```
> assume(n>0):with(LinearAlgebra):with(VectorCalculus):
```

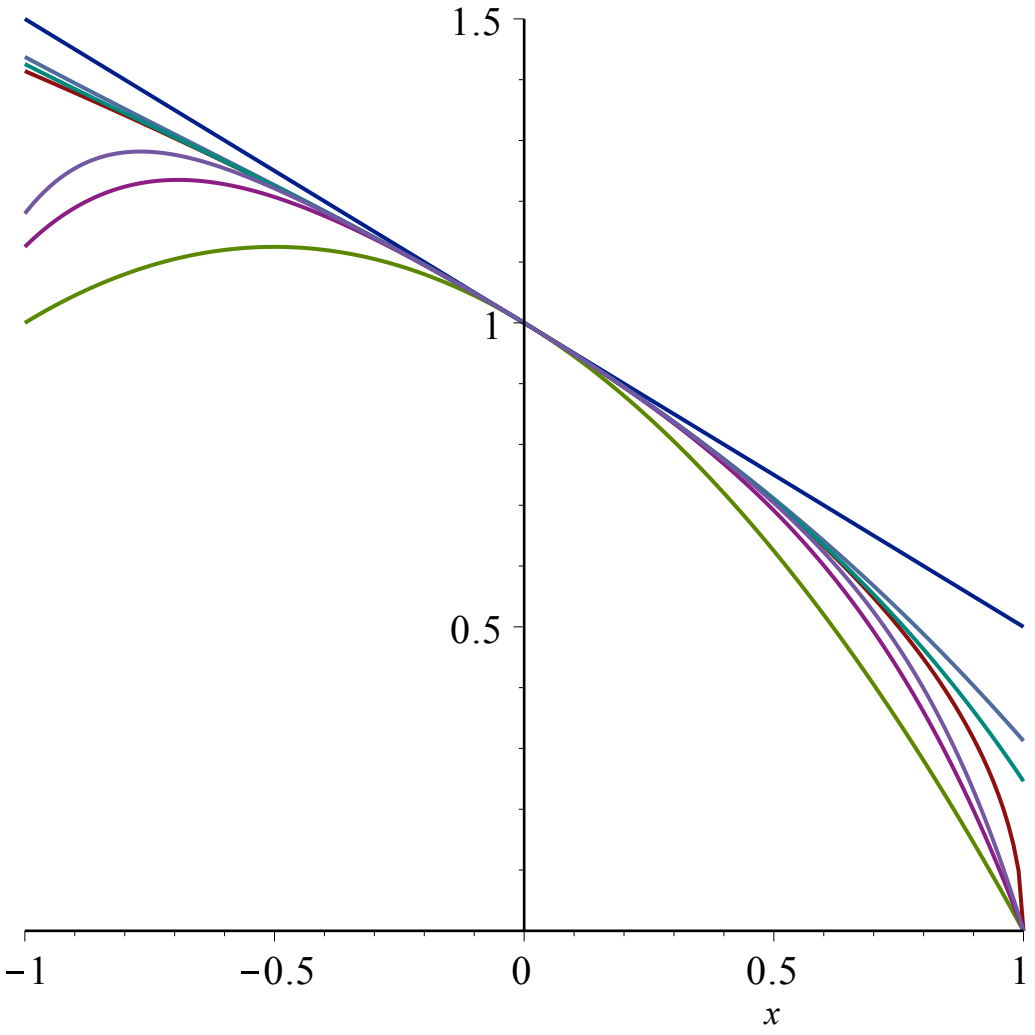
Length of the Voronoi Path in any Dimension

1. Picture

Upper and lower bound of

$\sqrt{1-x}$ *on* $[-1, 1]$ *using Taylor approximation*

```
> plot([      sqrt(1-x),
              1-x/2,
              1-x/2-x^2/2,
              1-x/2-x^2/8-x^3/16,
              1-x/2-x^2/8-x^3/16-5/16*x^4,
              1-x/2-x^2/8-x^3/16-5/128*x^4-7/256*x^5,
              1-x/2-x^2/8-x^3/16-5/128*x^4-7/256*x^5-63/256*x^6
       ],x=-1..1);
```



2. Formulas in General Dimension

```
> Lbound:= d-> GAMMA(d/2)^4*2^(4*d-5)*d/Pi^2/(2*d-2)!*(1-(d-1)/(4*d^2
  -1))*sqrt(2);
```

$$Lbound := d \to \Gamma\left(d \frac{1}{2}\right)^4 2^{4\,d + (-5)}\, d \frac{1}{\pi^2} \frac{1}{(2\,d + (-2))!} \left(1 + VectorCalculus:\text{-}\grave{}\text{-}\grave{}\left((d + (-1))\right) \frac{1}{4\,d^2 + (-1)}\right)\right) \sqrt{2} \tag{1}$$

```
> Ubound:= d-> GAMMA(d/2)^4*2^(4*d-5)*d/Pi^2/(2*d-2)!*(1+1/(4*d-2))*
  sqrt(2);
```

$$Ubound := d \to \Gamma\left(d \frac{1}{2}\right)^4 2^{4\,d + (-5)}\, d \frac{1}{\pi^2} \frac{1}{(2\,d + (-2))!} \left(1 + 1 \frac{1}{4\,d + (-2)}\right) \sqrt{2} \tag{2}$$

