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# Doubly probabilistic representation for the stochastic porous media type equation.

Viorel Barbu (1), Michael Röckner (2) and Francesco Russo (3)

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**Summary:** The purpose of the present paper consists in proposing and discussing a doubly probabilistic representation for a stochastic porous media equation in the whole space  $\mathbb{R}^1$  perturbed by a multiplicative colored noise. For almost all random realizations  $\omega$ , one associates a stochastic differential equation in law with random coefficients, driven by an independent Brownian motion.

**Key words:** stochastic partial differential equations; infinite volume; singular porous media type equation; doubly probabilistic representation; multiplicative noise; singular random Fokker-Planck type equation; filtering.

**2000 AMS-classification:** 35R60; 60H15; 60H30; 60H10; 60G46; 35C99; 58J65; 82C31.

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## 1 Introduction

We consider a function  $\psi : \mathbb{R} \rightarrow \mathbb{R}$  and real functions  $e^0, \dots, e^N$  on  $\mathbb{R}$ , for some strictly positive integer  $N$ . In the whole paper, the following assumption will be in force.

**Assumption 1.1.** •  $|\psi(u)| \leq \text{const}|u|$ ,  $u \geq 0$ . In particular,  $\psi(0) = 0$ .

- $\psi : \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function such that its restriction to  $\mathbb{R}_+$  is monotone increasing. Moreover we also suppose that  $\lim_{u \rightarrow 0} \frac{\psi(u)}{u}$  exists.
- Let  $e^i \in C_b^2(\mathbb{R})$ ,  $0 \leq i \leq N$ .

Let  $T > 0$  and  $(\Omega, \mathcal{F}, P)$ , be a fixed probability space. A generic element of  $\Omega$  will be denoted by  $\omega$ .  $(\mathcal{F}_t, t \in [0, T])$  will stand for a filtration, fulfilling the usual conditions and we suppose  $\mathcal{F} = \mathcal{F}_T$ . Let  $\mu(t, \xi), t \in [0, T], \xi \in \mathbb{R}$ , be a random field of the type

$$\mu(t, \xi) = \sum_{i=1}^N e^i(\xi) W_t^i + e^0(\xi)t, \quad t \in [0, T], \xi \in \mathbb{R},$$

where  $W^i, 1 \leq i \leq N$ , are independent continuous  $(\mathcal{F}_t)$ -Brownian motions on  $(\Omega, \mathcal{F}, P)$ , which are fixed from now on until the end of the paper.

For technical reasons we will sometimes set  $W_t^0 \equiv t$ . We focus on a stochastic partial differential equation of the following type:

$$\begin{cases} \partial_t X(t, \xi) &= \frac{1}{2} \partial_{\xi\xi}^2 (\psi(X(t, \xi)) + X(t, \xi) \partial_t \mu(t, \xi)), \\ X(0, d\xi) &= x_0(d\xi), \end{cases} \quad (1.1)$$

which holds in the sense of Definition 2.9, where  $x_0$  is a given probability measure on  $\mathbb{R}$ . The stochastic multiplication above is of Itô type. We look for a solution of (1.1) with time evolution in  $L^1(\mathbb{R})$ . Since  $\psi$  restricted to  $\mathbb{R}_+$  is non-negative, Assumption 1.1 implies  $\psi(u) = \Phi^2(u)u$ ,  $u \geq 0$ ,  $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}$  being a non-negative continuous function which is bounded on  $\mathbb{R}_+$ .

**Remark 1.2.** 1. In the sequel we will consider, without further comments extensions of  $\psi$  (and  $\Phi$ ) to the real line which fulfill the first two items of Assumption 1.1 for  $u \in \mathbb{R}$  instead of  $u \geq 0$ .

2. The restriction on  $u \mapsto \Phi(u)$  introduced in Assumption 1.1 to be continuous is not always necessary, but here we assume this for simplicity.

When  $\psi(u) = |u|^{m-1}u$ ,  $m > 1$ , (1.1) and  $\mu \equiv 0$ , (1.1) is nothing else but the classical porous media equation. When  $\psi$  is a general increasing function (and

$\mu \equiv 0$ ), there are several contributions to the analytical study of (1.1), starting from [12] for existence, [15] for uniqueness in the case of bounded solutions and [13] for continuous dependence on the coefficients. Those are the classical references when the space variable varies on the real line. For equations in a bounded domain and Dirichlet boundary conditions, for simplicity, we only refer to monographs, e.g. [28, 26, 1, 2].

As far as the stochastic porous media is concerned, most of the work for existence and uniqueness concerned the case of bounded domain, see for instance [4, 5, 3]. In the infinite volume case, i.e. when the underlying domain is  $\mathbb{R}^d$ , well-posedness was fully analyzed in [22], when  $\psi$  is polynomially bounded (including the fast diffusion case) when the space dimension is  $d \geq 3$ . [8] established existence and uniqueness for any dimension  $d \geq 1$  and the authors obtained estimates for finite time extinction. To the best of our knowledge, except for [22] and [8], this seems to be the only work concerning a stochastic porous type equation in infinite volume.

We provide a probabilistic representation of solutions to (1.1) extending the results of [14, 6] which treated the deterministic case  $\mu \equiv 0$ . In the deterministic case, it seems that the first author who considered a probabilistic representation (of the type studied in this paper) for the solutions of a non-linear deterministic PDE was McKean [19], particularly in relation with the so called propagation of chaos. In his case, however, the coefficients were smooth. From then on the literature steadily grew and nowadays there is a vast amount of contributions to the subject, see the reference list of [14, 6]. A probabilistic representation when  $\psi(u) = |u|u^{m-1}$ ,  $m > 1$ , was provided for instance in [11], in the case of the classical porous media equation. When  $m < 1$ , i.e. in the case of the fast diffusion equation, [9] provides a probabilistic representation of the so called **Barenblatt solution**, i.e. the solution whose initial condition is concentrated at zero.

[14, 6] discussed the probabilistic representation when  $\mu = 0$  in the so called non-degenerate and degenerate case respectively (see Definition 6.1), where  $\psi$  also may have jumps.

In the case  $\mu = 0$ , the equation (1.1) models a non-linear phenomenon macroscopically. Let us denote by  $u : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  the solution of that equation. The idea of the probabilistic representation is to find a process  $(Y_t, t \in [0, T])$  whose law at time  $t$  has  $u(t, \cdot)$  for its density. In this case the equation (1.1) is conservative, in the sense that the integral (mass) of the solution is conserved along the time.

The process  $Y$  turns out to be the weak solution of the non-linear stochastic differ-

ential equation

$$\begin{cases} Y_t &= Y_0 + \int_0^t \Phi(u(s, Y_s)) dB_s, \\ \text{Law}(Y_t) &= u(t, \cdot), \quad t \geq 0, \end{cases} \quad (1.2)$$

where  $B$  is a classical Brownian motion. The behavior of  $Y$  is the microscopic counterpart of the phenomenon described by (1.1), describing the evolution of a single particle, whose law behaves according to (1.1).

The idea of this paper is to consider the case when  $\mu \neq 0$ . This includes the case when  $\mu$  is not vanishing but it is deterministic; it happens when only  $e^0$  is non-zero, and  $e^i \equiv 0, 1 \leq i \leq n$ . In this case our technique gives a sort of forward Feynman-Kac formula for a non-linear PDE. One of the main interests of this paper is that it provides a (forward) probabilistic representation for *non conservative* (random) PDE.

We introduce a doubly stochastic representation on which one can represent the solution of (1.1) as the weighted-law with respect to the random field  $\mu$  (or simply the  $\mu$ -weighted law) of a solution to a non-linear SDE.

Intuitively, it describes the microscopic aspect of the SPDE (1.1) for almost all quenched  $\omega$ . The terminology strongly refers to the case where the probability space  $(\Omega, \mathcal{F}, P)$  on which the SPDE is defined, remains fixed.

We represent a solution  $X$  to (1.1) making use of another independent source of randomness described by another probability space based on some set  $\Omega_1$ .

The analog of the process  $Y$ , obtained when  $\mu$  is zero in [6, 14], is a doubly stochastic process, still denoted by  $Y$  defined on  $(\Omega_1 \times \Omega, Q)$ , for which,  $X$  constitutes the so-called family of  $\mu$ -marginal weighted laws of  $Y$ , see Definition 2.4.  $Y$  is the solution of a *doubly stochastic non-linear diffusion* problem, see Definition 3.1. It will be a (doubly) stochastic process  $(\omega_1, \omega) \mapsto Y(\omega_1, \omega)$  solution of

$$Y_t = Y_0 + \int_0^t \Phi(X(s, Y_s, \omega)) dB_s, \quad (1.3)$$

and  $B(\cdot, \omega)$  is a Brownian motion on  $\Omega_1$  for almost any fixed  $\omega \in \Omega$ . The solution of (1.3) is in the following sense: fixing a realization  $\omega \in \Omega$ ,  $Y(\cdot, \omega)$  is a weak solution to the first line of (1.2) with  $u(t, \xi) = X(t, \xi, \omega)$ . Moreover  $X(t, \xi, \omega)$  is the  $\mu$ -marginal weighted law of  $Y_t(\cdot, \omega)$ .

The paper includes the following main achievements.

1. If we replace in (1.3)  $a(s, \xi, \omega) = \Phi(X(s, \xi, \omega))$  and  $a$  is bounded and non-degenerate, we show existence and uniqueness of the solution, strongly in  $\omega$ , weakly in  $\omega_1 \in \Omega_1$ , see Proposition 4.1. We also show the existence of law densities, for  $P$ -almost all quenched  $\omega$ , see Proposition 4.4.

