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Vertex-Coloring Edge-Weighting of Bipartite Graphs with Two Edge Weights*

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Let G be a graph and S be a subset of Z . A vertex-coloring S -edge-weighting of G is an assignment of weights by the elements of S to each edge of G so that adjacent vertices have different sums of incident edges weights.

It was proved that every 3-connected bipartite graph admits a vertex-coloring S -edge-weighting for $S = \{1, 2\}$ (H. Lu, Q. Yu and C. Zhang, Vertex-coloring 2-edge-weighting of graphs, *European J. Combin.*, **32** (2011), 22-27). In this paper, we show that every 2-connected and 3-edge-connected bipartite graph admits a vertex-coloring S -edge-weighting for $S \in \{\{0, 1\}, \{1, 2\}\}$. These bounds we obtain are tight, since there exists a family of infinite bipartite graphs which are 2-connected and do not admit vertex-coloring S -edge-weightings for $S \in \{\{0, 1\}, \{1, 2\}\}$.

Keywords: edge-weighting, vertex-coloring, 2-connected, bipartite graph

1 Introduction

In this paper, we consider only finite, undirected and simple connected graphs. For a vertex v of graph $G = (V, E)$, $N_G(v)$ denotes the set of vertices which are adjacent to v and $d_G(v) = |N_G(v)|$ is called the *degree* of vertex v . Let $\delta(G)$ and $\Delta(G)$ denote the minimum degree and maximum degree of graph G , respectively. For $v \in V(G)$ and $r \in Z^+$, let $N_G^r(v) = \{u \in N(v) \mid d_G(u) = r\}$. If $v \in V(G)$ and $e \in E(G)$, we use $v \sim e$ to denote that v is an end-vertex of e . For two disjoint subsets S, T of $V(G)$, let $E_G(S, T)$ denote the subset of edges of $E(G)$ with one end in S and other end in T and let $e_G(S, T) = |E_G(S, T)|$. Let $G = (U, W, E)$ denote a bipartite graph with bipartition (U, W) and edge set E .

Let S be a subset of Z . An S -edge-weighting of a graph G is an assignment $w : E(G) \rightarrow S$. An S -edge-weighting w of a graph G induces a coloring of the vertices of G , where the color of vertex v , denoted by $c(v)$, is $\sum_{e \sim v} w(e)$. An S -edge-weighting of a graph G is a *vertex-coloring* if for every edge $e = uv$, $c(u) \neq c(v)$ and we say that G admits a *vertex-coloring S -edge-weighting*. If $S = \{1, 2, \dots, k\}$, then a vertex-coloring S -edge-weighting of a graph G is usually called a *vertex-coloring k -edge-weighting*.

For vertex-coloring edge-weighting, Karoński et al. (2004) posed the following conjecture:

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Conjecture 1.1 *Every graph without isolated edges admits a vertex-coloring 3-edge-weighting.*

This conjecture is still wide open. Karoński et al. (2004) showed that Conjecture 1.1 is true for 3-colorable graphs. Recently, Kalkowski et al. (2010) showed that every graph without isolated edges admits a vertex-coloring 5-edge-weighting. This result is an improvement on the previous bounds on k established by Addario-Berry et al. (2007), Addario-Berry et al. (2008), and Wang and Yu (2008), who obtained the bounds $k = 30$, $k = 16$, and $k = 13$, respectively.

Many graphs actually admit a vertex-coloring 2-edge-weighting (in fact, experiments suggest (see Addario-Berry et al. (2008)) that almost all graphs admit a vertex-coloring 2-edge-weighting), however it is not known which ones do not. Khatirinejad et al. (2012) explored the problem of classifying those graphs which admit a vertex-coloring 2-edge-weighting. Chang et al. (2011) had made some progress in determining which classes of graphs admit vertex-coloring 2-edge-weighting, and proved that there exists a family of infinite bipartite graphs (e.g., the generalized θ -graphs) which are 2-connected and admit a vertex-coloring 3-edge-weighting but not vertex-coloring 2-edge-weightings. Lu et al. (2011) showed that every 3-connected bipartite graph admits a vertex-coloring 2-edge-weighting.

