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Uncertainty and sensitivity analysis of functional risk curves based on Gaussian processes

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Abstract

A functional risk curve gives the probability of an undesirable event in function of the value of a critical parameter of a considered physical system. In several applicative situations, this curve is built using phenomenological numerical models which simulate complex physical phenomena. Facing to cpu-time expensive numerical models, we propose to use the Gaussian process regression model to build functional risk curve. An algorithm is given to provide confidence bounds due to this approximation. Two methods of global sensitivity analysis of the model random input parameters on the functional risk curve is also studied. As important information is given by the PLI sensitivity indices which allow to understand the effect of misjudgment on the input parameters' probability density functions.

Keywords: Computer experiments, Metamodel, Gaussian process, Sobol' indices, Structural Reliability, Non Destructive Testing, Probability of Detection

1. Introduction

In industrial practice, the estimation of a functional risk curve (FRC) is often required as a quantitative measure of a system safety [1]. A FRC (also called “functional risk criterion” or “fragility curve” in some applicative domains [2]) gives the probability of an undesirable event in function of the value of a critical parameter of a considered physical system. The estimation of this curve sometimes relies on deterministic phenomenological computer models which simulate

complex physical phenomena. Uncertain input parameters of this computer code are modeled as random variables while one of its scalar output variable becomes the studied random variable of interest.

As it is based on a probabilistic modeling of uncertain physical variables and their propagation through a numerical model, this problem can be directly related to the uncertainty management methodology in numerical simulation [1, 3]. This methodology proposes a generic framework of modeling, calibrating, propagating and prioritizing uncertainty sources through a numerical model (or computer code). Indeed, investigation of complex computer code experiments has remained an important challenge in all domain of science and technology, in order to make simulations as well as predictions, uncertainty analysis or sensitivity studies [4, 5]. In this framework, the physical numerical model G just writes

$$Y = G(X) = G(X_1, \dots, X_d) , \quad (1)$$

with $X \in \mathbb{R}^d$ the random input vector of dimension d and $Y \in \mathbb{R}$ a scalar model output.

However, standard uncertainty treatment techniques require many model evaluations and a major algorithmic difficulty arises when the computer code under study is too time expensive to be directly used. For cpu-time expensive models, one solution consists in replacing the numerical model by a mathematical approximation, called a response surface or a metamodel. Several statistical tools based on numerical design of experiments, uncertainty propagation efficient algorithms and metamodeling concepts will then be useful [4]. In this paper, the Gaussian process regression [6] is used as a metamodel technique (surrogate model of the computer code) and applied in the particular context of a FRC as a quantity of interest. Indeed, numerous studies have shown that this interpolating model provides a powerful statistical framework to compute an efficient predictor of a deterministic computer code response [7, 8].

Associated to the estimation of the quantity of interest after the uncertainty propagation step, the sensitivity analysis step is performed to determine those

parameters that mostly influence on model response [9]. In particular, global sensitivity analysis methods take into account the overall uncertainty ranges of the input parameter (see [10] for a recent review). Several works have focused on estimating global sensitivity indices (especially the variance-based ones, known as the Sobol' indices) using the Gaussian process model [11, 12, 8]. In this paper, we propose new global sensitivity indices attached to the whole FRC, while showing how to develop them with a Gaussian process model. We also focus on a recently developed method, the Perturbed-Law based sensitivity Indices (PLI) of [13], which seems promising in terms of computational efficiency and interpretation.

Two examples of FRC largely used in some industrial safety practices are given in the following section. The third section describes the Gaussian process way to model and estimate a FRC. The fourth section develops the sensitivity indices adapted to FRC. A conclusion synthesizes this work.

2. Motivating examples and classical methods

2.1. Probability of detection curves

In several industries, the Probability Of Detection (POD) curve is a standard tool to evaluate the performance of Non Destructive Testing (NDT) procedures. The goal is to assess the quantification of inspection capability for the detection of harmful flaws for the inspected structure. For instance, new aeronautic regulations require for critical parts to perform appropriate damage tolerance assessments to address the potential for failure from material and manufacturing. This imposes enhanced expectations on POD sizing. Another example is the eddy current non destructive examination process which is used to ensure integrity of steam generators tubes in nuclear power plants [14, 15, 16]. Experimental campaigns and simulations experiments are used in order to provide POD as a tool to the process qualification by the regulatory authorities.

In practice, high costs of the implementation of experimental POD campaigns combined with continuous increase in the complexity of configuration

make them sometimes unaffordable. To overcome this problem, it is possible to resort to numerical simulation of NDT process. This approach has been called MAPOD for “Model Assisted Probability of Detection” [17, 18, 16].

More precisely, in the POD context, the problem is formulated as follow. Given a threshold $s > 0$, a flaw is considered to be detected when $Y > s$ with Y the signal amplitude and s a detection threshold. Therefore the one dimensional POD curve is denoted by:

$$\forall a > 0 \quad \text{POD}(a) = P(Y > s | a) , \quad (2)$$

where a is the POD parameter of interest (for example the size flaw) and X are the other input parameters. When simulated experiments are used to build the POD (see for example [16]), Y is a scalar output of a numerical model ($Y = G(a, X)$), where a is determined by its bounds and X is a random vector defined by its joint probability density function (pdf).