3. Dimension 3

```
> M3:=Jacobian([x+r*cos(a1),r*sin(a1)*cos(b1),
                                r*sin(a1)*sin(b1),
                x+r*cos(a2),r*sin(a2)*cos(b2),
                                r*sin(a2)*sin(b2)    ],
             [x,r,a1,b1,a2,b2]);
```

$$M3 := [[1, \cos(a1), -r\sin(a1), 0, 0, 0],$$
$$[0, \sin(a1)\cos(b1), r\cos(a1)\cos(b1), -r\sin(a1)\sin(b1), 0, 0],$$
$$[0, \sin(a1)\sin(b1), r\cos(a1)\sin(b1), r\sin(a1)\cos(b1), 0, 0],$$
$$[1, \cos(a2), 0, 0, -r\sin(a2), 0],$$
$$[0, \sin(a2)\cos(b2), 0, 0, r\cos(a2)\cos(b2), -r\sin(a2)\sin(b2)],$$
$$[0, \sin(a2)\sin(b2), 0, 0, r\cos(a2)\sin(b2), r\sin(a2)\cos(b2)]] \tag{3}$$

```
> detJ3 := simplify(Determinant(M3))/r^4;
```

$$detJ3 := \sin(a1)\sin(a2)(\cos(a1) - \cos(a2)) \tag{4}$$

```
> I3 := int(exp(-n*4/3*Pi*r^3)*r^5,r=0..infinity);
```

$$I3 := \frac{3}{16\, n\sim^2\, \pi^2} \tag{5}$$

```
> dist3 := sqrt((     cos(a1)-cos(a2))^2
                    +(sin(a2)*cos(b2)-sin(a1)*cos(b1))^2
                    +(sin(a2)*sin(b2)-sin(a1)*sin(b1))^2 );
```

$$dist3 := \left((\cos(a1) - \cos(a2))^2 + (\sin(a2)\cos(b2) - \sin(a1)\cos(b1))^2 + (\sin(a2)\sin(b2) - \sin(a1)\sin(b1))^2\right)^{1/2} \tag{6}$$

```
> delta3:= ( cos(a1)*cos(a2)
           +sin(a1)*sin(a2)*(    cos(b1-b2)));
```

$$\delta 3 := \cos(a1)\cos(a2) + \sin(a1)\sin(a2)\cos(b1 - b2) \tag{7}$$

```
> simplify(expand(dist3^2 -sqrt(2-2*delta3)^2));
```

$$0 \tag{8}$$

```
> k:=41;
      TAY(x):=taylor(sqrt(1-x),x,k+1):
      UB := sqrt(2) * eval(convert(TAY(x),polynom),x=delta3):
      LB:= sqrt(2) * ( eval(convert(TAY(x),polynom),x=delta3)-eval
  (convert(TAY(x),polynom),x=1)*delta3^(k+1)):
      int(int(int(int(  detJ3 * UB ,a1=0..a2),a2=0..Pi),
                                          b1=0..2*Pi),b2=0..2*Pi)*
  2:
      print("borne sup");(n^2/2*I3*%);evalf(%);
      int(int(int(int(  detJ3 * LB ,a1=0..a2),a2=0..Pi),
                                          b1=0..2*Pi),b2=0..2*Pi)*
  2:
      print("borne inf");(n^2/2*I3*%);evalf(%);
```

41

"borne sup"

$$\frac{4523370364712510658076963509}{4264485828690604413776035840}\sqrt{2}$$

1.500066356

"borne inf"

$$\frac{7889842784702576406906971 43}{7450005363375152289126809 60}\sqrt{2}$$

1.497706663 **(9)**

Numerical integration

```
> region := [a1 = 0 .. Pi   , a2 = 0 .. Pi,
             b1 = 0 .. 2*Pi  , b2 = 0 .. 2*Pi]:
  evalf(Int( abs(detJ3) * dist3, region, 'epsilon' = 0.0001, 'method
  = _CubaDivonne', 'digits = 80')):
  evalf(n^2/2*I3*%);
  evalf(Int( abs(detJ3) * dist3, region, 'epsilon' = 0.00001, 'method
  = _CubaDivonne', 'digits = 80')):
  evalf(n^2/2*I3*%);
  evalf(Int( abs(detJ3) * dist3, region, 'epsilon' = 0.000001,
  'method = _CubaDivonne', 'digits = 80')):
  evalf(n^2/2*I3*%);
```

1.499885393

1.500003309

1.499999985 **(10)**

4. Dimension 4

```
> M4:=Jacobian([x+r*cos(a1),r*sin(a1)*cos(b1),
                              r*sin(a1)*sin(b1)*cos(c1),
                              r*sin(a1)*sin(b1)*sin(c1),
                 x+r*cos(a2),r*sin(a2)*cos(b2),
                              r*sin(a2)*sin(b2)*cos(c2),
                              r*sin(a2)*sin(b2)*sin(c2)    ],
               [x,r,a1,b1,c1,a2,b2,c2]);
```