2. Theorem 3.3 states that the  $\mu$ -marginal weighted laws  $X$  of a solution  $Y$  of a *doubly stochastic non-linear diffusion* problem constitute a solution of the stochastic porous media equation (1.1).
3. Conversely, given a solution  $X$  of (1.1), under suitable conditions, there is a solution  $Y$  of the doubly stochastic non-linear diffusion. This is discussed in Theorem 6.3 and in Theorem 7.1, distinguishing respectively the cases when  $\psi$  is non-degenerate and degenerate, see Definition 6.1.
4. When  $\psi$  is non-degenerate, then the doubly stochastic non-linear diffusion problem also admits uniqueness, see Theorem 6.3.
5. Section 3.2 illustrates a filtering interpretation for a solution of SPDE (1.1). Indeed, the  $\mu$ -marginal weighted laws  $X$  of a solution  $Y$  of a doubly stochastic non-linear diffusion problem (1.3) can be seen as *conditional densities* of  $Y_t, t \in [0, T]$  with respect to some probability measure.
6. Uniqueness of the stochastic Fokker-Planck equation obtained replacing  $\Phi^2$  by a function  $a(t, \omega, \xi)$  in (1.1), see Theorem 5.1.
7. Existence of a density to the solution of (1.3), see Proposition 4.4.

## 2 Preliminaries

### 2.1 Basic notations

First we introduce some basic recurrent notations.  $\mathcal{M}(\mathbb{R})$  denotes the space of finite real measures.

We recall that  $\mathcal{S}(\mathbb{R})$  is the space of the Schwartz fast decreasing test functions.  $\mathcal{S}'(\mathbb{R})$  is its dual, i.e. the space of Schwartz tempered distributions. On  $\mathcal{S}'(\mathbb{R})$ , the map  $(I - \Delta)^{\frac{s}{2}}, s \in \mathbb{R}$ , is well-defined. For  $s \in \mathbb{R}$ ,  $H^s(\mathbb{R})$  denotes the classical Sobolev space consisting of all functions  $f \in \mathcal{S}'(\mathbb{R})$  such that  $(I - \Delta)^{\frac{s}{2}}f \in L^2(\mathbb{R})$ . We introduce the norm

$$\|f\|_{H^s} := \|(I - \Delta)^{\frac{s}{2}}f\|_{L^2},$$

where  $\|\cdot\|_{L^p}$  is the classical  $L^p(\mathbb{R})$ -norm for  $1 \leq p \leq \infty$ . In the sequel, we will often simply denote  $H^{-1}(\mathbb{R})$ , by  $H^{-1}$  and  $L^2(\mathbb{R})$  by  $L^2$ . Furthermore,  $W^{r,p}$  denote the classical Sobolev space of order  $r \in \mathbb{N}$  in  $L^p(\mathbb{R})$  for  $1 \leq p \leq \infty$ .

**Definition 2.1.** *Given a function  $e$  belonging to  $L^1_{\text{loc}}(\mathbb{R}) \cap \mathcal{S}'(\mathbb{R})$ , we say that it is an  $H^{-1}$ -multiplier, if the map  $\varphi \mapsto \varphi e$  is continuous from  $\mathcal{S}(\mathbb{R})$  to  $H^{-1}$  with respect to the  $H^{-1}$ -topology on both spaces.*

In the following lines we give some other sufficient conditions on a function  $e$  to be an  $H^{-1}$ -multiplier.

**Lemma 2.2.** *Let  $e : \mathbb{R} \rightarrow \mathbb{R}$ . If  $e \in W^{1,\infty}$  (for instance if  $e \in W^{2,1}$ ), then  $e$  is a  $H^{-1}(\mathbb{R})$ -multiplier. In particular the functions  $e^i, 0 \leq i \leq N$  of Definition 1.1 are  $H^{-1}(\mathbb{R})$ -multipliers.*

*Proof.* By duality arguments, we observe that it is enough to show the existence of a constant  $\mathcal{C}(e)$  such that

$$\|eg\|_{H^1} \leq \mathcal{C}(e) \|g\|_{H^1}, \quad \forall g \in \mathcal{S}(\mathbb{R}). \quad (2.1)$$

(2.1) follows by product derivation rules, with for instance  $\mathcal{C}(e) = \sqrt{2} \left( \|e\|_\infty^2 + \|e'\|_\infty^2 \right)^{\frac{1}{2}}$ .  $\square$

With respect to the random field  $\mu$ , we introduce a notation for the Itô type stochastic integral below.

Let  $Z = (Z(s, \xi), s \in [0, T], \xi \in \mathbb{R})$  be a random field on  $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$  such that  $\int_0^T \left( \int_{\mathbb{R}} |Z(s, \xi)| d\xi \right)^2 ds < \infty$  a.s. and it is an  $L^1(\mathbb{R})$ -valued  $(\mathcal{F}_s)$ -progressively measurable process. Then, the stochastic integral

$$\int_{[0,t] \times \mathbb{R}} Z(s, \xi) \mu(ds, \xi) d\xi := \sum_{i=0}^N \int_0^t \left( \int_{\mathbb{R}} Z(s, \xi) e^i(\xi) d\xi \right) dW_s^i,$$

is well-defined.

More generally, if  $s \mapsto Z(s, \cdot)$  is a measurable map  $[0, T] \times \Omega \mapsto \mathcal{M}(\mathbb{R})$ , where  $\mathcal{M}(\mathbb{R})$  is the space of signed finite measures, such that  $\int_0^T \|Z(s, \cdot)\|_{\text{var}}^2 ds < \infty$ , then the stochastic integral

$$\int_{[0,t] \times \mathbb{R}} Z(s, \xi) \mu(ds, \xi) d\xi := \sum_{i=0}^N \int_0^t \left( \int_{\mathbb{R}} Z(s, d\xi) \right) e^i(\xi) dW_s^i,$$

is well-defined.

We specify now better the filtration  $(\mathcal{F}_t)_{t \in [0, T]}$  of the introduction. We will consider a fixed filtered probability space  $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \in [0, T]})$ , where  $(\mathcal{F}_t)_{t \in [0, T]}$  is the canonical filtration of a standard Brownian motion  $(W^1, \dots, W^N)$  enlarged with the  $\sigma$ -field generated by  $x_0$ . We also suppose that  $\mathcal{F}_0$  contains the  $P$ -null sets and  $\mathcal{F} = \mathcal{F}_T$ .

Let  $(\Omega_1, \mathcal{H})$  be a measurable space. In the sequel, we will also consider another filtered probability space  $(\Omega_0, \mathcal{G}, \mathbf{Q}, (\mathcal{G}_t)_{t \in [0, T]})$ , where  $\Omega_0 = \Omega_1 \times \Omega$ ,  $\mathcal{G} = \mathcal{H} \otimes \mathcal{F}$ .

Clearly any random element  $Z$  on  $(\Omega, \mathcal{F})$  will be implicitly extended to  $(\Omega_0, \mathcal{G})$  setting  $Z(\omega_1, \omega) = Z(\omega)$ . In particular  $W^i, i = 1 \dots N$  will be extended in that way.

Here we fix some conventions concerning measurability. Any topological space  $E$  is naturally equipped with its Borel  $\sigma$ -algebra  $\mathcal{B}(E)$ . For instance  $\mathcal{B}(\mathbb{R})$  (resp.  $\mathcal{B}([0, T])$ ) denotes the Borel  $\sigma$ -algebra of  $\mathbb{R}$  (resp.  $[0, T]$ ).

Given any probability space  $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ , the  $\sigma$ -field  $\mathcal{F}$  will always be omitted. When we will say that a map  $T : \Omega \times E \rightarrow \mathbb{R}$  is measurable, we will implicitly suppose that the corresponding  $\sigma$ -algebras are  $\mathcal{F} \otimes \mathcal{B}(E)$  and  $\mathcal{B}(\mathbb{R})$ .

All the processes on any generic measurable space  $(\Omega_2, \mathcal{F}_2)$  will be considered to be measurable with respect to both variables  $(t, \omega)$ . In particular any processes on  $\Omega_1 \times \Omega$  is supposed to be measurable with respect to  $([0, T] \times \Omega_1 \times \Omega, \mathcal{B}([0, T]) \otimes \mathcal{H} \otimes \mathcal{F})$ .

A function  $(A, \omega) \mapsto Q(A, \omega)$  from  $\mathcal{H} \times \Omega \rightarrow \mathbb{R}_+$  is called **random kernel** (resp. **random probability kernel**) if for each  $\omega \in \Omega$ ,  $Q(\cdot, \omega)$  is a finite positive (resp. probability) measure and for each  $A \in \mathcal{H}$ ,  $\omega \mapsto Q(A, \omega)$  is  $\mathcal{F}$ -measurable. The finite measure  $Q(\cdot, \omega)$  will also be denoted by  $Q^\omega$ . To that random kernel we can associate a specific finite measure (resp. probability) denoted by  $\mathbf{Q}$  on  $(\Omega_0, \mathcal{G})$  setting  $\mathbf{Q}(A \times F) = \int_F Q(A, \omega) P(d\omega) = \int_F Q^\omega(A) P(d\omega)$ , for  $A \in \mathcal{H}, F \in \mathcal{F}$ . The probability  $Q$  from above will be supposed here and below to be associated with a random probability kernel.

**Definition 2.3.** *If there is a measurable space  $(\Omega_1, \mathcal{H})$  and a random kernel  $Q$  as before, then the probability space  $(\Omega_0, \mathcal{G}, \mathbf{Q})$  will be called **suitable enlarged probability space** (of  $(\Omega, \mathcal{F}, P)$ ).*

As said above, any random variable on  $(\Omega, \mathcal{F})$  will be considered as a random variable on  $\Omega_0 = \Omega_1 \times \Omega$ . Then, obviously,  $W^1, \dots, W^N$  are independent Brownian motions also  $(\Omega_0, \mathcal{G}, Q)$ .