In the Sections 3 and 4 of this paper, we will illustrate the mathematical methods for FRC building and analysis on the same case study considered in [15, 16]. We will not give all the details of this case study here. We just recall that the NDT process is the examination under wear anti-vibration bars (AVB) of steam generator tubes inside a nuclear power plant [14]. The computer code that is used, called C3D, simulates by the finite-element method, the eddy-current propagation phenomena. The critical input parameter a is the sum of the two flaw sizes while five other input parameters (vector X) of the computer code are considered (with their associated pdf): E (pipe thickness), $ebav_1$ and $ebav_2$ (lengths of the gap between the AVB and the first and second flaws), h_{11} and h_{12} (first and second flaw heights). The output Y of the computer code is the signal amplitude and the threshold of detection s is fixed at a specific value.

2.2. Seismic fragility curves

The “fragility curve” is a popular functional risk criterion, commonly used in many engineering fields [2]. It describes the probability that the actual damage to a structure exceeds a damage criterion, when the structure is assigned to a

specified load intensity. The seismic fragility curve concerns systems subject to earthquake, whose interest is of particular significance in nuclear safety studies for instance [19].

In the case of seismic risk assessment the load is usually expressed as a scalar characteristic of a seismic signal, typically the horizontal Peak Ground Acceleration (PGA), common choice in civil engineering. Several parameters and phenomena, distinct from the PGA, also influence the load: for each PGA value, the occurrence of the damage event is random. Then, the fragility curve may be interpreted as the cumulative distribution function of the “structural capability”, i.e. the maximum load the structure under investigation can bear without damage [20, 21]. Fragility curves are then useful tools in structural analysis as they provide a more complete information than the usual failure probability (established for a reference value of the load only).

In the standard practice, the assessment is made either following an approach entirely based on the expertise or by the statistical analysis of actually observed or simulated data. As actual damage data may be scarce due to the rarity of severe earthquakes liable to generate damages on highly safe structures, observations are generated by mock-up or (most often) numerical experiments.

In this context, a fragility function is expressed as

$$\forall a > 0 \quad \text{Frag}(a) = P(Y > s \mid a) , \quad (3)$$

where Y is the variable which characterizes the structural damage, $s = Y_{\max}$ is the maximal capacity of structure and a is an intensity measure characterizing the ground motion severity (*e.g.* peak ground acceleration). When simulated experiments are used to build the fragility curve (see for example [19]), Y is a scalar output of a numerical model ($Y = G(\alpha, X)$), where a is determined by its bounds and X is a random vector defined by its joint pdf.

2.3. Classical methods

The standard practice for establishing a fragility curve consists in assuming a lognormal distribution for Y [22]. The classical approach for determining a

POD curve is called the Berens model and relies on simple linear relation (with Gaussian residuals) between $\log(Y)$ and a [23]. In fact, both cases are equivalent and lead to the following form for the FRC:

$$P(Y > s|a) = \Phi\left(\frac{a - \alpha}{\beta}\right), \quad (4)$$

where Φ is the standard Gaussian distribution function, and α and β are two constants to be estimated (by maximum likelihood estimation for example) in function of the Y -data sample and s . The parameter a can also be expressed in terms of logarithm in this expression.

As shown in [16], this model can be easily improved by considering a Box-Cox transformation [24] instead of a simple logarithmic one on Y : $\tilde{Y} = \frac{Y^\lambda - 1}{\lambda}$. λ is determined by maximum likelihood as the real number that offers the finest linear regression of \tilde{Y} regarding the parameter a . The model is now based on \tilde{Y} and is defined as

$$\tilde{Y}(a) = \beta_0 + \beta_1 a + \epsilon, \quad (5)$$

with ϵ the model error such as $\epsilon \sim \mathcal{N}(0, \sigma_\epsilon^2)$. Maximum likelihood method provides the estimators $\hat{\beta}_0, \hat{\beta}_1$ and $\hat{\sigma}_\epsilon$. Hence the model implies the following result: $\forall a > 0, \tilde{Y}(a) \sim \mathcal{N}(\hat{\beta}_0 + \hat{\beta}_1 a, \hat{\sigma}_\epsilon^2)$. Then the value of the FRC can be estimated from $\hat{\beta}_0, \hat{\beta}_1$ and $\hat{\sigma}_\epsilon$, as its confidence interval from the property of the maximum likelihood estimators. Note that it is possible to relax the Gaussian hypothesis on ϵ by using non-parametric approaches [25, 16]

3. Gaussian-process based functional risk curve estimation

Classical methods described in Section 2.3 use a simple model between Y and a without modeling the functional dependence of Y on the other uncertain variables X . As in our case data on Y are obtained by the way of numerical models, the input variables X are controlled and can be introduced in the FRC determination process. However, with cpu-time expensive numerical models, model evaluations can be somewhat limited and only small samples of Y can be obtained. As shown in [26, 20, 16], the use of a metamodel, which is a

mathematical approximation of the computer code [4], is useful. We propose in this section a general formulation for FRC that uses the Gaussian process regression metamodel [27, 6].

