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Adaptive Physically-Based Models in Computer Graphics

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Abstract

One of the major challenges in physically-based modeling is making simulations efficient. Adaptive models provide an essential solution to these efficiency goals. These models are able to self-adapt in space and time, attempting to provide the best possible compromise between accuracy and speed. This survey reviews the adaptive solutions proposed so far in computer graphics. Models are classified according to the strategy they use for adaptation, from time-stepping and freezing techniques to geometric adaptivity in the form of structured grids, meshes, and particles. Applications range from fluids, through deformable bodies, to articulated solids.

Categories and Subject Descriptors (according to ACM CCS): I.3.7 [Computer Graphics]: Computer Graphics—Three-Dimensional Graphics and Realism-Animation

1 **Keywords:** adaptivity, physically-based animation

2 1. Introduction

3 The ability of physically-based animation to generate highly
4 complex and realistic motion, as well as react to unpre-
5 dictable input, has made it increasingly indispensable in
6 movies, video games and simulators since its introduction in
7 1987 by Terzopoulos et al. [TPBF87]. Complex real-world
8 behaviors exhibit multiscale phenomena in space and time.
9 Sophisticated physical models involving a tremendous num-
10 ber of degrees of freedom are required to accurately repro-
11 duce these phenomena. This comes at a high memory and
12 computational price, thus practitioners and researchers have
13 been striving to achieve good trade-offs between accuracy
14 and computational resources. This is especially relevant in
15 computer graphics, where visual plausibility is the main
16 quality criterion, and high-resolution results with rich details
17 are sought. Trade-offs are typically obtained by carefully
18 choosing the constitutive laws of the modeled objects, while

19 sampling time and space appropriately. However, a model
20 suitable for a given viewpoint may become overly coarse
21 when the object comes closer to the camera, or waste com-
22 putational resources when it is no longer visible. Variations
23 in external loading, large deformations, or topology changes
24 may also require model updates during the simulation. Car-
25 rying on a simulation across a wide range of viewpoints and
26 changes to the object, while simultaneously optimizing effi-
27 ciency and visual realism, requires the simulation model to
28 adapt to the changing circumstances of the simulation. These
29 adaptive simulation techniques are the focus of this report.

30 We call a model or simulation method *adaptive* if it au-
31 tomatically adapts the underlying mathematical representa-
32 tion, data structure and/or algorithm at run time, based on
33 the evolving state of the simulated system. The adaptation
34 is designed for meeting a given criteria which depends on
35 the application. Examples of frequently used criteria include
36 reducing the overall computational complexity without loss
37 of quality, improving the quality of real-time simulation, or
38 simulating more precisely the parts of the scene with which
39 the user is currently interacting.

40 Computer animation has come a long way since Carl-
41 son and Hodgins proposed to switch between dynamic,
42 kinematic and hybrid models for jumping creatures in
43 1997 [CH97], and the variety of approaches and sub-

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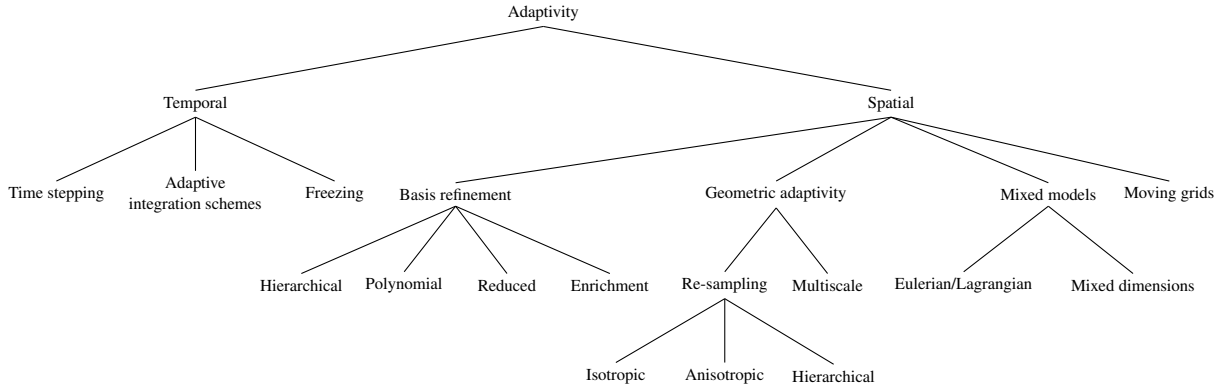


Figure 1: Taxonomy of adaptive physically-based models in Computer Graphics. In this report, we discuss in detail the two most widely used adaptive techniques: Temporal adaptivity in Section 2 and geometric adaptivity in Section 3. We also discuss less common spatial adaptive techniques that proved to be very promising: Basis refinement in Section 4.1, moving grids in Section 4.2 and mixed models in Section 4.3.

1 domains makes it difficult to sort. Rather than iterating on
 2 the different subdomains of physically based computer anima-
 3 tion (rigid bodies, deformable solids, liquids, smoke, *etc.*)
 4 to present their adaptive approaches, we have chosen to re-
 5 view the different families of adaptive methods, and present
 6 the way they have been applied in the different domains. We
 7 hope that this organization will give the reader a broader
 8 picture of adaptivity, by showing how variations of each ap-
 9 proach have been developed in different domains.

10 In Section 2, we present time adaptive techniques such as
 11 dynamic time stepping and freezing techniques. Then, we fo-
 12 cus on spatially adaptive techniques. Section 3 discusses the
 13 most popular approach of spatial adaptivity: geometric adap-
 14 tivity (h -adaptivity where h classically refers to the size of el-
 15 ements in the finite element method), which refers to varying
 16 the discretization resolution via refinement and coarsening
 17 strategies. This section is subdivided according to the types
 18 of adaptive spatial discretization. Section 4 covers other spa-
 19 tial adaptivity approaches: basis refinement which adapts the
 20 number of bases, their order (p -adaptivity where p stands
 21 for polynomial), the basis functions themselves using enrich-
 22 ment, or, in subspace simulation, the deformation modes of
 23 the basis; moving grids methods (r -adaptivity where r stands
 24 for relocation) which relocate nodes without changing their
 25 connectivity; and mixed models which selectively apply a
 26 combination of different computational models. This organiza-
 27 tion reflects the taxonomy we propose in Figure 1. We
 28 finally conclude and sketch future research avenues in adap-
 29 tive simulation.

30 2. Temporal adaptivity

31 Animating any simulated object requires integrating its equa-
 32 tions of motion over time. There are many reasons why
 33 this integration procedure may need to adapt to the circum-
 34 stances of the simulation, whether for accuracy, consistency,

35 or stability. For instance, a system entering a highly non-
 36 linear regime, such as during fracture, typically requires
 37 smaller time steps to maintain the desired degree of accu-
 38 racy. Alternatively, adaptive time steps may be necessary to
 39 prevent the system from entering an invalid state: for exam-
 40 ple, many collision resolution schemes maintain the invari-
 41 ant that the system is never allowed to enter an interpenetrat-
 42 ing configuration, which can require adjusting the time step
 43 according to the frequency of collisions. Finally, the stabil-
 44 ity of continuum-mechanical simulations is closely tied to
 45 the relationship between the time step length, the spatial re-
 46 solution, and the speed of propagation of information in the
 47 system, so if either of the latter change (in the presence of
 48 *spatial* adaptivity or fast-moving flow) the time step must be
 49 adapted as well.

50 Techniques for temporal adaptivity fall into two main cate-
 51 gories. One approach focuses on time resolution, that is, how
 52 to choose an appropriate step length for time integration. The
 53 second focuses on integration techniques themselves, seek-
 54 ing to switch between different integration schemes depend-
 55 ing on the local context. This section explores both of these
 56 possibilities for temporal adaptivity.

57 2.1. Adaptive time step selection

58 **Time step criteria** First, let us focus on the simulation of
 59 continuous media, like fluids and elastic solids, whose mod-
 60 els are governed largely by hyperbolic partial differential
 61 equations. Here the most prominent criterion for time step
 62 selection is the Courant-Friedrichs-Lewy (CFL) condition
 63 [CFL28]. To understand this condition, we first note that
 64 the solution of a partial differential equation at some point
 65 depends on a particular subset of initial or boundary data;
 66 we call this subset of data the *domain of dependence*. The
 67 CFL condition states simply that for any numerical scheme
 68 to converge to the true solution, the domain of dependence of

1 the numerical scheme must, in the limit, contain the true do-
 2 main of dependence of the underlying differential equation.
 3 (Otherwise, one could perturb the initial data in the region
 4 outside the numerical domain of dependence and change the
 5 true solution without affecting the computed one.) The CFL
 6 condition can also serve as a stability criterion thanks to the
 7 Lax-Richtmeyer equivalence theorem [LR56, Str04], which
 8 states that a consistent finite difference method for a well-
 9 posed initial value problem is convergent if and only if it is
 10 stable.

11 For example, for the classical wave equation $\partial_t^2 f =$
 12 $c^2 \nabla^2 f$, the domain of dependence of any point (\mathbf{x}, t) in-
 13 cludes only points (\mathbf{x}_0, t_0) with $\|\mathbf{x} - \mathbf{x}_0\| \leq c(t - t_0)$, be-
 14 cause information propagates only at the wave speed c . Con-
 15 sequently, the time step Δt must be small enough to prevent
 16 information from propagating outside the spatial stencil over
 17 a single time step. In particular, for the first-order explicit fi-
 18 nite difference scheme applied to the wave equation, where
 19 over each time step a grid cell is only affected by adjacent
 20 grid cells (separated by Δx), the CFL condition requires that
 21 $c\Delta t \leq \Delta x$.

In general, the CFL condition typically takes the form

$$C \equiv \frac{c\Delta t}{\Delta x} \leq C_{\max}, \quad (1)$$

22 where c is the speed of information propagation, and C_{\max}
 23 is a method-dependent constant which depends on the size
 24 of the finite-difference stencils. The dimensionless ratio C
 25 is known as the CFL number or the Courant number. Note
 26 that implicit methods are not restricted by the CFL condi-
 27 tion because the solution at any point at the end of the time
 28 step depends on the values at *all* the points at the beginning
 29 of the time step, so the numerical domain of dependence is
 30 effectively infinite.

31 The CFL condition can be a useful heuristic even in situ-
 32 ations where it does not directly apply. In computer graph-
 33 ics, semi-Lagrangian advection [Sta99] has been a popu-
 34 lar scheme for solving the advection equation, as it is un-
 35 conditionally stable and not restricted by the CFL condi-
 36 tion [Bri08]. Nevertheless, excessively large time steps can
 37 lead to undesirable artifacts such as numerical dissipation
 38 and volume loss. Foster and Fedkiw [FF01] advocate limit-
 39 ing the time step size using (1) with c being the maximum
 40 of the flow speed $\|\mathbf{u}\|_{\infty}$ over the domain, and $C_{\max} = 5$.

In smoothed particle hydrodynamics (SPH), each particle
 is affected by other particles within its smoothing radius h ,
 analogous to the grid separation Δx in finite-difference meth-
 ods. Consequently, time step selection criteria based on the
 CFL condition can be applied. For pressure waves in com-
 pressible SPH, we continue to have

$$\Delta t \leq C_{\max} \frac{h}{c}, \quad (2)$$

and values of C_{\max} between 0.25 and 0.4 have been used

[Mon92, DC99]. Fast relative motion of particles, which oc-
 curs especially during fluid-fluid or fluid-solid collisions,
 can also produce artifacts due to interactions not considered
 in (2). Therefore, one can introduce additional time step con-
 straints based on the instantaneous acceleration \mathbf{a} and diver-
 gence of velocity $\nabla \cdot \mathbf{v}$ [Mon92, DC99]:

$$\Delta t \leq \Lambda \sqrt{\frac{h}{\|\mathbf{a}\|}}, \quad (3)$$

$$\Delta t \leq \frac{\Gamma}{|\nabla \cdot \mathbf{v}|}. \quad (4)$$

41 Λ and Γ are dimensionless constants which Desbrun et al.
 42 [DC99] set to 0.5 and 0.005 respectively. Equation (3) can
 43 in fact be interpreted as a CFL-type condition comparing the
 44 size of the spatial neighborhood, h , to a particle's relative
 45 displacement $\frac{1}{2}\|\mathbf{a}\|\Delta t^2$ due to acceleration over time Δt . Fi-
 46 nally, we also point out that these criteria (2)–(4) may either
 47 be applied globally by taking the minimum of all particles'
 48 allowed time steps, or locally on a per-particle basis (as we
 49 discuss below).

50 When adapting the time step based on higher-order deriva-
 51 tives such as acceleration (3), care must be taken because
 52 such quantities can be much noisier than the system vari-
 53 ables themselves, causing large fluctuations in Δt . Ihmsen
 54 et al. [IAGT10] have observed that in incompressible SPH
 55 simulations, these time step fluctuations can lead to spurious
 56 density shocks that destroy the convergence of the simula-
 57 tion. A solution is to change Δt gradually rather than instan-
 58 taneously: if the system violates any of the desired time step
 59 criteria, Δt is decreased by a small amount ([IAGT10] use
 60 0.2%), otherwise, if it is well within all the criteria, Δt is
 61 increased by the same amount. This strategy causes the time
 62 step to change smoothly towards the ideal step length. On the
 63 other hand, it may also prevent the time step from changing
 64 quickly enough to resolve sudden shocks, such as those from
 65 high-velocity impacts. Therefore, Ihmsen et al. introduce an
 66 additional procedure that detects shocks if the density error
 67 increases suddenly; if so, the simulation is rewound two time
 68 steps and resumed with a sufficiently small Δt .

69 Shock detection can be considered an example of an *a*
 70 *posteriori* time step adaptation strategy, where undesirably
 71 large time steps are detected and rewound. This kind of ap-
 72 proach is useful whenever it is costly or difficult to estimate
 73 the right time step length in advance. Instead, we simply per-
 74 form a time step with the current estimate of Δt , and then
 75 test whether it adequately resolves the motion of the system.
 76 If not, we reject the step, decrease the time step by setting
 77 $\Delta t \leftarrow \Delta t / \alpha$ for some factor $\alpha > 1$, and try again. When sev-
 78 eral time steps in a row are successful, we increase the time
 79 step, $\Delta t \leftarrow \alpha \Delta t$.