Given a local martingale  $M$  on any filtered probability space, the process  $Z := \mathcal{E}(M)$  denotes its Doléans exponential, which is a local martingale. In particular it is the unique solution of  $dZ_t = Z_{t-} dM_t$ ,  $Z_0 = 1$ . When  $M$  is continuous we have  $Z_t = e^{M_t - \frac{1}{2}\langle M \rangle_t}$ .

## 2.2 The concept of marginal weighted laws

Let us consider a suitably enlarged probability space as in Definition 2.3.

**Definition 2.4.** *Let  $Y : \Omega_1 \times \Omega \times [0, T] \rightarrow \mathbb{R}$  be a measurable process, progressively measurable on  $(\Omega_0, \mathcal{G}, \mathbf{Q}, (\mathcal{G}_t))$ , where  $(\mathcal{G}_t)$  is some filtration on  $(\Omega_0, \mathcal{G}, \mathbf{Q})$  such that  $W^1, \dots, W^N$  are  $(\mathcal{G}_t)$ -Brownian motions on  $(\Omega_0, \mathcal{G}, \mathbf{Q})$ . We will make use of the*

stochastic integral notation

$$\int_0^t \mu(ds, Y_s) = \sum_{i=0}^N \int_0^t e^i(Y_s) dW_s^i, t \in [0, T]. \quad (2.2)$$

As we shall see below in Proposition 2.6, for every  $t \in [0, T]$

$$E^{\mathbf{Q}} \left( \mathcal{E}_t \left( \int_0^\cdot \mu(ds, Y_s) \right) \right) < \infty. \quad (2.3)$$

To  $Y$ , we will associate its **family of  $\mu$ -marginal weighted laws**, (or simply **family of  $\mu$ -weighted laws**) i.e. the family of random kernels ( $t \in [0, T]$ ),

$$\Gamma_t = (\Gamma_t^Y(A, \omega), A \in \mathcal{B}(\mathbb{R}), \omega \in \Omega)$$

defined by

$$\varphi \mapsto E^{\mathbf{Q}^\omega} \left( \varphi(Y_t(\cdot, \omega)) \mathcal{E}_t \left( \int_0^\cdot \mu(ds, Y_s)(\cdot, \omega) \right) \right) = \int_{\mathbb{R}} \varphi(r) \Gamma_t^Y(dr, \omega), \quad (2.4)$$

where  $\varphi$  is a generic bounded real Borel function. We will also say that for fixed  $t \in [0, T]$ ,  $\Gamma_t$  is the  **$\mu$ -marginal weighted law** of  $Y_t$ .

**Remark 2.5.** i) If  $\Omega$  is a singleton  $\{\omega_0\}$ ,  $e^i = 0$ ,  $1 \leq i \leq N$ , the  $\mu$ -marginal weighted laws coincide with the weighted laws

$$\varphi \mapsto E^{\mathbf{Q}} \left( \varphi(Y_t) \exp \left( \int_0^t e^0(Y_s) ds \right) \right),$$

with  $\mathbf{Q} = \mathbf{Q}^{\omega_0}$ . In particular if  $\mu \equiv 0$  then the  $\mu$ -marginal weighted laws are the classical laws.

ii) By (2.3), for any  $t \in [0, T]$ , for  $P$  almost all  $\omega \in \Omega$ ,

$$E^{\mathbf{Q}^\omega} \left( \mathcal{E}_t \left( \int_0^\cdot \mu(ds, Y_s)(\cdot, \omega) \right) \right) < \infty.$$

iii) The function  $(t, \omega) \mapsto \Gamma_t(A, \omega)$  is measurable, for any  $A \in \mathcal{B}(\mathbb{R})$ , because  $Y$  is a measurable process.

iv) In the case  $e^0 = 0$ , the situation is the following. For each fixed  $\omega \in \Omega$ , (2.4) is a (random) non-negative measure which is not a probability. However the expectation of its total mass is indeed 1.

**Proposition 2.6.** Consider the situation of Definition 2.4. Then we have the following.

i) The process  $M_t := \mathcal{E}_t \left( \sum_{i=1}^N \int_0^t e^i(Y_s) dW_s^i \right)$  is a martingale. We emphasize that the sum starts indeed at  $i = 1$ .

- ii) The quantity (2.3) is bounded by  $\exp(T \|e^0\|_\infty)$ .
- iii)  $E^{\mathbf{Q}}(M_t^2) \leq \exp(3T \sum_{i=1}^N \|e^i\|_\infty^2)$ ,  $t \in [0, T]$ . Consequently  $M$  is a uniformly integrable martingale.
- iv) For  $P$ -a.e.  $\omega \in \Omega$ ,  $\sup_{0 \leq t \leq T} \|\Gamma_t(\cdot, \omega)\|_{\text{var}} < \infty$ , where we remind that  $\|\cdot\|_{\text{var}}$  stands for the total variation.

**Remark 2.7.** Proposition 2.6 ii) yields in particular that  $Y$  always admits  $\mu$ -marginal weighted laws.

*Proof.* i) The result follows since the Novikov condition  $E^{\mathbf{Q}}\left(\exp\left(\frac{1}{2} \sum_{i=1}^N \int_s^t e^i(Y_s)^2 ds\right)\right) < \infty$  is verified, because the functions  $e^i$ ,  $i = 1 \dots N$ , are bounded.

ii) This follows because  $E^{\mathbf{Q}}(M_t) = 1, \forall t \in [0, T]$ .

iii)  $M_t^2$  is equal to  $N_t \exp\left(3 \sum_{i=1}^N \int_0^t (e^i)^2(Y_s) ds\right)$ , where  $N$  is a positive martingale with  $N_0 = 1$ .

iv) For  $t \in [0, T]$ ,

$$\begin{aligned} \sup_{t \leq T} \|\Gamma_t(\cdot, \omega)\|_{\text{var}} &= \sup_{t \leq T} E^{Q^\omega} \left( M_t \exp \left( \int_0^t e^0(Y_s) ds \right) \right) \\ &\leq \exp(T \|e^0\|_\infty) \sup_{t \leq T} E^{Q^\omega}(M_t). \end{aligned}$$

Taking the expectation with respect to  $P$  it implies

$$\begin{aligned} E^P \left( \sup_{t \leq T} \|\Gamma_t^Y(\cdot, \omega)\|_{\text{var}} \right) &\leq \exp(T \|e^0\|_\infty) E^P \left( \sup_{t \leq T} E^{Q^\omega}(M_t) \right) \\ &\leq \exp(T \|e^0\|_\infty) E^P \left( E^{Q^\omega} \left( \sup_{t \leq T} M_t \right) \right). \end{aligned}$$

By the Burkholder-Davis-Gundy (BDG) inequality this is bounded by

$$\begin{aligned} 3 \exp(T \|e^0\|_\infty) E^{\mathbf{Q}} \left( \langle M \rangle_T^{\frac{1}{2}} \right) &\leq 3 \exp(T \|e^0\|_\infty) E^{\mathbf{Q}} \left( \left[ \int_0^T ds \sum_{i=1}^N M_s^2 e^i(Y_s)^2 \right]^{\frac{1}{2}} \right) \\ &\leq C(e, N, T) \left\{ E^{\mathbf{Q}} \left( \int_0^T ds M_s^2 \right) \right\}^{\frac{1}{2}}, \end{aligned}$$

where the last inequality is due to Jensen's inequality;  $C(e, N, T)$  is a constant depending on  $N, T$  and  $e^i$ ,  $i = 0 \dots N$ . By Fubini's Theorem and item iii), we have

$$E^{\mathbf{Q}} \left( \int_0^T ds M_s^2 \right) \leq T \exp(3T \sum_{i=1}^N \|e^i\|_\infty^2).$$

□

The lemma below gives a characterization of the  $\mu$ -weighted laws of a process  $Y$  living on an enlarged probability space.

**Lemma 2.8.** *Let  $Y$  (resp.  $\tilde{Y}$ ) be a process on a suitable enlarged probability space  $(\Omega_0, \mathcal{G}, \mathbf{Q})$  (resp.  $(\tilde{\Omega}_0, \tilde{\mathcal{G}}, \tilde{\mathbf{Q}})$ ). Set  $W = (W^1, \dots, W^N)$ . Suppose that the law of  $(Y, W)$  under  $\mathbf{Q}$  and the law of  $(\tilde{Y}, W)$  under  $\tilde{\mathbf{Q}}$  are the same. Then, the  $\mu$ -marginal weighted laws of  $Y$  under  $\mathbf{Q}$  coincide a.s. with the  $\mu$ -marginal weighted laws of  $\tilde{Y}$  under  $\tilde{\mathbf{Q}}$ .*

*Proof.* Let  $0 \leq t \leq T$ . Using the assumption, we deduce that for any bounded continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , and every  $F \in \mathcal{F}_t$ , we have

$$E^{\mathbf{Q}} \left( 1_F f(Y_t) \mathcal{E}_t \left( \sum_{i=0}^N \int_0^\cdot e^i(Y_s) dW_s^i \right) \right) = E^{\tilde{\mathbf{Q}}} \left( 1_F f(\tilde{Y}_t) \mathcal{E}_t \left( \sum_{i=0}^N \int_0^\cdot e^i(\tilde{Y}_s) dW_s^i \right) \right). \quad (2.5)$$

To show this, using classical regularization properties of Itô integral, see e.g. Theorem 2 in [25], and uniform integrability arguments, we first observe that

$$\mathcal{E}_t \left( \sum_{i=0}^N \int_0^\cdot e^i(Y_s) dW_s^i \right)$$

is the limit in  $L^2(\Omega_0, \mathbf{Q})$  of

$$\mathcal{E}_t \left( \sum_{i=0}^N \int_0^\cdot e^i(Y_s) \frac{W_{s+\varepsilon}^i - W_s^i}{\varepsilon} ds \right).$$

A similar approximation property arises replacing  $Y$  with  $\tilde{Y}$  and  $\mathbf{Q}$  with  $\tilde{\mathbf{Q}}$ . Then (2.5) easily follows.