Let X be a random vector of influent and uncertain parameter inside the computer model $G(\cdot)$ and a be the parameter of interest (the abscissa of the FRC). The prior knowledge on $G(a, X)$ is modeled by $Y(a, X)$ and defined as follows .

$$Y(a, X) = \beta_0 + \beta_1 a + Z(a, X), \quad (6)$$

where Z is a centered Gaussian process. We make the assumption that Z is second order stationary with variance σ^2 and a parametric covariance (for example the Matérn 5/2 parametrized by its lengthscale θ). Thanks to the maximum likelihood method, we can estimate the values of the so far-unknown parameters: $\beta_0, \beta_1, \sigma^2$ and θ (see for instance [7] for more details).

Gaussian process regression (also known as the kriging process) provides an estimator of $G(a, x)$ which is called the kriging predictor and written $\hat{Y}(a, x)$. In addition to the kriging predictor, the kriging variance $\sigma_Y^2(a, x)$ quantifies the uncertainty induced by estimating $Y(a, x)$ with $\hat{Y}(a, x)$. The predictive distribution is given by $Y(a, x)$ conditioned by y^N :

$$\forall x \quad (Y(a, x) | y^N) = \mathcal{N}\left(\hat{Y}(a, x), \sigma_Y^2(a, x)\right) \quad (7)$$

where $\hat{Y}(a, x)$ (the kriging mean) and $\sigma_Y^2(a, x)$ (the kriging variance) can both be explicitly estimated.

Obtaining the FRC $\Psi(a)$ consists in replacing $Y = G(a, X)$ by its Gaussian process metamodel (7) in (2). Hence we can estimate the value of $\Psi(a)$, for $a > 0$ from:

$$\Psi(a) = P(Y^N(a, X) > s|a), \quad (8)$$

where $Y^N(a, X)$ is a Gaussian process which follows the distribution $(Y(a, X) | y^N)$ and s is a fixed threshold value. Two sources of uncertainty have to be taken into account in (8): the first coming from the parameter X and the second coming from the Gaussian distribution in (7).

From (8), the following estimate for $\Psi(a)$ can be deduced:

$$\Psi(a) = E_X \left[1 - \Phi \left(\frac{s - \widehat{Y}(a, X)}{\sigma_Y(a, X)} \right) \right]. \quad (9)$$

This equation corresponds to the mean FRC with respect to the Gaussian process metamodel. The expectation in (9) is estimated using a classical Monte Carlo integration procedure.

In order to estimate the uncertainty on the FRC estimation, we start from its integral expression:

$$\Psi(a) = P_X(y(a, X) > s|a) = \int \mathbf{1}_{y(a,x) > s} f(x) dx, \quad (10)$$

where $f(x)$ is the joint pdf of X (independent on a). The first uncertainty source on $\Psi(a)$ comes from the numerical evaluation of the integral inside the $\Psi(a)$ definition. This evaluation is done by the Monte Carlo method:

$$\Psi(a) \approx \Psi_{MC}(a) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}_{y(a, x^{(i)}) > s}, \quad (11)$$

where $(x^{(i)})_{i=1, \dots, n}$ is a sample of the random variable X . The size of this sample has to be large, then we replace the code by its Gaussian process approximation $Y^N(a, X)$:

$$\Psi_{MC}(a) \approx \Psi_{MC,PG}(a) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}_{Y^N(a, X) > s}. \quad (12)$$

Let us recall that $Y^N(a, X)$ is a Gaussian process that we know the mean and variance. To compute the integration error, the central limit theorem is used and gives for $n \rightarrow \infty$:

$$\sqrt{n} (\Psi_{MC,PG}(a) - \Psi_{PG}(a)) \longrightarrow \mathcal{N}(0, \Psi_{PG}(a)(1 - \Psi_{PG}(a))), \quad (13)$$

where

$$\Psi_{PG}(a) = \int \mathbf{1}_{Y^N(a, X) > s} f(x) dx. \quad (14)$$

$\Psi_{PG}(a)$ corresponds to the FRC for the process $Y^N(a, X)$. The algorithm consists in taking a large n in order to deal with a valid Gaussian approximation.

The second uncertainty source on $\Psi(a)$ estimation comes from the Gaussian process approximation. We simulate m realizations $(y^{(j)}(a, x))_{j=1, \dots, m}$ from

$(Y(a, x)|y^N)$ in order to evaluate the variability of $\Psi_{MC,PG}(a)$ which comes from the Gaussian process approximation. Thus, we compute:

$$\psi_{MC,PG}^{(j)}(a) = \frac{1}{n} \sum_{i=1}^n \mathbf{1}_{y^{(j)}(a, x^{(i)}) > s}. \quad (15)$$

Then, for each $\psi_{MC,PG}^{(j)}(a)$, $j = 1, \dots, m$, we compute the Monte Carlo error thanks to the central limit theorem cited before in (13):

$$\psi_{PG}^{(j)}(a) \sim \mathcal{N}\left(\psi_{MC,PG}^{(j)}(a), \Upsilon^{(j)}(a)\right), \quad (16)$$

where $\Upsilon^{(j)}(a) = \frac{\psi_{MC,PG}^{(j)}(a) \left(1 - \psi_{MC,PG}^{(j)}(a)\right)}{n}$.