80 Bridson et al. [BFA02] use this approach for efficient col-
 81 lision handling in cloth simulation, using a combination of
 82 inelastic repulsion forces and a robust collision resolution
 83 algorithm. Inelastic repulsions are much more inexpensive

than full collision resolution, but as they only check proximity at discrete points in time, they can easily fail to prevent interpenetrations if the cloth moves too far in a single time step. If this happens, the time step is rejected and Δt is reduced; collision resolution is only triggered if after multiple failures the time step falls to a specified minimum size. The collision resolution step incorporates rigid impact zones [Pro97], which can be seen as a freezing technique and is discussed in Section 2.2. Bargteil et al. [BWH07] simulate plastic flow using a finite element mesh, where excessive deformation of elements can be problematic. They reject a time step if any edge changes significantly in length, or a sudden acceleration takes place. Both methods discussed here use $\alpha = 2$, that is, time steps are halved or doubled as needed.

The main drawback of this *a posteriori* strategy is that the whole system must be globally rolled back to its previous safe state, even if it was caused by a localized event. In rigid bodies simulation, this challenge was addressed by Mirtich [Mir00] to efficiently handle collisions. Inspired by the work of Jefferson et al. [Jef85], he proposed a *time warp* algorithm to asynchronously handle collision events. Here, the integration of a rigid body is interrupted only when resolving an event that concerns it. This technique can be seen as a local time stepping technique, and inspired works on *asynchronous variational integrators* (AVIs) which we discuss later in this section.

Global time stepping The simplest way to perform adaptive time stepping is to choose the time step that is safe for the entire simulation domain, and perform integration for the entire system using that time step. That is to say, given a time step criterion (or criteria) such as (2)–(4) that can be evaluated locally, one evaluates the permissible time step Δt_i at all simulation points i , and steps the entire system forward by a time step of length $\Delta t = \min_i \Delta t_i$. For methods that use implicit integration or other globally coupled schemes, such as grid-based fluids with a global pressure solve, this is typically the only possible approach. For this reason as well as for its conceptual and practical simplicity, global time stepping is probably the most widely used form of temporal adaptivity in practice.

It is worth pointing out here that adaptive time stepping is not a free lunch for all time integration schemes. The Verlet, or leapfrog, scheme is second-order accurate, and has excellent energy conservation properties thanks to its symplectic nature, but both these features rely on the time step being fixed. To maintain second-order accuracy with a variable time step, Bridson et al. [BMF03] proposed a time integration scheme that combines a leapfrog scheme for position with an implicit trapezoidal rule for velocity:

$$\begin{array}{ll}
 1: \tilde{v}^{n+1/2} = v^n + \frac{\Delta t}{2} a(t^n, x^n, \tilde{v}^{n+1/2}) & \triangleright \text{implicit} \\
 2: x^{n+1} = x^n + \Delta t \tilde{v}^{n+1/2} & \triangleright \text{explicit} \\
 3: v^{n+1/2} = v^n + \frac{\Delta t}{2} a(t^n, x^n, v^{n+1/2}) & \triangleright \text{explicit} \\
 4: v^{n+1} = v^{n+1/2} + \frac{\Delta t}{2} a(t^{n+1}, x^{n+1}, v^{n+1}) & \triangleright \text{implicit}
 \end{array}$$

Maintaining symplecticity is much more challenging, as naively varying the time step can lead to instabilities and inconsistent energy behavior [HVS*09, Sec. 3]. Much more elaborate time stepping schemes are needed to recover energy preservation, such as the asynchronous variational integrators discussed below.

Local time stepping In complex simulation scenarios, different regions of the simulation domain may have very different time step requirements. For example, resolving challenging collision and contact scenarios requires careful time stepping, but only for the parts of the system that are affected by the contact. Similarly, in simulations with adaptive spatial resolution, the CFL condition requires finer-resolution regions to take smaller time steps. It can become intractable to simulate the whole model in lockstep using the most conservative time step. Instead, it is desirable to perform local time stepping, integrating each element of the simulation at its own pace.

Explicit integration schemes can readily incorporate local time stepping. We illustrate this with a generic two-dimensional system,

$$q_1'(t) = f_1(t, q_1(t), q_2(t)), \quad (5)$$

$$q_2'(t) = f_2(t, q_1(t), q_2(t)). \quad (6)$$

If at time t , q_2 requires a very small time step Δt_2 , one can still integrate q_1 with its own time step Δt_1 , giving

$$q_1(t + \Delta t_1) = q_1(t) + \Delta t_1 f_1(t, q_1(t), q_2(t)). \quad (7)$$

Meanwhile, as q_2 takes multiple steps to cover the same time interval, it will require values of q_1 at intermediate times $t + \Delta t_2$, $t + 2\Delta t_2$, and so on; these can be linearly interpolated from $q_1(t)$ and $q_1(t + \Delta t_1)$. Equivalently, to simplify bookkeeping, one can take the same small time steps Δt_2 for both q_1 and q_2 , but only evaluate q_1' at the first step and hold it fixed until a time Δt_1 has been covered. This approach has the same computational advantage because evaluation of f is typically the most expensive part in explicit methods.

Early work on SPH [DC96, DC99] recommended this approach for local time stepping, using the CFL condition as the time step criterion. The same technique was also used to simulate elastic bodies using spatially adaptive finite element meshes [DDCB01], discussed in more detail in Section 3.2. For a linear elastic material with density ρ and Lamé coefficients λ and μ , the upper bound on the time step for an element can be approximated by

$$\Delta t \leq h \sqrt{\frac{\rho}{\lambda + 2\mu}} \quad (8)$$

where h is a measure of the size of the element, such as its inradius: skinnier elements or stiffer materials require smaller time steps. To improve the parallelizability of local time stepping for SPH fluids, Goswami and Batty [GB14] divide the computational domain into blocks and choose time steps independently per block.

Asynchronous variational integrators (AVIs) studied by Lew et al. [LMOW04] are a family of time integration schemes that provide excellent energy conservation behavior while allowing different elements of the system to use different time steps. Unlike the local time stepping model described above, AVIs associate a time step with each *force* rather than each variable. Thus each force term applies a series of impulses to its associated nodes. The time step for a particular term must remain constant throughout the simulation, but different forces may have very different time steps. This approach is typically implemented as an event-driven simulation loop, using a priority queue to schedule the updates for all the forces in order. Thomaszewski et al. [TPS08] applied this approach to cloth simulation with a finite element triangle mesh, choosing time steps independently for each element using the CFL criterion (8). AVIs are known to exhibit “resonance instabilities” that can potentially cause the energy to increase without bound; however, these instabilities tend to be extremely weak in solid mechanics problems [FDL07] and have not been observed to cause difficulties in computer graphics [HVS*09].

The energy conservation properties of AVIs hinge on the regular spacing of the impulses applied by each force term, which makes collisions challenging to incorporate: naively applying contact forces at the moment of collision breaks the periodicity and destroys energy conservation, while applying contact forces at regular intervals risks missing collisions. Recent work [HVS*09, AVGT12] addresses this problem by replacing each contact force with a sum of stiffer and stiffer penalty layers with smaller and smaller time steps, which together are guaranteed to prevent interpenetration. While the number of penalty layers is conceptually infinite, any given collision can only activate a finite number of penalty layers, so the actual amount of computation is finite. Initial work by Harmon et al. [HVS*09] used kinetic data structures to detect all collision events in advance. Ainsley et al. [AVGT12] instead adopt a speculative approach based on the time warp algorithm [Jef85, Mir00], analogous to the *a posteriori* time step adaptation techniques discussed previously. An interval of time is first simulated without considering any new collisions; then, collision detection over the simulated interval is performed, and if new collisions are found, those force terms are activated and the interval is resimulated. Speculative simulation greatly reduces the computation time spent in bookkeeping and collision detection, and also allows for easy parallelization.

2.2. Adaptive integration

Adaptive choice of time integration scheme In some cases, such as when the system involves highly stiff modes or poorly conditioned elements, an adaptive choice of time step is no longer the most efficient strategy. The time step requirements for explicit integration may become extremely restrictive. Instead, one may locally change the integration

scheme to deal with the problematic components, for example by switching to an implicit method or a nonlinear one. By doing so adaptively, one can continue to use an inexpensive explicit integration scheme for the remainder of the system.

In the finite element method, ill-shaped elements impose severe restrictions on the allowed time step for explicit integration (8). When the object undergoes topological changes such as cutting or fracture, it can be difficult to avoid introducing such elements. As an alternative to local time stepping, where ill-shaped elements would have to be simulated with extremely small time steps, Fierz et al. [FSH11] propose an element-wise implicit-explicit (IMEX) scheme. Here the same time step Δt is used for the entire system, but ill-shaped elements that would be unstable if explicitly integrated over Δt are instead simulated with an unconditionally stable implicit scheme. Nodes that are not adjacent to any ill-shaped element are integrated explicitly, then held fixed as boundary conditions for implicit integration of the remaining nodes. As long as the number of ill-shaped elements is low, this relaxes the time step restriction faced by explicit integration, while minimizing the computational cost and numerical dissipation associated with implicit integration.

Thin materials such as hair exhibit a high stiffness in their stretching modes, but pose the additional challenge that their collision response is highly nonlinear due to the presence of rotation. Therefore, depending on the amount of bending, an implicit first-order model for collisions may fail to capture the correct response and lead to instabilities. Kaufman et al. [KTS*14] propose an algorithm that adaptively chooses the *degree* of nonlinearity in each contact resolution step to safely resolve the collision. Specifically, they adapt the number of constrained Newton iterations used to solve the nonlinear contact model, terminating when the stretch over all affected edges is sufficiently reduced. This allows for large simulation time steps in the face of many energetic collisions, while efficiency is maintained because most collisions require only a single iteration (equivalent to a linearly implicit step).

Freezing techniques Freezing techniques, also called sleeping techniques, lie in between temporal and spatial adaptability: the degrees of freedom that are considered unimportant in the simulation are kept constant in time for a specified duration. This can be seen as animating them at a much larger time scale, or as temporarily deactivating them. No memory is saved while doing so, but computation time is reduced. These techniques are useful in situations where the spatial domain is large and filled with many quiescent objects. Therefore, game environments and surgical simulations are perfect candidates for these methods. Conversely, freezing techniques may not be useful in highly dynamic situations where most of the degrees of freedom are active most of the time. In addition to defining good freezing criteria, the main challenge of freezing techniques is to design a reactive

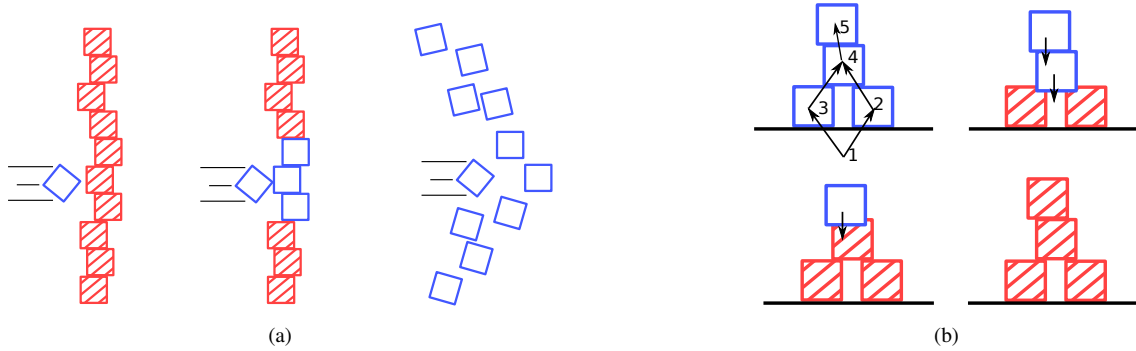


Figure 2: Freezing techniques applied to rigid bodies stacking. At the stage of contact solving, those methods can save computational time while ensuring a plausible motion. In (a), quiescent rigid bodies are frozen (in hatched red) in a stacking configuration. A collision event then awakes one of the rigid bodies, and the active rigid body (in blue) propagates the information to its neighbors. In (b), a contact graph stores stack ordering. During shock propagation, a bottom-up traversal is performed and objects at each level are frozen by assigning an infinite mass.

1 vation process of the frozen degrees of freedom that ensures
2 plausible subsequent motion.

3 In rigid objects simulation (see the survey of Bender et
4 al. [BETC12]), important computational resources are dedi-
5 cated to solving contacts between the different objects of a
6 scene. This process becomes unnecessarily expensive when
7 stacking occurs and nothing moves. In these cases, freezing
8 techniques prove to be useful for saving computational time
9 without compromising the plausibility of the simulation.

10 Schmidl [Sch02] uses a heuristic based on kinetic energy
11 to determine whether to freeze a body or not:

$$\frac{1}{2}m\mathbf{v}^2 + \frac{1}{2}\boldsymbol{\omega}^T \mathbf{I}\boldsymbol{\omega} < \frac{\mathbf{p}_g^2}{2m} \quad (9)$$

12 where \mathbf{v} and $\boldsymbol{\omega}$ respectively denote the linear and angular ve-
13 locity of the rigid body of mass m and inertia tensor \mathbf{I} , and
14 $\mathbf{p}_g = m\mathbf{g}\Delta t$ is the momentum that the body accumulates dur-
15 ing a time step Δt from gravity. If condition (9) is fulfilled by
16 the body during a user-defined number of consecutive time
17 steps, then it is frozen. Note that a single frozen body by
18 itself does not save computation time. Time is saved when
19 multiple neighboring objects become frozen, as their con-
20 tact forces no longer need to be computed. The only way
21 for a frozen rigid body to be reactivated is when it receives
22 a large impulse during a collision. Once it is reactivated, the
23 information is propagated to all its direct neighbors and a
24 given number of indirect neighbors, potentially awakening
25 them (see Figure 2a).