We write

$$\begin{aligned}\mathcal{G}_{12} &= \{G \mid G \text{ admits a vertex-coloring } \{1, 2\}\text{-edge-weighting}\}; \\ \mathcal{G}_{01} &= \{G \mid G \text{ admits a vertex-coloring } \{0, 1\}\text{-edge-weighting}\}; \\ \mathcal{G}_{12}^* &= \{G \mid G \text{ is bipartite and admits a vertex-coloring } \{1, 2\}\text{-edge-weighting}\}; \\ \mathcal{G}_{01}^* &= \{G \mid G \text{ is bipartite and admits a vertex-coloring } \{0, 1\}\text{-edge-weighting}\}.\end{aligned}$$

Dudek and Wajc (2011) showed that determining whether a graph belongs to \mathcal{G}_{12} or \mathcal{G}_{01} is NP-complete. Moreover, they showed that $\mathcal{G}_{12} \neq \mathcal{G}_{01}$. The counterexamples constructed by Dudek and Wajc (2011) are non-bipartite.

Now we construct a bipartite graph, which admits a vertex-coloring 2-edge-weighting but not vertex-coloring $\{0, 1\}$ -edge-weightings. Let C_6 be a cycle of length six and Γ be a graph obtained by connecting an isolated vertex to one of the vertices of C_6 . Take two disjoint copies of Γ . Connect two vertices of degree one of the two copies and this gives a connected bipartite graph G . It is easy to prove that G admits a vertex-coloring 2-edge-weighting but not vertex-coloring $\{0, 1\}$ -edge-weighting. Hence $\mathcal{G}_{01}^* \neq \mathcal{G}_{12}^*$. Next we would like to propose the following problem.

Problem 1 *Determining whether a graph $G \in \mathcal{G}_{12}^*$ or $G \in \mathcal{G}_{01}^*$ is polynomial?*

In this paper, we characterize bipartite graphs which admit a vertex-coloring \mathcal{S} -edge-weighting for $\mathcal{S} \in \{\{0, 1\}, \{1, 2\}\}$, and obtain the following result.

Theorem 1.2 *Let G be a 3-edge-connected bipartite graph $G = (U, W, E)$ with minimum degree $\delta(G)$. If G contains a vertex u of degree $\delta(G)$ such that $G - u$ is connected, then G admits a vertex-coloring \mathcal{S} -edge-weighting for $\mathcal{S} \in \{\{0, 1\}, \{1, 2\}\}$.*

By Theorem 1.2, it is easy to obtain the following result, which improves and extends the result obtained by Lu et al. (2011).

Theorem 1.3 *Every 2-connected and 3-edge-connected bipartite graph $G = (U, W, E)$ admits a vertex-coloring \mathcal{S} -edge-weighting for $\mathcal{S} \in \{\{0, 1\}, \{1, 2\}\}$.*

So far, all known counterexamples of bipartite graphs, which do not have vertex-coloring $\{0, 1\}$ -edge-weightings or vertex-coloring $\{1, 2\}$ -edge-weightings are graphs with minimum degree 2. So we would like to propose the following problem.

Problem 2 *Does every bipartite graph with $\delta(G) \geq 3$ admit a vertex-coloring \mathcal{S} -edge-weighting, where $\mathcal{S} \in \{\{0, 1\}, \{1, 2\}\}$.*

A *factor* of a graph G is a spanning subgraph. For a graph G , there is a close relationship between 2-edge-weighting and graph factors. Namely, a 2-edge-weighting problem is equivalent to finding special factors of graphs (see Addario-Berry et al. (2007) and Addario-Berry et al. (2008)). So to find factors with pre-specified degree is an important part of edge-weighting.

Let $g, f : V(G) \rightarrow Z$ be two integer-valued functions such that $g(v) \leq f(v)$ and $g(v) \equiv f(v) \pmod{2}$ for all $v \in V(G)$. A factor F of G is called (g, f) -parity factor if $g(v) \leq d_F(v) \leq f(v)$ and $d_F(v) \equiv f(v) \pmod{2}$ for all $v \in V(G)$. For $X \subseteq V(G)$, we write $g(X) = \sum_{x \in X} g(x)$ and $f(X)$ is defined similarly. For (g, f) -parity factors, Lovász obtained a sufficient and necessary condition.