From this double Monte Carlo method, we are able to compute the FRC uncertainty due to the Gaussian process metamodel and due to the numerical integration process. Figure 1 illustrates the algorithm on a POD curve estimation in a NDT application (see section 2.1). A Gaussian process metamodel has been built on $N = 100$ numerical simulations of eddy current non destructive examination on steam generators tube [16]. We visualize the confidence interval induced by the Monte Carlo (MC) estimation ($n = 10000$), the one induced by the Gaussian process approximation ($m = 3000$) and the total confidence interval (including both approximations: PG+MC).

4. Global sensitivity analysis of functional risk curves

The objective of sensitivity analysis is to determine those input variables that mostly influence on model response [9, 28, 29]. Global sensitivity analysis methods take into account the overall uncertainty ranges of the input parameter. Previous works, on the POD [30] and seismic fragility context [31], have only considered sensitivity analysis on the variance and distribution of model output variable Y . In this section, we propose new global sensitivity indices attached to the whole FRC. Therefore, the influence of the parameter of interest a , as it is the FRC abscissa, will be not considered as an input variable whose sensitivity index has to be computed.

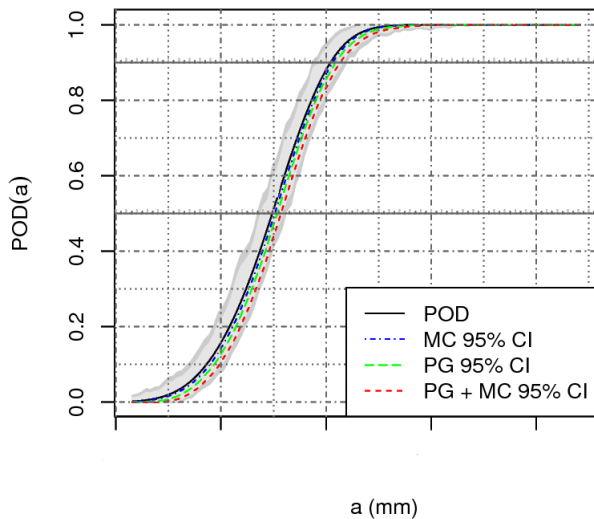


Figure 1: Example of FRC estimated with a Gaussian process model. The present curve is a POD curve.

4.1. Aggregated Sobol' indices

We first focus on the variance-based sensitivity formulation [32], which is one of the most popular tools that has been proved robust and easily interpretable [10]. The associated sensitivity indices are often called Sobol' indices.

In the case of independent inputs, thanks to the Sobol-Hoeffding decomposition [33], the variance of the numerical model $Y = G(X_1, \dots, X_d)$ is decomposed in the following sum:

$$\text{Var}(Y) = V = \sum_{i=1}^d V_i + \sum_{i < j} V_{ij} + \dots + V_{1\dots d} \quad (17)$$

with $V_i = \text{Var}[E(Y|X_i)]$, $V_{ij} = \text{Var}[E(Y|X_i X_j)] - V_i - V_j$, etc. Then, $\forall i, j = 1 \dots d$, $i < j$, the Sobol' indices write

$$S_i = \frac{V_i}{V}, S_{ij} = \frac{V_{ij}}{V}, \dots, \text{ and } T_i = S_i + S_{ij} + \dots \quad (18)$$

The first-order Sobol' index S_i measures the individual effect of the input X_i on the variance of the output Y , while the total Sobol' index T_i measures the X_i effect and all the interaction effects between X_i and the other inputs (as

the second-order effect S_{ij}). T_i can be reinterpreted as $T_i = 1 - \frac{V_{-i}}{V}$ with $V_{-i} = \text{Var}[E(Y|X_{-i})]$ and X_{-i} is the vector of all inputs except X_i .

In order to define similar sensitivity indices for the whole FRC (Eq. (9)), we first define the following quantities:

$$\begin{aligned} \Psi_X(a) &= P_X(Y > s \mid a, X) = 1 - \Phi\left(\frac{s - \hat{Y}(a, X)}{\sigma_Y(a, X)}\right), \\ \Psi_{X_i}(a) &= P_X(Y > s \mid a, X_i), \\ \Psi_{X_{-i}}(a) &= P_X(Y > s \mid a, X_{-i}), \\ D &= E_X \|\Psi(a) - \Psi_X(a)\|^2 \end{aligned} \tag{19}$$

with $\|\cdot\|$ the euclidean norm. As in [34] (see also [35]) who deal with functional model outputs, our objective is to obtain a single sensitivity index for each input by synthesizing the variability of the function, here the FRC depending on a . The specificity here is that we deal with a FRC function. Therefore, we define the FRC aggregated Sobol' indices by:

$$\begin{aligned} S_i^{\text{FRC}} &= \frac{E_X \|\Psi(a) - \Psi_{X_i}(a)\|^2}{D}, \\ T_i^{\text{FRC}} &= \frac{E_X \|\Psi_X(a) - \Psi_{X_{-i}}(a)\|^2}{D}. \end{aligned} \tag{20}$$

S_i^{FRC} (resp. T_i^{FRC}) gives the first-order (resp. total) effect of X_i on the mean FRC.