26 Freezing techniques have been used as a failsafe pro-
27 cedure for resolving collisions and contact between rigid
28 or deformable bodies. When many interacting contacts are
29 present, iterative processes for collision response can require
30 an excessively large number of iterations to converge, and
31 early termination leaves the system in an interpenetrating
32 state. Freezing procedures are attractive in this context as
33 they can guarantee elimination of interpenetrations. How-

34 ever, they also introduce loss of kinetic energy by treating
35 contacts as fully inelastic, and are therefore only invoked as
36 a last resort after multiple iterations of a physically correct
37 solver have failed to resolve the collisions. For rigid body
38 contact, Guendelman et al. [GBF03] propose a shock propa-
39 gation strategy which freezes bodies progressively from the
40 ground up. They start by building a directed acyclic graph
41 consisting of “levels” of objects that are resting on objects
42 of lower levels (cyclic dependencies are grouped into the
43 same level). A single bottom-up traversal of the graph re-
44 solves contacts level by level, assigning infinite mass to
45 lower-level objects for which contact is solved (see Figure
46 2b). Rigid impact zones, described by Provot [Pro97] and
47 Bridson et al. [BFA02], are extensively used to resolve col-
48 lisions in cloth. Nodes involved in multiple interfering col-
49 lisions are collected into disjoint sets called “impact zones”,
50 constructed by merging zones if their nodes are involved
51 in the same node-face or edge-edge collision. Each impact
52 zone is rigidified by replacing the motion of its nodes with a
53 rigid body motion while preserving the total linear and angu-
54 lar momentum. This process eliminates collisions within the
55 zone, but may introduce new collisions with nearby elements
56 outside the zone. Therefore, one must perform collision de-
57 tection again and grow the impact zones if new collisions are
58 detected, iterating this process until no new collisions occur.

59 Freezing techniques were also used for articulated
60 rigid bodies, both for dynamics [RGL05] and quasi-
61 statics [RL06]: joints are activated or deactivated based on
62 user-defined error metrics, leading to a simplification of the
63 whole model and a sub-linear computational complexity.

64 Denoting by $\ddot{\mathbf{q}}^C = (\ddot{\mathbf{q}}_1, \dots, \ddot{\mathbf{q}}_{N_C})^T$ the composite accel-
65 eration of an articulated body C , where N_C is the number of
66 joints in C , the *acceleration metric value* of C is defined as

$$A(C) = \sum_{i \in C} \ddot{\mathbf{q}}_i^T \mathbf{A}_i \ddot{\mathbf{q}}_i,$$

67 where \mathbf{A}_i , $i \in C$, are symmetric, positive definite $d_{J_i} \times d_{J_i}$

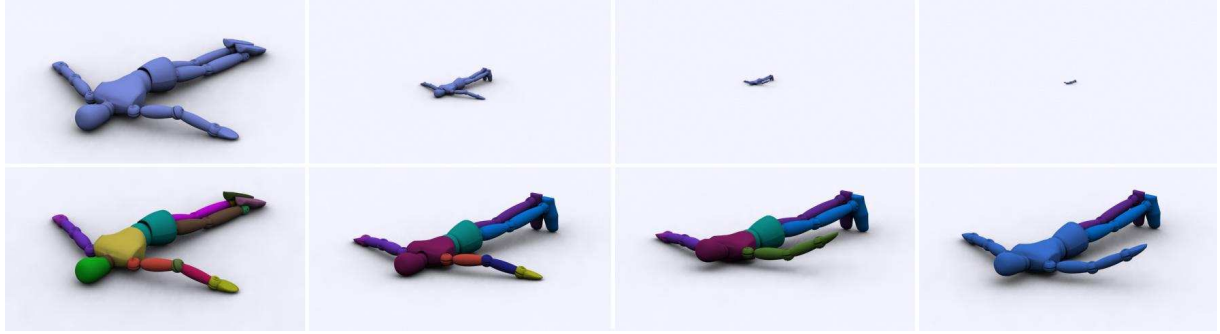


Figure 3: View-dependent dynamics of articulated bodies. Top: the algorithm proposed by Kim et al. [KRK08a] automatically simplifies the dynamics of a falling character as its distance to the viewer increases. Bottom: corresponding rigidification at this time step (one color per rigid group).

1 *weight matrices*, and d_i is the number of degrees of freedom of joint i in C . The weight matrices A_i are required to depend at most on joint positions, and the simplest choice is the identity matrix. The key to the sub-linear complexity is the demonstration that the acceleration metric of each sub-articulated body is a quadratic function of the forces applied to its handles (i.e. the free rigid bodies used to assemble sub-articulated bodies in Featherstone’s methodology [RGL05]). As a result, the acceleration metric may be computed *before computing the joint accelerations themselves*. This is used during the top-down pass of Featherstone’s divide-and-conquer algorithm to determine which joint accelerations should be computed because they are sufficiently large [RGL05]. For example, if the acceleration metric of the complete articulated body is small, then all joint accelerations are small and joint velocities are constant for the current time step. A similar metric is built for joint velocities to determine which joint positions should remain constant, and thus which inter-body forces and inertial terms should be updated. Once all position-dependent terms are up to date, the motion metrics are available for the next time step, and a new set of active joints can be determined.

23 It was shown that this approach could also help to speed up continuous collision detection for articulated bodies [KRK08a] and haptics rendering [MR07], as well as enable view-dependent articulated-body dynamics by combining motion metrics with visibility estimates [KRK08b] (see Figure 3). Gayle et al. [GLM06] demonstrate how to perform contact handling with such an adaptive articulated-body method, which is used as the core of a physics-based sampling algorithm for highly articulated chains [GRS*07] and cable route planning [KGL07].

33 Recent work has sought to apply freezing techniques to accelerate SPH fluid simulation. In these methods, particles are divided into two sets: a set of *active* particles following a classical simulation step, and a set of *inactive* particles that are skipped in the simulation. As physical quantities in SPH are interpolated from neighbors, inactive neighbors of active particles also need to be updated to ensure that active parti-

cles compute correct physical quantities. Computation time is saved for inactive particles that only have inactive neighbors. The main challenge in these methods is the definition of the process to transform a particle from the inactive set to the active one and vice versa. Indeed, as SPH is sensitive to particle distribution, an inadequate transition of state can directly lead to instabilities. Additionally, a judicious choice of criterion to decide whether a particle should be active or not is essential.

Goswami and Pajarola [GP11] propose a simple method that evaluates the active status of particles at each time step. Particles are marked as active if they are close to the boundary or if their velocity exceeds a threshold. Inactive neighbors of such particles are also added to the active set, and continue with their last active velocity. All other particles are considered inactive. Unfortunately, the resulting method does not obey Newton’s third law, resulting in some loss of momentum.

In the context of molecular simulation, Artemova and Redon [AR12] propose a fundamentally different approach which ensures momentum conservation. This approach, called *Adaptively Restrained Particle Simulation (ARPS)*, builds a modified equation of motion which adapts itself depending on the state of the particles. It continuously interpolates the behavior of a particle switching from one state to another. Here is the modified equation of motion for a particle with \mathbf{p} as momentum, \mathbf{q} as position, $V(\mathbf{q})$ as potential energy, m as mass and ρ a *restraining function*:

$$\begin{aligned} \dot{\mathbf{p}} &= -\frac{\partial V(\mathbf{q})}{\partial \mathbf{q}} \\ \dot{\mathbf{q}} &= m^{-1} [1 - \rho(\mathbf{p})] \mathbf{p} - \frac{1}{2} \mathbf{p}^T m^{-1} \frac{\partial \rho(\mathbf{p})}{\partial \mathbf{p}} \mathbf{p} \end{aligned} \quad (10)$$

68 The *restraining function* $\rho(\mathbf{p}) \in [0, 1]$ is a twice differentiable function that can be customized to incorporate different freezing criteria such as kinetic energy. We can observe that if $\rho = 0$ the dynamics of the particle will follow the clas-

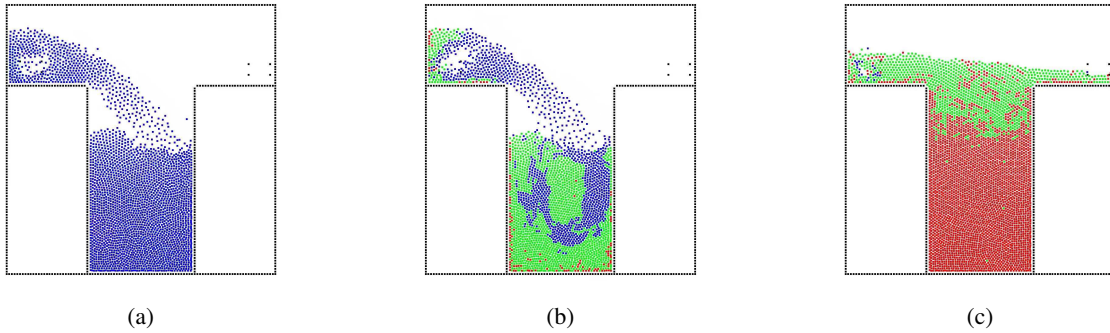


Figure 4: (a) shows a classic SPH simulation of a 2D scenario with inflow and outflow boundary conditions at the walls above a chasm. (b) shows at the same time the effect of the ARPS method from Manteaux et al. [MFRC13]. Blue particles are active, red particles are inactive, and green particles are transitive. Large regions quickly become inactive (c), allowing to save significant computational time as numerous particles' neighborhoods do not need to be updated anymore.

1 sical equation of motion:

$$\begin{aligned} \dot{\mathbf{p}} &= -\frac{\partial V(\mathbf{q})}{\partial \mathbf{q}} \\ \dot{\mathbf{q}} &= m^{-1}\mathbf{p} \end{aligned} \quad (11)$$

2 Whereas if $\rho = 1$ then the particle will still accumulate mo- 3
mentum but will remain still:

$$\begin{aligned} \dot{\mathbf{p}} &= -\frac{\partial V(\mathbf{q})}{\partial \mathbf{q}} \\ \dot{\mathbf{q}} &= 0 \end{aligned} \quad (12)$$

4 In this case, even if particles' momentum are still correctly 5
computed, forces do not need to be recomputed as positions 6
are held fixed. Therefore substantial computational time can 7
be saved. Moreover, by using an incremental algorithm to 8
update interaction forces, inactive particles with active neigh- 9
bors can be efficiently updated while staying aware of their 10
neighborhood. Finally, if $0 < \rho < 1$ then the particle fol- 11
lows the interpolated dynamics of equation (10). The partic- 12
le is then called *transitive*. Manteaux et al. [MFRC13] ex- 13
tend their work to SPH (see Figure 4) and to deformable 14
model simulation, in order to better match computer graph- 15
ics needs.

16 3. Geometric adaptivity

17 Geometric adaptivity describes various techniques that adapt 18
the spatial resolution of a model by refining and coarsening 19
its discretization. These techniques are also referred in the 20
literature as *adaptive spatial refinement*. However, as they 21
include coarsening as well, we adopted a more general term.

22 Geometric adaptive techniques have two major compo- 23
nents: a refinement criterion that determines where higher 24
resolution is needed, and a refinement/coarsening scheme 25
that modifies the discretization to match the desired resolu- 26
tion. Both physical and visual criteria have been employed 27
in existing work, and we discuss them in more detail be- 28
low. The refinement scheme itself essentially depends on the

29 type of spatial discretization. In the following subsections, 30
we will deal with the three major kinds of discretization sep- 31
arately: structured meshes and grids, unstructured meshes, 32
and meshless models.

33 **Refinement criteria** The choice of refinement criteria plays 34
a major role in the quality of the resulting simulation. Many 35
techniques use simple heuristics such as the distance to 36
boundaries, surface curvature, and the presence of contacts. 37
However, some authors have shown that employing criteria 38
that are more closely tied to the dynamics of the system can 39
be important in many contexts.

40 In elastic and plastic solids, the stress and strain in an el- 41
ement characterize the amount of local deformation. There- 42
fore, the values and gradients of these quantities are often 43
used to control refinement. Wu et al. [WDGT01] describe 44
several different error estimators of this type and discuss 45
their relative advantages, drawbacks, and performance. Go- 46
ing beyond estimating a scalar error, Wicke et al. [WRK*10] 47
define a metric tensor \mathbf{M} that approximates the spatial vari- 48
ation of strain by comparing deformation gradients of adja- 49
cent elements. The matrix \mathbf{M} is defined in such a way that 50
it has large eigenvalues in directions in which the deforma- 51
tion changes most rapidly, and can thus be used to control 52
anisotropic refinement (see Figure 5).

53 If a multigrid algorithm is used, the difference between 54
the solution at the current level, \mathbf{x}^j , and the one prolonged 55
from the next coarser level, $\mathbf{P}^j \mathbf{x}^{j+1}$, provides a natural mea- 56
sure of the quality of \mathbf{x}^{j+1} . Otaduy et al. [OGRG07] weigh 57
the error by the local system matrix \mathbf{A} to reduce possible 58
popping in stiff scenarios, leading to the error metric

$$e = \|\mathbf{A}^j(\mathbf{x}^j - \mathbf{P}^j \mathbf{x}^{j+1})\|, \quad (13)$$

59 and perform refinement if e exceeds a predefined threshold.

60 Lower-dimensional bodies, like wires, strands, cloth, and 61
thin sheets, undergo not just stretching but also bending (and 62
torsion, in the case of one-dimensional strands). While the

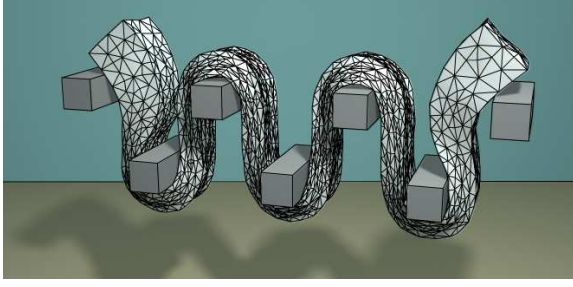


Figure 5: Elastoplastic simulation with dynamic local remeshing [WRK*10]. By using the strain gradient as the refinement criterion, regions undergoing severe deformation are refined locally.

1 stretching forces within the material are much stiffer than
 2 the bending forces, the bending deformation can often be
 3 more visually important, and also introduces geometric non-
 4 linearities that must be carefully resolved. In this context, ge-
 5 ometrical curvature and the presence of contacts (which are
 6 likely to induce bending) are the most commonly used re-
 7 finement criteria. However, the interaction between stretch-
 8 ing and bending leads to additional considerations.

