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► **To cite this version:**

Hongliang Lu. Vertex-Coloring Edge-Weighting of Bipartite Graphs with Two Edge Weights. Discrete Mathematics and Theoretical Computer Science, 2016, Vol. 17 no. 3 (3), pp.1-12. 10.46298/dmtcs.2162 . hal-01352856

**HAL Id: hal-01352856**

**<https://inria.hal.science/hal-01352856>**

Submitted on 16 Aug 2016

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# Vertex-Coloring Edge-Weighting of Bipartite Graphs with Two Edge Weights\*

Hongliang Lu<sup>†</sup>

*School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an, PR China*

*received 28<sup>th</sup> Feb. 2015, revised 29<sup>th</sup> Oct. 2015, accepted 9<sup>th</sup> Dec. 2015.*

Let  $G$  be a graph and  $S$  be a subset of  $Z$ . A vertex-coloring  $S$ -edge-weighting of  $G$  is an assignment of weights by the elements of  $S$  to each edge of  $G$  so that adjacent vertices have different sums of incident edges weights.

It was proved that every 3-connected bipartite graph admits a vertex-coloring  $S$ -edge-weighting for  $S = \{1, 2\}$  (H. Lu, Q. Yu and C. Zhang, Vertex-coloring 2-edge-weighting of graphs, *European J. Combin.*, **32** (2011), 22-27). In this paper, we show that every 2-connected and 3-edge-connected bipartite graph admits a vertex-coloring  $S$ -edge-weighting for  $S \in \{\{0, 1\}, \{1, 2\}\}$ . These bounds we obtain are tight, since there exists a family of infinite bipartite graphs which are 2-connected and do not admit vertex-coloring  $S$ -edge-weightings for  $S \in \{\{0, 1\}, \{1, 2\}\}$ .

**Keywords:** edge-weighting, vertex-coloring, 2-connected, bipartite graph

## 1 Introduction

In this paper, we consider only finite, undirected and simple connected graphs. For a vertex  $v$  of graph  $G = (V, E)$ ,  $N_G(v)$  denotes the set of vertices which are adjacent to  $v$  and  $d_G(v) = |N_G(v)|$  is called the *degree* of vertex  $v$ . Let  $\delta(G)$  and  $\Delta(G)$  denote the minimum degree and maximum degree of graph  $G$ , respectively. For  $v \in V(G)$  and  $r \in Z^+$ , let  $N_G^r(v) = \{u \in N(v) \mid d_G(u) = r\}$ . If  $v \in V(G)$  and  $e \in E(G)$ , we use  $v \sim e$  to denote that  $v$  is an end-vertex of  $e$ . For two disjoint subsets  $S, T$  of  $V(G)$ , let  $E_G(S, T)$  denote the subset of edges of  $E(G)$  with one end in  $S$  and other end in  $T$  and let  $e_G(S, T) = |E_G(S, T)|$ . Let  $G = (U, W, E)$  denote a bipartite graph with bipartition  $(U, W)$  and edge set  $E$ .

Let  $S$  be a subset of  $Z$ . An  $S$ -edge-weighting of a graph  $G$  is an assignment  $w : E(G) \rightarrow S$ . An  $S$ -edge-weighting  $w$  of a graph  $G$  induces a coloring of the vertices of  $G$ , where the color of vertex  $v$ , denoted by  $c(v)$ , is  $\sum_{e \sim v} w(e)$ . An  $S$ -edge-weighting of a graph  $G$  is a *vertex-coloring* if for every edge  $e = uv$ ,  $c(u) \neq c(v)$  and we say that  $G$  admits a *vertex-coloring  $S$ -edge-weighting*. If  $S = \{1, 2, \dots, k\}$ , then a vertex-coloring  $S$ -edge-weighting of a graph  $G$  is usually called a *vertex-coloring  $k$ -edge-weighting*.

For vertex-coloring edge-weighting, Karoński et al. (2004) posed the following conjecture:

\*This work was supported by the National Natural Science Foundation of China No. 11471257 and the Fundamental Research Funds for the Central Universities.