These FRC aggregated Sobol' indices are easily computed by the metamodels. In particular, the kriging metamodel allows to replace $P_X(Y > s \mid a)$ by the expectation $E_X \left[1 - \Phi\left(\frac{s - \hat{Y}(a, X)}{\sigma_Y(a, X)}\right) \right]$ in the FRC expressions of (19). The sensitivity analysis results on our use case are given in Figure 2 with the FRC aggregated Sobol' indices of the five physical input parameters (called E, ebav₁, ebav₂, h₁₁ and h₁₂).

The FRC aggregated Sobol' indices quantify the sensitivity of each input on the overall curve. However, we could be interested by the sensitivities on the FRC at a specific value of the parameter of interest a . This can be directly done by replacing Y by $\Psi_X(a)$ in the equations (17) and (18). If we are now interested by the sensitivities at a specific probability value of the FRC, we have to study the inverse function of the FRC: $\Psi_X^{-1}(p)$ with p a given probability.

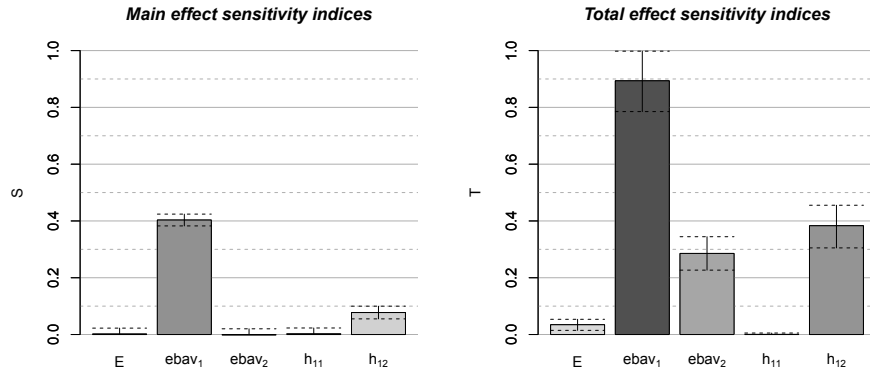


Figure 2: First order (left) and total (right) FRC aggregated Sobol' indices.

Similarly to the previous case, the Sobol' indices can be obtained by replacing Y by $\Psi_X^{-1}(p)$ in the equations (17) and (18). Figure 3 gives these sensitivity indices on our data for $p = 0.90$ which corresponds to the quantile of a at a 90% order (noted a_{90}).

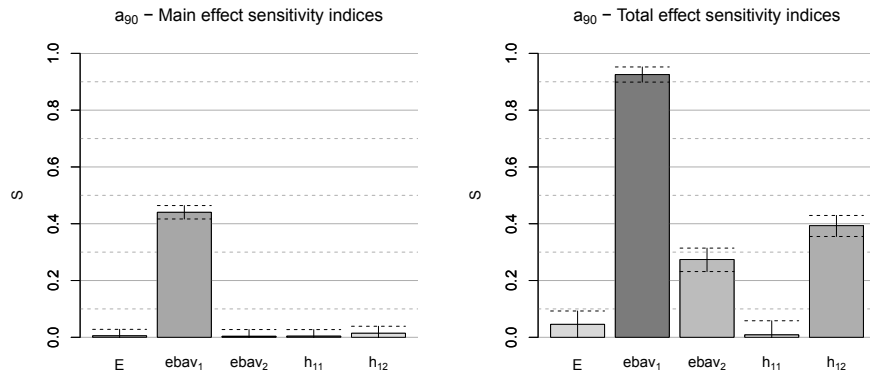


Figure 3: First order (left) and total (right) Sobol' indices on a_{90} .

4.2. Perturbed-law based sensitivity indices

We propose now to quantify the impact, on the FRC, of a perturbation of the input parameters pdfs by answering to the following question: “What would be the FRC if the pdf of the i^{th} input X_i had been modified?”. In this approach, all the input parameters are modeled by random variables and their input probability densities are supposed to be uncertain. Let us remark that a negligible sensitivity index of an input will not allow to fix this input but just to say that its pdf has no influence on the FRC.