To conclude, it will be enough to show the existence of a countable family  $(f_j)_{j \in \mathbb{N}}$  of bounded continuous real functions for which, for  $P$  almost all  $\omega \in \Omega$ , for any  $j \in \mathbb{N}$ , we have  $R_j = \tilde{R}_j$  where

$$\begin{aligned} R_j(\omega) &= E^{\mathbf{Q}^\omega} \left( f_j(Y_t(\cdot, \omega)) \mathcal{E}_t \left( \sum_{i=0}^N \int_0^\cdot e^i(Y_s(\cdot, \omega)) dW_s^i \right) \right) \\ \tilde{R}_j(\omega) &= E^{\tilde{\mathbf{Q}}^\omega} \left( f_j(\tilde{Y}_t(\cdot, \omega)) \mathcal{E}_t \left( \sum_{i=0}^N \int_0^\cdot e^i(\tilde{Y}_s(\cdot, \omega)) dW_s^i \right) \right). \end{aligned}$$

This will follow, since applying (2.5), for any  $F \in \mathcal{F}_t$ , we have  $E^P(1_F R_j) = E^P(1_F \tilde{R}_j)$ .  $\square$

### 2.3 SPDE, weak-strong existence of SDEs

In this section we introduce the basic concepts related to the stochastic porous media equation and the related non-linear diffusion.

**Definition 2.9.** A random field  $X = (X(t, \xi, \omega), t \in [0, T], \xi \in \mathbb{R}, \omega \in \Omega)$  is said to be a solution to (1.1) if  $P$  a.s. we have the following.

1.  $X \in C([0, T]; \mathcal{S}'(\mathbb{R})) \cap L^2([0, T]; L^1_{\text{loc}}(\mathbb{R}))$ .
2.  $X$  is an  $\mathcal{S}'(\mathbb{R})$ -valued  $(\mathcal{F}_t)$ -progressively measurable process.
3. For any test function  $\varphi \in \mathcal{S}(\mathbb{R})$  with compact support,  $t \in ]0, T]$  a.s. we have

$$\begin{aligned} \int_{\mathbb{R}} X(t, \xi) \varphi(\xi) d\xi &= \int_{\mathbb{R}} x_0(d\xi) \varphi(\xi) + \frac{1}{2} \int_0^t ds \int_{\mathbb{R}} \psi(X(s, \xi, \cdot)) \varphi''(\xi) d\xi \\ &+ \int_{[0, t] \times \mathbb{R}} X(s, \xi) \varphi(\xi) \mu(ds, \xi) d\xi. \end{aligned}$$

At Definition 3.1, we will present the concept of *double stochastic non-linear diffusion* which is a McKean type equation with a supplementary source of randomness. Before this, as a first step, we will introduce a particular the case of simple *double stochastic differential equation* (DSDE). Let  $\gamma : [0, T] \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$  be an  $(\mathcal{F}_t)$ -progressively measurable random fields and  $x_0$  be a probability on  $\mathcal{B}(\mathbb{R})$ .

**Definition 2.10.** a) We say that  $(DSDE)(\gamma, x_0)$  admits **weak-strong existence** if there is a suitable extended probability space  $(\Omega_0, \mathcal{G}, \mathbf{Q})$ , i.e. a measurable space  $(\Omega_1, \mathcal{H})$ , a probability kernel  $(Q(\cdot, \omega), \omega \in \Omega)$  on  $\mathcal{H} \times \Omega$ , two  $\mathbf{Q}$ -a.s. continuous processes  $Y, B$  on  $(\Omega_0, \mathcal{G})$  where  $\Omega_0 = \Omega_1 \times \Omega$ ,  $\mathcal{G} = \mathcal{H} \otimes \mathcal{F}$  such that the following holds.

- 1) For almost all  $\omega$ ,  $Y(\cdot, \omega)$  is a (weak) solution to

$$\begin{cases} Y_t(\cdot, \omega) = Y_0 + \int_0^t \gamma(s, Y_s(\cdot, \omega), \omega) dB_s(\cdot, \omega), \\ \text{Law}(Y_0) = x_0, \end{cases} \quad (2.7)$$

with respect to  $Q^\omega$ , where  $B(\cdot, \omega)$  is a  $Q^\omega$ -Brownian motion for almost all  $\omega$ .

- 2) We denote  $(\mathcal{Y}_t)$  the canonical filtration associated with  $(Y_s, 0 \leq s \leq t)$  and  $\mathcal{G}_t = \mathcal{Y}_t \vee (\{\emptyset, \Omega_1\} \otimes \mathcal{F}_t)$ .  $W^1, \dots, W^N$  is a  $(\mathcal{G}_t)$ -martingale under  $\mathbf{Q}$ .
- 3) For every  $0 \leq s \leq T$ , for every bounded continuous  $\mathcal{A} : C([0, s]) \rightarrow \mathbb{R}$ , the r.v.  $\omega \mapsto E^{Q^\omega}(\mathcal{A}(Y_r(\cdot, \omega), r \in [0, s]))$  is  $\mathcal{F}_s$ -measurable.

b) We say that  $(DSDE)(\gamma, x_0)$  admits **weak-strong uniqueness** if the following holds. Consider a measurable space  $(\Omega_1, \mathcal{H})$  (resp.  $(\tilde{\Omega}_1, \tilde{\mathcal{H}})$ ), a probability kernel  $(Q(\cdot, \omega), \omega \in \Omega)$  (resp.  $(\tilde{Q}(\cdot, \omega), \omega \in \Omega)$ ), with processes  $(Y, B)$  (resp.  $(\tilde{Y}, \tilde{B})$ ) such that (2.7) holds (resp. (2.7) holds with  $(\Omega_0, \mathcal{G}, \mathbf{Q})$  replaced with

( $\tilde{\Omega}_0, \tilde{\mathcal{G}}_0, \tilde{\mathbf{Q}}, \tilde{\mathbf{Q}}$  being associated with  $(\tilde{Q}(\cdot, \omega))$ ). Moreover we suppose that item 2) is verified for  $Y$  and  $\tilde{Y}$ .

Then  $(Y, W^1, \dots, W^N)$  and  $(\tilde{Y}, W^1, \dots, W^N)$  have the same law.

c) A process  $Y$  fulfilling items 1) and 2) under (a) will be called **weak-strong solution of (DSDE)** $(\gamma, x_0)$ .

**Remark 2.11.** Let  $Y$  be a weak-strong solution of (DSDE) $(\gamma, x_0)$  with corresponding  $B$ .

a) Since for almost all  $\omega \in \Omega$ ,  $B(\cdot, \omega)$  is a Brownian motion under  $Q^\omega$ , it is clear that  $B$  is a Brownian motion under  $Q$ , which is independent of  $\mathcal{F}_T$ , i.e. independent of  $W^1, \dots, W^N$ .

Indeed let  $A : C([0, T]) \rightarrow \mathbb{R}$  be a continuous bounded functional, and denote by  $\mathcal{W}$  the Wiener measure on  $C([0, T])^N$ . Let  $F$  be a bounded  $\mathcal{F}_T$ -measurable r.v. Since for each  $\omega$ ,  $B(\cdot, \omega)$  is a Wiener process with respect to  $Q^\omega$ , we get

$$\begin{aligned} E^{\mathbf{Q}}(FA(B)) &= \int_{\Omega} FE^{Q^\omega}(\mathcal{A}(B(\cdot, \omega)))dP(\omega) = \int_{\Omega} F(\omega)dP(\omega) \int_{\Omega_1} \mathcal{A}(\omega_1)d\mathcal{W}(\omega_1) \\ &= \int_{\Omega_0} F(\omega)d\mathbf{Q}(\omega_0) \int_{\Omega_0} \mathcal{A}(\omega_1)d\mathbf{Q}(\omega_0). \end{aligned}$$

This shows that  $(W^1, \dots, W^N)$  and  $B$  are independent. Taking  $F = 1_{\Omega}$  in previous expression, the equality between the left-hand side and the third term, shows that  $B$  is a Brownian motion under  $Q$ .

b) Since for any  $1 \leq i, j \leq N$ ,

$$[W^i, W^j]_t = \delta_{ij}t, \quad [W^i, B] = 0, \quad [B, B]_t = t,$$

Lévy's characterization theorem, implies that  $(W^1, \dots, W^N, B)$  is a  $\mathbf{Q}$ -Brownian motion.

c) An equivalent formulation to 1) in item a) of Definition 2.10 is the following. For  $P$  a.e.,  $\omega \in \Omega$ ,  $Y(\cdot, \omega)$  solves the  $Q^\omega$ -martingale problem with respect to the (random) PDE operator

$$L_t^\omega f(\xi) = \frac{1}{2} \gamma^2(t, \xi, \omega) f''(\xi),$$

and initial distribution  $x_0$ . Indeed, we remark that the construction can be performed on the canonical space  $\Omega_1 = C([0, T]; \mathbb{R})$ .

**Proposition 2.12.** Let  $Y$  be a process as in Definition 2.10 a). We have the following.

1.  $Y$  is a  $(\mathcal{G}_t)$ -martingale on the product space  $(\Omega_0, \mathcal{G}, \mathbf{Q})$ .
2.  $[Y, W^i] = 0, \forall 1 \leq i \leq N$ .