$M4 := [[1, \cos(a1), -r\sin(a1), 0, 0, 0, 0, 0],$ **(11)**

$[0, \sin(a1)\cos(b1), r\cos(a1)\cos(b1), -r\sin(a1)\sin(b1), 0, 0, 0, 0],$

$$[0, \sin(a1)\sin(b1)\cos(c1), r\cos(a1)\sin(b1)\cos(c1), r\sin(a1)\cos(b1)\cos(c1),$$
$$-r\sin(a1)\sin(b1)\sin(c1), 0, 0, 0],$$
$$[0, \sin(a1)\sin(b1)\sin(c1), r\cos(a1)\sin(b1)\sin(c1), r\sin(a1)\cos(b1)\sin(c1),$$
$$r\sin(a1)\sin(b1)\cos(c1), 0, 0, 0],$$
$$[1, \cos(a2), 0, 0, 0, -r\sin(a2), 0, 0],$$
$$[0, \sin(a2)\cos(b2), 0, 0, 0, r\cos(a2)\cos(b2), -r\sin(a2)\sin(b2), 0],$$
$$[0, \sin(a2)\sin(b2)\cos(c2), 0, 0, 0, r\cos(a2)\sin(b2)\cos(c2),$$
$$r\sin(a2)\cos(b2)\cos(c2), -r\sin(a2)\sin(b2)\sin(c2)],$$
$$[0, \sin(a2)\sin(b2)\sin(c2), 0, 0, 0, r\cos(a2)\sin(b2)\sin(c2),$$
$$r\sin(a2)\cos(b2)\sin(c2), r\sin(a2)\sin(b2)\cos(c2)]]$$

```
> detJ4 := -simplify(Determinant(M4))/r^6;
```

$$detJ4 := \sin(a1)^2 \sin(b1) \sin(a2)^2 \sin(b2) (\cos(a1) - \cos(a2)) \quad (12)$$

```
> I4 := int(exp(-n*Pi^2/2*r^4)*r^7,r=0..infinity);
```

$$I4 := \frac{1}{n\sim^2 \pi^4} \quad (13)$$

```
> dist4 := sqrt((     cos(a1)-cos(a2))^2
                    +(sin(a2)*cos(b2)-sin(a1)*cos(b1))^2
                    +(sin(a2)*sin(b2)*cos(c2)-sin(a1)*sin(b1)*cos(c1)
  )^2
                    +(sin(a2)*sin(b2)*sin(c2)-sin(a1)*sin(b1)*sin(c1)
  )^2);
```

$$dist4 := \left((\cos(a1) - \cos(a2))^2 + (\sin(a2)\cos(b2) - \sin(a1)\cos(b1))^2 + (\sin(a2)\sin(b2)\cos(c2) - \sin(a1)\sin(b1)\cos(c1))^2 + (\sin(a2)\sin(b2)\sin(c2) - \sin(a1)\sin(b1)\sin(c1))^2\right)^{1/2} \quad (14)$$

```
> delta4:= ( cos(a1)*cos(a2)
           +sin(a1)*sin(a2)*( cos(b1)*cos(b2)
                             +sin(b1)*sin(b2)*(  cos(c1-c2))));
```

$$\delta 4 := \cos(a1)\cos(a2) + \sin(a1)\sin(a2)(\cos(b1)\cos(b2) + \sin(b1)\sin(b2)\cos(c1 - c2)) \quad (15)$$

```
> simplify(expand(dist4^2 -sqrt(2-2*delta4)^2));
```

$$0 \quad (16)$$

```
> k:=7;
      TAY(x):=taylor(sqrt(1-x),x,k+1):
      UB := sqrt(2) * eval(convert(TAY(x),polynom),x=delta4):
      LB:= sqrt(2) * ( eval(convert(TAY(x),polynom),x=delta4)-eval
  (convert(TAY(x),polynom),x=1)*delta4^(k+1)):
      int(int(int(int(int(int(  detJ4 * UB ,a1=0..a2),a2=0..Pi),
                                          b1=0..Pi),b2=0..Pi),
                                          c1=0..2*Pi),c2=0..2*Pi)*
  2:
      print("borne sup");(n^2/2*I4*%);evalf(%);
      int(int(int(int(int(int(  detJ4 * LB ,a1=0..a2),a2=0..Pi),
                                          b1=0..Pi),b2=0..Pi),
                                          c1=0..2*Pi),c2=0..2*Pi)*
  2:
```

```
print("borne inf");(n^2/2*I4*%);evalf(%);
```

$$k := 7$$

"borne sup"

$$\frac{121774997}{10270260}\,\frac{\sqrt{2}}{\pi^2}$$

1.698994377

"borne inf"

$$\frac{102494570}{8729721}\,\frac{\sqrt{2}}{\pi^2}$$

1.682347651 **(17)**

Numerical integration

```
> region := [a1 = 0 .. Pi   , a2 = 0 .. Pi,
             b1 = 0 .. Pi   , b2 = 0 .. Pi,
             c1 = 0 .. 2*Pi   , c2 = 0 .. 2*Pi]:
  evalf(Int( abs(detJ4) * dist4, region, 'epsilon' = 0.0001, 'method
  = _CubaDivonne', 'digits = 80')):
  evalf(n^2/2*I4*%);
  evalf(Int( abs(detJ4) * dist4, region, 'epsilon' = 0.00001, 'method
  = _CubaDivonne', 'digits = 80')):
  evalf(n^2/2*I4*%);
  evalf(Int( abs(detJ4) * dist4, region, 'epsilon' = 0.000001,
  'method = _CubaDivonne', 'digits = 80')):
  evalf(n^2/2*I4*%);
```