We use the so-called Perturbed-Law based sensitivity Indices (PLI) measures recently introduced in [13] (see also [36]). We start from the integral-form of the FRC:

$$P_X(y(a, X) > s) = \int 1_{y(a, x) > s} f(x) dx,$$

where $f(x)$ is the joint pdf of X . Modifying the pdf $f_i(x_i)$ of X_i gives us $f_{i, \delta}(x_i)$, the perturbed pdf of X_i . After this perturbation, the FRC, noticed $P_{X_i, \delta}(\cdot)$ instead of $P_X(\cdot)$, can be written as:

$$P_{X_i, \delta}(y(a, X) > s) = \int 1_{y(a, x) > s} \frac{f_{i, \delta}(x_i)}{f_i(x_i)} f(x) dx, \quad (21)$$

The PLI measures only consist in the comparison of the FRCs between and after the perturbation, and are defined by:

$$S_{i, \delta} = \begin{cases} \frac{P_{X_i, \delta}(y(a, X) > s) - P_X(y(a, X) > s)}{P_{X_i, \delta}(y(a, X) > s)} \\ \text{if } P_{X_i, \delta}(y(a, X) > s) \geq P_X(y(a, X) > s), \\ \\ \frac{P_{X_i, \delta}(y(a, X) > s) - P_X(y(a, X) > s)}{P_X(y(a, X) > s)} \\ \text{if } P_{X_i, \delta}(y(a, X) > s) < P_X(y(a, X) > s). \end{cases} \quad (22)$$

A negative $S_{i, \delta}$ means that the FRC is smaller after the perturbation, while a positive $S_{i, \delta}$ means that the FRC has increased. The estimation of $P_{X_i, \delta}(y(a, X) > s)$ is based on reverse importance sampling [37]. Asymptotical properties of the estimators give also some confidence intervals on the PLI measures.

In [13], the numerical model is directly used to estimate the PLI measures by large Monte Carlo samples. We propose here to estimate the PLI measures by

using the Gaussian process metamodel and integrating its error in the estimates.

The mean FRC that we consider is given by (9):

$$\Psi(a) = E_X \left[1 - \Phi \left(\frac{s - \widehat{Y}(a, X)}{\sigma_Y(a, X)} \right) \right] = \int 1 - \Phi \left(\frac{s - \widehat{Y}(a, x)}{\sigma_Y(a, x)} \right) f(x) dx. \quad (23)$$

After the perturbation of the pdf of the i^{th} input, the mean FRC is given by:

$$\Psi_{i,\delta}(a) = \int 1 - \Phi \left(\frac{s - \widehat{Y}(a, x)}{\sigma_Y(a, x)} \right) \frac{f_{i,\delta}(x_i)}{f_i(x_i)} f(x) dx. \quad (24)$$

PLI measures are then given by:

$$S_{i,\delta}(a) = \begin{cases} \frac{\Psi_{i,\delta}(a) - \Psi(a)}{\Psi_{i,\delta}(a)} & \text{if } \Psi_{i,\delta}(a) \geq \Psi(a), \\ \frac{\Psi_{i,\delta}(a) - \Psi(a)}{\Psi(a)} & \text{if } \Psi_{i,\delta}(a) < \Psi(a). \end{cases} \quad (25)$$

The last element of the PLI method is the definition of the perturbations $f_{i,\delta}(x_i)$. [13] choose to perturb a statistical characteristic (for example the mean, or the variance, or a quantile, ...) of X_i in order to be “as close as possible” to the initial pdf $f_i(x_i)$. The dissimilarity measure between $f_{i,\delta}(x_i)$ and $f_i(x_i)$, which contains the required properties, is the Kullback-Leibler divergence:

$$KL(f_{i,\delta}(x_i), f_i(x_i)) = \int_{-\infty}^{+\infty} f_{i,\delta}(x_i) \log \frac{f_{i,\delta}(x_i)}{f_i(x_i)} dx_i. \quad (26)$$

It implies that $f_{i,\delta}(X_i)$ and $f_i(X)$ have the same definition domain (as a consequence, the input domain bounds cannot be changed). The Kullback-Leibler divergence has the advantage to be easily minimized, in order to obtain explicit solutions for $f_{i,\delta}(X_i)$ for classical pdf (*e.g.* Gaussian).

Figure 4 gives some examples of perturbations for the uniform pdf on $[0, 1]$ (mean is 0.5 and variance is 1/12). Figure 5 gives some examples of perturbations for the standard Gaussian pdf (mean is 0 and variance is 1).

Figure 6 gives an example of a PLI-based sensitivity analysis on our NDT test case (see section 2.1) which aims to estimate POD curve and the associated sensitivity indices to its five physical input parameters (called E, ebav₁, ebav₂, h₁₁ and h₁₂), the defect size a being fixed here at a given value. The input

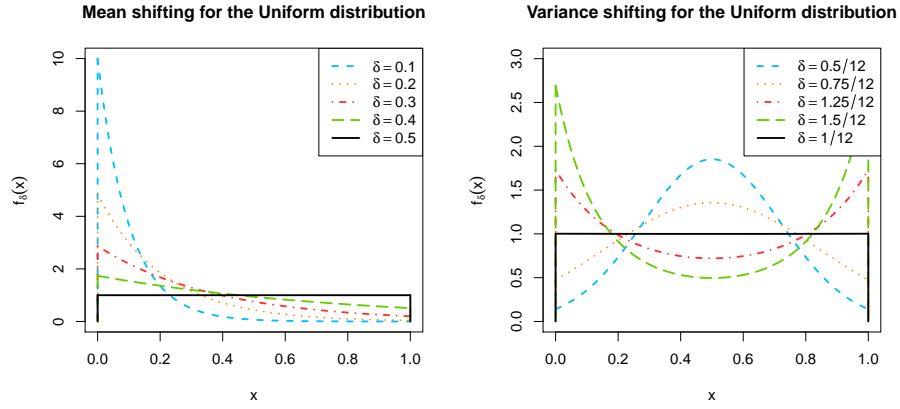


Figure 4: Examples of perturbations for the uniform pdf on $[0, 1]$. Left: the mean of the pdf is perturbed (δ is the new mean value). Right: the variance of the pdf is perturbed (δ is the new variance value).