*Proof.* Let  $0 \leq s < t \leq T$ ,  $F_s \in \mathcal{F}_s$  and  $G : C([0, s]) \rightarrow \mathbb{R}$  be continuous and bounded. We will prove below that, for  $1 \leq i \leq N + 1$ , setting  $W_t^{N+1} = 1$ , for all  $t \geq 0$ ,

$$E^{\mathbf{Q}}(Y_t W_t^i G(Y_r, r \leq s) 1_{F_s}) = E^{\mathbf{Q}}(Y_s W_s^i 1_{F_s} G(Y_r, r \leq s)). \quad (2.8)$$

Then (2.8) with  $i = N + 1$  shows item 1. Considering (2.8) with  $1 \leq i \leq N$ , shows that  $YW^i$  is a  $(\mathcal{G}_t)$ -martingale, which shows item 2. Therefore, it remains to show (2.8).

The left-hand side of that equality gives

$$\begin{aligned} \int_{\Omega} dP(\omega) \quad & W_t^i(\omega) 1_{F_s}(\omega) E^{Q^\omega}(Y_t(\cdot, \omega) G(Y_r(\cdot, \omega), r \leq s)) \\ &= \int_{\Omega} dP(\omega) 1_{F_s}(\omega) W_t^i(\omega) E^{Q^\omega}(Y_s(\cdot, \omega) G(Y_r(\cdot, \omega), r \leq s)), \end{aligned}$$

because  $Y(\cdot, \omega)$  is a  $Q^\omega$ -martingale for  $P$ -almost all  $\omega$ . To obtain the right-hand side of (2.8) it is enough to remember that  $W^i$  are  $(\mathcal{G}_t)$ -martingales and that item a) 3) in Definition 2.10 holds. This concludes the proof of Proposition 2.12.  $\square$

**Remark 2.13.** *Lemma 2.8 shows that, whenever weak-strong uniqueness holds, then the  $\mu$ -weighted marginal laws of any weak solution  $Y$  are uniquely determined.*

## 3 The concept of doubly probabilistic representation

### 3.1 The doubly stochastic non-linear diffusion.

We come back to the notations and conventions of the introduction and of Section 2. Let  $x_0$  be a probability on  $\mathbb{R}$ . The doubly probabilistic representation is based on the following idea. Let  $Y : \Omega_0 \times [0, T] \rightarrow \mathbb{R}$  be a measurable process where  $\Omega_0 = \Omega_1 \times \Omega$  is the usual enlarged probability space as introduced in Definition 2.3. Let  $\mathbf{Q}$  be a probability inherited from a random kernel  $Q^\omega$  as before Definition 2.3. Let  $(\mathcal{G}_t)$ , where  $(\mathcal{G}_t)$  is some filtration on  $(\Omega_0, \mathcal{G})$  such that  $W^1, \dots, W^N$  are  $(\mathcal{G}_t)$ -Brownian motions on  $(\Omega_0, \mathcal{G}, \mathbf{Q})$ .

Suppose that

$$\begin{cases} Y_t &= Y_0 + \int_0^t \Phi(X(s, Y_s)) dB_s, \\ \mu - \text{Weighted Law}(Y_t) &= X(t, \xi) d\xi, \quad t \in ]0, T], \\ \mu - \text{Weighted Law}(Y_0) &= x_0(d\xi), \end{cases} \quad (3.1)$$

where  $B$  is a  $Q$ -standard Brownian motion with respect to  $(\mathcal{G}_t)$ . Then  $X$  solves the SPDE (1.1). This will be the object of Theorem 3.3. Vice versa, if  $X$  is a solution of (1.1) then there is a process  $Y$  solving (3.1), see Theorem 7.1.

**Definition 3.1.** 1) We say that the doubly stochastic non-linear diffusion (DSNLD) driven by  $\Phi$  (on the space  $(\Omega, \mathcal{F}, P)$  with initial condition  $x_0$ , related to the random field  $\mu$  (shortly (DSNLD) $(\Phi, \mu, x_0)$ ) admits **weak existence** if there is a measurable random field  $X : [0, T] \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$  with the following properties.

a) The problem (DSDE) $(\gamma, x_0)$  with  $\gamma(t, \xi, \omega) = \Phi(X(t, \xi, \omega))$  admits weak-strong existence.

b)  $X = X(t, \xi, \cdot) d\xi, t \in ]0, T]$ , is the family of  $\mu$ -marginal weighted laws of  $Y$ , where  $Y$  is the solution of (2.7) in Definition 2.10. In other words  $X$  constitutes the densities of those  $\mu$ -marginal weighted laws.

2) A couple  $(Y, X)$ , such that  $Y$  is a (weak-strong) solution to the (DSDE) $(\gamma, x_0)$ , is called **weak solution** to the (DSNLD) $(\Phi, \mu, x_0)$ .  $Y$  is also called doubly stochastic representation of the random field  $X$ .

3) Suppose that, given two measurable random fields  $X^i : [0, T] \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}, i = 1, 2$  on  $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$ , and  $Y^i$ , on extended probability space  $(\Omega^i, \mathbf{Q}^i), i = 1, 2$ , such that  $(Y^i, X^i)$  is a weak-strong solution of (DSDE) $(\Phi(X^i), x_0), i = 1, 2$ , where we always have that  $(Y^1, W^1, \dots, W^N)$  and  $(Y^2, W^1, \dots, W^N)$  have the same law. Then we say that the (DSNLD) $(\Phi, \mu, x_0)$  admits **weak uniqueness**.

**Remark 3.2.** If (DSNLD) $(\Phi, \mu, x_0)$  admits **weak uniqueness** then the  $\mu$ -marginal weighted laws of  $Y$  are uniquely determined,  $P$ -a.s., see Lemma 2.8.

**Theorem 3.3.** Let  $(Y, X)$  be a solution of (DSNLD) $(\Phi, \mu, x_0)$ . Then  $X$  is a solution to the SPDE (1.1).

**Remark 3.4.** 1. Let  $t \in [0, T]$ . Let  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  be Borel and bounded. Then

$$\int_{\mathbb{R}} \varphi(\xi) X(t, \xi, \omega) d\xi = E^{Q^\omega} \left( \varphi(Y_t(\omega)) \mathcal{E}_t \left( \int_0^t \mu(ds, Y_s(\omega)) \right) \right).$$

So

$$\int_{\mathbb{R}} X(t, \xi, \omega) d\xi = E^{Q^\omega} \left( \mathcal{E}_t \left( \int_0^t \mu(ds, Y_s(\omega)) \right) \right).$$

Even though for a.e.  $\omega \in \Omega$ , the previous expression is not necessarily a probability measure, of course,

$$\nu_\omega : \varphi \mapsto \frac{\int_{\mathbb{R}} \varphi(\xi) X(t, \xi, \omega) d\xi}{\int_{\mathbb{R}} X(t, \xi, \omega) d\xi}$$

is one. It can be expressed as

$$\nu_\omega(A) = \frac{E^{Q^\omega}(1_A(Y_t) \mathcal{E}_t(M(\cdot, \omega)))}{E^{Q^\omega} \mathcal{E}_t(M(\cdot, \omega))},$$

where  $M_t(\cdot, \omega) = \int_0^t \mu(ds, Y_s(\cdot, \omega))$ ,  $t \in [0, T]$ , is defined in (2.2).

2. Consider the particular case  $e_0 = 0, e_1 = c$ ,  $c$  being some constant. In this case, the  $\mu$ -marginal laws are given by

$$A \mapsto E^{Q^\omega}(1_A(Y_t) c \mathcal{E}_t(W)) = c \mathcal{E}_t(W) E^{Q^\omega}(1_A(Y_t)) = c \mathcal{E}_t(W) \nu_\omega(t, A)$$

and  $\nu_\omega(t, \cdot)$  is the law of  $Y_t(\cdot, \omega)$  under  $Q^\omega$ .

*Proof.* Let  $B$  denote the Brownian motion associated to  $Y$  as a solution to (DSDE)( $\gamma, x_0$ ), mentioned in item a)1) of Definition 3.1. For  $t \in [0, T]$ , we set

$$Z_t = \mathcal{E}_t \left( \int_0^t \mu(ds, Y_s) \right), \quad M_t = Z_t \exp \left( - \int_0^t e^0(Y_s) ds \right), \quad t \in [0, T].$$

1. Proof of Definition 2.9 1. By Proposition 2.6,  $(M_t, t \in [0, T])$  is a uniformly integrable martingale. Consequently  $t \mapsto Z_t$  is continuous in  $L^1(\Omega_0, \mathbf{Q})$ . On the other hand the process  $Y$  is continuous. This implies that  $P$  a.e.  $\omega \in \Omega$ ,  $X \in C([0, T]; \mathcal{M}(\mathbb{R}))$ , where  $\mathcal{M}(\mathbb{R})$  is equipped with the weak topology. This implies that  $X \in C([0, T]; \mathcal{S}'(\mathbb{R}))$ . Furthermore, for  $P$  a.e.  $\omega \in \Omega$ , and  $t \in ]0, T]$ ,  $X(t, \cdot, \omega) \in L^1(\mathbb{R})$  and  $\int_{\mathbb{R}} X(t, \xi, \omega) d\xi = \|\Gamma(t, \cdot, \omega)\|_{\text{var}}$ . By Proposition 2.6 iv), it follows that  $P$ -a.s.  $X(\cdot, \cdot, \omega) \in L^\infty([0, T]; L^1(\mathbb{R})) \subset L^2([0, T]; L^1_{\text{loc}}(\mathbb{R}))$ .