<sup>†</sup>Corresponding email: luhongliang215@sina.com

**Conjecture 1.1** *Every graph without isolated edges admits a vertex-coloring 3-edge-weighting.*

This conjecture is still wide open. Karoński et al. (2004) showed that Conjecture 1.1 is true for 3-colorable graphs. Recently, Kalkowski et al. (2010) showed that every graph without isolated edges admits a vertex-coloring 5-edge-weighting. This result is an improvement on the previous bounds on  $k$  established by Addario-Berry et al. (2007), Addario-Berry et al. (2008), and Wang and Yu (2008), who obtained the bounds  $k = 30$ ,  $k = 16$ , and  $k = 13$ , respectively.

Many graphs actually admit a vertex-coloring 2-edge-weighting (in fact, experiments suggest (see Addario-Berry et al. (2008)) that almost all graphs admit a vertex-coloring 2-edge-weighting), however it is not known which ones do not. Khatirinejad et al. (2012) explored the problem of classifying those graphs which admit a vertex-coloring 2-edge-weighting. Chang et al. (2011) had made some progress in determining which classes of graphs admit vertex-coloring 2-edge-weighting, and proved that there exists a family of infinite bipartite graphs (e.g., the generalized  $\theta$ -graphs) which are 2-connected and admit a vertex-coloring 3-edge-weighting but not vertex-coloring 2-edge-weightings. Lu et al. (2011) showed that every 3-connected bipartite graph admits a vertex-coloring 2-edge-weighting.

We write

$$\begin{aligned}\mathcal{G}_{12} &= \{G \mid G \text{ admits a vertex-coloring } \{1, 2\}\text{-edge-weighting}\}; \\ \mathcal{G}_{01} &= \{G \mid G \text{ admits a vertex-coloring } \{0, 1\}\text{-edge-weighting}\}; \\ \mathcal{G}_{12}^* &= \{G \mid G \text{ is bipartite and admits a vertex-coloring } \{1, 2\}\text{-edge-weighting}\}; \\ \mathcal{G}_{01}^* &= \{G \mid G \text{ is bipartite and admits a vertex-coloring } \{0, 1\}\text{-edge-weighting}\}.\end{aligned}$$

Dudek and Wajc (2011) showed that determining whether a graph belongs to  $\mathcal{G}_{12}$  or  $\mathcal{G}_{01}$  is NP-complete. Moreover, they showed that  $\mathcal{G}_{12} \neq \mathcal{G}_{01}$ . The counterexamples constructed by Dudek and Wajc (2011) are non-bipartite.

Now we construct a bipartite graph, which admits a vertex-coloring 2-edge-weighting but not vertex-coloring  $\{0, 1\}$ -edge-weightings. Let  $C_6$  be a cycle of length six and  $\Gamma$  be a graph obtained by connecting an isolated vertex to one of the vertices of  $C_6$ . Take two disjoint copies of  $\Gamma$ . Connect two vertices of degree one of the two copies and this gives a connected bipartite graph  $G$ . It is easy to prove that  $G$  admits a vertex-coloring 2-edge-weighting but not vertex-coloring  $\{0, 1\}$ -edge-weighting. Hence  $\mathcal{G}_{01}^* \neq \mathcal{G}_{12}^*$ . Next we would like to propose the following problem.

**Problem 1** *Determining whether a graph  $G \in \mathcal{G}_{12}^*$  or  $G \in \mathcal{G}_{01}^*$  is polynomial?*

In this paper, we characterize bipartite graphs which admit a vertex-coloring  $\mathcal{S}$ -edge-weighting for  $\mathcal{S} \in \{\{0, 1\}, \{1, 2\}\}$ , and obtain the following result.

