k} \left( u_k^n \left( \mathcal{F}_f^{n+1} \cdot N_f^k \right)_+ - u_{k_f}^n \left( \mathcal{F}_f^{n+1} \cdot N_f^k \right)_- \right) \mu_f^k \end{aligned}$$

with the numerical flux

$$(16) \quad \mathcal{F}_f^{n+1} = (h^{n+1} u^n)_f - \gamma \Delta t \left( \frac{h^{n+1}}{l} \right)_f [\phi^{n+1}]_k^{k_f} N_f^k.$$

The numerical flux is estimated using the water height at time iteration  $n + 1$  and the approximation of the velocity at time iteration  $n$ . This corresponds to a classical Implicit-Explicit time discretization.

**3.2. Numerical analysis.** The regularizing term in the numerical flux gives a stability result in the sense of a dissipation of mechanical energy. Furthermore as a centered discretization for the non conservative forces is used, the stability of the steady state at rest can be proved. The more, the conservation of the total water volume as well as the positivity of the water height and the consistency is proved. The local discrete kinetic (resp. potential) energy at time  $t^n$  is denoted  $\mathcal{K}_k^n = \mathcal{K}_\lambda(h_k^n, u_k^n)$  (resp.  $\mathcal{P}_k^n = \mathcal{P}(h_k^n)$ ). Thus the discrete mechanical energy reads  $\mathcal{E}_k^n = \mathcal{K}_k^n + \mathcal{P}_k^n$ .

**Proposition 8** (Entropic stability). *For  $\gamma \geq 1$  and under the CFL condition*

$$(17) \quad \left( |u_f^n \cdot N_f^k| + \sqrt{\frac{\gamma}{2}} \sqrt{|\phi^{n+1}]_k^{k_f}|} \right) \Delta t \leq \frac{\min(h_k^{n+1}, h_{k_f}^{n+1})}{h_k^{n+1} + h_{k_f}^{n+1}} \min(l_k, l_{k_f})$$

*the discrete mechanical energy satisfies the following entropy inequality (discrete version of (13) without dissipative effects, i.e.  $\kappa = 0$  and  $\mu^0 = 0$ )*

$$(18) \quad \mathcal{E}_k^{n+1} \leq \mathcal{E}_k^n - \frac{\Delta t}{l_k} \sum_{f \in \mathbb{F}_k} \left( (\mathcal{G}_{\mathcal{K},f}^{n+1} + \mathcal{G}_{\mathcal{P},f}^{n+1}) \cdot N_f^k + \mathcal{H}_{\mathcal{K},f}^{n+1} + \mathcal{H}_{\mathcal{P},f}^{n+1} \right) \mu_f^k + \mathcal{S}_k^{n+1}$$

*with the fluxes of the kinetic energy*

$$\begin{aligned} \mathcal{G}_{\mathcal{K},f}^{n+1} \cdot N_f^k &= \frac{1}{2} \|u_k^n\|^2 \left( \mathcal{F}_f^{n+1} \cdot N_f^k \right)_+ - \frac{1}{2} \|u_{k_f}^n\|^2 \left( \mathcal{F}_f^{n+1} \cdot N_f^k \right)_- \\ \mathcal{H}_{\mathcal{K},f}^{n+1} &= \frac{\Delta t}{2} \left( \frac{h_{k_f}^{n+1}}{l_{k_f}} - \frac{h_k^{n+1}}{l_k} \right) \left| [\phi^{n+1}]_k^{k_f} \right|^2 \end{aligned}$$

*the fluxes of the potential energy*

$$\begin{aligned} \mathcal{G}_{\mathcal{P},f}^{n+1} \cdot N_f^k &= \phi_f^{n+1} \mathcal{F}_f^{n+1} \cdot N_f^k \\ \mathcal{H}_{\mathcal{P},f}^{n+1} &= \frac{h_k^{n+1} u_k^n - h_{k_f}^{n+1} u_{k_f}^n}{2} \cdot [\phi^{n+1}]_k^{k_f} N_f^k \end{aligned}$$

and the source term

$$\mathcal{S}_k^{n+1} = (p_k^{n+1} - P_k^{n+1}) \left( \overline{H}_k^n - \overline{H}_k^{n+1} \right) - h_k^n (P_k^n - P_k^{n+1}) - gh_k^n (B_k^n - B_k^{n+1}).$$