Theorem 1.4 (Lovász (1972)) *A graph G contains a (g, f) -parity factor if and only if for any two disjoint subsets S and T of $V(G)$, it follows that*

$$\eta(S, T) = f(S) - g(T) + \sum_{x \in T} d_{G-S}(x) - \tau(S, T) \geq 0,$$

where $\tau(S, T)$ denotes the number of components C , called g -odd components of $G - S - T$ such that $g(V(C)) + e_G(V(C), T) \equiv 1 \pmod{2}$.

In the proof of the main theorems, we also need the following three lemmas.

Theorem 1.5 (Chang et al. (2011)) *Every non-trivial connected bipartite graph $G = (A, B, E)$ with $|A|$ even, admits a vertex-coloring 2-edge-weighting w such that $c(u)$ is odd for $u \in A$ and $c(v)$ is even for $v \in B$.*

Theorem 1.6 (Chang et al. (2011)) *Let $r \geq 3$ be an integer. Every r -regular bipartite graph G admits a vertex-coloring 2-edge-weighting.*

Theorem 1.7 (Khatirinejad et al. (2012)) *Every r -regular graph G admits a vertex-coloring 2-edge-weighting if and only if it admits a vertex-coloring $\{0, 1\}$ -edge-weighting.*

2 Proof of Theorem 1.2

Corollary 2.1 *Every non-trivial connected bipartite graph $G = (A, B, E)$ with $|A|$ even admits a vertex-coloring $\{0, 1\}$ -edge-weighting.*

Proof: By Theorem 1.5, G admits a vertex-coloring 2-edge-weighting w such that $c(u)$ is odd for $u \in A$ and $c(v)$ is even for $v \in B$. Let $w'(e) = 0$ if $w(e) = 2$ and $w'(e) = 1$ if $w(e) = 1$. Then w' is a vertex-coloring $\{0, 1\}$ -edge-weighting of graph G . \square

For completing the proof of Theorem 1.2, we need the following two technical lemmas.

Lemma 2.2 *Let G be a bipartite graph with bipartition (U, W) , where $|U| \equiv |W| \equiv 1 \pmod{2}$. Let $\delta(G) = \delta$ and $u \in U$ such that $d_G(u) = \delta$. If one of the following two conditions holds, then G contains a factor F such that $d_F(u) = \delta$, $d_F(x) \equiv \delta + 1 \pmod{2}$ for all $x \in U - u$, $d_F(y) \equiv \delta \pmod{2}$ for all $y \in W$ and $d_F(y) \leq \delta - 2$ for all $y \in N_G^\delta(u)$.*

(i) $\delta(G) \geq 4$, G is 3-edge-connected and $G - u$ is connected.

(ii) $\delta(G) = 3$, G is 3-edge-connected and $|N_G^\delta(u)| \leq 2$.

Proof: Let M be an integer such that $M \geq \Delta(G)$ and $M \equiv \delta \pmod{2}$. Let $m \in \{0, -1\}$ such that $m \equiv \delta \pmod{2}$. Let $g, f : V(G) \rightarrow \mathbb{Z}$ such that

$$g(x) = \begin{cases} \delta & \text{if } x = u, \\ m - 1 & \text{if } x \in U - u, \\ m & \text{if } x \in W, \end{cases}$$

and

$$f(x) = \begin{cases} M + 1 & \text{if } x \in U - u, \\ M & \text{if } x \in (W \cup \{u\}) - N_G^\delta(u), \\ \delta - 2 & \text{if } x \in N_G^\delta(u). \end{cases}$$