9 First, in the context of cloth simulation, Simnett et al.
 10 [SLD09] and Narain et al. [NSO12] point out that elements
 11 under compression are likely to buckle and should there-
 12 fore also be refined, otherwise the wrinkles that would arise
 13 in subsequent time steps will fail to be represented on the
 14 coarse mesh. By considering the trade-off between bending
 15 and stretching energy, Narain et al. estimate that a sheet un-
 16 der compressive strain ϵ is likely to form wrinkles of width
 17 proportional to $\sqrt{k_b/(k_s\epsilon)}$ where k_b and k_s are the bending
 18 and stretching stiffnesses respectively, providing an estimate
 19 of the extent of refinement necessary. In the physics litera-
 20 ture, Cerda and Mahadevan [CM03] have provided a detailed
 21 semi-analytical analysis of wrinkle geometry that is valid far
 22 from the small-deformation limit, and may be useful for fu-
 23 ture work in graphics.

24 Second, finer meshes allow higher-frequency modes of
 25 transverse oscillation, leading to time step constraints and
 26 stability problems that can be fatal for interactive applica-
 27 tions. To ensure stability, Servin et al. [SLN08] propose a
 28 *coarsening* criterion, reducing the resolution of the mesh so
 29 that only those oscillations whose frequency is lower than
 30 the time stepping rate can be represented. For simulation
 31 of systems with stiff wires, they estimate the maximum fre-
 32 quency of oscillations in a wire discretized with n nodes as
 33

$$\omega_{\max} \approx 2(n+1)\sqrt{\frac{f}{mL}}, \quad (14)$$

34 where f is the tension in the wire and m and L are its mass
 35 and length. Requiring this frequency to be lower than $1/\Delta t$
 36 gives an upper bound on n for each wire.

37 In fracture simulation, the stress forms a singularity at
 38 the crack tip, which must be resolved accurately with a fine
 39 resolution mesh for realistic crack paths to be obtained. If
 40 the crack origin is known *a priori*, one can track the crack
 41 path as the fracture proceeds and refine the mesh based on
 42 the distance from the crack [BDW13]. However, if fracture
 43 is allowed to originate anywhere, it is necessary to refine
 44 the mesh wherever stress is sufficiently high, i.e. close to
 45 the material’s strength τ , because such regions may generate
 46 new cracks. Koschier et al. [KLB14] refine elements whose
 47 tensile stress σ exceeds a specified fraction of τ . Pfaff et
 48 al. [PNdJO14] choose the desired resolution of the mesh to
 49 be proportional to tensile stress by requiring the length of
 50 each edge \mathbf{e} to satisfy

$$\|\mathbf{e}\| \leq \max\left(\frac{\tau}{2\sigma}, 1\right) \ell_{\min} \quad (15)$$

51 where ℓ_{\min} is a user-specified refinement limit. This ap-
 52 proach allows the mesh to be coarsened again after the crack
 53 tip has passed.

54 It is also possible to perform view-dependent adaptivity
 55 by modulating the refinement criteria based on visibility and
 56 distance from the camera. Such techniques have been ap-
 57 plied to the simulation of wires [SLN08] and cloth [KNO14].
 58 New issues arise in such contexts, such as preventing ar-
 59 tifacts when coarser regions come into the field of view
 60 and must be refined: Koh et al. [KNO14] achieve this by
 61 smoothly increasing the resolution in advance based on the
 62 known camera path.

63 Many adaptive methods for simulation of liquids have re-
 64 lied on the distance to the free surface as the sole refinement
 65 criterion. However, much greater adaptivity is possible by
 66 observing that in regions where the surface is flat and does
 67 not exhibit detailed motion, it can also be coarsened without
 68 significantly affecting the flow. Adams et al. [APKG07] pro-
 69 pose a purely geometrical criterion based on an “extended
 70 local feature size” which measures the distance to the sur-
 71 face and to the medial axis of the fluid volume. This criterion
 72 assigns coarse resolution both far from the surface and near
 73 thick, flat surfaces. However, it does not take the motion of
 74 the fluid into account. To detect regions with significant flow
 75 detail, Hong et al. [HHK08] use a “deformation factor” that
 76 locally estimates the Reynolds number,

$$Df = \frac{(\mathbf{u} \cdot \nabla)\mathbf{u}}{v\nabla^2\mathbf{u}}, \quad (16)$$

77 where \mathbf{u} is the fluid’s velocity field and v is its viscosity.
 78 Ando et al. [ATW13] use a flexible sizing function that com-
 79 bine multiple criteria to set the desired resolution. Among
 80 them the depth of the liquid, the camera viewpoint, the fluid
 81 surface curvature and the norm of the strain rate, $e = \|\nabla\mathbf{u}\|_F$,
 82 in order to preserve detailed motions.

3.1. Structured meshes and grids

Spatially adaptive simulation techniques can often benefit from symmetry or structure, at the expense of flexibility. Techniques using hexahedral elements or finite differences can use quadtrees or octrees to add spatial adaptivity. Alternatively, techniques that use a volumetric tetrahedral mesh, like some finite element methods or mass-spring systems, can easily take advantage of structured meshes based on lattices. Special techniques can also combine grids with “tall cells” or far-field grid structures, as discussed at the end of this section.

The techniques described in this section are useful when one wishes to speed up simulations by using local spatial adaptivity without completely committing to an unstructured mesh technique. Structured meshes often allow more code to be re-used when converting from a regular grid to an adaptive one, and they are often more cache-coherent than fully unstructured meshes. However, structured meshes do not have as much flexibility as unstructured ones.

Quadtrees and Octrees The spatially adaptive schemes of Hutchinson et al. [HPH96] and Villard and Borouchaki [VB05] use quadtrees to directly connect masses and springs, introducing T-junctions in the process. Ganovelli et al. [GCS99] propose an octree-based multi-resolution method for determining connectivity in a mass-spring system. De-bunne et al. [DDBC99] simulate elastic models using control nodes based on approximate finite differences operators, and they use an octree-based refinement to increase detail based on a Laplacian deformation metric. As discussed in several of the aforementioned works, the main difficulty with using mass-spring models instead of approaches based on continuum mechanics is the notion of *convergence*. Convergence is well-studied in the finite element method, and it is trivial to show that increasing spatial resolution will lead to a more accurate solution. Mass-spring models can also converge under refinement if spring stiffness parameters are chosen appropriately, but these stiffness values are not as straightforward to derive compared to the stiffness matrix in a finite element method.

The works of Dick et al. [DGW11] and Seiler et al. [SSSH11] use an octree to define a set of hexahedral finite elements in an elasticity simulation, which is specifically used for simulating cuts in a deformable body. Dick et al. also leverage the hierarchy provided by the octree in a geometric multigrid method for solving the elastodynamics. The work was subsequently extended to interactively animate high resolution boundary surfaces [WDW11], to improve collision handling [WDW13], and to simulate at haptic rates [WWD14b]. For more details on methods for cutting deformable bodies, please see the state of the art report by Wu et al. [WWD14a].

Researchers have also developed octree-based discretizations of the Navier-Stokes equations, which lead to efficient

animations of smoke and liquid [SY04, LGF04, LFO05]. These approaches also inspired spatially adaptive works that handle discontinuities across free surfaces [HK05], resolve extremely thin surfaces in bubbles and foam without losing volume [KLL*07], and animate multi-phase fluids using regional level-sets [Kim10]. More recent work [FWD14] combines an adaptive hexahedral finite element method based on octrees with a multigrid Poisson solver to animate highly detailed liquids. Bargteil et al. [BGOS06] use octree-based spatial adaptivity to track detailed deforming liquid surfaces using semi-Lagrangian contouring.

It is worth noting that quadtree and octree grid refinement can have subtle side-effects when discretizing partial differential equations. In particular, the regular staggered-grid discretization of the Poisson equation (which is used for enforcing incompressibility in fluid flows [Bri08]) happens to satisfy Stokes’ theorem and exactly integrates fluid fluxes — it doubles as a “finite volume method” and can be alternatively derived using discrete exterior calculus [CDGDS13]. When one replaces the regular grid with an octree, however, it is unsafe to assume that such useful properties will still hold in the presence of T-junctions. The previously-mentioned octree-based fluid simulation methods counteracted this particular problem by carefully designing a new divergence operator, and some researchers observed the emergence of spurious rotational flows when refining an octree near liquid surfaces.

These subtle problems help explain the large number of adaptive BCC-mesh-based liquid solvers discussed below, which do not exhibit T-junctions.

Adaptive BCC lattices A standard way to efficiently generate a tetrahedral mesh with spatial adaptivity is to combine a spatial hierarchy with predefined lattice-based stencils. One particularly popular strategy combines an octree with the body-centered cubic (BCC) lattice tetrahedralization. To give some context, a *regular* BCC lattice is defined by first inserting vertices at the corner of a regular cubic grid, inserting additional vertices at the center of every grid cell, and then creating a Delaunay tetrahedralization of these vertices. A *spatially adaptive* tetrahedral mesh is created similarly by first creating a weakly-balanced octree (instead of a regular grid), inserting vertices at the corners and centers of the octree cells, and then tetrahedralizing the set of vertices. Please see the work of Molino et al. [MBTF03] and Labelle and Shewchuk [LS07] for some examples of how to create such an octree-based adaptive BCC mesh. This particular meshing strategy has several benefits: the average tetrahedral element has a nearly optimal shape, the quality of the worst element is bounded and completely acceptable in practice, and the computation time required to build the mesh is orders of magnitude faster than unstructured meshes, because it makes use of trees and precomputed stencils.

Employing these ideas, Wojtan and Turk [WT08] use an adaptive non-conforming BCC tetrahedralization to simulate

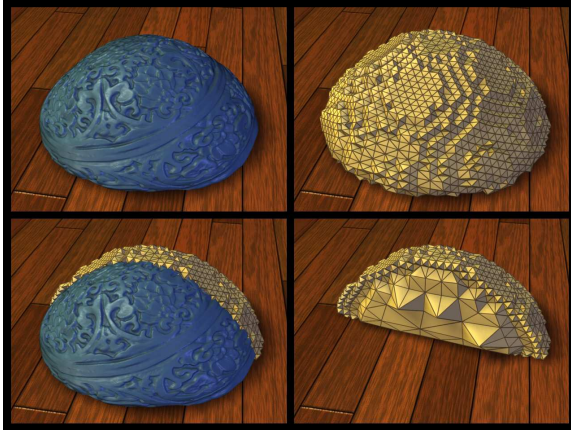


Figure 6: A high resolution surface mesh (blue) embedded into a deforming low resolution spatially-adaptive tetrahedral mesh based on a BCC lattice (gold). The bottom row shows the cross section of the tetrahedral mesh, illustrating the BCC structure. Image from Wojtan and Turk [WT08].

1 highly plastic materials while efficiently remeshing when-
 2 ever element quality degrades (See Figure 6). Batty et al.
 3 [BXH10] use a similar meshing strategy to simulate inviscid
 4 fluids using a finite-volume discretization. The method
 5 also handles embedded solid and free-surface boundary
 6 conditions, even though the tetrahedral mesh does not conform
 7 to the domain boundaries. Batty and Houston [BH11] extend
 8 this work by adding an implicit viscosity model. As
 9 explained later in the Section 3.3, Ando et al. [ATW13] also
 10 use an adaptive BCC mesh for simulating liquids. Sifakis et
 11 al. [SSIF07] introduce the concept of “hard bindings” to create
 12 an adaptive BCC lattice with T-junctions for the purposes
 13 of elastic solid simulation.

14 In the authors’ experience, an adaptively-refined BCC lat-
 15 tice is exceptionally useful in simulations requiring tetrahe-
 16 dral meshes, because the structured lattice makes the typi-
 17 cally expensive operations of remeshing and resampling simu-
 18 lation data relatively insignificant. Although the structure
 19 of the mesh removes some control over the shapes and na-
 20 ture of the spatial adaptivity, the structure also eliminates
 21 typical issues, such as degenerate tetrahedra.

22 **Tall cell grids** Simulations of deep water also benefit from
 23 adaptive “tall cell” techniques [IGLF06, CM11], which use
 24 regular grid cells near the water surface (where detail and
 25 accuracy is important), and tall rectangular fluid cells far-
 26 ther below the surface. This strategy effectively assumes
 27 that the behavior in regions located deep underwater is sim-
 28 pler, in that the simulation variables cannot change arbitrar-
 29 ily with depth. While we specifically discuss tall cell grids
 30 here because of their use of geometric adaptivity, we will
 31 also address some related height-field methods in Section
 32 4.3. These methods seem to work very well when the as-
 33 sumptions of the methods hold. However, there is reason to

34 believe that artifacts (like spurious reflecting waves or dissi-
 35 pating vortices) can occur when the tall cells fail to properly
 36 resolve important dynamics.

37 **Far-field grid structures** Zhu et al. [ZLC*13] propose
 38 a method for fluid simulation that maintains an efficient
 39 Cartesian grid structure but allows non-uniform spacing be-
 40 tween the nodes. This approach makes it easy to concentrate
 41 smaller cells where the domain is more interesting and retain
 42 larger cells farther away from areas of interest. In addition to
 43 concentrating detail in important regions, this method also
 44 approximates non-reflecting boundary conditions by greatly
 45 extending the boundaries of the simulation domain.

46 3.2. Unstructured meshes

47 Adaptivity on unstructured meshes is closely related to the
 48 problem of remeshing. Considered purely as a geometrical
 49 problem, remeshing has been studied extensively in com-
 50 putational geometry [CDS12] and in a modeling context
 51 in computer graphics [AUGA08]. Indeed, the techniques
 52 adopted for adaptive simulation often build on the frame-
 53 work of geometrical remeshing methods, and extend them
 54 to a simulation context. A number of challenges arise when
 55 performing remeshing during simulation, though. First, the
 56 mesh elements must remain well-conditioned to avoid de-
 57 grading the stability of the simulation. Second, modifying
 58 the discretization on the fly risks introducing errors in trans-
 59 ferring energy and momentum to the new mesh, such as
 60 numerical diffusion due to resampling, and discontinuous
 61 “popping” artifacts in thin strands or sheets. Third, remov-
 62 ing degrees of freedom must be done with care, as a mesh
 63 with fewer DOFs cannot represent the previous system state
 64 exactly. Depending on the characteristics of the simulated
 65 material—whether it is elastic, plastic, or fluid; whether volu-
 66 metric or lower-dimensional—different remeshing methods
 67 are found to be appropriate.