**Theorem 1.2** *Let  $G$  be a 3-edge-connected bipartite graph  $G = (U, W, E)$  with minimum degree  $\delta(G)$ . If  $G$  contains a vertex  $u$  of degree  $\delta(G)$  such that  $G - u$  is connected, then  $G$  admits a vertex-coloring  $\mathcal{S}$ -edge-weighting for  $\mathcal{S} \in \{\{0, 1\}, \{1, 2\}\}$ .*

By Theorem 1.2, it is easy to obtain the following result, which improves and extends the result obtained by Lu et al. (2011).

**Theorem 1.3** *Every 2-connected and 3-edge-connected bipartite graph  $G = (U, W, E)$  admits a vertex-coloring  $\mathcal{S}$ -edge-weighting for  $\mathcal{S} \in \{\{0, 1\}, \{1, 2\}\}$ .*

So far, all known counterexamples of bipartite graphs, which do not have vertex-coloring  $\{0, 1\}$ -edge-weightings or vertex-coloring  $\{1, 2\}$ -edge-weightings are graphs with minimum degree 2. So we would like to propose the following problem.

**Problem 2** *Does every bipartite graph with  $\delta(G) \geq 3$  admit a vertex-coloring  $\mathcal{S}$ -edge-weighting, where  $\mathcal{S} \in \{\{0, 1\}, \{1, 2\}\}$ .*

A *factor* of a graph  $G$  is a spanning subgraph. For a graph  $G$ , there is a close relationship between 2-edge-weighting and graph factors. Namely, a 2-edge-weighting problem is equivalent to finding special factors of graphs (see Addario-Berry et al. (2007) and Addario-Berry et al. (2008)). So to find factors with pre-specified degree is an important part of edge-weighting.

Let  $g, f : V(G) \rightarrow Z$  be two integer-valued functions such that  $g(v) \leq f(v)$  and  $g(v) \equiv f(v) \pmod{2}$  for all  $v \in V(G)$ . A factor  $F$  of  $G$  is called  $(g, f)$ -parity factor if  $g(v) \leq d_F(v) \leq f(v)$  and  $d_F(v) \equiv f(v) \pmod{2}$  for all  $v \in V(G)$ . For  $X \subseteq V(G)$ , we write  $g(X) = \sum_{x \in X} g(x)$  and  $f(X)$  is defined similarly. For  $(g, f)$ -parity factors, Lovász obtained a sufficient and necessary condition.

**Theorem 1.4 (Lovász (1972))** *A graph  $G$  contains a  $(g, f)$ -parity factor if and only if for any two disjoint subsets  $S$  and  $T$  of  $V(G)$ , it follows that*

$$\eta(S, T) = f(S) - g(T) + \sum_{x \in T} d_{G-S}(x) - \tau(S, T) \geq 0,$$

where  $\tau(S, T)$  denotes the number of components  $C$ , called  $g$ -odd components of  $G - S - T$  such that  $g(V(C)) + e_G(V(C), T) \equiv 1 \pmod{2}$ .

In the proof of the main theorems, we also need the following three lemmas.

**Theorem 1.5 (Chang et al. (2011))** *Every non-trivial connected bipartite graph  $G = (A, B, E)$  with  $|A|$  even, admits a vertex-coloring 2-edge-weighting  $w$  such that  $c(u)$  is odd for  $u \in A$  and  $c(v)$  is even for  $v \in B$ .*

**Theorem 1.6 (Chang et al. (2011))** *Let  $r \geq 3$  be an integer. Every  $r$ -regular bipartite graph  $G$  admits a vertex-coloring 2-edge-weighting.*

**Theorem 1.7 (Khatirinejad et al. (2012))** *Every  $r$ -regular graph  $G$  admits a vertex-coloring 2-edge-weighting if and only if it admits a vertex-coloring  $\{0, 1\}$ -edge-weighting.*

## 2 Proof of Theorem 1.2

**Corollary 2.1** *Every non-trivial connected bipartite graph  $G = (A, B, E)$  with  $|A|$  even admits a vertex-coloring  $\{0, 1\}$ -edge-weighting.*

**Proof:** By Theorem 1.5,  $G$  admits a vertex-coloring 2-edge-weighting  $w$  such that  $c(u)$  is odd for  $u \in A$  and  $c(v)$  is even for  $v \in B$ . Let  $w'(e) = 0$  if  $w(e) = 2$  and  $w'(e) = 1$  if  $w(e) = 1$ . Then  $w'$  is a vertex-coloring  $\{0, 1\}$ -edge-weighting of graph  $G$ .  $\square$

For completing the proof of Theorem 1.2, we need the following two technical lemmas.