2. Definition 2.9 2. follows from Remark 3.4 1) and Definition 2.10 a) 3).

3. Proof of Definition 2.9 3. Let  $\varphi \in \mathcal{S}(\mathbb{R})$  with compact support. Taking into account Proposition 2.12, we apply Itô's formula to get

$$\begin{aligned} \varphi(Y_t) Z_t &= \varphi(Y_0) + \int_0^t \varphi'(Y_s) Z_s dY_s + \int_0^t \varphi(Y_s) Z_s \left( \mu(ds, Y_s) - \frac{1}{2} \sum_{i=1}^N (e^i(Y_s))^2 ds \right) \\ &+ \frac{1}{2} \int_0^t \varphi''(Y_s) \Phi^2(X(s, Y_s)) Z_s ds + \frac{1}{2} \int_0^t \varphi(Y_s) Z_s \left( \sum_{i=1}^N (e^i(Y_s))^2 \right) ds. \end{aligned}$$

Indeed we remark that  $\int_0^t \varphi'(Y_s) d[Z, Y]_s = 0$ , because  $[Z, Y]_t = \sum_{i=1}^N \int_0^t e^i(Y_s) Z_s d[W^i, Y]_s = 0$ ; in fact  $[W^i, Y] = 0$  by Proposition 2.12. So

$$\begin{aligned} \varphi(Y_t) Z_t &= \varphi(Y_0) + \int_0^t \varphi'(Y_s) Z_s \Phi(X(s, Y_s)) dB_s \\ &\quad + \int_0^t \varphi(Y_s) Z_s \mu(ds, Y_s) + \frac{1}{2} \int_0^t \varphi''(Y_s) \Phi^2(X(s, Y_s)) Z_s ds. \end{aligned}$$

Taking the expectation with respect to  $Q^\omega$  we get  $dP$ -a.s.,

$$\begin{aligned} \int_{\mathbb{R}} d\xi \varphi(\xi) X(t, \xi) &= \int_{\mathbb{R}} \varphi(\xi) x_0(d\xi) + \sum_{i=0}^N \int_0^t dW_s^i \left( \int_{\mathbb{R}} d\xi \varphi(\xi) e^i(\xi) X(s, \xi) \right) \\ &\quad + \frac{1}{2} \int_0^t ds \int_{\mathbb{R}} d\xi \varphi''(\xi) \Phi^2(X(s, \xi)) X(s, \xi), \end{aligned}$$

which implies the result. Indeed, in the previous equality, we have used Lemma 3.5 below. □

**Lemma 3.5.** *Let  $1 \leq i \leq N$ . For  $P$  a.e.  $\omega \in \Omega$ , we have*

$$E^{Q^\omega} \left( \int_0^t \varphi(Y_s) Z_s e^i(Y_s) dW_s^i \right) (\cdot, \omega) = \int_0^t dW_s^i(\omega) \int_{\mathbb{R}} \varphi(\xi) e^i(\xi) X(s, \xi, \omega) d\xi.$$

*Proof.* Since the Brownian motions  $W^i$  are not random for  $Q^\omega$ , it is possible to justify the permutation of the stochastic integral with respect to  $W^i$  and  $E^{Q^\omega}$  by a Fubini argument approximating the stochastic integrals via Lebesgue integral, see e.g. Theorem 2 of [25]. A complete proof is given in [7]. □

## 3.2 Filtering interpretation

Item 1. of Remark 3.4 has an interpretation in the framework of filtering theory, see e.g. [20] for a comprehensive introduction on that subject.

Suppose  $e^0 = 0$ . Let  $\hat{\mathbf{Q}}$  be a probability on some probability space  $(\Omega_0, \mathcal{G}_T)$ , and consider the non-linear diffusion problem (1.2) as a basic dynamical phenomenon. We suppose now that there are  $N$  observations  $Y^1, \dots, Y^N$  related to the process  $Y$  generating a filtration  $(\mathcal{F}_t)$ . We suppose in particular that  $dY_t^i = dW_t^i + e^i(Y_t) dt$ ,  $1 \leq i \leq N$ , and  $W^1, \dots, W^N$  be  $(\mathcal{F}_t)$ -Brownian motions. Consider the following dynamical system of non-linear diffusion type:

$$\begin{cases} Y_t &= Y_0 + \int_0^t \Phi(X(s, Y_s)) dB_s \\ dY_t^i &= dW_t^i + e^i(Y_t) dt, 1 \leq i \leq N, \\ X(t, \cdot) &: \text{conditional law of } Y_t \text{ under } \mathcal{F}_t. \end{cases} \quad (3.1)$$

The third equality of (3.1) means, under  $\hat{\mathbf{Q}}$ , that we have,

$$\int_{\mathbb{R}} \varphi(\xi) X(t, \xi) d\xi = E(\varphi(Y_t) | \mathcal{F}_t). \quad (3.2)$$

We remark that, under the new probability  $Q$  defined by  $d\mathbf{Q} = d\hat{\mathbf{Q}}\mathcal{E}(\int_0^T \mu(ds, Y_s))$ ,  $Y^1, \dots, Y^N$  are standard  $(\mathcal{F}_t)$ -independent Brownian motions. Then (3.2) becomes

$$\int_{\mathbb{R}} \varphi(\xi) X(t, \xi) d\xi = E^{\hat{\mathbf{Q}}}(\varphi(Y_t) | \mathcal{F}_t) = \frac{E^{\mathbf{Q}}(\varphi(Y_t) \mathcal{E}_t(\int_0^T \mu(ds, Y_s) | \mathcal{F}_t))}{E^{\mathbf{Q}}(\mathcal{E}_t(\int_0^T \mu(ds, Y_s) | \mathcal{F}_t))}.$$

Consequently, by Theorem 3.3,  $X$  will be the solution of the SPDE (1.1), with  $x_0$  being the law of  $Y_0$ ; so (1.1) constitutes the Zakai type equation associated with our filtering problem.

## 4 The densities of the $\mu$ -marginal weighted laws

This section constitutes an important step towards the doubly probabilistic representation of a solution to (1.1), when  $\psi$  is non-degenerate. Let  $x_0$  be a fixed probability on  $\mathbb{R}$ . We recall that a process  $Y$  (on a suitable enlarged probability space  $(\Omega_0, \mathcal{G}, \mathbf{Q})$ ), which is a weak solution to the (DSNLD) $(\Phi, \mu, x_0)$ , is in particular a weak-strong solution of a (DSDE) $(\gamma, x_0)$  where  $\gamma : [0, T] \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$  is some suitable progressively measurable random field on  $(\Omega, \mathcal{F}, P)$ . The aim of this section is twofold.

- A) To show that whenever  $\gamma$  is a.s. bounded and non-degenerate, (DSDE) $(\gamma, x_0)$  admit weak-strong existence and uniqueness.
- B) The marginal  $\mu$ -laws of the solution to (DSDE) $(\gamma, x_0)$  admit a density for  $P$  a.s.
- A) We start discussing well-posedness.

**Proposition 4.1.** *We suppose the existence of random variables  $A_1, A_2$  such that*

$$0 < A_1(\omega) \leq \gamma(t, \xi, \omega) \leq A_2(\omega), \quad \forall (t, \xi) \in [0, T] \times \mathbb{R}, \quad dP\text{-a.s.}$$

*Then (DSDE) $(\gamma, x_0)$  admits weak-strong existence and uniqueness.*

*Proof. Uniqueness.* This is the easy part. Let  $Y$  and  $\tilde{Y}$  be two solutions. Then for  $\omega$  outside a  $P$ -null set  $N_0$ ,  $Y(\cdot, \omega)$  and  $\tilde{Y}(\cdot, \omega)$  are solutions to the same one-dimensional classical SDE with measurable bounded and non-degenerate (i.e. greater than a strictly positive constant) coefficients. Then, by Exercise 7.3.3 of [27]

the law of  $Y(\cdot, \omega)$  equals the law of  $\tilde{Y}(\cdot, \omega)$ . Then obviously the law of  $Y$  equals the law of  $\tilde{Y}$ .

*Existence.* This point is more delicate. In fact one needs to solve the random SDE for  $P$  almost all  $\omega$  but in such a way that the solution produces bimeasurable processes  $Y$  and  $B$ .

First we regularize the coefficient  $\gamma$ . Let  $\phi$  be a mollifier with compact support; we set  $\phi_n(x) = n\phi(nx)$ ,  $x \in \mathbb{R}$ ,  $n \in \mathbb{N}$ . We consider the random fields  $\gamma_n : [0, T] \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$  by  $\gamma_n(t, x, \omega) := \int_{\mathbb{R}} \gamma(t, x - y, \omega) \phi_n(y) dy$ .

Let  $(\tilde{\Omega}_1, \tilde{\mathcal{H}}_1, \tilde{P})$  be a probability space where we can construct a random variable  $Y_0$  distributed according to  $x_0$  and an independent Brownian motion  $B$ .

In this way on  $(\tilde{\Omega}_1 \times \Omega, \tilde{\mathcal{H}}_1 \otimes \mathcal{F}, \tilde{P} \otimes P)$  we dispose of a random variable  $Y_0$  and a Brownian motion independent of  $\{\emptyset, \tilde{\Omega}_1\} \otimes \mathcal{F}$ . By usual fixed point techniques, it is possible to exhibit a (strong) solution of (DSDE) $(\gamma_n, x_0)$  on the over mentioned product probability space. We can show that there is a unique solution  $Y = Y^n$  of  $Y_t = Y_0 + \int_0^t \gamma_n(s, Y_s, \cdot) dB_s$ . In fact, the maps  $\Gamma_n : Z \mapsto \int_0^t \gamma_n(s, Z_s, \omega) dB_s + Y_0$ , where  $\Gamma_n : L^2(\tilde{\Omega}_1 \times \Omega; \tilde{P} \otimes P) \rightarrow L^2(\tilde{\Omega}_1 \times \Omega, \tilde{P} \otimes P)$  are Lipschitz; by usual Picard fixed point arguments one can show the existence of a unique solution  $Z = Z^n$  in  $L^2(\tilde{\Omega}_1 \times \Omega; \tilde{P} \otimes P)$ . We observe that, by usual regularization arguments for Itô integral as in Lemma 3.5, for  $\omega$ -a.s.,  $Y(\cdot, \omega)$  solves for  $P$  a.e.  $\omega \in \Omega$ , equation

$$Y_t(\cdot, \omega) = Y_0 + \int_0^t \gamma_n(s, Y_s(\cdot, \omega), \omega) dB_s, \quad (4.1)$$

on  $(\tilde{\Omega}_1, \tilde{\mathcal{H}}_1, \tilde{P})$ . We consider now the measurable space  $\Omega_0 = \Omega_1 \times \Omega$ , where  $\Omega_1 = C([0, T], \mathbb{R})$ , equipped with product  $\sigma$ -field  $\mathcal{G} = \mathcal{B}(\Omega_1) \otimes \mathcal{F}$ . On that measurable space, we introduce the probability measures  $\mathbf{Q}_n$  where  $\mathbf{Q}_n(d\omega_1, \omega) = Q_n(d\omega_1, \omega)P(d\omega)$  and  $Q_n(\cdot, \omega)$  being the law of  $Y^n(\cdot, \omega)$  for almost all fixed  $\omega$ .