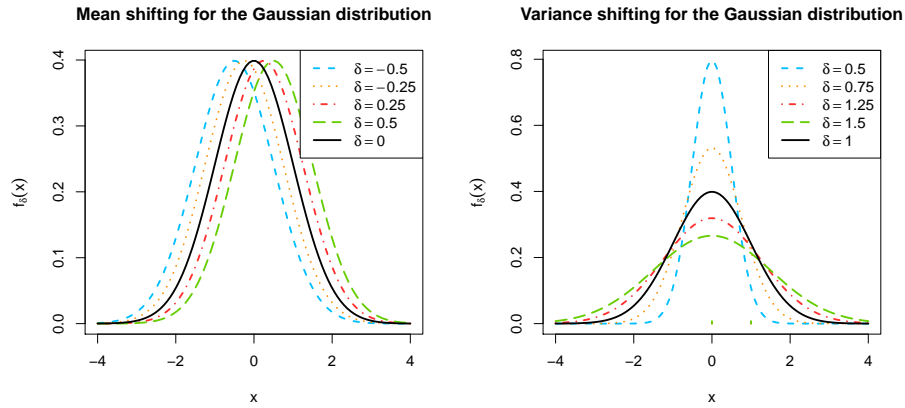


Figure 5: Examples of perturbations for the standard Gaussian pdf. Left: the mean of the pdf is perturbed (δ is the new mean value). Right: the standard-deviation of the pdf is perturbed (δ is the new standard-deviation value).

parameters pdfs are all uniform on $[0, 1]$. Their non-perturbed mean is then $1/2$. Then, the mean of each input is modified in the range of $\delta \in [0.1, 0.9]$. For each δ value and each influent parameter, the graph 6 gives the PLI estimates \hat{S}_δ . Large absolute value of \hat{S}_δ imply a large impact of the perturbation on the FRC. Moreover, the sign of \hat{S}_δ indicates if the new probability has decreased

(negative case) or has increased (positive case). Confidence intervals illustrated on the Figure 6 are obtained thanks to a Gaussian asymptotical property of the estimates $\hat{S}_{i,\delta}$ [13]. In Figure 6, one can see for instance that increasing the h_{12} input mean largely increases the FRC, while decreasing the $ebav_1$ input mean largely decreases the FRC. In contrary, perturbations of the E mean have no effect on the FRC.

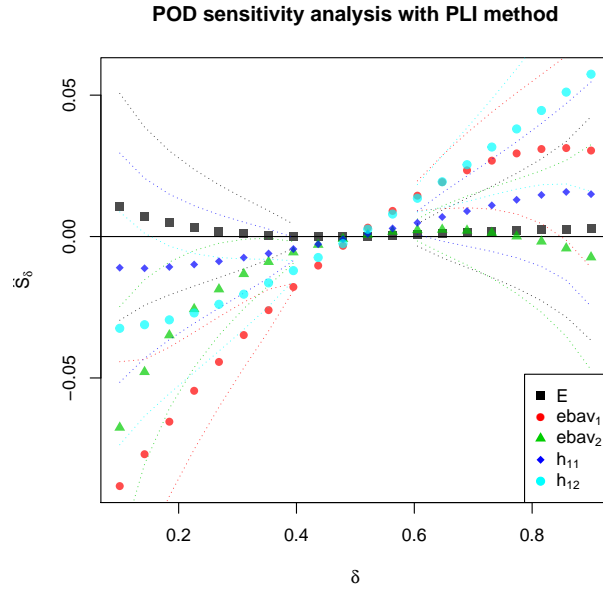


Figure 6: Graphical example for the PLI measures. The present study is on the POD curves.

In order to have a global view of the FRC sensitivity for different defect sizes a , Figure 7 give the PLI graphs for each input, with variations of a and the δ perturbation. Clearly, the inputs $ebav_1$ and h_{12} have strong impacts on the FRC, but especially in the $a < 0.3$ range. In contrary, the inputs E and h_{11} have no influence. These elements are key points for the engineers to understand the effects of the physical parameters on the POD curve. It is also an help for the presentation of the result before for the safety regulation.