**Lemma 2.2** *Let  $G$  be a bipartite graph with bipartition  $(U, W)$ , where  $|U| \equiv |W| \equiv 1 \pmod{2}$ . Let  $\delta(G) = \delta$  and  $u \in U$  such that  $d_G(u) = \delta$ . If one of the following two conditions holds, then  $G$  contains a factor  $F$  such that  $d_F(u) = \delta$ ,  $d_F(x) \equiv \delta + 1 \pmod{2}$  for all  $x \in U - u$ ,  $d_F(y) \equiv \delta \pmod{2}$  for all  $y \in W$  and  $d_F(y) \leq \delta - 2$  for all  $y \in N_G^\delta(u)$ .*

(i)  $\delta(G) \geq 4$ ,  $G$  is 3-edge-connected and  $G - u$  is connected.

(ii)  $\delta(G) = 3$ ,  $G$  is 3-edge-connected and  $|N_G^\delta(u)| \leq 2$ .

**Proof:** Let  $M$  be an integer such that  $M \geq \Delta(G)$  and  $M \equiv \delta \pmod{2}$ . Let  $m \in \{0, -1\}$  such that  $m \equiv \delta \pmod{2}$ . Let  $g, f : V(G) \rightarrow \mathbb{Z}$  such that

$$g(x) = \begin{cases} \delta & \text{if } x = u, \\ m - 1 & \text{if } x \in U - u, \\ m & \text{if } x \in W, \end{cases}$$

and

$$f(x) = \begin{cases} M + 1 & \text{if } x \in U - u, \\ M & \text{if } x \in (W \cup \{u\}) - N_G^\delta(u), \\ \delta - 2 & \text{if } x \in N_G^\delta(u). \end{cases}$$

By definition, we have  $g(v) \equiv f(v) \pmod{2}$  for all  $v \in V(G)$ . It is sufficient for us to show that  $G$  contains a  $(g, f)$ -parity factor. Indirectly, suppose that  $G$  contains no  $(g, f)$ -parity factors. By Theorem 1.4, there exist two disjoint subsets  $S$  and  $T$  such that

$$\eta(S, T) = f(S) - g(T) + \sum_{x \in T} d_{G-S}(x) - \tau(S, T) < 0,$$

where  $\tau(S, T)$  denotes the number of  $g$ -odd components of  $G - S - T$ . Since  $f(V(G))$  is even, by parity, we have

$$\eta(S, T) = f(S) - g(T) + \sum_{x \in T} d_{G-S}(x) - \tau(S, T) \leq -2. \quad (1)$$

Hence  $S \cup T \neq \emptyset$ . We choose  $S$  and  $T$  such that  $S \cup T$  is minimal. Let  $A = V(G) - S - T$ .

**Claim 1.**  $T \subseteq \{u\}$ .

Otherwise, let  $v \in T - u$  and  $T' = T - v$ . We have

$$\begin{aligned}
\eta(S, T') &= f(S) - g(T') + \sum_{x \in T'} d_{G-S}(x) - \tau(S, T') \\
&= f(S) - (g(T) - g(v)) + \left( \sum_{x \in T} d_{G-S}(x) - d_{G-S}(v) \right) - \tau(S, T') \\
&\leq f(S) - g(T) + \left( \sum_{x \in T} d_{G-S}(x) - d_{G-S}(v) \right) - (\tau(S, T) - e_G(v, A)) + g(v) \\
&\leq f(S) - g(T) + \sum_{x \in T} d_{G-S}(x) - \tau(S, T) + (g(v) - d_{G-S}(v) + e_G(v, A)) \\
&= \eta(S, T) + (g(v) - d_{G-S}(v) + e_G(v, A)) \\
&\leq \eta(S, T) - (d_{G-S}(v) - e_G(v, A)) \\
&\leq \eta(S, T) \leq -2,
\end{aligned}$$

contradicting the choice of  $S$  and  $T$ .