68 **Hierarchical schemes** The simplest remeshing strategy is
 69 to use a fixed hierarchical scheme for refinement, which
 70 provides guaranteed bounds on element quality. Two such
 71 schemes are illustrated in Figure 7a, 7b. In cloth simu-
 72 lation, triangle subdivision schemes such as 1-to-4 splits,
 73 $\sqrt{3}$ refinement, and edge bisection have been employed
 74 [LV05, SLD09, BD12]. A similar scheme for subdivision of
 75 tetrahedra was used by Koschier et al. [KLB14] for volu-
 76 metric fracture. Wu et al. [WDGT01] build on the concept
 77 of progressive meshes [Hop96] to precompute FEM param-
 78 eters, allowing adaptive simulation of deformable bodies at
 79 interactive rates.

80 However, subdivision schemes still degrade element qual-
 81 ity by a moderate extent compared to an optimized mesh.
 82 When this is undesirable, an alternative is to use a hierarchy
 83 of “non-nested meshes” at different resolutions [DCDB00,
 84 DDCB01]; see Figure 7c. Each level of the hierarchy is a

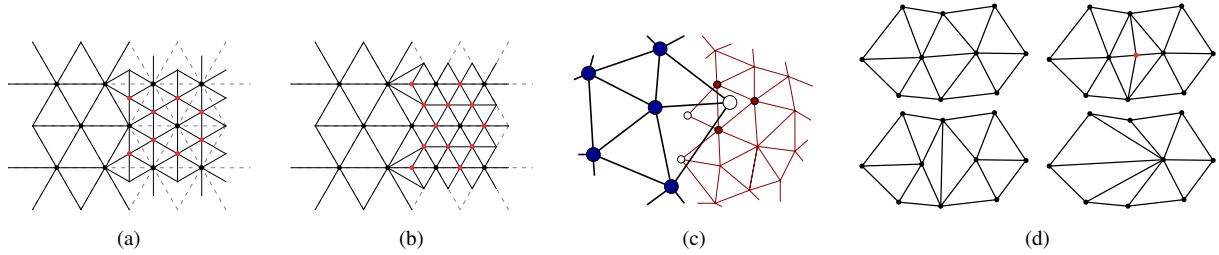


Figure 7: Overview of refinement schemes for unstructured meshes. In (a) and (b), the right half of a mesh is refined using (a) $\sqrt{3}$ refinement and (b) hierarchical edge bisection, with inserted nodes highlighted in red. In (c), we illustrate two levels of non-nested meshes. In (d), the three primitive operations of local remeshing are applied to the middle edge: in reading order, we show the original mesh, after an edge split, after an edge flip, and after one of two possible edge collapses.

1 complete mesh that does not necessarily share any nodes
 2 with meshes at other levels, and can be independently opti-
 3 mized *a priori*. At run time, regions at different levels of detail
 4 use subsets of different meshes. Coupling is achieved by
 5 allowing the regions to overlap slightly; nodes in the overlap
 6 region in each mesh are treated as inactive “ghost” nodes that
 7 are embedded in the containing element of the other mesh.
 8 The work of Otaduy et al. [OGRG07] seamlessly integrates
 9 such adaptive non-nested meshes with a multigrid algorithm
 10 and an adaptivity-aware collision detection technique.

11 Hair simulation can benefit from adaptivity, as the contact
 12 interactions between hair strands lead to the formation
 13 of emergent clusters. Bertails et al. [BKCN03] introduce a
 14 hierarchical structure called an adaptive wisp tree (AWT),
 15 which represents hair clusters that can progressively split
 16 into smaller clusters from the base to the tip. Refinement
 17 is performed by splitting a node if its size and acceleration
 18 are large, while coarsening is performed by merging sibling
 19 nodes if they have similar positions and velocities.

20 **Nearly regular meshes** Some techniques for liquid simu-
 21 lation use a structured mesh in the interior of the volume,
 22 but allow irregularities at boundaries to better capture the
 23 dynamics of the free surface. These techniques benefit from
 24 many of the advantages of structured meshes discussed in
 25 Section 3.1, while still retaining much of the flexibility of
 26 unstructured meshes, such as the ability to accurately match
 27 boundary conditions.

28 In such methods, one typically maintains a high-
 29 resolution representation of the surface as a triangle mesh
 30 that is updated at each time step. The surface is superim-
 31 posed on a regular mesh structure such as an octree or a
 32 uniform grid, and elements near the surface are modified,
 33 or new elements inserted, to better conform to the surface.
 34 In particular, Chentanez et al. [CFL*07] use the isosurface
 35 stuffing algorithm [LS07] that generates an adaptive BCC
 36 lattice whose surface tetrahedra are warped and possibly sub-
 37 divided to conform to the surface geometry. Using an octree
 38 to construct the lattice allows for coarser resolution away
 39 from the free surface. Brochu et al. [BBB10] use a uniform
 40 background lattice and introduce additional pressure sam-

41 ples along both sides of the free surface to ensure that all
 42 surface features are resolved. The pressure projection is then
 43 performed on a mesh consisting of the Voronoi cells of these
 44 sample points.

45 **Local remeshing** In some contexts, it is desirable to al-
 46 low the connectivity structure of the mesh to be modified
 47 freely during the course of the simulation. Such a require-
 48 ment arises when anisotropic elements are needed to resolve
 49 strongly directional features, or when the material exhibits
 50 both elastic properties and unbounded deformation, such as
 51 in plastic flow.

52 While it is possible to perform global remeshing—that is,
 53 simply creating a new simulation mesh from scratch when-
 54 ever needed [KFCO06, BWHT07]—this approach can lead
 55 to undesirable diffusion of stored physical quantities such as
 56 plasticity information. An increasingly popular alternative is
 57 to remesh locally using a set of local operations that refine,
 58 coarsen, and reshape existing elements. In local remeshing
 59 techniques, any elements in the mesh that do not satisfy the
 60 desired size and shape criteria are improved by careful appli-
 61 cation of these operations. This process is repeated until all
 62 mesh elements are satisfactory.

63 When performing remeshing, it is important to ensure that
 64 the mesh remains well-conditioned for simulation, through
 65 the use of various element quality measures [She02]. In 2D,
 66 the Delaunay triangulation optimizes several important no-
 67 tions of mesh quality, including the minimum angle and
 68 the maximum circumradius of any triangle, but the Delau-
 69 nay tetrahedralization in 3D provides few such guarantees,
 70 and more sophisticated mesh improvement strategies may
 71 be required [WRK*10]. If anisotropic remeshing is desired,
 72 the remeshing criterion is often expressed through a spa-
 73 tially varying metric tensor \mathbf{M} , with the goal being that
 74 each edge \mathbf{e} should have $\mathbf{e}^T \mathbf{M} \mathbf{e} \approx 1$. Equivalently, we desire
 75 $\|\mathbf{M}^{1/2} \mathbf{e}\| \approx 1$; that is, we want edges to be of unit length and
 76 elements to be equilateral in the transformed space of $\mathbf{M}^{1/2}$.
 77 This viewpoint allows element quality metrics and Delaunay
 78 properties defined for isotropic meshes to be carried over to
 79 the anisotropic setting.



Figure 8: The anisotropic remeshing algorithm of Narain et al. [NSO12] allows detailed wrinkles in cloth to be resolved accurately with fine elements (red), while much coarser elements (blue/green) are used in flat regions. Long, narrow folds are best represented using anisotropic elements (yellow) aligned with the curvature direction.

1 For manifold triangle meshes, high-quality remeshing can
 2 be accomplished using only the three simple operations of
 3 edge split, edge collapse, and edge flip (see Figure 7d).
 4 Narain et al. [NSO12] use these operations in cloth simu-
 5 lation, generating an adaptive anisotropic mesh that resolves
 6 detailed wrinkles and folds (see Figure 8). First, all edges
 7 that are unacceptably long according to the refinement cri-
 8 terion are split, then edge collapses are attempted as long
 9 as they do not create new unacceptable edges. During both
 10 steps, edge flips are performed to maintain an approximately
 11 Delaunay mesh relative to the anisotropic metric.

12 Local remeshing of tetrahedral meshes is significantly
 13 more involved, requiring several different local operations
 14 and a complex schedule for the order in which to apply
 15 them [KS07]. This technique was first applied to simulation
 16 by Wicke et al. [WRK*10], who used it to minimize artificial
 17 diffusion in elastoplastic flow. Subsequent work has applied
 18 such remeshing techniques to simulation of incompressible
 19 liquids [MB12, MEB*12, CWSO13].

20 Misztal et al. [MB12, MEB*12] build a mesh over the
 21 entire simulation domain, with some elements belonging
 22 to the fluid and the rest to the exterior, while Clausen et
 23 al. [CWSO13] mesh only the fluid volume. The former ap-
 24 proach allows topological changes like collisions to be han-
 25 dled automatically without special treatment, although at the
 26 cost of maintaining a mesh over a potentially much larger do-
 27 main. Clausen et al. also describe techniques for guarant-
 28 eeing incompressibility and momentum conservation that are
 29 applicable to both approaches. Two key advantages offered
 30 by these methods are that (i) the advection step causes no
 31 numerical diffusion, because physical quantities move with
 32 the mesh nodes, and that (ii) surface tension can be modeled
 33 accurately thanks to an explicit surface representation tied
 34 directly to the simulation mesh.

35 Apart from simply adding or removing vertices, surface

36 tracking algorithms in fluid dynamics may also move ver-
 37 tices along the surface in order to optimize mesh shapes. A
 38 process called “null-space smoothing”, which slides vertices
 39 within the tangent space of a meshed surface, is used in sev-
 40 eral works [Jia07, BB09, BBB10, WRK*10, CWSO13]. This
 41 strategy improves the quality of simulation elements without
 42 changing the shape of the tracked surface.

43 **Additional challenges and techniques** When refinement is
 44 performed, the position of the newly inserted node has to be
 45 chosen carefully. Simply placing it at the midpoint of the
 46 original element can cause physical quantities such as bend-
 47 ing to change discontinuously, injecting artificial energy into
 48 the system and leading to instabilities. Instead, it is better to
 49 adjust the mesh locally to bring it into an energy-minimizing
 50 configuration. Spillmann and Teschner [ST08] consider the
 51 positions of the new node \mathbf{x}_i and its neighbors as variables
 52 and perform an optimization to minimize the total energy,

$$U(\mathbf{x}_1, \dots, \mathbf{x}_n) - \sum_{j=1}^n \mathbf{f}_j^T \mathbf{x}_j, \quad (17)$$

53 where U is the internal energy due to elastic forces, and
 54 \mathbf{f}_j is the external force acting on node j . A similar ap-
 55 proach has been used for simulating the behavior of stiff
 56 two-dimensional sheets such as paper and metal [NPO13,
 57 PNJO14], but with \mathbf{f}_j replaced with an acceleration-
 58 corrected term $\mathbf{f}_j - m_j \mathbf{a}_j$ to preserve the instantaneous ac-
 59 celeration of each node.

60 Liquids with surface tension may freely transition be-
 61 tween volumes, thin films, filaments, and droplets; repre-
 62 senting these transitions is a challenge for most mesh-based
 63 techniques. Zhu et al. [ZQC*14] address this problem using
 64 non-manifold meshes of mixed dimensionality, composed of
 65 tetrahedra, triangles, segments, and points. Beyond the tra-
 66 ditional remeshing operations that work within a single di-
 67 mensionality, they also provide operations for dimensionality
 68 transitions via element collapse (e.g. transforming a thin
 69 triangle to a segment) and merging (e.g. generating a tetra-
 70 hedron to connect two adjacent triangles with small dihedral
 71 angle).

72 Finally, we point out the recent “power particles” tech-
 73 nique of de Goes et al. [dGWH*15], which builds an unstruc-
 74 tured mesh at each time step using a Voronoi-style power di-
 75 agram. This can be viewed as a global remeshing approach
 76 like the ones mentioned above [KFCO06, BWHT07], but this
 77 method stores physical quantities on Lagrangian particles
 78 without maintaining an explicit connectivity, and thus avoids
 79 numerical diffusion due to resampling. While this work is
 80 not specifically an adaptive strategy, it is a form of spatial
 81 discretization that makes adaptivity very easy to implement,
 82 and could be a fruitful basis for future work in adaptive sim-
 83 ulation.

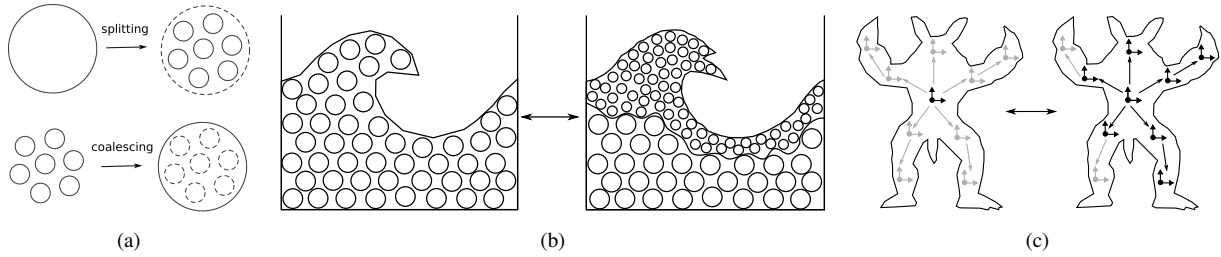


Figure 9: Overview of adaptive meshless techniques. (a) Dynamic local resampling applies splitting and coalescing operators to degrees of freedom in order to locally refine and coarsen regions of interests (b) Multiscale methods couple several simulations with different resolutions. Coarser simulations are used as boundary conditions for the finer resolutions. Feedbacks from the finer resolutions are used to avoid divergences between two different resolutions. (c) Hierarchical refinement dynamically activates or deactivates levels of a precomputed hierarchy between degrees of freedom.

3.3. Meshless models

In the last two decades, numerous meshless models have been extended to perform adaptive physically-based animation. They include smoothed particle hydrodynamics (SPH) (see the survey of Ihmsen et al. [IOS*14]), fluid-implicit particle (FLIP) (see the seminal work of Zhu and Bridson [ZB05]), moving least squares (MLS) (see Muller et al. [MKN*04]) and frame-based models (see Gilles et al. [GBFP11]). These models were used to describe a wide range of phenomena, from fluids (SPH, FLIP) to solids (MLS, frame-based).