By definition, we have $g(v) \equiv f(v) \pmod{2}$ for all $v \in V(G)$. It is sufficient for us to show that G contains a (g, f) -parity factor. Indirectly, suppose that G contains no (g, f) -parity factors. By Theorem 1.4, there exist two disjoint subsets S and T such that

$$\eta(S, T) = f(S) - g(T) + \sum_{x \in T} d_{G-S}(x) - \tau(S, T) < 0,$$

where $\tau(S, T)$ denotes the number of g -odd components of $G - S - T$. Since $f(V(G))$ is even, by parity, we have

$$\eta(S, T) = f(S) - g(T) + \sum_{x \in T} d_{G-S}(x) - \tau(S, T) \leq -2. \quad (1)$$

Hence $S \cup T \neq \emptyset$. We choose S and T such that $S \cup T$ is minimal. Let $A = V(G) - S - T$.

Claim 1. $T \subseteq \{u\}$.

Otherwise, let $v \in T - u$ and $T' = T - v$. We have

$$\begin{aligned}
\eta(S, T') &= f(S) - g(T') + \sum_{x \in T'} d_{G-S}(x) - \tau(S, T') \\
&= f(S) - (g(T) - g(v)) + \left(\sum_{x \in T} d_{G-S}(x) - d_{G-S}(v) \right) - \tau(S, T') \\
&\leq f(S) - g(T) + \left(\sum_{x \in T} d_{G-S}(x) - d_{G-S}(v) \right) - (\tau(S, T) - e_G(v, A)) + g(v) \\
&\leq f(S) - g(T) + \sum_{x \in T} d_{G-S}(x) - \tau(S, T) + (g(v) - d_{G-S}(v) + e_G(v, A)) \\
&= \eta(S, T) + (g(v) - d_{G-S}(v) + e_G(v, A)) \\
&\leq \eta(S, T) - (d_{G-S}(v) - e_G(v, A)) \\
&\leq \eta(S, T) \leq -2,
\end{aligned}$$

contradicting the choice of S and T .

Claim 2. $S \subseteq N_G^\delta(u)$.

Otherwise, suppose that $S - N_G^\delta(u) \neq \emptyset$ and let $v \in S - N_G^\delta(u)$. Let $S' = S - v$. We have

$$\begin{aligned}
\eta(S', T) &= f(S') - g(T) + \sum_{x \in T} d_{G-S'}(x) - \tau(S', T) \\
&= (f(S) - f(v)) - g(T) + \left(\sum_{x \in T} d_{G-S}(x) + e_G(v, T) \right) - \tau(S', T) \\
&\leq f(S) - g(T) + \sum_{x \in T} d_{G-S}(x) - (\tau(S, T) - e_G(v, A)) - f(v) + e_G(v, T) \\
&= f(S) - g(T) + \sum_{x \in T} d_{G-S}(x) - \tau(S, T) + (e_G(v, T) + e_G(v, A) - f(v)) \\
&\leq \eta(S, T) + (d_G(v) - f(v)) \\
&\leq \eta(S, T) \leq -2,
\end{aligned}$$

contradicting the choice of S and T again.

We write $\tau(S, T) = \tau$. By Claims 1 and 2, we have

$$\begin{aligned}
\eta(S, T) &= f(S) - g(T) + \sum_{x \in T} d_{G-S}(x) - \tau \\
&= (\delta - 2)|S| - \delta|T| + |T|(\delta - |S|) - \tau \quad (\text{by Claims 1 and 2}) \\
&= (\delta - 2 - |T|)|S| - \tau \quad (\text{by Claim 1}) \\
&\leq -2,
\end{aligned}$$

i.e.,

$$(\delta - 2 - |T|)|S| + 2 \leq \tau, \quad (2)$$

which implies $\tau \geq 2$ since $|T| \leq 1$.