**Claim 2.**  $S \subseteq N_G^\delta(u)$ .

Otherwise, suppose that  $S - N_G^\delta(u) \neq \emptyset$  and let  $v \in S - N_G^\delta(u)$ . Let  $S' = S - v$ . We have

$$\begin{aligned}
\eta(S', T) &= f(S') - g(T) + \sum_{x \in T} d_{G-S'}(x) - \tau(S', T) \\
&= (f(S) - f(v)) - g(T) + \left( \sum_{x \in T} d_{G-S}(x) + e_G(v, T) \right) - \tau(S', T) \\
&\leq f(S) - g(T) + \sum_{x \in T} d_{G-S}(x) - (\tau(S, T) - e_G(v, A)) - f(v) + e_G(v, T) \\
&= f(S) - g(T) + \sum_{x \in T} d_{G-S}(x) - \tau(S, T) + (e_G(v, T) + e_G(v, A) - f(v)) \\
&\leq \eta(S, T) + (d_G(v) - f(v)) \\
&\leq \eta(S, T) \leq -2,
\end{aligned}$$

contradicting the choice of  $S$  and  $T$  again.

We write  $\tau(S, T) = \tau$ . By Claims 1 and 2, we have

$$\begin{aligned}
\eta(S, T) &= f(S) - g(T) + \sum_{x \in T} d_{G-S}(x) - \tau \\
&= (\delta - 2)|S| - \delta|T| + |T|(\delta - |S|) - \tau \quad (\text{by Claims 1 and 2}) \\
&= (\delta - 2 - |T|)|S| - \tau \quad (\text{by Claim 1}) \\
&\leq -2,
\end{aligned}$$

i.e.,

$$(\delta - 2 - |T|)|S| + 2 \leq \tau, \quad (2)$$

which implies  $\tau \geq 2$  since  $|T| \leq 1$ .

Since  $G - u$  is connected, we may see that  $S \neq \emptyset$ . Note that  $G$  is 3-edge-connected, by Claims 1 and 2, we have

$$\begin{aligned} 3\tau &\leq e_G(A, S \cup T) \\ &= e_G(A, S) + e_G(A, T) \\ &\leq (\delta - |T|)|S| + |T|(\delta - |S|) \quad (\text{by Claims 1 and 2}) \\ &= (\delta - 2|T|)|S| + |T|\delta, \end{aligned}$$

i.e.,

$$3\tau \leq (\delta - 2|T|)|S| + |T|\delta. \quad (3)$$

Combining (2) and (3), we may see that

$$\delta|T| \geq (2\delta - |T| - 6)|S| + 6. \quad (4)$$

If  $\delta \geq 4$ , then we have

$$\begin{aligned} \delta &\geq \delta|T| \quad (\text{since } |T| \leq 1) \\ &\geq (2\delta - |T| - 6)|S| + 6 \quad (\text{since } |S| \geq 1) \\ &\geq 2\delta - |T| \\ &\geq 2\delta - 1, \end{aligned}$$

a contradiction. So we may assume that  $\delta = 3$ . Note that  $|S| \leq |N_G^\delta(u)| \leq 2$ . By (4), we have

$$3 = \delta \geq \delta|T| \geq -|T||S| + 6 \geq 4, \quad (5)$$

a contradiction again.

This completes the proof.  $\square$

**Lemma 2.3** *Let  $G$  be a bipartite graph with bipartition  $(U, W)$ , where  $|U| \equiv |W| \equiv 1 \pmod{2}$ . Let  $\delta(G) = \delta$  and  $u \in U$  such that  $d_G(u) = \delta$ . If one of the following two conditions holds, then  $G$  contains a factor  $F$  such that  $d_F(u) = 0$ ,  $d_F(x) \equiv 1 \pmod{2}$  for all  $x \in U - u$ ,  $d_F(y) \equiv 0 \pmod{2}$  for all  $x \in W$  and  $d_F(y) \geq 2$  for all  $y \in N_G(u)$ .*

(i)  $\delta(G) \geq 4$ ,  $G$  is 3-edge-connected and  $G - u$  is connected.