Note that the terms  $\mathcal{G}_{\mathcal{K},f}^{n+1} \cdot N_f^k$ ,  $\mathcal{H}_{\mathcal{K},f}^{n+1}$ ,  $\mathcal{G}_{\mathcal{P},f}^{n+1} \cdot N_f^k$  and  $\mathcal{H}_{\mathcal{P},f}^{n+1}$  are anti-symmetric with respect to the control volume. Thus summing (18) over  $k \in \mathbb{T}$ , we get a decrease of the global discrete energy.

*Proof.* Let's start by an estimation of the discrete potential energy evolution. Setting  $A^{n+1} = \left( h_k^{n+1} - \overline{H}_k^{n+1} \right)$ , with the definition of the positive and negative part functions

$$A_+^{n+1} (A^{n+1} - A_n) = A_+^{n+1} (A_+^{n+1} - A_+^n) + A_+^{n+1} A_-^n$$

and using  $2a(a-b) = a^2 - b^2 + |a-b|$ , simple calculations lead to

$$\phi_k^{n+1} (h_k^{n+1} - h_k^n) = \mathcal{P}_k^{n+1} - \mathcal{P}_k^n + \mathcal{Q}_k^{n+1} - \mathcal{S}_k^{n+1}$$

where

$$\begin{aligned} \mathcal{Q}_k^{n+1} &= \frac{g}{2} |h_k^{n+1} - h_k^n|^2 + \frac{g}{2\lambda^2} \left| \left( h_k^{n+1} - \overline{H}_k^{n+1} \right)_+ - \left( h_k^n - \overline{H}_k^n \right)_+ \right|^2 \\ &\quad + \frac{g}{\lambda^2} \left( h_k^{n+1} - \overline{H}_k^{n+1} \right)_+ \left( h_k^n - \overline{H}_k^n \right)_-. \end{aligned}$$

Now multiplying (14) by  $\phi_k^{n+1}$  and taking into account that  $\mathcal{Q}_k^{n+1} \geq 0$ ,

$$(19) \quad \mathcal{P}_k^{n+1} \leq \mathcal{P}_k^n - \frac{\Delta t}{l_k} \sum_{f \in \mathbb{F}_k} \left( \mathcal{G}_{\mathcal{P},f}^{n+1} \cdot N_f^k + \mathcal{H}_{\mathcal{P},f}^{n+1} - \mathcal{R}_{\mathcal{P},f}^{n+1} \right) \mu_f^k + \Delta t \mathcal{W}_k^{n+1} + \mathcal{S}_k^{n+1}$$

with the rest coming from the space discretisation

$$\mathcal{R}_{\mathcal{P},f}^{n+1} = \left( \mathcal{F}_f^{n+1} - h_f^{n+1} u_f^n \right) \cdot \left[ \phi^{n+1} \right]_k^{k_f} N_f^k = -\gamma \Delta t \left( \frac{h^{n+1}}{l} \right)_f \left| \left[ \phi^{n+1} \right]_k^{k_f} \right|^2$$

and the work of the conservative forces defined by

$$\mathcal{W}_k^{n+1} = \frac{h_k^{n+1} u_k^n}{l_k} \cdot \sum_{f \in \mathbb{F}_k} \left[ \phi^{n+1} \right]_k^{k_f} N_f^k \mu_f^k.$$