Since $G - u$ is connected, we may see that $S \neq \emptyset$. Note that G is 3-edge-connected, by Claims 1 and 2, we have

$$\begin{aligned} 3\tau &\leq e_G(A, S \cup T) \\ &= e_G(A, S) + e_G(A, T) \\ &\leq (\delta - |T|)|S| + |T|(\delta - |S|) \quad (\text{by Claims 1 and 2}) \\ &= (\delta - 2|T|)|S| + |T|\delta, \end{aligned}$$

i.e.,

$$3\tau \leq (\delta - 2|T|)|S| + |T|\delta. \quad (3)$$

Combining (2) and (3), we may see that

$$\delta|T| \geq (2\delta - |T| - 6)|S| + 6. \quad (4)$$

If $\delta \geq 4$, then we have

$$\begin{aligned} \delta &\geq \delta|T| \quad (\text{since } |T| \leq 1) \\ &\geq (2\delta - |T| - 6)|S| + 6 \quad (\text{since } |S| \geq 1) \\ &\geq 2\delta - |T| \\ &\geq 2\delta - 1, \end{aligned}$$

a contradiction. So we may assume that $\delta = 3$. Note that $|S| \leq |N_G^\delta(u)| \leq 2$. By (4), we have

$$3 = \delta \geq \delta|T| \geq -|T||S| + 6 \geq 4, \quad (5)$$

a contradiction again.

This completes the proof. \square

Lemma 2.3 *Let G be a bipartite graph with bipartition (U, W) , where $|U| \equiv |W| \equiv 1 \pmod{2}$. Let $\delta(G) = \delta$ and $u \in U$ such that $d_G(u) = \delta$. If one of the following two conditions holds, then G contains a factor F such that $d_F(u) = 0$, $d_F(x) \equiv 1 \pmod{2}$ for all $x \in U - u$, $d_F(y) \equiv 0 \pmod{2}$ for all $x \in W$ and $d_F(y) \geq 2$ for all $y \in N_G(u)$.*

(i) $\delta(G) \geq 4$, G is 3-edge-connected and $G - u$ is connected.

(ii) $\delta(G) = 3$, G is 3-edge-connected and there exists a vertex $v \in N_G(u)$ such that $d_G(v) > 3$.

Proof: Let M be an even integer such that $M \geq \Delta(G)$. Let $g, f : V(G) \rightarrow Z$ such that

$$g(x) = \begin{cases} 0 & \text{if } x \in (\{u\} \cup W) - N_G(u), \\ 2 & \text{if } x \in N_G(u), \\ -1 & \text{if } x \in U - u, \end{cases}$$

and

$$f(x) = \begin{cases} M & \text{if } x \in W \\ 0 & \text{if } x = u, \\ M + 1 & \text{if } x \in U - u. \end{cases}$$

Clearly, $g(v) \equiv f(v) \pmod{2}$ for all $v \in V(G)$ and $g(V(G))$ is even. It is also sufficient for us to show that G contains a (g, f) -parity factor.

Indirectly, suppose that G contains no (g, f) -parity factors. By Theorem 1.4, there exist two disjoint subsets S and T such that

$$\eta(S, T) = f(S) - g(T) + \sum_{x \in T} d_{G-S}(x) - \tau(S, T) \leq -2, \quad (6)$$

where $\tau(S, T)$ denotes the number of g -odd components of $G - S - T$. We choose S and T such that $S \cup T$ is minimal. Let $A = V(G) - S - T$.

Claim 1. $S \subseteq \{u\}$.

Otherwise, suppose that there exists a vertex $v \in S - u$. Let $S' = S - v$. Then we have

$$\begin{aligned} \eta(S', T) &= f(S') - g(T) + \sum_{y \in T} d_{G-S'}(y) - \tau(S', T) \\ &= (f(S) - f(v)) - g(T) + \left(\sum_{y \in T} d_{G-S}(y) + e_G(v, T) \right) - \tau(S', T) \\ &\leq f(S) - g(T) + \sum_{y \in T} d_{G-S}(y) - f(v) + e_G(v, T) - (\tau(S, T) - e_G(v, A)) \\ &= \eta(S, T) - (f(v) - e_G(v, T) - e_G(v, A)) \\ &\leq \eta(S, T) - (f(v) - d_G(v)) \\ &\leq \eta(S, T) \leq -2, \end{aligned}$$

contradicting the the choice of $S \cup T$.

Claim 2. $T \subseteq N_G(u)$.