(ii)  $\delta(G) = 3$ ,  $G$  is 3-edge-connected and there exists a vertex  $v \in N_G(u)$  such that  $d_G(v) > 3$ .

**Proof:** Let  $M$  be an even integer such that  $M \geq \Delta(G)$ . Let  $g, f : V(G) \rightarrow Z$  such that

$$g(x) = \begin{cases} 0 & \text{if } x \in (\{u\} \cup W) - N_G(u), \\ 2 & \text{if } x \in N_G(u), \\ -1 & \text{if } x \in U - u, \end{cases}$$

and

$$f(x) = \begin{cases} M & \text{if } x \in W \\ 0 & \text{if } x = u, \\ M + 1 & \text{if } x \in U - u. \end{cases}$$

Clearly,  $g(v) \equiv f(v) \pmod{2}$  for all  $v \in V(G)$  and  $g(V(G))$  is even. It is also sufficient for us to show that  $G$  contains a  $(g, f)$ -parity factor.

Indirectly, suppose that  $G$  contains no  $(g, f)$ -parity factors. By Theorem 1.4, there exist two disjoint subsets  $S$  and  $T$  such that

$$\eta(S, T) = f(S) - g(T) + \sum_{x \in T} d_{G-S}(x) - \tau(S, T) \leq -2, \quad (6)$$

where  $\tau(S, T)$  denotes the number of  $g$ -odd components of  $G - S - T$ . We choose  $S$  and  $T$  such that  $S \cup T$  is minimal. Let  $A = V(G) - S - T$ .

**Claim 1.**  $S \subseteq \{u\}$ .

Otherwise, suppose that there exists a vertex  $v \in S - u$ . Let  $S' = S - v$ . Then we have

$$\begin{aligned} \eta(S', T) &= f(S') - g(T) + \sum_{y \in T} d_{G-S'}(y) - \tau(S', T) \\ &= (f(S) - f(v)) - g(T) + \left( \sum_{y \in T} d_{G-S}(y) + e_G(v, T) \right) - \tau(S', T) \\ &\leq f(S) - g(T) + \sum_{y \in T} d_{G-S}(y) - f(v) + e_G(v, T) - (\tau(S, T) - e_G(v, A)) \\ &= \eta(S, T) - (f(v) - e_G(v, T) - e_G(v, A)) \\ &\leq \eta(S, T) - (f(v) - d_G(v)) \\ &\leq \eta(S, T) \leq -2, \end{aligned}$$

contradicting the the choice of  $S \cup T$ .

**Claim 2.**  $T \subseteq N_G(u)$ .



Otherwise, suppose that  $T - N_G(u) \neq \emptyset$ . Let  $x \in T - N_G(u)$  and let  $T' = T - x$ . Then we have

$$\begin{aligned}
\eta(S, T') &= f(S) - g(T') + \sum_{y \in T'} d_{G-S}(y) - \tau(S, T') \\
&= f(S) - (g(T) - g(x)) + \left( \sum_{y \in T} d_{G-S}(y) - d_{G-S}(x) \right) - \tau(S, T') \\
&\leq f(S) - (g(T) - g(x)) + \left( \sum_{y \in T} d_{G-S}(y) - d_{G-S}(x) \right) - (\tau(S, T) - e_G(x, A)) \\
&= f(S) - g(T) + \sum_{y \in T} d_{G-S}(y) - \tau(S, T) - (d_{G-S}(x) - e_G(x, A)) + g(x) \\
&= \eta(S, T) - (d_{G-S}(x) - e_G(x, A)) + g(x) \\
&\leq \eta(S, T) \leq -2,
\end{aligned}$$

contradicting the choice of  $S \cup T$ .