Due to the absence of fixed connectivity, meshless models are among the most flexible models for spatial adaptivity. This flexibility, combined with the variety of models, leads to an impressive number of resampling strategies, developed to resolve details near splashes, large deformations, viscoplastic flows and fractures. We classify these strategies into three categories: (1) dynamic local resampling, (2) multiscale methods, and (3) hierarchical refinement (see Figure 9).

These different strategies make meshless models particularly useful for material that undergo large irreversible deformations such as the one cited above. However, it is important to keep in mind that the flexibility of meshless models come with expensive nearest-neighbor search algorithms to determine the connectivity between material samples at run time.

We inform the reader that complementary information about SPH adaptive techniques can be found in the state-of-the-art report by Ihmsen et al. [IOS*14].

Dynamic local resampling The idea is to dynamically subdivide or merge particles to fit a desired resolution (see Figure 9a). The success of this strategy mainly relies on the resampling scheme’s ability to ensure stability, to accurately represent boundaries, and to prevent popping artefacts. Depending on the underlying model (SPH, FLIP, MLS) and its sensitivity to intense resampling, different strategies have been proposed.

As a full particle-based method, the SPH model is a per-

fect candidate to dynamic resampling. Yet, its sensitivity to particle distribution makes resampling strategies challenging. First, the interaction between particles with different sizes increases the error in the pressure term, leading to instabilities. Therefore smooth grading of resolution is required to minimize this error. Second, the change of positions during resampling can create a sudden change in density which will result in violent pressure forces, which again lead to instabilities (see Orthmann and Kolb [OK12]). Several methods that can be combined were proposed to avoid these local change in density. First, instead of computing the density based on positions, one can use the continuity equation as done by Desbrun and Cani [DC99]. In order to avoid integration error to be accumulated along the simulation, the density still needs to be re-computed based on positions at a user-defined interval. Then, one can perform position optimization to minimize errors during resampling [APKG07] and use quantity blending over time to smooth out inevitable sampling error [OK12]. Also, it is important to keep in mind that another challenge is to efficiently retain the parallel nature of SPH in the adaptive scheme. Zhang et al. [ZSP08] and Yan et al. [YWH*09] propose two different methods to make splitting and merging operators parallel.

More recently, dynamic resampling has been applied to FLIP. As FLIP is a combination of grid and particles, two levels of adaptivity are possible: one on the grid resolution, and the other on the particle sampling. Also, only advection operations are performed on the particles, which results in a less position-sensitive simulation and allows much more flexibility than SPH. More precisely, FLIP does not apply density-based forces to the particles. Consequently, sudden density changes due to particle splitting or merging do not have the same catastrophic consequences as in SPH. However, damping is introduced once particles are merged. This can be taken into account by changing blending parameters of the FLIP simulator as suggested by Ando et al. [ATT12]. Early works on adaptive FLIP perform adaptivity only on particles based on a deformability criterion and the distance to surface [HHK08, ATT12]. Ando et al. use the flexibility of FLIP regarding the particles’ positions in order to preserve

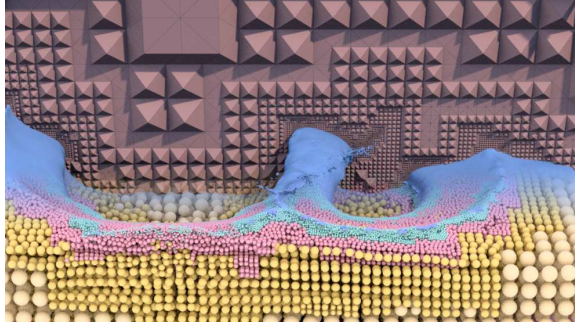


Figure 10: Ando et al. [ATW13] simulate liquid by combining adaptively-sized FLIP particles (bottom, front) with an adaptive tetrahedral mesh for the pressure solve (top, back).

1 fluid sheets by creating additional particles. In both methods,
 2 the largest particle size is bounded by the cell size of the
 3 underlying grid, which precludes aggressive adaptive sam-
 4 pling and the use of a fully adaptive FLIP simulator. Ando
 5 et al. [ATW13] combine an adaptive BCC mesh (see Section
 6 3.1) with adaptive particle sampling to handle highly differ-
 7 ent resolutions. (see Figure 10).

8 When simulating solids, local re-sampling is essential in
 9 describing phenomena such as large deformations and frac-
 10 ture. First, like mesh-based methods, large deformations cre-
 11 ate poorly sampled regions leading to ill-conditioned defor-
 12 mation gradients and instabilities. Second, as explained in
 13 Section 3, a challenge in fracture simulation is to reach a suf-
 14 ficiently fine discretization near the tip of the crack in order
 15 to obtain a realistic crack path. In these cases, local resam-
 16 pling can greatly improve accuracy and stability while retain-
 17 ing efficiency. However, there are two main challenges that
 18 require special care.

19 The first one consists in accurately describing material dis-
 20 continuities. Most of the time, shape functions are spherical
 21 and they must be modified so that sharp boundaries can be
 22 represented. In their work on large viscoplastic deformation
 23 and fracture, Pauly et al. [PKA*05] model discontinuities us-
 24 ing extended shape functions with transparency criteria, and
 25 locally modify a sparse neighborhood graph to update con-
 26 nectivity. Another strategy was proposed by Steinemann et
 27 al. [SOG09] to address the high cost of shape functions up-
 28 date. It consists in using a visibility graph to efficiently han-
 29 dle connectivity and approximate material distances used in
 30 computing shape functions.

31 The second one comes from rendering artifacts that can
 32 occur at the surface due to splitting. In this context, Jones
 33 et al. [JWJ*14] propose a strategy to resample elastoplastic
 34 simulation while alleviating popping artifacts. They base
 35 their method on the evaluation of each particle neighbor’s
 36 density. This evaluation is performed using a weighted co-
 37 variance matrix computed in rest space for each particle i ,
 38 where \mathbf{u}_{ij} denotes the vector between particle i and particle

39 j , its neighbor.

$$\mathbf{B}_i = \sum_j \frac{\mathbf{u}_{ij}\mathbf{u}_{ij}^T}{\|\mathbf{u}_{ij}\|^4} \quad (18)$$

40 If the maximum eigenvalue of \mathbf{B}_i is too small, then there
 41 are too few particles in the neighborhood and the particle
 42 is split in two. New particles are positioned along each side
 43 of the eigenvector with the minimum eigenvalue. In order
 44 to prevent from rendering artifacts, splittings which are not
 45 tangent to the surface are rejected, and particles near the sur-
 46 face are split along the middle eigenvector whose direction
 47 is tangent to the surface. Conversely, if the minimum eigen-
 48 value of \mathbf{B}_i is too large, then there are too many particles
 49 in the neighborhood and the particle is merged with its closest
 50 neighbor. The new particle is positioned halfway between
 51 the two merged particles.

52 **Multiscale methods** In multiscale methods, several simu-
 53 lations with different resolutions are coupled in a hierarchi-
 54 cal way (see Figure 9b). At the coarsest simulation level L_0 ,
 55 the whole domain is discretized. Then, each finer simulation
 56 level L_r discretizes a subset of L_{r-1} with a finer scale. This
 57 finer subset is defined to match regions of interests that can
 58 be physically or visually motivated. Transitions between two
 59 scales are bilateral: the coarsest simulation level L_r is used to
 60 build boundary conditions for the finer simulation level L_{r+1}
 61 and feedbacks from a finer simulation level L_{r+1} to a coarser
 62 simulation level L_r are applied, in order to prevent dynam-
 63 ics of two different levels from diverging. Solenthaler and
 64 Gross [SG11] and Horvath and Solenthaler [HS13] apply
 65 this idea to SPH fluid simulation. Compared to merging and
 66 splitting particles, the main advantage is that interactions be-
 67 tween different resolutions are not direct anymore. Thus, sta-
 68 bility can be more easily ensured and large differences in res-
 69 olution can be handled. In Solenthaler and Gross’s approach,
 70 a two-scale simulation is performed. High-resolution regions
 71 are defined based on the distance to the surface and the view
 72 frustum. In these regions, low-resolution particles emit finer
 73 particles according to a cubic pattern which ensures a uni-
 74 form space sampling. Relaxation steps are performed when
 75 particles enter the high-resolution region in order to avoid
 76 large pressure forces. Horvath and Solenthaler extend the
 77 two-scale simulation to multiscale simulation, and avoid pre-
 78 vious artifacts such as mass loss due to particle removal and
 79 instabilities due to oversampling near boundaries.

80 **Hierarchical refinement** Dynamic resampling techniques
 81 were also used in frame-based methods to simulate elastic
 82 deformations, see [GBFP11] for a full description of the
 83 method. Tournier et al. [TNFG14] use a hierarchical ap-
 84 proach to achieve simplifications during deformation with-
 85 out popping artifacts (see Figure 9c). The material is de-
 86 formed using physically-based control frames organized in
 87 a generalized hierarchy. The model can be simplified by at-
 88 taching frames to their parents at any time in their current rel-

1 ative positions. Activation and deactivation of nodes is per-
 2 formed based on relative velocity and user-specified metrics,
 3 while integration points are updated according to the hier-
 4 archy. These hierarchical techniques take advantage of their
 5 structure to improve efficiency, but may lack of flexibility,
 6 especially regarding topological changes.

7 4. Miscellaneous techniques for spatial adaptivity

8 By far the most popular approach to spatial adaptivity in
 9 computer graphics is to add more computational elements
 10 where more accuracy or detail is desired, as surveyed in
 11 Section 3. This type of spatial adaptivity is often called
 12 h -refinement, because the length of an edge in a mesh is
 13 typically indicated by the letter h . In addition to this tried-
 14 and-true strategy, there are other fundamentally different ap-
 15 proaches for achieving spatial adaptivity. In the next sections,
 16 we will discuss strategies that refine the basis within a single
 17 element (a superset of p -refinement, which refers specifically
 18 to the order of a polynomial basis) and during a subspace
 19 simulation, strategies that use multiple grids that move and
 20 overlap to track locations where more detail is desired, and
 21 strategies that mix different reference frames in order to use
 22 the computational degrees of freedom optimally.

23 4.1. Basis refinement

24 If we wish to achieve spatial adaptivity without explicitly re-
 25 meshing (perhaps because it is difficult to control element
 26 quality when re-meshing, or because a particular application
 27 requires that we preserve the original mesh), then we can
 28 perform basis refinement instead. The concept of basis re-
 29 finement can be a difficult one to grasp for newcomers to
 30 the field. One of the best ways to understand basis refine-
 31 ment is in the context of finite element methods (FEM). In
 32 generic terms, FEM attempts to approximate a function (typ-
 33 ically the solution to a partial differential equation) with a
 34 very limited, very specific subset of all possible functions.
 35 Most methods in computer graphics use linear interpolation
 36 within each element, which essentially restricts the solution
 37 to a piecewise linear function. For the purposes of this dis-
 38 cussion, we would say that the elements are using linear ba-
 39 sis functions, and that the overall solution is expressed in a
 40 piecewise-linear basis. However, we can actually represent
 41 the solution more accurately (in the sense that the solution
 42 converges more quickly under refinement) by using more
 43 elaborate bases, like piecewise quadratic functions instead
 44 of piecewise linear ones.

45 This section discusses four types of basis refinement: hier-
 46 archical basis refinement, polynomial basis refinement, basis
 47 enrichment, and adaptive reduced basis functions. The first
 48 three topics discuss adaptivity at the level of basis functions,
 49 whereas the fourth is about the adaptive creation of reduced
 50 basis in reduced model simulations.

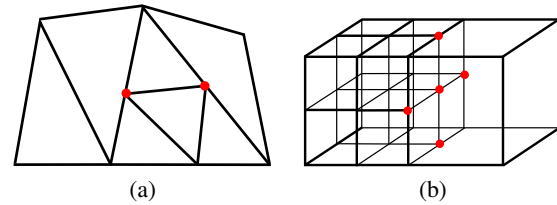


Figure 11: Refinement by element subdivision is attractive by its simplicity for 2D and 3D meshes, however it introduces T-junctions (in red) at interface between resolutions with different sizes. Bookkeeping or extra-remeshing operations are required to take care of these non-independent degrees of freedom.

51 **Hierarchical basis refinement** In computer graphics, hier-
 52 archical basis refinement has mainly been applied to finite
 53 element simulations (FEM) of solids and shells [CGC*02,
 54 GKS02]. The idea is to refine computational basis functions
 55 instead of elements. From a theoretical point of view, there
 56 are no differences between hierarchical basis refinement and
 57 hierarchical element refinement. Both adaptively add more
 58 degrees of freedom with increasingly local support in order
 59 to improve accuracy where needed. Both use hierarchi-
 60 cal schemes in order to efficiently sample the simulation do-
 61 main. The main differences are practical. By refining basis
 62 functions instead of elements, compatibility between regions
 63 with different resolutions are implicitly handled. This makes
 64 adaptivity much easier and general.

65 For instance, hierarchical basis refinement allows a simple
 66 handling of T -junctions. In FEM, each element's node
 67 carries a basis and builds a local stiffness matrix from its
 68 node's stiffness which are then assemble into a global stiff-
 69 ness matrix. During this process, only independent degrees
 70 of freedom should add their contribution to the global matrix.
 71 However, when using hierarchical element refinement, non-
 72 independent degrees of freedom are added during the subdivi-
 73 sion of the simulation mesh. They are called T -junctions or
 74 T -nodes (see Figure 11) and require specific handling. Sud-
 75 denly, a simple subdivision scheme becomes dependent on
 76 the dimensionality, the element type and the basis order, thus
 77 requiring important implementation work. In contrast, hier-
 78 archical basis refinement handles T -nodes at the basis level
 79 by making sure that no bases are redundant. The hierarchical
 80 structure makes this process simple and thus offers a more
 81 general framework for adaptivity which can handle arbitrary
 82 resolution differences.

83 Capell et al. [CGC*02] embed a high-resolution mesh in
 84 a hexahedral complex and precompute a hierarchy of bases
 85 up to a given level of refinement. During the simulation, de-
 86 pending on the amount of deformation, each level of the hi-
 87 erarchy will refine or coarsen, thus updating the current set
 88 of active bases. Basically, it is the same idea of [DDCB01]
 89 but from a basis point of view.