1.697720998

1.697649613

1.697649318 **(18)**

5. Dimension 5

```
> M5:=Jacobian([x+r*cos(a1),r*sin(a1)*cos(b1),
                              r*sin(a1)*sin(b1)*cos(c1),
                              r*sin(a1)*sin(b1)*sin(c1)*cos(d1),
                              r*sin(a1)*sin(b1)*sin(c1)*sin(d1),
                  x+r*cos(a2),r*sin(a2)*cos(b2),
                              r*sin(a2)*sin(b2)*cos(c2),
                              r*sin(a2)*sin(b2)*sin(c2)*cos(d2),
                              r*sin(a2)*sin(b2)*sin(c2)*sin(d2)    ],
               [x,r,a1,b1,c1,d1,a2,b2,c2,d2]);
```

$$M5 := [[1, \cos(a1), -r\sin(a1), 0, 0, 0, 0, 0, 0, 0],$$
$$[0, \sin(a1)\cos(b1), r\cos(a1)\cos(b1), -r\sin(a1)\sin(b1), 0, 0, 0, 0, 0, 0],$$
$$[0, \sin(a1)\sin(b1)\cos(c1), r\cos(a1)\sin(b1)\cos(c1), r\sin(a1)\cos(b1)\cos(c1),$$
$$-r\sin(a1)\sin(b1)\sin(c1), 0, 0, 0, 0, 0],$$
$$[0, \sin(a1)\sin(b1)\sin(c1)\cos(d1), r\cos(a1)\sin(b1)\sin(c1)\cos(d1),$$
$$r\sin(a1)\cos(b1)\sin(c1)\cos(d1), r\sin(a1)\sin(b1)\cos(c1)\cos(d1),$$
$$-r\sin(a1)\sin(b1)\sin(c1)\sin(d1), 0, 0, 0, 0],$$ **(19)**

$$[0, \sin(a1)\sin(b1)\sin(c1)\sin(d1), r\cos(a1)\sin(b1)\sin(c1)\sin(d1),$$
$$r\sin(a1)\cos(b1)\sin(c1)\sin(d1), r\sin(a1)\sin(b1)\cos(c1)\sin(d1),$$
$$r\sin(a1)\sin(b1)\sin(c1)\cos(d1), 0, 0, 0, 0],$$
$$[1, \cos(a2), 0, 0, 0, 0, -r\sin(a2), 0, 0, 0],$$
$$[0, \sin(a2)\cos(b2), 0, 0, 0, 0, r\cos(a2)\cos(b2), -r\sin(a2)\sin(b2), 0, 0],$$
$$[0, \sin(a2)\sin(b2)\cos(c2), 0, 0, 0, 0, r\cos(a2)\sin(b2)\cos(c2),$$
$$r\sin(a2)\cos(b2)\cos(c2), -r\sin(a2)\sin(b2)\sin(c2), 0],$$
$$[0, \sin(a2)\sin(b2)\sin(c2)\cos(d2), 0, 0, 0, 0, r\cos(a2)\sin(b2)\sin(c2)\cos(d2),$$
$$r\sin(a2)\cos(b2)\sin(c2)\cos(d2), r\sin(a2)\sin(b2)\cos(c2)\cos(d2),$$
$$-r\sin(a2)\sin(b2)\sin(c2)\sin(d2)],$$
$$[0, \sin(a2)\sin(b2)\sin(c2)\sin(d2), 0, 0, 0, 0, r\cos(a2)\sin(b2)\sin(c2)\sin(d2),$$
$$r\sin(a2)\cos(b2)\sin(c2)\sin(d2), r\sin(a2)\sin(b2)\cos(c2)\sin(d2),$$
$$r\sin(a2)\sin(b2)\sin(c2)\cos(d2)]]$$

```
> detJ5 := simplify(Determinant(M5))/r^8;
```

$$detJ5 := \sin(a1)^3 \sin(b1)^2 \sin(a2)^3 \sin(b2)^2 \sin(c2)\sin(c1)\,(\cos(a1) - \cos(a2)) \quad (20)$$

```
> I5 := int(exp(-n*8/15*Pi^2*r^5)*r^9,r=0..infinity);
```

$$I5 := \frac{45}{64\, n\sim^2\, \pi^4} \quad (21)$$

```
> dist5 := sqrt((     cos(a1)-cos(a2))^2
                    +(sin(a2)*cos(b2)-sin(a1)*cos(b1))^2
                    +(sin(a2)*sin(b2)*cos(c2)-sin(a1)*sin(b1)*cos(c1)
  )^2
                    +(sin(a2)*sin(b2)*sin(c2)*cos(d2)-sin(a1)*sin(b1)
  *sin(c1)*cos(d1))^2
                    +(sin(a2)*sin(b2)*sin(c2)*sin(d2)-sin(a1)*sin(b1)
  *sin(c1)*sin(d1))^2);
```