Let's now focus on the kinetic energy. Lemma 2.3 in [11] gives an estimation of the discrete kinetic energy evolution under the CFL-condition (17). It reads

$$(20) \quad \mathcal{K}_k^{n+1} \leq \mathcal{K}_k^n - \frac{\Delta t}{l_k} \sum_{f \in \mathbb{F}_k} \left( \mathcal{G}_{\mathcal{K},f}^{n+1} \cdot N_f^k + \mathcal{H}_{\mathcal{K},f}^{n+1} - \mathcal{R}_{\mathcal{K},f}^{n+1} \right) \mu_f^k - \Delta t \mathcal{W}_k^{n+1}$$

with the rest coming from the space discretisation

$$\mathcal{R}_{\mathcal{K},f}^{n+1} = \Delta t \left( \frac{h^{n+1}}{l} \right)_f \left| \left[ \phi^{n+1} \right]_k^{k_f} \right|^2.$$

To conclude the proof, let's follow the proof of theorem 2.2 in [11]. For  $\gamma \geq 1$ ,  $\mathcal{R}_{\mathcal{K},f}^{n+1} + \mathcal{R}_{\mathcal{P},f}^{n+1} \leq 0$  and summing the two energy estimations (19) and (20) gives the result.  $\square$

**Proposition 9** (Consistency). *For smooth enough solutions and a regular cartesian grid with a space step  $\text{dx}$ , i.e. the distance between the center of two neighboring cells, the scheme (14)-(15)-(16) with  $\gamma$  bounded with respect to  $\text{dx}$  and  $\Delta t$  is consistent with the relaxed model (12) without dissipative effects, i.e.  $\mu^0 = 0$  and  $\kappa = 0$ . More precisely,*

$$|h_k^{n+1} - h(x_k, t^{n+1})| = O(\Delta t, \text{dx}^2)$$

and

$$\|h_k^{n+1} u_k^{n+1} - h(x_k, t^{n+1}) u(x_k, t^{n+1})\| = O(\Delta t, \text{dx}).$$

*Proof.* The proof is done in [11].  $\square$

**Proposition 10.** *The scheme (14)-(15)-(16) satisfies the following properties*

- (1) (Positivity) *Assume that the water height at the initial condition is positive, i.e.  $h_k^0 > 0$  for any  $k \in \mathbb{T}$  and that the following CFL condition is satisfied*

$$\left( |u_f^n \cdot N_f^k| + \sqrt{\frac{\gamma}{2}} \sqrt{|\phi^{n+1}|_k^{k_f}} \right) \Delta t \leq \frac{\min(h_k^{n+1}, h_{k_f}^{n+1})}{h_k^{n+1} + h_{k_f}^{n+1}} \min(l_k, l_{k_f}).$$

*Then the positivity of the water height is preserved, i.e.  $h_k^n > 0$  for any  $k \in \mathbb{T}$  and  $n \in \mathbb{N}$ .*

- (2) (Discrete conservation of the total water volume) *The total water volume at the discrete level is conserved, i.e. for any  $n \in \mathbb{N}$*

$$\sum_{k \in \mathbb{T}} l_k h_k^n = \sum_{k \in \mathbb{T}} l_k h_k^0.$$

- (3) (Well-balanced character) *The steady state at rest given in Proposition 5 is preserved at the discrete level.*

*Proof.* The proof is done in [11].  $\square$

**Remark 5.** *Note that the CFL condition (17) does not allow the simulation of dry areas as the time step vanishes.*

## 4. SIMULATION

In the current section, illustrations in one space dimension, i.e.  $d = 1$ , of analytical solutions are given. The first section is devoted to classical stationary solutions, also called equilibrium states [1] and in the second section, dynamical systems are considered. For all the test cases, the gravity acceleration is taken equal to 9.81.

### 4.1. Stationary solutions.

*The lake at rest.* The steady state at rest corresponds to the lake at rest situation, i.e.  $h_k^n = \text{constant}$  and  $\phi_k^n = \text{constant}$ . The numerical validation of the steady state at rest, given in Proposition 5, was done with various bottom elevations in the case where  $h_k^n > 0$ , i.e. where no dry areas occur. The error obtained is the error of the resolution method of the linear system.