We set  $Y_t(\omega_1, \omega) = \omega_1(t)$ , where  $\omega_1 \in C([0, T]; \mathbb{R})$ . We denote by  $(\mathcal{Y}_t, t \in [0, T])$  (resp.  $(\mathcal{Y}_t^1)$ ) the canonical filtration associated with  $Y$  on  $\Omega_0$  (resp.  $\Omega_1$ ). The next step will be the following.

**Lemma 4.2.** *For almost all  $\omega$  dP a.s.  $Q_n(\omega, \cdot)$  converges weakly to  $Q(\omega, \cdot)$ , where under  $Q(\cdot, \omega)$ ,  $Y(\cdot, \omega)$  solves the SDE*

$$Y_t(\cdot, \omega) = Y_0 + \int_0^t \gamma(s, Y_s(\cdot, \omega), \omega) dB_s(\cdot, \omega),$$

where  $B(\cdot, \omega)$  is an  $(\mathcal{Y}_t^1)$ -Brownian motion on  $\Omega_1$ .

*Proof.* It follows directly from Proposition A.1 of the Appendix.  $\square$

This shows the validity of 1) if Definition 2.10 a).

**Remark 4.3.** 1) Since  $Q_n(\cdot, \omega)$  converges weakly to  $Q(\cdot, \omega)$ ,  $\omega$  dP a.s., then the limit (up to an obvious modification) is a measurable random kernel.

2) This also implies that  $Y_n(\cdot, \omega)$  converges stably to  $Q(\cdot, \omega)$ . For details about the stable convergence the reader can consult [17, section VIII 5. c] and the recent monograph [16].

The considerations above allow to complete the proof of Proposition 4.1. By Lemma 4.2,  $Q^\omega = Q(\cdot, \omega)$  is a random kernel, being a limit of random kernels. Let us consider the associated probability measure on the suitable enlarged probability space  $(\Omega_0, \mathcal{G}, Q)$ . We observe that  $Y$  on  $(\Omega_0, \mathcal{G})$  is obviously measurable, because it is the canonical process  $Y(\omega_1, \omega) = \omega_1$ . Setting

$$B_t(\cdot, \omega) = \int_0^t \frac{dY_s(\cdot, \omega)}{\gamma(s, Y_s(\cdot, \omega), \omega)},$$

we get  $[B]_t(\cdot, \omega) = t$  under  $Q(\cdot, \omega)$ , so, by Lévy characterization theorem, it is a Brownian motion. Moreover  $B$  is bimeasurable.

Let  $G = \mathcal{A}(Y_r(\cdot, \omega), r \in [0, s])$ , where  $\mathcal{A}$  is a bounded functional  $C([0, s]) \rightarrow \mathbb{R}$ . We first observe that the r.v.  $\omega \mapsto E^{Q^\omega}(G)$  is  $\mathcal{F}_s$ -measurable. This happens because  $Y$  is, under  $Q^\omega$ , a martingale with quadratic variation  $\left(\int_0^t \gamma^2(s, Y_s(\cdot, \omega), \omega) ds, 0 \leq t \leq T\right)$ , i.e. with (random) coefficient which is  $(\mathcal{F}_t)$ -progressively measurable. This shows item 3) of Definition 2.10 a).

The last point to check is that  $W^1, \dots, W^N$  are  $(\mathcal{G}_t)$ -martingales, where  $\mathcal{G}_t = \mathcal{Y}_t \vee (\{\emptyset, \Omega_1\} \otimes \mathcal{F}_t)$ ,  $0 \leq t \leq T$ , i.e. item 2) of Definition 2.10.

Indeed, we justify this immediately. Consider  $0 \leq s \leq t \leq T$ . Taking into account monotone class arguments, given  $F \in \mathcal{F}_s$ ,  $G \in \mathcal{Y}_s^1$ ,  $1 \leq i \leq N$ , it is enough to prove that

$$E^{\mathbf{Q}}(FGW_t^i) = E^{\mathbf{Q}}(FGW_s^i). \quad (4.2)$$

Using the fact that  $W^i$  is an  $(\mathcal{F}_t)$ -martingale and that  $E^{Q^\omega}(G)$  is  $\mathcal{F}_s$ -measurable by item a) 3) of Definition 2.10 (established above), the left-hand side of (4.2) gives

$$E^P(FW_t^i E^{Q^\omega}(G)) = E^P(FW_s^i E^{Q^\omega}(G)),$$

which constitutes the right-hand side of (4.2). This concludes the proof of the proposition.  $\square$

We go on now with step B) of the beginning of Section 4.

**Proposition 4.4.** *We suppose the existence of r.v.  $A_1, A_2$  such that*

$$0 < A_1(\omega) \leq \gamma(t, \xi, \omega) \leq A_2(\omega), \forall (t, \xi) \in [0, T] \times \mathbb{R}, \quad a.s.$$

Let  $Y$  be a weak-strong solution to (DSDE)( $\gamma, x_0$ ) and we denote by  $(\nu_t(dy, \cdot), t \in [0, T])$ , the  $\mu$ -marginal weighted laws of process  $Y$ .

1. *There is a measurable function  $q : [0, T] \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}_+$  such that  $dtdP$  a.e.,  $\nu_t(dy, \cdot) = q_t(y, \cdot)dy$ . In other words the  $\mu$ -marginal weighted laws admit densities.*
2.  $\int_{[0, T] \times \mathbb{R}} q_t^2(y, \cdot) dtdy < \infty \quad dP$ -a.s..
3.  *$q$  is an  $L^2(\mathbb{R})$ -valued progressively measurable process.*

*Proof.* By 3) of Definition 2.10, the  $\mu$ -marginal laws constitute an  $S'(\mathbb{R})$ -valued progressively measurable process. Consequently 3. holds if 1. and 2. hold.

Let

$$B_t(\cdot, \omega) := \int_0^t \frac{dY_s(\cdot, \omega)}{\gamma(s, Y_s(\cdot, \omega), \omega)}.$$

We denote again  $Q^\omega := Q(\cdot, \omega)$  according to Definition 2.10,  $\omega \in \Omega$ .

Let  $\omega \in \Omega$  be fixed. Let  $\varphi : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function with compact support. We need to evaluate

$$E^{Q^\omega} \left( \int_0^T \varphi(s, Y_s) Z_s ds \right), \quad (4.3)$$

where  $Z_s = M_s \exp \left( \int_0^s e^0(Y_r) dr \right)$  where  $M_s = \mathcal{E}_s \left( \sum_{i=1}^N \int_0^s e^i(Y_r) dW_r^i \right)$ .

$M_s$  is smaller or equal than  $\exp \left( \sum_{j=1}^N \int_0^s e^j(Y_r) dW_r^j \right)$  which equals

$$\exp \left( \sum_{j=1}^N \left\{ W_s^j e^j(Y_s) - \int_0^s W_r^j (e^j)'(Y_r) dY_r \right\} - \frac{1}{2} \int_0^s \left\{ \sum_{j=1}^N W_r^j (e^j)''(Y_r) \gamma^2(r, Y_r, \cdot) \right\} dr \right), \quad (4.4)$$

taking into account the fact that  $[Y, W^j] = 0$  for any  $1 \leq j \leq n$ , by Proposition 2.12.