$$dist5 := \Big((\cos(a1) - \cos(a2))^2 + (\sin(a2)\cos(b2) - \sin(a1)\cos(b1))^2 + (\sin(a2)\sin(b2)\cos(c2) - \sin(a1)\sin(b1)\cos(c1))^2 + (\sin(a2)\sin(b2)\sin(c2)\cos(d2) - \sin(a1)\sin(b1)\sin(c1)\cos(d1))^2 + (\sin(a2)\sin(b2)\sin(c2)\sin(d2) - \sin(a1)\sin(b1)\sin(c1)\sin(d1))^2\Big)^{1/2} \quad (22)$$

```
> delta5:= ( cos(a1)*cos(a2)
           +sin(a1)*sin(a2)*( cos(b1)*cos(b2)
                             +sin(b1)*sin(b2)*( cos(c1)*cos(c2)
                                               +sin(c1)*sin(c2) *
  cos(d1-d2))));
```

$$\delta5 := \cos(a1)\cos(a2) + \sin(a1)\sin(a2)\,(\cos(b1)\cos(b2) + \sin(b1)\sin(b2)\,(\cos(c1)\cos(c2) + \sin(c1)\sin(c2)\cos(d1 - d2))) \quad (23)$$

```
> simplify(expand(dist5^2 -sqrt(2-2*delta5)^2));
```

$$0 \quad (24)$$

```
> k:=3;
      TAY(x):=taylor(sqrt(1-x),x,k+1):
      UB := sqrt(2) * eval(convert(TAY(x),polynom),x=delta5):
```

```
      LB:= sqrt(2) * ( eval(convert(TAY(x),polynom),x=delta5)-eval
(convert(TAY(x),polynom),x=1)*delta5^(k+1)):
      int(int(int(int(int(int(int(int(  detJ5 * UB ,a1=0..a2),a2=0.
.Pi),
                                              b1=0..Pi),b2=0..Pi),
                                              c1=0..Pi),c2=0..Pi),
                                              d1=0..2*Pi),d2=0..2*Pi)*
2:
      print("borne sup");(n^2/2*I5*%);evalf(%);
      int(int(int(int(int(int(int(int(  detJ5 * LB ,a1=0..a2),a2=0.
.Pi),
                                              b1=0..Pi),b2=0..Pi),
                                              c1=0..Pi),c2=0..Pi),
                                              d1=0..2*Pi),d2=0..2*Pi)*
2:
      print("borne inf");(n^2/2*I5*%);evalf(%);
```

$k := 3$

"borne sup"

$$\frac{21305}{16016}\sqrt{2}$$

1.881232514

"borne inf"

$$\frac{135}{104}\sqrt{2}$$

1.835757989 **(25)**

Numerical integration

```
> region := [a1 = 0 .. Pi   , a2 = 0 .. Pi,
             b1 = 0 .. Pi   , b2 = 0 .. Pi,
             c1 = 0 .. Pi   , c2 = 0 .. Pi,
             d1 = 0 .. 2*Pi   , d2 = 0 .. 2*Pi]:
  evalf(Int( abs(detJ5) * dist5, region, 'epsilon' = 0.0001, 'method
  = _CubaDivonne', 'digits = 80')):
  evalf(n^2/2*I5*%);
  evalf(Int( abs(detJ5) * dist5, region, 'epsilon' = 0.00001, 'method
  = _CubaDivonne', 'digits = 80')):
  evalf(n^2/2*I5*%);
  evalf(Int( abs(detJ5) * dist5, region, 'epsilon' = 0.000001,
  'method = _CubaDivonne', 'digits = 80')):
  evalf(n^2/2*I5*%);
```

1.875747647

1.874989353

1.875057472 **(26)**

6. Dimension 6

```
> M6:=Jacobian([x+r*cos(a1),r*sin(a1)*cos(b1),
                            r*sin(a1)*sin(b1)*cos(c1),
                            r*sin(a1)*sin(b1)*sin(c1)*cos(d1),
                            r*sin(a1)*sin(b1)*sin(c1)*sin(d1)*cos
  (e1),
                            r*sin(a1)*sin(b1)*sin(c1)*sin(d1)*sin
  (e1),
```

```
                x+r*cos(a2),r*sin(a2)*cos(b2),
                            r*sin(a2)*sin(b2)*cos(c2),
                            r*sin(a2)*sin(b2)*sin(c2)*cos(d2),
                            r*sin(a2)*sin(b2)*sin(c2)*sin(d2)*cos
  (e2),
                            r*sin(a2)*sin(b2)*sin(c2)*sin(d2)*sin(e2)
     ],
          [x,r,a1,b1,c1,d1,e1,a2,b2,c2,d2,e2]);
```

$$M6 := \begin{bmatrix} \textit{12 x 12 Matrix} \\ \textit{Data Type: anything} \\ \textit{Storage: rectangular} \\ \textit{Order: Fortran\_order} \end{bmatrix} \tag{27}$$

```
> detJ6 := -simplify(Determinant(M6))/r^10;
```

$$detJ6 := \sin(a1)^4 \sin(c1)^2 \sin(b1)^3 \sin(a2)^4 \sin(b2)^3 \sin(c2)^2 \sin(d2) \sin(d1) \left(\cos(a1) - \cos(a2)\right) \tag{28}$$

```
> I6 := int(exp(-n*1/6*Pi^3*r^6)*r^11,r=0..infinity);
```