90 The Conforming Hierarchical Adaptive Refinement Meth-

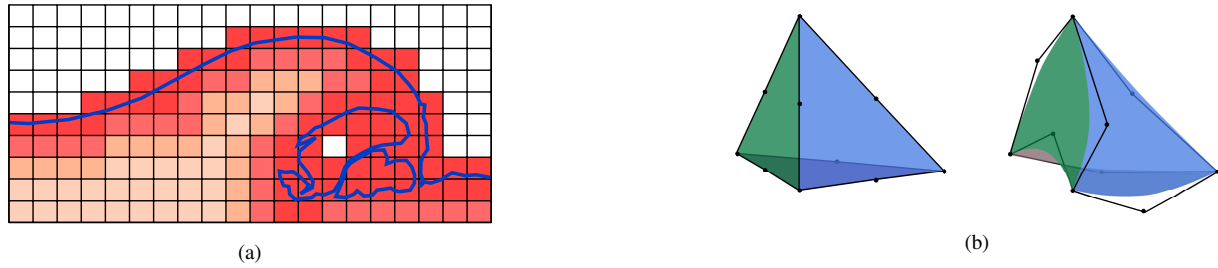


Figure 12: Illustrations of p -adaptive techniques for (a) fluids and (b) deformable solids. In (a) Each fluid cell uses a different approximation space depending on its distance to surface (blue line). Surface cells (in red) use fourth order polynomial to precisely approximate pressure. A smooth grading of the approximation is performed inside the fluid (lighter cells) that allows to save computational time. In (b) the canonical tetrahedron with quadratic control points and next to it the quadratic deformation induced by the deformed control mesh. Such a deformation would require many linear elements.

1 ods (CHARMS) framework of Grinspun et al. [GKS02] gen- 39
 2 eralizes the idea of spatially refining the bases instead of the 40
 3 elements. They provide an in-depth explanation of the conce- 41
 4 pt and describe numerous results of basis refinement ap- 42
 5 plied to shells, solids and electrocardiography simulations. 43
 6 In fact, it is quite surprising that this method was not more 44
 7 studied or extended in the last decade. A possible explana- 45
 8 tion is the fact that, in the last few years, adaptivity proved 46
 9 to be essential for large and complex deformations such 47
 10 as visco-elastic, visco-plastic flows. In those cases, hierar- 48
 11 chical refinement is not sufficient anymore to ensure well- 49
 12 conditioning of the system matrices.

13 **Polynomial basis refinement** Polynomial basis refinement 52
 14 methods, also called p -adaptivity, increase or decrease the 53
 15 order of the basis functions. For a given spatial resolution, 54
 16 this allows to improve the quality of the deformation without 55
 17 remeshing. In computer graphics, using high order approxi- 56
 18 mation to resolve fine details is not new. However mixing dif- 57
 19 ferent orders of approximation to adaptively resolve details 58
 20 was only recently applied in the context of fluid simulation
 21 and elastic deformations.

22 For fluid simulation, the smoothness of velocity and pres- 60
 23 sure makes p -adaptivity potentially much more efficient than 61
 24 geometric adaptivity, because the error per degree of free- 62
 25 dom decreases exponentially with the approximation order 63
 26 but only geometrically with the spatial resolution. In this 64
 27 favorable context, Edwards and Bridson [EB12, EB14] use 65
 28 polynomial basis refinement in a Discontinuous Galerkin 66
 29 FEM framework in order to simulate detailed water with 67
 30 coarse grids. They use low-order bases deep inside the liq- 68
 31 uid and increase the basis order closer to the liquid surface, 69
 32 where more visual detail is desired (see Figure 12a). By us- 70
 33 ing basis refinement instead of element refinement, they can 71
 34 keep the simple structure of a low resolution Cartesian grid 72
 35 while pushing back the limit on the scale of details in one 73
 36 cell. In terms of cost, their method is approximately as ex- 74
 37 pensive as a classical high spatial resolution simulation but 75
 38 it provides much more details such as extremely thin sheets. 76
 77

Bargteil and Cohen [BC14] animate deformable bodies 39
 40 by combining linear and quadratic Bézier elements. The 41
 42 main advantage of their method is the ability to locally in- 43
 44 crease degrees of freedom and to simulate nonlinear geome- 44
 45 try without remeshing (see Figure 12b). To decide whether 45
 46 an element should be linear or quadratic, they compare the 46
 47 linear and quadratic predicted positions of the midpoint of 47
 48 each edge of the element. If the difference between the two 48
 49 positions is larger than a threshold then the edge becomes 49
 50 quadratic. If the difference is less than another threshold then 50
 51 it becomes linear. Thus, some elements can have linear and 51
 52 quadratic edges, usually in transition regions. Bargteil and 52
 53 Cohen observe that, as the number of degrees of freedom 53
 54 increases, visual differences between linear and quadratic el- 54
 55 ements become difficult to discern. Moreover, the additional 55
 56 cost remains important and local deformations on the surface 56
 57 of a quadratic element due to collisions still cannot be re- 57
 58 solved without element refinement. Therefore, their method 58
 59 is particularly efficient on low-resolution models, where it 59
 60 provides smoother geometry and better dynamics quality.

61 In both cases, the differences of resolution between two 60
 62 regions that can be achieved only using p -adaptivity are lim- 61
 63 ited. An important avenue for research would be to combine 62
 64 those methods with geometric adaptive techniques. Such 63
 65 methods have been well studied in engineering fields and 64
 66 are called hp -adaptive methods.

65 **Basis enrichment** Another way to add spatial detail to a 65
 66 physical model without re-meshing is by using “basis enrich- 66
 67 ment.” The main idea behind basis enrichment is to adap- 67
 68 tively add carefully-chosen basis functions that are specifi- 68
 69 cally designed for the phenomena being modeled. (This is in 69
 70 contrast to p -refinement, which is restricted to polynomial 70
 71 functions, regardless of the phenomena being simulated.) 71
 72 For an example of basis refinement, consider an object is 72
 73 being fractured; some material that used to be connected to- 73
 74 gether will have to be split in two. A simple linear basis func- 74
 75 tion could be split into two functions that are linear on one 75
 76 side of the fracture and zero on the other, as illustrated in 76
 77 Figure 13. This type of basis replacement effectively avoids

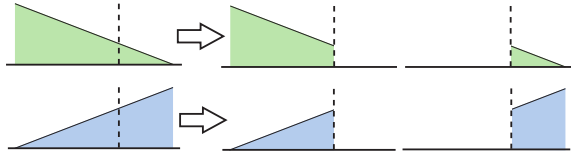


Figure 13: A one-dimensional element with linear basis functions can simulate a fracture by enriching its basis. Here, the new basis functions drop off to zero on the other side of the fracture site indicated by the dashed line, directly encoding the severed connection.

1 re-meshing by inserting a fracture directly into the basis it-
 2 self. Note that in this case, we opted to enrich the basis with
 3 these particular “step” functions which drop to zero after a
 4 certain point, instead of trying to fit some polynomial that
 5 might overshoot or otherwise imperfectly capture the desired
 6 physics. Such basis enrichment techniques have been termed
 7 the “general finite element method” (GFEM) or “extended fi-
 8 nite element method” (XFEM) [BGV09].

9 Adaptive basis enrichment methods have been recently ap-
 10 plied in many computer graphics contexts. The virtual node
 11 algorithm uses basis enrichment to stabilize fracture and cut-
 12 ting simulations [MBF05, HJST13]. Instead of re-meshing
 13 and potentially inserting poorly-conditioned elements, the
 14 virtual node algorithm essentially copies the entire element
 15 and adapts its basis functions to the fracture site. The vir-
 16 tual node algorithm has recently been improved to robustly
 17 allow multiple cuts in a single element [SDF07, WJST14].
 18 One drawback to adaptive basis enrichment is that compli-
 19 cated cuts can require arbitrarily complicated basis func-
 20 tions that may not be simple to compute. In the case of cut-
 21 ting thin shells, Kaufmann and colleagues [KMB*09] note
 22 that the most physically-appropriate enrichment functions re-
 23 quire the solution of a Laplace equation with time-varying
 24 boundary conditions. Similar basis enrichment techniques
 25 may not be computationally feasible for the adaptive sim-
 26 ulation of more complicated volumetric phenomena.

27 **Adaptive reduced bases** As pointed out by Bargteil and
 28 Cohen [BC14], volumetric elastic effects are usually low-
 29 resolution in space, because we often care about longer
 30 time scales in computer graphics. Model reduction methods,
 31 also called subspace simulation, exploit this fact to turn ex-
 32 tremely costly nonlinear finite element simulations into inter-
 33 active ones. The idea is to solve equations of motion in a re-
 34 duced basis that is usually computed using modal analysis or
 35 from a database using principal component analysis (PCA).
 36 Thus, instead of having a complexity dependent on the simu-
 37 lation mesh resolution, it only depends on the size of the re-
 38 duced basis (the number of deformation modes) which is typi-
 39 cally much smaller. A cubature scheme is used to perform
 40 integration on only a reduced sets of well-chosen samples.
 41 In the end, this allows simulation of nonlinear deformable
 42 models with orders of magnitude speed-up.

43 The drawbacks are that the creation of the basis requires
 44 heavy precomputation, and that the motion of the model is
 45 constrained to lie in this basis. Furthermore, while the full
 46 finite element simulations often have local compact basis
 47 functions and sparse system matrices (because each degree
 48 of freedom is a vertex that only influences its neighbors lo-
 49 cally), reduced simulations tend to have global basis func-
 50 tions and dense system matrices (because each degree of
 51 freedom impacts all points in space simultaneously). Small
 52 dense systems are often more efficient than large sparse
 53 ones, but the increased storage requirement limits how many
 54 modes can be feasibly included in model-reduced simula-
 55 tions, and it limits the use of model reduction in settings
 56 where large numbers of modes are essential for visually-
 57 plausible behavior (like cloth and fluids). Therefore model
 58 reduction is only truly efficient for smooth global deforma-
 59 tions and predictable scenarios where it is possible to build a
 60 basis that remains constant over time. It is also only useful in
 61 situations where a limited number of modes can adequately
 62 describe the system.

63 Several strategies have been proposed to overcome some
 64 of these drawbacks by adaptively improving the reduced bas-
 65 is. Common components to those strategies are the differ-
 66 ent processes to update the basis and the criteria used to de-
 67 cide when the basis should be adapted. Moreover, a com-
 68 mon challenge is to ensure temporal coherence while adapt-
 69 ing the basis. Kim and James [KJ09] combine a full nonlin-
 70 ear simulation with subspace simulation in order to exploit
 71 coherence in the global motion of the model. They incre-
 72 mentally build a reduced-order nonlinear deformable model
 73 as the full nonlinear simulation progresses. When possible,
 74 full nonlinear steps are skipped with subspace steps result-
 75 ing in significantly cheaper steps. The challenge of this strat-
 76 egy is to provide efficient operators to update the reduced-
 77 basis and to robustly choose which steps can be reduced.
 78 We briefly describe the two main operations that compose
 79 the incremental construction of the basis: the updating op-
 80 eration and the downdating operation. The updating opera-
 81 tion adds a new vector to the basis only if it is significant.
 82 To do so, a displacement vector received from a full simu-
 83 lation step is orthogonalized against the existing basis. If its
 84 norm is above a given threshold then it is concatenated to
 85 the basis. The downdating operator is applied when the bas-
 86 is reached its maximal size r and a new significant vector
 87 needs to be added. The basis is then modified so that the
 88 $r/2$ most significant directions are preserved. Finally, as the
 89 cubature (reduced integration) is dependent on the basis, it
 90 also needs to be updated. The updating operation is trig-
 91 gered through different criteria depending on the application.
 92 Kim and James describe criteria for quasistatic and dynamic sim-
 93 ulations. The dynamic case is particularly challenging as the
 94 error is history dependent, which means that taking full steps
 95 after reduced steps do not correct errors from the subspace
 96 simulation. This method presents impressive speed up for

1 nonlinear simulations. However, the performance is highly
2 dependent of the rate of expansion of the basis.

3 As subspace simulation tends to reproduce the global motion
4 of an object, it is particularly challenging to produce
5 very local deformations in this framework. A direct consequence
6 is a simplification of the dynamic behavior. Harmon
7 and Zorin [HZ13] tackle this problem by including *a priori*
8 knowledge in the building of the basis. More precisely, they
9 augment a standard precomputed basis with a dynamic basis.
10 This dynamic basis is built with custom functions derived
11 from analytic solutions to static load. Unpredictable local
12 deformations that arise due to collisions and contacts can then
13 be handled. Moreover, as the change in the basis is very local,
14 they can ensure temporal coherence by projecting the
15 current subspace coordinate vector into the new basis whenever
16 it changes. However, a limitation of the addition of local
17 modes is a restriction on the time step size in order to properly
18 represent the dynamics. Furthermore, the size of local
19 displacements is limited.

20 Pushing further the idea of using *a priori* knowledge in
21 the building of a reduced basis, Hahn et al. [HTC*14] perform
22 adaptive subspace simulation of cloth. They start with a
23 large amount of high-resolution simulation data from multiple
24 training animations, and convert it into a database of low-
25 dimensional bases associated with poses. Then at each time
26 step, they adaptively choose a subset of low-dimensional
27 bases in the data base depending on the pose of a clothed
28 character. Highly nonlinear folds and wrinkles, which typically
29 are very hard problems for subspace simulation, can then
30 be reproduced. Dynamics is damped near tightly constrained
31 regions such as sleeves, but this may be acceptable
32 in practice for animating tight clothing.

33 Recently, Teng et al. [TMDK15] propose the use of sub-
34 space condensation to locally switch between subspace and
35 fullspace simulation at run time. When dealing with localized
36 deformations, this allows the behavior not to be limited
37 by *a priori* knowledge such as in [HZ13] and [HTC*14].