A channel of length 1 and a space step in order of  $10^{-2}$  is considered. The regularization parameter of the CPR scheme is taken equal to 1 and  $\lambda$  is taken one order lower than the space step, i.e. in order of  $10^{-3}$ . We impose the flux at the left border of the channel and the water height on the right border. The more,

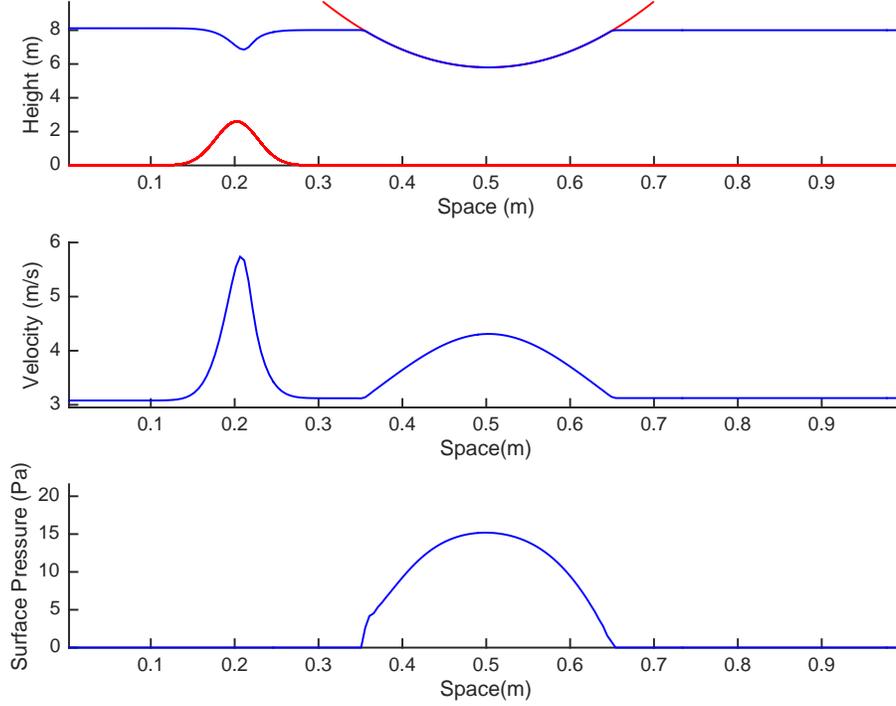


FIGURE 2. Bernoulli's equilibrium in fluvial regime

for  $x$  between 0.3 and 0.7, a roof under the form  $R(x, t) = 100(x - 0.5)^2 + 5.8$  is considered.

*Bernoulli's equilibrium (fluvial regime).* Taking a topography with a bump under the form  $B(x, t) = 2.6 \exp(-800(x - 0.2)^2)$ , figure 2 shows the water height, the velocity and the surface pressure at the stationary state.

*Bernoulli's equilibrium (transcritical case).* Defining the topography as  $B(x, t) = 1$  for  $x \leq 0.65$  and  $B(x, t) = \exp(-100(x - 0.65)^2)$  elsewhere, figure 3 shows the water height, the velocity and the surface pressure at the stationary state.

*Bernoulli's equilibrium (hydraulic jump).* Taking a topography with a bump under the form  $B(x, t) = 3.5 \exp(-100(x - 0.35)^2)$ , figure 4 shows the water height, the velocity and the surface pressure at the stationary state.