$$I6 := \frac{6}{n\sim^2 \pi^6} \tag{29}$$

```
> dist6:= sqrt((     cos(a1)-cos(a2))^2
                   +(sin(a2)*cos(b2)-sin(a1)*cos(b1))^2
                   +(sin(a2)*sin(b2)*cos(c2)-sin(a1)*sin(b1)*cos(c1)
  )^2
                   +(sin(a2)*sin(b2)*sin(c2)*cos(d2)-sin(a1)*sin(b1)
  *sin(c1)*cos(d1))^2
                   +(sin(a2)*sin(b2)*sin(c2)*sin(d2)*cos(e2)-sin(a1)
  *sin(b1)*sin(c1)*sin(d1)*cos(e1))^2
                   +(sin(a2)*sin(b2)*sin(c2)*sin(d2)*sin(e2)-sin(a1)
  *sin(b1)*sin(c1)*sin(d1)*sin(e1))^2);
```

$$dist6 := \Big(\left(\cos(a1) - \cos(a2)\right)^2 + \left(\sin(a2)\cos(b2) - \sin(a1)\cos(b1)\right)^2 + \left(\sin(a2)\sin(b2)\cos(c2) - \sin(a1)\sin(b1)\cos(c1)\right)^2 + \left(\sin(a2)\sin(b2)\sin(c2)\cos(d2) - \sin(a1)\sin(b1)\sin(c1)\cos(d1)\right)^2 + \left(\sin(a2)\sin(b2)\sin(c2)\sin(d2)\cos(e2) - \sin(a1)\sin(b1)\sin(c1)\sin(d1)\cos(e1)\right)^2 + \left(\sin(a2)\sin(b2)\sin(c2)\sin(d2)\sin(e2) - \sin(a1)\sin(b1)\sin(c1)\sin(d1)\sin(e1)\right)^2\Big)^{1/2} \tag{30}$$

```
> [Lbound(6),Ubound(6)]; evalf(%);
```

$$\left[\frac{3014656}{225225}\,\frac{\sqrt{2}}{\pi^2}, \frac{753664}{51975}\,\frac{\sqrt{2}}{\pi^2}\right]$$

$$[1.917946193, 2.077775042] \tag{31}$$

Numerical integration

```
> region := [a1 = 0 .. Pi  , a2 = 0 .. Pi,
             b1 = 0 .. Pi  , b2 = 0 .. Pi,
```

```
            c1 = 0 .. Pi  , c2 = 0 .. Pi,
            d1 = 0 .. Pi  , d2 = 0 .. Pi,
            e1 = 0 .. 2*Pi  , e2 = 0 .. 2*Pi]:
evalf(Int( abs(detJ6) * dist6, region, 'epsilon' = 0.0001, 'method
= _CubaDivonne', 'digits = 80')):
evalf(n^2/2*I6*%);
evalf(Int( abs(detJ6) * dist6, region, 'epsilon' = 0.00001, 'method
= _CubaDivonne', 'digits = 80')):
evalf(n^2/2*I6*%);
evalf(Int( abs(detJ6) * dist6, region, 'epsilon' = 0.000001,
'method = _CubaDivonne', 'digits = 80')):
evalf(n^2/2*I6*%);
```

2.035006460

2.037025066

2.050798852 **(32)**

7. Dimension 7

```
> M7:=Jacobian([x+r*cos(a1),r*sin(a1)*cos(b1),
                            r*sin(a1)*sin(b1)*cos(c1),
                            r*sin(a1)*sin(b1)*sin(c1)*cos(d1),
                            r*sin(a1)*sin(b1)*sin(c1)*sin(d1)*cos
  (e1),
                            r*sin(a1)*sin(b1)*sin(c1)*sin(d1)*sin(e1)
  *cos(f1),
                            r*sin(a1)*sin(b1)*sin(c1)*sin(d1)*sin(e1)
  *sin(f1),
              x+r*cos(a2),r*sin(a2)*cos(b2),
                            r*sin(a2)*sin(b2)*cos(c2),
                            r*sin(a2)*sin(b2)*sin(c2)*cos(d2),
                            r*sin(a2)*sin(b2)*sin(c2)*sin(d2)*cos
  (e2),
                            r*sin(a2)*sin(b2)*sin(c2)*sin(d2)*sin(e2)
  *cos(f2),
                            r*sin(a2)*sin(b2)*sin(c2)*sin(d2)*sin(e2)
  *sin(f2)    ],
            [x,r,a1,b1,c1,d1,e1,f1,a2,b2,c2,d2,e2,f2]);
```

$$M7 := \begin{bmatrix} 14 \text{ x } 14 \text{ Matrix} \\ \text{Data Type: anything} \\ \text{Storage: rectangular} \\ \text{Order: Fortran\_order} \end{bmatrix} \quad \textbf{(33)}$$

```
> detJ7 := -simplify(Determinant(M7))/r^12;
```

$$detJ7 := \quad \textbf{(34)}$$

$$-\sin(b1)^4 \sin(c1)^3 \sin(a1)^5 \sin(d1)^2 \sin(e1) \sin(a2)^5 \sin(b2)^4 \sin(c2)^3 \sin(d2)^2 \sin(e2) (\cos(a1) - \cos(a2))$$

```
> I7 := int(exp(-n*16/105*Pi^3*r^7)*r^13,r=0..infinity);
```