By Claims 1 and 2, we may see that  $f(S) = 0$  and  $g(T) = 2|T|$ . For simplicity, we write  $\tau(S, T) = \tau$ . By (6), we see that

$$\tau \geq \sum_{x \in T} (d_G(x) - |S|) - 2|T| + 2, \quad (7)$$

which implies

$$\tau \geq \sum_{x \in T} (\delta - 1) - 2|T| + 2 \geq 2. \quad (8)$$

Note that  $G - u$  is connected, so we have  $|T| \geq 1$ . Since  $G$  is 3-edge-connected, we have

$$\begin{aligned}
3\tau &\leq \sum_{x \in T} (d_G(x) - |S|) + (\delta - |T|)|S| \\
&= \sum_{x \in T} d_G(x) + (\delta - 2|T|)|S|,
\end{aligned}$$

i.e.,

$$3\tau \leq \sum_{x \in T} d_G(x) + (\delta - 2|T|)|S|. \quad (9)$$

Inequalities (7) and (9) implies

$$\begin{aligned}
2 \sum_{x \in T} d_G(x) + 6 &\leq |S||T| + 6|T| + \delta|S| \quad (\text{since } |S| \leq 1 \text{ and } |T| \geq 1) \\
&\leq 7|T| + \delta,
\end{aligned}$$

i.e.,

$$7|T| \geq 2 \sum_{x \in T} d_G(x) + 6 - \delta. \quad (10)$$

If  $\delta \geq 4$ , by (10), it follows

$$7|T| \geq 6 + \delta(2|T| - 1) \geq 8|T| + 2,$$

a contradiction. So we may assume that  $\delta = 3$ . By condition (ii),  $\sum_{x \in T} d_G(x) \geq 3|T| + 1$ . Combining (10),

$$\begin{aligned} 7|T| &\geq 2 \sum_{x \in T} d_G(x) + 6 - \delta \\ &\geq 2(3|T| + 1) + 3, \end{aligned}$$

which implies  $|T| \geq 5$ , a contradiction since  $|T| \leq |N_G(u)| \leq 3$ .

This completes the proof.  $\square$

**Proof of Theorem 1.2:** By Theorem 1.5 and Corollary 2.1, we can assume that both  $|A|$  and  $|B|$  are odd.

Firstly, we consider  $\mathcal{S} = \{0, 1\}$ . If  $G$  is 3-regular, by Theorem 1.6, then  $G$  admits a vertex-coloring 2-edge-weighting. By Theorem 1.7,  $G$  also admits a vertex-coloring  $\{0, 1\}$ -edge-weighting. So we can assume that  $\delta(G) \geq 3$  and  $G$  is not 3-regular. If  $\delta(G) = 3$ , since  $G$  is 3-edge-connected, then  $G - x$  is connected for every vertex  $x$  of  $G$  with degree three. Hence there exists a vertex  $v$  with degree three such that  $N_G(v)$  contains a vertex with degree at least four. Let

$$u^* = \begin{cases} u & \text{if } \delta \geq 4, \\ v & \text{if } \delta = 3. \end{cases}$$

Without loss generality, we may assume that  $u^* \in U$  and so it is a vertex satisfying the conditions of Lemma 2.3. Hence by Lemma 2.3,  $G$  contains a factor  $F$ , which satisfies the following three conditions.

- (i)  $d_F(u^*) = 0$ ;
- (ii)  $d_F(x) \equiv 1 \pmod{2}$  for all  $x \in U - u^*$ ;
- (iii)  $d_F(y) \equiv 0 \pmod{2}$  for all  $y \in W$  and  $d_F(y) \geq 2$  for all  $y \in N_G(u^*)$ .

Clearly,  $d_F(x) \neq d_F(y)$  for all  $xy \in E(G)$ . We assign weight 1 for each edge of  $E(F)$  and weight 0 for each edge of  $E(G) - E(F)$ . Then we obtain a vertex-coloring  $\{0, 1\}$ -edge-weighting of graph  $G$ .