## 5. CONCLUSION

[Conclusion here](#)

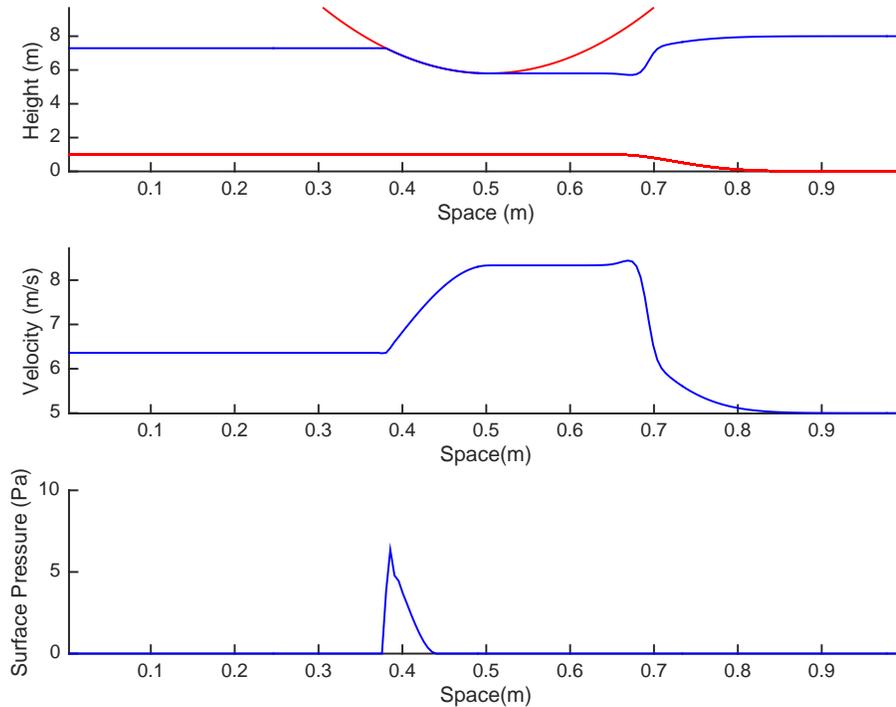


FIGURE 3. Bernoulli's equilibrium in the transcritical case

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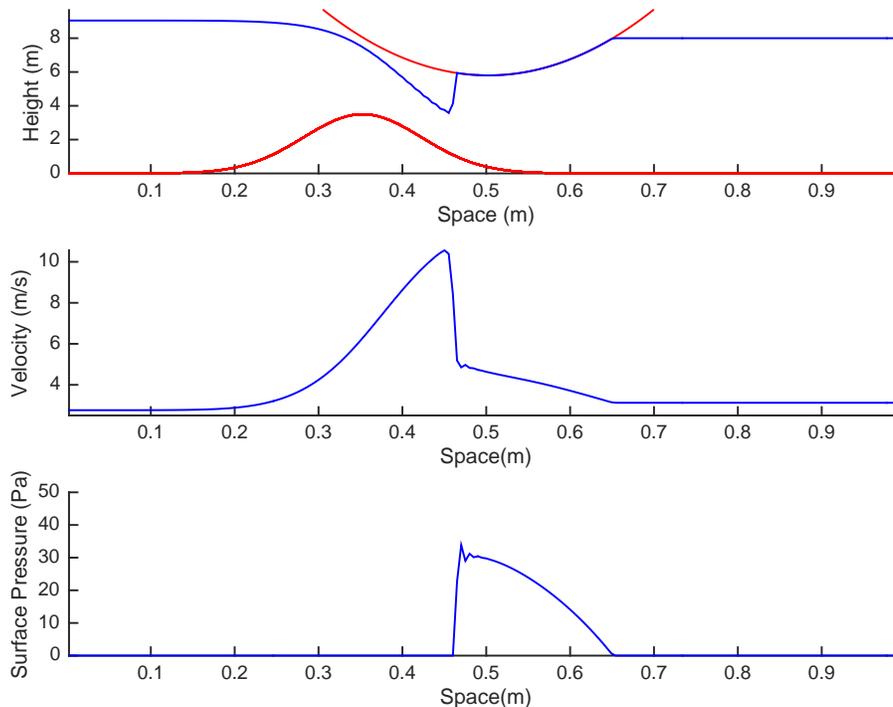


FIGURE 4. Hydraulic jump

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