$$I7 := \frac{1575}{256\, n\sim^2\, \pi^6} \quad \textbf{(35)}$$

```
> dist7:= sqrt((     cos(a1)-cos(a2))^2
                   +(sin(a2)*cos(b2)-sin(a1)*cos(b1))^2
                   +(sin(a2)*sin(b2)*cos(c2)-sin(a1)*sin(b1)*cos(c1)
```

```
  )^2
                    +(sin(a2)*sin(b2)*sin(c2)*cos(d2)-sin(a1)*sin(b1)
  *sin(c1)*cos(d1))^2
                    +(sin(a2)*sin(b2)*sin(c2)*sin(d2)*cos(e2)-sin(a1)
  *sin(b1)*sin(c1)*sin(d1)*cos(e1))^2
                    +(sin(a2)*sin(b2)*sin(c2)*sin(d2)*sin(e2)*cos(f2)
  -sin(a1)*sin(b1)*sin(c1)*sin(d1)*sin(e1)*cos(f1))^2
                    +(sin(a2)*sin(b2)*sin(c2)*sin(d2)*sin(e2)*sin(f2)
  -sin(a1)*sin(b1)*sin(c1)*sin(d1)*sin(e1)*sin(f1))^2);
```

$$dist7 := \left((\cos(a1)-\cos(a2))^2+(\sin(a2)\cos(b2)-\sin(a1)\cos(b1))^2+(\sin(a2)\sin(b2)\cos(c2)-\sin(a1)\sin(b1)\cos(c1))^2+(\sin(a2)\sin(b2)\sin(c2)\cos(d2)-\sin(a1)\sin(b1)\sin(c1)\cos(d1))^2+(\sin(a2)\sin(b2)\sin(c2)\sin(d2)\cos(e2)-\sin(a1)\sin(b1)\sin(c1)\sin(d1)\cos(e1))^2+(\sin(a2)\sin(b2)\sin(c2)\sin(d2)\sin(e2)\cos(f2)-\sin(a1)\sin(b1)\sin(c1)\sin(d1)\sin(e1)\cos(f1))^2+(\sin(a2)\sin(b2)\sin(c2)\sin(d2)\sin(e2)\sin(f2)-\sin(a1)\sin(b1)\sin(c1)\sin(d1)\sin(e1)\sin(f1))^2\right)^{1/2} \quad (36)$$

```
> [Lbound(7),Ubound(7)]; evalf(%);
```

$$\left[\frac{210}{143}\sqrt{2},\frac{225}{143}\sqrt{2}\right]$$

$$[2.076817120, 2.225161198] \quad (37)$$

Numerical integration

```
> region := [a1 = 0 .. Pi  , a2 = 0 .. Pi,
             b1 = 0 .. Pi  , b2 = 0 .. Pi,
             c1 = 0 .. Pi  , c2 = 0 .. Pi,
             d1 = 0 .. Pi  , d2 = 0 .. Pi,
             e1 = 0 .. Pi  , e2 = 0 .. Pi,
             f1 = 0 .. 2*Pi  , f2 = 0 .. 2*Pi]:
  evalf(Int( abs(detJ7) * dist7, region, 'epsilon' = 0.0001, 'method
  = _CubaDivonne', 'digits = 80')):
  evalf(n^2/2*I7*%);
  evalf(Int( abs(detJ7) * dist7, region, 'epsilon' = 0.00001, 'method
  = _CubaDivonne', 'digits = 80')):
  evalf(n^2/2*I7*%);
  evalf(Int( abs(detJ7) * dist7, region, 'epsilon' = 0.000001,
  'method = _CubaDivonne', 'digits = 80')):
  evalf(n^2/2*I7*%);
```

2.173345690

2.187087657

2.215605108 (38)

8. Dimension 8

```
> M8:=Jacobian([x+r*cos(a1),r*sin(a1)*cos(b1),
```

```
                              r*sin(a1)*sin(b1)*cos(c1),
                              r*sin(a1)*sin(b1)*sin(c1)*cos(d1),
                              r*sin(a1)*sin(b1)*sin(c1)*sin(d1)*cos
  (e1),
                              r*sin(a1)*sin(b1)*sin(c1)*sin(d1)*sin(e1)
  *cos(f1),
                              r*sin(a1)*sin(b1)*sin(c1)*sin(d1)*sin(e1)
  *sin(f1)*cos(g1),
                              r*sin(a1)*sin(b1)*sin(c1)*sin(d1)*sin(e1)
  *sin(f1)*sin(g1),
                 x+r*cos(a2),r*sin(a2)*cos(b2),
                              r*sin(a2)*sin(b2)*cos(c2),
                              r*sin(a2)*sin(b2)*sin(c2)*cos(d2),
                              r*sin(a2)*sin(b2)*sin(c2)*sin(d2)*cos
  (e2),
                              r*sin(a2)*sin(b2)*sin(c2)*sin(d2)*sin(e2)
  *cos(f2),
                              r*sin(a2)*sin(b2)*sin(c2)*sin(d2)*sin(e2)
  *sin(f2)*cos(g2) ,
                              r*sin(a2)*sin(b2)*sin(c2)*sin(d2)*sin(e2)
  *sin(f2)*sin(g2)    ],
              [x,r,a1,b1,c1,d1,e1,f1,g1,a2,b2,c2,d2,e2,f2,g2]);
```