38 4.2. Moving grids

39 As mentioned in Section (3.1), grids are an efficient and simple
40 data structure compared to unstructured meshes. However,
41 this simplicity is counterbalanced by a severe lack of
42 flexibility. Firstly, simulating fluids on very large domains
43 requires a prohibitive amount of memory. Secondly,
44 focusing computational resources on regions of interest remains
45 a challenge. While octrees and other adaptive structured
46 meshes discussed in Section 3.1 address these challenges,
47 they lose the cache-coherent structure that makes uniform
48 grids so efficient. Moving grids methods, also called
49 Chimera grids, allow more flexibility while keeping the
50 advantages of Cartesian grids. The main idea is to use one or
51 more computational grids and allow them to move at each
52 time step to follow the region(s) of interest.

53 Shah et al. [SCP*04] propose a simple approach in which
54 a single grid is used whose location and size changes to en-
55 close the region containing significant flow. This strategy is
56 useful when there is only a single region of interest in the
57 fluid, such as when simulating explosions.

58 A more versatile approach is to use multiple indepen-
59 dently moving grids, typically centered at each moving ob-
60 ject in the scene. The grids may undergo pure translation
61 [CTG10], or may also rotate with the object [DMYN08,
62 EQYF13]; in the latter case, centrifugal and Coriolis forces
63 may also need to be taken into account. A coarser back-
64 ground grid can also be used to represent the global flow
65 in the remainder of the domain not covered by any local grid.
66 The major question that arises in these approaches is how to
67 couple the degrees of freedom in regions where two or more
68 grids overlap.

69 For interactive smoke simulation, Cohen et al. [CTG10]
70 omit the coupling step entirely. Instead, smoke particles that
71 lie within multiple overlapping grids are simply advected
72 with a weighted average of the flow velocities indicated by
73 different grids. The resulting motion is not physically valid,
74 but works well for interactive applications.

75 To perform correct coupling for globally incompressible
76 flow, there are two possible strategies: solve for incompress-
77 ibility as usual in each grid and transfer data between them
78 in an outer loop, or build a single discretization that couples
79 all the grids together. Dobashi et al. [DMYN08] use
80 the former approach to efficiently simulate interaction be-
81 tween smoke and rigid objects. Pressure is solved using a
82 modified Gauss-Seidel solver where each iteration follows
83 three steps. First, data from coarser grids is copied to the
84 boundary cells of finer grids for use as boundary conditions.
85 Then, pressure is computed independently on each grid. Fi-
86 nally, data from the interior cells of finer grids is copied
87 to overlapping coarser grids. This process is repeated until
88 it converges; unfortunately, convergence can be slow. More
89 recently, English et al. [EQYF13] developed a full moving
90 Cartesian grids model. Instead of solving for pressure on
91 each grid separately, they combine all grids in a single dis-
92 cretization. Coarse grid cells that contain the cell center of a
93 finer cell are removed, and a Voronoi mesh is built using the
94 remaining cell centers. An monolithic Poisson solver then
95 computes the pressure over the entire mesh.

96 In summary, moving grids are a good solution for accel-
97 erating Eulerian fluid simulation. In the applied math vocabu-
98 lary, they belong to dynamic domain decomposition meth-
99 ods. These methods tackle with success the challenge of in-
100 creasing the local accuracy of Cartesian grids methods while
101 keeping their natural efficiency. However, while such meth-
102 ods are extremely efficient for environments where regions
103 of interests are known or easy to compute, they are not well
104 suited for interactive scenarios where any number of new re-
105 gions of interest may pop up at any location and time, and
106 where adaptive particle simulation remain more appropriate.

4.3. Mixed models

Another important strategy for adaptively focusing computation in computer animation is to selectively apply a mixture of different computational models. The motivation is that every model has its own strengths and weaknesses, and some are better suited to some situations than others. In particular, many methods combine Eulerian reference frames (which describe motions relative to a fixed point in space) with Lagrangian reference frames (which describe motions relative to a physically important moving trajectory). The driving goal is to judiciously combine techniques in a way that leverages the strengths of each model and suffers none of the drawbacks.

While most of these mixed models represent a clever combination of techniques whose whole is greater than the sum of its parts, the mixed models which could not be classified as “adaptive” have been omitted from this paper. This section only discusses mixed models that adaptively change from one model to another when the situation calls for it. This section is separated into methods used to simulate solid objects and methods used to simulate fluids.

We note that numerous other techniques use two-way coupling between different phenomena ([CMT04, RMSG*08, SSF08, RK13], to name a few). However, while these approaches are adaptive in the sense that they modify computation depending on the phenomena being simulated, we feel these two-way coupling methods are outside of the scope of this report. Instead of surveying all possible combinations of different phenomena-specific discretizations, we only discuss here methods that combine different discretizations of the *same phenomena* in order to gain a computational speedup.

4.3.1. Solids

While most mixed models for simulating solid dynamics do not quite adapt their models to their environment, both Sueda et al. [SJLP11] and Servin et al. [SLNB11] successfully address the challenging problem of simulating stiff elastic strands in a collision-heavy scenario. They accomplished this by introducing Eulerian nodes into a largely Lagrangian strand simulation. The Eulerian nodes sit still at important contact points, while the standard Lagrangian nodes sample the strands as normal. These models are “adaptive” under our definition, because the Eulerian nodes add local detail and their location is decided during run-time.

4.3.2. Fluids

The large memory and computation requirements of 3D fluid discretizations are undesirable, so 2D simplifications are often preferred when applicable. In addition, an Eulerian reference frame is popular for guaranteeing a uniform mesh-spacing, maintaining cache-coherence, avoiding remeshing, and describing swirling flows without explicitly sampling

complicated trajectories. However, Eulerian methods are often inferior to their Lagrangian counterparts when sampling fine individual features like droplets and bubbles, or for explicitly tracking many individual vortices.

This section describes many techniques that adaptively combine 2D/3D and Eulerian/Lagrangian techniques in order to get the most out of a fluid simulation. We first survey various methods that combine Eulerian techniques with Lagrangian particles (droplets, bubbles, vortices, etc.). Next, we discuss how some Eulerian models also use Lagrangian particles to couple directly with SPH solvers. After that, we review some methods which combine 3D solvers with 2D techniques or with surface physics.

Eulerian simulation & Lagrangian particles Several early methods combined Eulerian fluid simulations with Lagrangian particles to animate splashing droplets. O’Brien and Hodgins [OH95] combine a 2D pipe-based fluid model with particle-based droplets. Holmberg and Wünsche [HW04] create an Eulerian waterfall model and used Lagrangian particles to animate spray, and Kim et al. [KCC*06] used particles from a 3D surface tracker to fill in missing splash details. Chentanez and Müller [CM10] combine an Eulerian discretization of the shallow water equations with a Lagrangian simulation of spray, splash, and foam particles. The particles add important missing details to the simulations and are allocated dynamically at run-time.

Researchers also use Lagrangian particles to capture bubble behavior in Eulerian simulations. Mould and Yang [MY97] augment a height-field model with Lagrangian particles for droplets and bubbles. Many simulation methods [GH04, HLYK08, PAKF13] compute a 3D Eulerian fluid simulation, and they represent bubbles that are too small to be resolved on the grid with Lagrangian particles. The differences in these methods lie in the varying bubble dynamics and the subtleties of how to transition between the Eulerian grid bubbles and the Lagrangian particle bubbles.

The main concept for these methods is to use an Eulerian representation for the bulk of the flow, but to adaptively turn to a Lagrangian particle representation whenever the Eulerian model is insufficient. This switching point is often easy to detect, because it occurs exactly when the diameter of a water droplet or bubble falls below the Eulerian grid resolution. Not only does this strategy conserve mass and momentum better than simply deleting small features, but it fills in visually important information by animating sprays as a collection of small Lagrangian droplets and foams as a collection of Lagrangian bubbles. Lagrangian droplets and bubbles are practically indispensable in a production workflow, because the small expense of adding additional point geometry with simple physics pays off with enhanced visual realism.

Eulerian simulation & SPH Eulerian methods can also be combined with Lagrangian particles in other ways beyond droplets and bubbles. Losasso et al. [LTKF08] combine SPH

1 with a FLIP simulation, Wang et al. [WZKQ13] combine
 2 SPH with a Lattice-Boltzmann simulation and Chentanez et
 3 al. [CMK14] combine SPH with 3D Eulerian grid. This idea
 4 of adaptively switching between Eulerian simulations and
 5 SPH is still an active research topic. Using SPH instead of
 6 simple passive or ballistic particles is clearly more realistic,
 7 but it comes with the expense of additional neighborhood oper-
 8 ations and more delicate numerical calculations in general.
 9 It is not clear yet whether the realism gained by augment-
 10 ing an Eulerian simulation with SPH particles is worth the
 11 computational expense.

12 **Combining 2D & 3D** Several techniques like discretizing
 13 the shallow water equations [LvdP02,HHL*05] or linearized
 14 wave equations [KM90, Tes04, KB14] are useful for reduc-
 15 ing computational degrees of freedom, but we do not believe
 16 they are inherently *adaptive* by themselves, and we do not
 17 discuss them in detail in this document. However, several
 18 techniques utilize these 2D discretizations in ways that we
 19 would classify as an adaptive “mixed-model” approach.

20 The work of Thürey et al. [TRS06] combines a 3D Lattice-
 21 Boltzmann simulation with a 2D simulation of the shallow
 22 water equations, allowing a local region that to adapt to 3D
 23 phenomena while distant motion remains a simple height
 24 field. Chentanez et al. [CMK14] combine a 3D Eulerian
 25 solver with both a 2D shallow water solver and a particle-
 26 based fluid simulation. Mixing three models allows them
 27 to simulate extremely detailed water interactions at efficient
 28 frame rates. These methods fit our definition of adaptivity,
 29 because they both locally increase the computational deg-
 30 rees of freedom in interesting regions, and these decisions
 31 of where to place the new degrees of freedom are decided
 32 at run-time as the simulation progresses. As these papers
 33 suggest, adaptively switching simulation dimensions will
 34 clearly make animations more efficient, because the compu-
 35 tational complexity plummets as the simulation transitions
 36 from 3D to 2D. The only reservations here are that there is
 37 a significant implementation expense to maintaining two or
 38 more solvers (one for each dimension), and the seamless cou-
 39 pling between dimensions is a sensitive process that is still
 40 being actively researched.

41 5. Discussion

42 Adaptive physically-based models are becoming ubiquitous
 43 in computer graphics. In the last decade, various techniques
 44 for almost all types of deformable models have been pro-
 45 posed and extended. In this survey, we have classified adap-
 46 tive techniques into five different categories: (1) temporal
 47 adaptivity, (2) geometric adaptivity, (3) basis refinement, (4)
 48 moving grids and (5) mixed models. For each category, we
 49 have described the variants that were developed for differ-
 50 ent applications and have discussed their strengths and weak-
 51 nesses.

52 Among those different categories, geometric adaptive

53 techniques are the most studied, and are perhaps the most in-
 54 tuitive due to their geometrical nature. The many variations
 55 in application contexts such as dimensionality and discretiza-
 56 tion have led to a proliferation of different innovative tech-
 57 niques. Yet, as we have tried to show, there are also many
 58 commonalities between aspects of disparate techniques, for
 59 example in terms of refinement criteria, and there is potential
 60 for consolidation of the many approaches in this area.

61 In the opposite direction, mixed models represent an im-
 62 portant area of work. Unfortunately, as they rely on the spe-
 63 cific characteristics of each model, it is more difficult to ex-
 64 tract general patterns and strategies. Even so, they perfectly
 65 represent the idea and the versatility of adaptive techniques
 66 by combining the strengths of different approaches as the
 67 simulation evolves.

68 For now, polynomial basis refinement represents only a
 69 small fraction of the methods studied. In computer graphics,
 70 this is a very recent topic, but which can build on strong found-
 71 ations from engineering and applied mathematics where it
 72 has been extensively studied. Results in solid and fluid sim-
 73 ulation show that polynomial basis refinement can indeed
 74 produce impressive animations. One of the most exciting av-
 75 enues of future research is the combination of this technique
 76 with geometric adaptivity.

77 In subspace simulation, adaptive reduced bases can
 78 greatly extend the range of deformation that can be achieved
 79 by introducing local and non-linear deformations such as
 80 wrinkles. Nevertheless, there is still a large room for im-
 81 provement and innovation, especially regarding the possibili-
 82 ty to handle topological changes and couple different sub-
 83 space simulations such as deformable solids and fluids.

84 Even if there are only a few works that focus on temporal
 85 adaptivity, these methods are widely used and play a crucial
 86 role in ensuring stability and efficiency. Their importance is
 87 due to two main reasons. First, spatial and temporal resolu-
 88 tion are often strongly related. Secondly, the necessary tem-
 89 poral resolution is inherently dependent on the events occur-
 90 ring during a simulation and cannot always be predicted in
 91 advance, necessitating adaptive techniques. As we seek to
 92 resolve details at increasingly finer time scales, further re-
 93 search will be required to capture them without paying an
 94 exorbitant computational cost.

95 Adaptive methods are not without limitations, and we
 96 briefly summarize the major ones. At present, setting up
 97 a new adaptive method is quite difficult: adaptive methods
 98 have often been application-specific so far, which makes the
 99 study of existing solutions quite intricate. Adaptivity usu-
 100 ally makes the implementation of a model much more com-
 101 plex and may ruin the regularity of computations, causing
 102 incompatibilities with GPU implementations. Evaluating the
 103 future overhead due to the online adaptation process is of-
 104 ten difficult, which may make such techniques unusable in
 105 performance-constrained contexts such as interactive appli-
 106 cations. There are also potential concerns relating to simula-

1 tion fidelity. If not tackled with care, popping between differ-
 2 ent spatial and temporal resolutions may cause instabilities
 3 and visual artifacts. Furthermore, the energy diffusion that
 4 necessarily occurs when permanently adapting a model may
 5 be an issue when an accurate simulation is required.

6 Nevertheless, the space of adaptive simulation techniques
 7 is vast and fruitful, and many compelling benefits have been
 8 uncovered so far. There is much room for future work in
 9 developing new adaptive methods which are both easier to
 10 implement and still generic enough to be used in different
 11 applications. This challenge requires methods which, for ex-
 12 ample, minimize the overhead due to additional structures
 13 while making it possible to integrate different adaptation cri-
 14 teria. Many other avenues of future research remain, includ-
 15 ing combinations of different forms of adaptivity and tech-
 16 niques that adapt between different dimensionalities, differ-
 17 ent formulations, and other characteristics that have tradi-
 18 tionally remained separate.

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