Otherwise, suppose that $T - N_G(u) \neq \emptyset$. Let $x \in T - N_G(u)$ and let $T' = T - x$. Then we have

$$\begin{aligned}
\eta(S, T') &= f(S) - g(T') + \sum_{y \in T'} d_{G-S}(y) - \tau(S, T') \\
&= f(S) - (g(T) - g(x)) + \left(\sum_{y \in T} d_{G-S}(y) - d_{G-S}(x) \right) - \tau(S, T') \\
&\leq f(S) - (g(T) - g(x)) + \left(\sum_{y \in T} d_{G-S}(y) - d_{G-S}(x) \right) - (\tau(S, T) - e_G(x, A)) \\
&= f(S) - g(T) + \sum_{y \in T} d_{G-S}(y) - \tau(S, T) - (d_{G-S}(x) - e_G(x, A)) + g(x) \\
&= \eta(S, T) - (d_{G-S}(x) - e_G(x, A)) + g(x) \\
&\leq \eta(S, T) \leq -2,
\end{aligned}$$

contradicting the choice of $S \cup T$.

By Claims 1 and 2, we may see that $f(S) = 0$ and $g(T) = 2|T|$. For simplicity, we write $\tau(S, T) = \tau$. By (6), we see that

$$\tau \geq \sum_{x \in T} (d_G(x) - |S|) - 2|T| + 2, \quad (7)$$

which implies

$$\tau \geq \sum_{x \in T} (\delta - 1) - 2|T| + 2 \geq 2. \quad (8)$$

Note that $G - u$ is connected, so we have $|T| \geq 1$. Since G is 3-edge-connected, we have

$$\begin{aligned}
3\tau &\leq \sum_{x \in T} (d_G(x) - |S|) + (\delta - |T|)|S| \\
&= \sum_{x \in T} d_G(x) + (\delta - 2|T|)|S|,
\end{aligned}$$

i.e.,

$$3\tau \leq \sum_{x \in T} d_G(x) + (\delta - 2|T|)|S|. \quad (9)$$

Inequalities (7) and (9) implies

$$\begin{aligned}
2 \sum_{x \in T} d_G(x) + 6 &\leq |S||T| + 6|T| + \delta|S| \quad (\text{since } |S| \leq 1 \text{ and } |T| \geq 1) \\
&\leq 7|T| + \delta,
\end{aligned}$$

i.e.,

$$7|T| \geq 2 \sum_{x \in T} d_G(x) + 6 - \delta. \quad (10)$$

If $\delta \geq 4$, by (10), it follows

$$7|T| \geq 6 + \delta(2|T| - 1) \geq 8|T| + 2,$$

a contradiction. So we may assume that $\delta = 3$. By condition (ii), $\sum_{x \in T} d_G(x) \geq 3|T| + 1$. Combining (10),

$$\begin{aligned} 7|T| &\geq 2 \sum_{x \in T} d_G(x) + 6 - \delta \\ &\geq 2(3|T| + 1) + 3, \end{aligned}$$

which implies $|T| \geq 5$, a contradiction since $|T| \leq |N_G(u)| \leq 3$.

This completes the proof. \square

Proof of Theorem 1.2: By Theorem 1.5 and Corollary 2.1, we can assume that both $|A|$ and $|B|$ are odd.

Firstly, we consider $\mathcal{S} = \{0, 1\}$. If G is 3-regular, by Theorem 1.6, then G admits a vertex-coloring 2-edge-weighting. By Theorem 1.7, G also admits a vertex-coloring $\{0, 1\}$ -edge-weighting. So we can assume that $\delta(G) \geq 3$ and G is not 3-regular. If $\delta(G) = 3$, since G is 3-edge-connected, then $G - x$ is connected for every vertex x of G with degree three. Hence there exists a vertex v with degree three such that $N_G(v)$ contains a vertex with degree at least four. Let

$$u^* = \begin{cases} u & \text{if } \delta \geq 4, \\ v & \text{if } \delta = 3. \end{cases}$$

Without loss generality, we may assume that $u^* \in U$ and so it is a vertex satisfying the conditions of Lemma 2.3. Hence by Lemma 2.3, G contains a factor F , which satisfies the following three conditions.