$$M8 := \begin{bmatrix} 16 \text{ x } 16 \text{ Matrix} \\ \text{Data Type: anything} \\ \text{Storage: rectangular} \\ \text{Order: Fortran\_order} \end{bmatrix} \tag{39}$$

```
> detJ8 := -simplify(Determinant(M8))/r^14;
```

$$detJ8 := \tag{40}$$

$$\sin(a1)^6 \sin(b1)^5 \sin(d1)^3 \sin(e1)^2 \sin(f1) \sin(c1)^4 \sin(a2)^6 \sin(b2)^5 \sin(c2)^4$$
$$\sin(d2)^3 \sin(f2) \sin(e2)^2 (\cos(a1) - \cos(a2))$$

```
> I8 := int(exp(-n*1/24*Pi^4*r^8)*r^15,r=0..infinity);
```

$$I8 := \frac{72}{n\sim^2 \pi^8} \tag{41}$$

```
> dist8:= sqrt((     cos(a1)-cos(a2))^2
                   +(sin(a2)*cos(b2)-sin(a1)*cos(b1))^2
                   +(sin(a2)*sin(b2)*cos(c2)-sin(a1)*sin(b1)*cos(c1)
  )^2
                   +(sin(a2)*sin(b2)*sin(c2)*cos(d2)-sin(a1)*sin(b1)
  *sin(c1)*cos(d1))^2
                   +(sin(a2)*sin(b2)*sin(c2)*sin(d2)*cos(e2)-sin(a1)
  *sin(b1)*sin(c1)*sin(d1)*cos(e1))^2
                   +(sin(a2)*sin(b2)*sin(c2)*sin(d2)*sin(e2)*cos(f2)
  -sin(a1)*sin(b1)*sin(c1)*sin(d1)*sin(e1)*cos(f1))^2
                   +(sin(a2)*sin(b2)*sin(c2)*sin(d2)*sin(e2)*sin(f2)
  *cos(g2)-sin(a1)*sin(b1)*sin(c1)*sin(d1)*sin(e1)*sin(f1)*cos(g1))^2
                   +(sin(a2)*sin(b2)*sin(c2)*sin(d2)*sin(e2)*sin(f2)
  *sin(g2)-sin(a1)*sin(b1)*sin(c1)*sin(d1)*sin(e1)*sin(f1)*sin(g1))
  ^2);
```

$$dist8 := \tag{42}$$

$$\left((\cos(a1) - \cos(a2))^2 + (\sin(a2)\cos(b2) - \sin(a1)\cos(b1))^2\right.$$
$$\left. + (\sin(a2)\sin(b2)\cos(c2) - \sin(a1)\sin(b1)\cos(c1))^2\right.$$

$$+ (\sin(a2)\sin(b2)\sin(c2)\cos(d2) - \sin(a1)\sin(b1)\sin(c1)\cos(d1))^2$$
$$+ (\sin(a2)\sin(b2)\sin(c2)\sin(d2)\cos(e2)$$
$$- \sin(a1)\sin(b1)\sin(c1)\sin(d1)\cos(e1))^2$$
$$+ (\sin(a2)\sin(b2)\sin(c2)\sin(d2)\sin(e2)\cos(f2)$$
$$- \sin(a1)\sin(b1)\sin(c1)\sin(d1)\sin(e1)\cos(f1))^2$$
$$+ (\sin(a2)\sin(b2)\sin(c2)\sin(d2)\sin(e2)\sin(f2)\cos(g2)$$
$$- \sin(a1)\sin(b1)\sin(c1)\sin(d1)\sin(e1)\sin(f1)\cos(g1))^2$$
$$+ (\sin(a2)\sin(b2)\sin(c2)\sin(d2)\sin(e2)\sin(f2)\sin(g2)$$
$$- \sin(a1)\sin(b1)\sin(c1)\sin(d1)\sin(e1)\sin(f1)\sin(g1))^2)^{1/2}$$

```
> [Lbound(8),Ubound(8)]; evalf(%);
```

$$\left[\frac{2080374784}{134008875}\,\frac{\sqrt{2}}{\pi^2}, \frac{130023424}{7882875}\,\frac{\sqrt{2}}{\pi^2}\right]$$

$$[2.224453195, 2.363481519] \quad (43)$$

Numerical integration

```
> region := [a1 = 0 .. Pi  , a2 = 0 .. Pi,
             b1 = 0 .. Pi  , b2 = 0 .. Pi,
             c1 = 0 .. Pi  , c2 = 0 .. Pi,
             d1 = 0 .. Pi  , d2 = 0 .. Pi,
             e1 = 0 .. Pi  , e2 = 0 .. Pi,
             f1 = 0 .. Pi  , f2 = 0 .. Pi,
             g1 = 0 .. 2*Pi, g2 = 0 .. 2*Pi]:
  evalf(Int( abs(detJ8) * dist8, region, 'epsilon' = 0.0001, 'method
  = _CubaDivonne', 'digits = 80')):
  evalf(n^2/2*I8*%);
  evalf(Int( abs(detJ8) * dist8, region, 'epsilon' = 0.00001, 'method
  = _CubaDivonne', 'digits = 80')):
  evalf(n^2/2*I8*%);
  evalf(Int( abs(detJ8) * dist8, region, 'epsilon' = 0.000001,
  'method = _CubaDivonne', 'digits = 80')):
  evalf(n^2/2*I8*%);
```

2.329702008

2.393338423

2.329423158 **(44)**