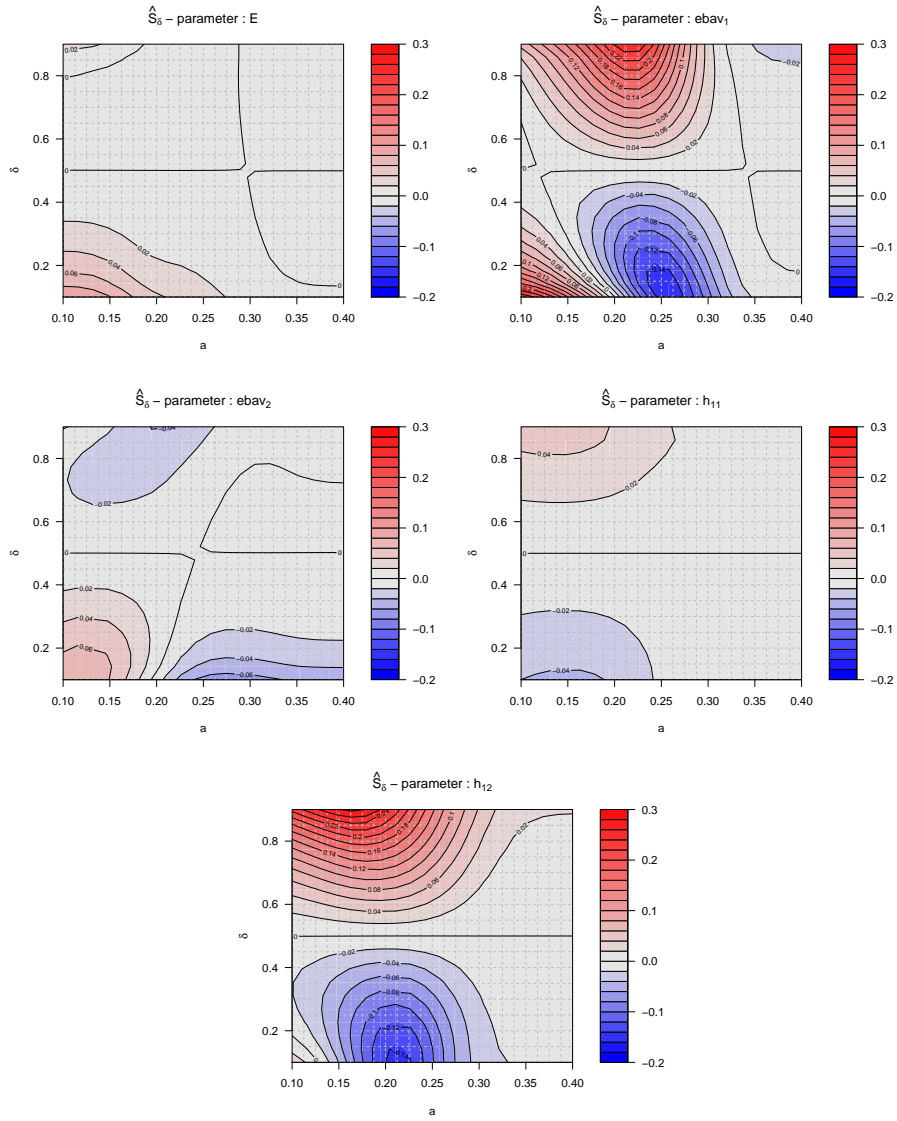


Figure 7: PLI measures on the POD curves for different perturbations on the pdf means and different values of a . $\delta = 0.5$ corresponds to the non perturbed case. The grey color corresponds to no change in the FRC, while the red color (resp. blue) corresponds to an increase (resp. decrease) of the FRC.

5. Conclusion

In this paper, several methods of uncertainty and sensitivity analysis of model outputs have been developed on the particular quantity of interest of the functional risk curve (FRC). A FRC gives the probability of an undesirable event in function of the value of a critical input parameter of a considered physical system. Focus has been made on the use of the Gaussian process metamodel in order to build FRCs from numerical experiments. This approach is useful when the computer model is expensive to evaluate in such a way that only a small sample of the model output can be obtained. In addition to the mean risk curve, the metamodel allows to obtain confidence bands via conditional Gaussian process simulations.

One important advantage of using a metamodel is that sensitivity analysis is facilitated. We have defined two kind of sensitivity indices related to the FRC as the quantity of interest. First, we have formulated the FRC aggregated Sobol' indices, which are variance-based measures. Second, based on perturbation of the pdf of each model input variables, sensitivity indices (called PLI) of the model inputs on the FRC are also proposed. They allow to understand the effect of misjudgment on the pdf of each input parameters'.

An example, coming from non destructive examination simulated experiments, shows the interest of the FRC, which correspond in this context to the probability of detection curves. For the analysis by some experts of the controlled materials, the two sensitivity analysis methods that are proposed reveal a strong complementarity. However, FRCs are used in many other engineering framework, e.g. in the seismic fragility assessment (as shown in section 2.2) and in the evaluation of hydraulic works reliability submitted to extremely high water level [2]. These tools would be also useful in works aiming to separate effects of aleatory input variables and epistemic input parameters. For instance, this problem is highlighted in [38] and [39].

Finally, two main mathematical perspectives are identified from this work. First, adaptive designs can be developed: In the spirit of [40], it would consist

by using the Gaussian process model in order to define some SUR (“Sequential Uncertainty Reduction”) criteria on the FRC as the quantity of interest. Optimizing a such criterion will provide a new set of input parameters values that would be run with the computer code in order to decrease at most the POD confidence interval. Second, [41] have started some works by considering FRC as a random distribution function. This framework allows to deal with stochastic computer code instead of the deterministic ones of this paper (see for example Kleijnen [42]).

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