- (i) $d_F(u^*) = 0$;
- (ii) $d_F(x) \equiv 1 \pmod{2}$ for all $x \in U - u^*$;
- (iii) $d_F(y) \equiv 0 \pmod{2}$ for all $y \in W$ and $d_F(y) \geq 2$ for all $y \in N_G(u^*)$.

Clearly, $d_F(x) \neq d_F(y)$ for all $xy \in E(G)$. We assign weight 1 for each edge of $E(F)$ and weight 0 for each edge of $E(G) - E(F)$. Then we obtain a vertex-coloring $\{0, 1\}$ -edge-weighting of graph G .

Secondly, we show that G admits a vertex-coloring 2-edge-weighting. By Theorem 1.6, we may assume that G is not 3-regular. If $\delta = 3$, since G is 3-edge-connected, then G contains a vertex v' such that $d_G(v') = 3$, $G - v'$ is connected and $|N_G^\delta(v')| \leq 2$. Let

$$u^* = \begin{cases} u & \text{if } \delta \geq 4, \\ v' & \text{if } \delta = 3. \end{cases}$$

Then u^* is a vertex satisfying the conditions of Lemma 2.2. Hence by Lemma 2.2, G contains a factor F such that

- (i) $d_F(u^*) = \delta$;
- (ii) $d_F(x) \equiv \delta \pmod{2}$ for all $x \in W$ and $d_F(x) \leq \delta - 2$ for all $x \in N_G^\delta(u^*)$;
- (iii) $d_F(y) \equiv \delta + 1 \pmod{2}$ for all $y \in U - u^*$.

Let $w : E(G) \rightarrow \{1, 2\}$ be a 2-edge-weighting such that $w(e) = 1$ for each $e \in E(F)$ and $w(e') = 2$ for each $e' \in E(G) - E(F)$. Clearly, $c(u^*) = \delta$. If $y \in N_G^\delta(u^*)$, since there exists an edge $e \sim y$ such that $e \notin E(F)$, then $c(y) = \sum_{e \sim y} w(e) > \delta$. If $y \in N_G(u^*) - N_G^\delta(u^*)$, then $c(y) \geq d_G(y) > \delta$. Hence $c(y) \neq c(u^*)$ for all $y \in N_G(u^*)$. For each $xy \in E(G)$, where $x \in U - u^*$ and $y \in W$, by the choice of F , we have $c(x) \equiv \delta + 1 \pmod{2}$ and $c(y) \equiv \delta \pmod{2}$. Hence w is a vertex-coloring $\{1, 2\}$ -edge-weighting of the graph G .

This completes the proof. \square

Corollary 2.4 *Let G be a 3-edge-connected bipartite graph. If $3 \leq \delta(G) \leq 5$, then G admits a vertex-coloring \mathcal{S} -edge-weighting for $\mathcal{S} \in \{\{0, 1\}, \{1, 2\}\}$.*

Proof: Since $3 \leq \delta \leq 5$ and G is 3-edge-connected, then for every vertex v of degree δ , $G - v$ is connected. By Lemma 2.2 and Theorem 1.2, with the same proof, G admits a vertex-coloring \mathcal{S} -edge-weighting for $\mathcal{S} \in \{\{0, 1\}, \{1, 2\}\}$. \square

3 Conclusions

In this paper, we prove that every 2-connected and 3-edge-connected bipartite graph admits a vertex-coloring \mathcal{S} -edge-weighting for $\mathcal{S} \in \{\{0, 1\}, \{1, 2\}\}$. The generalized θ -graphs is 2-connected and has a vertex-coloring 3-edge-weighting but not vertex-coloring $\{0, 1\}$ -edge-weighting or vertex-coloring 2-edge-weighting. So it is an interesting problem to classify all 2-connected bipartite graphs admitting a vertex-coloring \mathcal{S} -edge-weighting. Since the parity-factor problem is polynomial, then there exists a polynomial algorithm to find a vertex-coloring \mathcal{S} -edge-weighting of bipartite graphs satisfying the conditions of Theorem 1.2.

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