Secondly, we show that  $G$  admits a vertex-coloring 2-edge-weighting. By Theorem 1.6, we may assume that  $G$  is not 3-regular. If  $\delta = 3$ , since  $G$  is 3-edge-connected, then  $G$  contains a vertex  $v'$  such that  $d_G(v') = 3$ ,  $G - v'$  is connected and  $|N_G^\delta(v')| \leq 2$ . Let

$$u^* = \begin{cases} u & \text{if } \delta \geq 4, \\ v' & \text{if } \delta = 3. \end{cases}$$

Then  $u^*$  is a vertex satisfying the conditions of Lemma 2.2. Hence by Lemma 2.2,  $G$  contains a factor  $F$  such that

- (i)  $d_F(u^*) = \delta$ ;
- (ii)  $d_F(x) \equiv \delta \pmod{2}$  for all  $x \in W$  and  $d_F(x) \leq \delta - 2$  for all  $x \in N_G^\delta(u^*)$ ;
- (iii)  $d_F(y) \equiv \delta + 1 \pmod{2}$  for all  $y \in U - u^*$ .

Let  $w : E(G) \rightarrow \{1, 2\}$  be a 2-edge-weighting such that  $w(e) = 1$  for each  $e \in E(F)$  and  $w(e') = 2$  for each  $e' \in E(G) - E(F)$ . Clearly,  $c(u^*) = \delta$ . If  $y \in N_G^\delta(u^*)$ , since there exists an edge  $e \sim y$  such that  $e \notin E(F)$ , then  $c(y) = \sum_{e \sim y} w(e) > \delta$ . If  $y \in N_G(u^*) - N_G^\delta(u^*)$ , then  $c(y) \geq d_G(y) > \delta$ . Hence  $c(y) \neq c(u^*)$  for all  $y \in N_G(u^*)$ . For each  $xy \in E(G)$ , where  $x \in U - u^*$  and  $y \in W$ , by the choice of  $F$ , we have  $c(x) \equiv \delta + 1 \pmod{2}$  and  $c(y) \equiv \delta \pmod{2}$ . Hence  $w$  is a vertex-coloring  $\{1, 2\}$ -edge-weighting of the graph  $G$ .

This completes the proof.  $\square$

**Corollary 2.4** *Let  $G$  be a 3-edge-connected bipartite graph. If  $3 \leq \delta(G) \leq 5$ , then  $G$  admits a vertex-coloring  $\mathcal{S}$ -edge-weighting for  $\mathcal{S} \in \{\{0, 1\}, \{1, 2\}\}$ .*

**Proof:** Since  $3 \leq \delta \leq 5$  and  $G$  is 3-edge-connected, then for every vertex  $v$  of degree  $\delta$ ,  $G - v$  is connected. By Lemma 2.2 and Theorem 1.2, with the same proof,  $G$  admits a vertex-coloring  $\mathcal{S}$ -edge-weighting for  $\mathcal{S} \in \{\{0, 1\}, \{1, 2\}\}$ .  $\square$

### 3 Conclusions

In this paper, we prove that every 2-connected and 3-edge-connected bipartite graph admits a vertex-coloring  $\mathcal{S}$ -edge-weighting for  $\mathcal{S} \in \{\{0, 1\}, \{1, 2\}\}$ . The generalized  $\theta$ -graphs is 2-connected and has a vertex-coloring 3-edge-weighting but not vertex-coloring  $\{0, 1\}$ -edge-weighting or vertex-coloring 2-edge-weighting. So it is an interesting problem to classify all 2-connected bipartite graphs admitting a vertex-coloring  $\mathcal{S}$ -edge-weighting. Since the parity-factor problem is polynomial, then there exists a polynomial algorithm to find a vertex-coloring  $\mathcal{S}$ -edge-weighting of bipartite graphs satisfying the conditions of Theorem 1.2.

### Acknowledgements

The authors would like to thank the anonymous Reviewer for all valuable comments and suggestions to improve the quality of our paper.

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