Denoting  $\|g\|_\infty := \sup_{t \in [0, T]} |g(t)|$ , for a function  $g : [0, T] \rightarrow \mathbb{R}$ , (4.4) is smaller or equal than

$$\exp \left( \sum_{j=1}^N \|W^j\|_\infty (\|e^j\|_\infty + \frac{T}{2} \|(e^j)''\|_\infty A_2^2) \right) \exp \left( - \int_0^s \left[ \sum_{j=1}^N W_r^j (e^j)'(Y_r) \gamma(r, Y_r, \cdot) \right] dB_r \right).$$

So (4.3) is bounded by

$$\varrho(\omega) E^{Q^\omega} \left( \int_0^T |\varphi|(s, Y_s(\cdot, \omega)) R_s(\cdot, \omega) ds \right), \quad (4.5)$$

where

$$\begin{aligned} \varrho(\omega) &= \exp\left(T\|e_0\|_\infty + \sum_{i=1}^N \|W^i\|_\infty \|e^i\|_\infty\right. \\ &\quad \left.+ T\frac{A_2^2(\omega)}{2} \sum_{i=1}^N (\|W^i\|_\infty^2 \|(e^i)'\|_\infty^2 + \|W^i\|_\infty \|(e^i)''\|_\infty)\right) \end{aligned}$$

and  $R$  is the  $Q^\omega$ -exponential martingale

$$R_t(\cdot, \omega) = \exp\left(-\int_0^t \delta(r, \cdot, \omega) dB_r - \frac{1}{2} \int_0^t \delta^2(r, \cdot, \omega) dr\right).$$

where  $\delta(r, \cdot, \omega) = \sum_{j=1}^N W_r^j (e^j)'(Y_r(\cdot, \omega)) \gamma(r, Y_r(\cdot, \omega), \omega)$ . So there is a random (depending on  $\omega \in \Omega$ ) constant

$$\varrho_1(\omega) := \text{const}(T, W^j, \|e^j\|_\infty, \|(e^j)'\|_\infty, \|(e^j)''\|_\infty, 1 \leq j \leq N, A_2(\omega)), \quad (4.6)$$

so that (4.5) is smaller than

$$\varrho_1(\omega) E^{Q^\omega} \left( \int_0^T |\varphi(s, Y_s(\cdot, \omega))| ds R_T(\cdot, \omega) \right), \quad (4.7)$$

where we remind that  $R(\cdot, \omega)$  is a  $Q^\omega$ -martingale. By Girsanov theorem,  $\tilde{B}_t(\cdot, \omega) = B_t(\cdot, \omega) + \int_0^t \delta(r, \cdot, \omega) dr$  is a  $\tilde{Q}^\omega$ -Brownian motion with  $d\tilde{Q}^\omega = R_T(\cdot, \omega) dQ^\omega$ . At this point, the expectation in (4.7) gives

$$E^{\tilde{Q}^\omega} \left( \int_0^T |\varphi|(s, Y_s(\cdot, \omega)) ds \right), \quad (4.8)$$

where

$$Y_t(\cdot, \omega) = Y_0 + \int_0^t \gamma(s, Y_s(\cdot, \omega), \omega) d\tilde{B}_s - \int_0^t \gamma(s, Y_s(\cdot, \omega), \omega) \delta(s, \cdot, \omega) ds.$$

For fixed  $\omega \in \Omega$ ,  $\delta$  is bounded by a random constant  $\varrho_2(\omega)$  of the type (4.6). Moreover we keep in mind assumption (4.1) on  $\gamma$ . By Exercise 7.3.3 of [27], (4.8) is bounded by  $\varrho_3(\omega) \|\varphi\|_{L^2([0, T] \times \mathbb{R})}$ , where  $\varrho_3(\omega)$  again depends on the same quantities as in (4.6) and  $\Phi$ . So for  $\omega$  dP-a.s., the map  $\varphi \mapsto E^{Q^\omega} \left( \int_0^T \varphi(s, Y_s(\cdot, \omega)) Z_s(\cdot, \omega) ds \right)$  prolongs to  $L^2([0, T] \times \mathbb{R})$ . Using Riesz' theorem it is not difficult to show the existence of an  $L^2([0, T] \times \mathbb{R})$  function  $(s, y) \mapsto q_s(y, \omega)$  which constitutes indeed the density of the family of the  $\mu$ -marginal weighted laws.  $\square$

## 5 On the uniqueness of a Fokker-Planck type SPDE

The next result is an extension of Theorem 3.8 of [14] to the stochastic case. It has an independent interest since it is a Fokker-Planck SPDE with possibly degenerate measurable coefficients.

**Theorem 5.1.** *Let  $z_0$  be a distribution in  $\mathcal{S}'(\mathbb{R})$ . Let  $z^1, z^2$  be two measurable random fields belonging  $\omega$  a.s. to  $C([0, T], \mathcal{S}'(\mathbb{R}))$  such that  $z^1, z^2 : ]0, T] \times \Omega \rightarrow \mathcal{M}(\mathbb{R})$ . Let  $a : [0, T] \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}_+$  be a bounded measurable random field such that, for any  $t \in [0, T]$ ,  $a(t, \cdot)$  is  $\mathcal{B}([0, t]) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{F}_t$ -measurable. We suppose moreover the following.*

i)  $z^1 - z^2 \in L^2([0, T] \times \mathbb{R})$  a.s.

ii)  $t \mapsto (z^1 - z^2)(t, \cdot)$  is an  $(\mathcal{F}_t)$ -progressively measurable  $\mathcal{S}'(\mathbb{R})$ -valued process.

iii)  $\int_0^T \|z^i(s, \cdot)\|_{\text{var}}^2 ds < \infty$  a.s.

iv)  $z^1, z^2$  are solutions to

$$\begin{cases} \partial_t z(t, \xi) = \partial_{\xi\xi}^2((az)(t, \xi)) + z(t, \xi)\mu(dt, \xi), \\ z(0, \cdot) = z_0. \end{cases} \quad (5.1)$$

Then  $z^1 \equiv z^2$ .

**Remark 5.2.** *By solution of equation (5.1) we intend, as expected, the following: for every  $\varphi \in \mathcal{S}(\mathbb{R})$ ,  $\forall t \in [0, T]$ ,*

$$\int_{\mathbb{R}} \varphi(\xi) z(t, d\xi) = \langle z_0, \varphi \rangle + \int_0^t ds \int_{\mathbb{R}} a(s, \xi) \varphi''(\xi) z(s, d\xi) + \sum_{j=0}^N \int_0^t dW_s^j \int_{\mathbb{R}} \varphi(\xi) e^j(\xi) z(s, d\xi).$$

*Proof of Theorem 5.1.* The proof makes use of the similar arguments as in Theorem 3.8 of [14] or Theorem 3.1 in [10], in a randomized form. The full proof is given in [24] Theorem 4.2, see also [7].

□

## 6 The non-degenerate case

We are now able to discuss the doubly probabilistic representation of a solution to (1.1) when  $\psi$  is non-degenerate provided that its solution fulfills some properties.

**Definition 6.1.** • *We will say that equation (1.1) (or  $\psi$ ) is **non-degenerate** if on each compact, there is a constant  $c_0 > 0$  such that  $\Phi \geq c_0$ .*

• *We will say that equation (1.1) or  $\psi$  is **degenerate** if  $\lim_{u \rightarrow 0^+} \Phi(u) = 0$ .*

One of the typical examples of degenerate  $\psi$  is the case of  $\psi$  being **strictly increasing after some zero**. This notion was introduced in [6] and it means the

following. There is  $0 \leq u_c$  such that  $\psi_{[0, u_c]} \equiv 0$  and  $\psi$  is strictly increasing on  $]u_c, +\infty[$ .

- Remark 6.2.**
1.  $\psi$  is non-degenerate if and only if  $\lim_{u \rightarrow 0^+} \Phi(u) > 0$ .
  2. Of course, if  $\psi$  is strictly increasing after some zero, with  $u_c > 0$  then  $\psi$  is degenerate.
  3. If  $\psi$  is degenerate, then  $\psi^\kappa(u) = (\Phi^2(u) + \kappa)u$ , for every  $\kappa > 0$ , is non-degenerate.

As announced the theorem below also holds when  $\psi$  is multi-valued.

**Theorem 6.3.** *We suppose the following assumptions.*

1.  $x_0$  is a real probability measure.
2.  $\psi$  is non-degenerate.
3. There is only one random field  $X : [0, T] \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$  solution of (1.1) (see Definition 2.9) such that

$$\int_{[0, T] \times \mathbb{R}} X^2(s, \xi) ds d\xi < \infty \quad a.s. \quad (6.1)$$

Then there is a unique weak solution to the (DSNLD) $(\Phi, \mu, x_0)$ .

**Remark 6.4.** 1. An easy adaptation of Theorem 3.4 of [8] (taking into account  $e^0$ ), when  $\psi$  is Lipschitz and  $e^0, \dots, e^N$  belong to  $H^1$  allows to show that there is a solution to (1.1) such that

$$E \left( \int_{[0, T] \times \mathbb{R}} X^2(s, \xi) ds d\xi \right) < \infty. \text{ This holds even if } x_0 \text{ belongs to } H^{-1}(\mathbb{R}).$$

According to Theorem B.1, that solution is unique. In particular item 3. in Theorem 6.3 statement holds.

2. Theorem 6.3 constitutes the converse of Theorem 3.3 when  $\psi$  is non-degenerate.
3. The theorem also holds if  $\psi$  is multi-valued. For implementing this, we need to adapt the techniques of [14].
4. As side-effect of the proof of the weak-strong existence Proposition 4.1, the space  $(\Omega_0, \mathcal{G}, \mathbf{Q})$  can be chosen as  $\Omega_0 = \Omega_1 \times \Omega$ ,  $\Omega_1 = C([0, T]; \mathbb{R}) \times \mathbb{R}$ ,  $\mathcal{G} = \mathcal{B}(\Omega_1) \times \mathcal{F}$ ,  $\mathbf{Q}(H \times F) = \int_{\Omega_1 \times \Omega} dP(\omega) 1_F(\omega) Q(d\omega_1, \omega)$ .

*Proof.* 1) We set  $\gamma(t, \xi, \omega) = \Phi(X(t, \xi, \omega))$ . According to Proposition 4.1 there is a weak-strong solution  $Y$  to (DSDE)( $\gamma, x_0$ ). By Proposition 4.4  $\omega$  a.s. the  $\mu$ -marginal weighted laws of  $Y$  admit densities  $(q_t(\xi, \omega), t \in ]0, T], \xi \in \mathbb{R}, \omega \in \Omega)$  such that  $dP$ -a.s.  $\int_{[0, T] \times \mathbb{R}} ds d\xi q_s^2(\xi, \cdot) < \infty$  a.s.

2) Setting

$$\nu_t(\xi, \omega) = \begin{cases} q_t(\xi, \omega) d\xi & : t \in ]0, T], \\ x_0 & : t = 0, \end{cases}$$

$\nu$  is a solution to (5.1) with  $\nu_0 = x_0$ ,  $a(t, \xi, \omega) = \Phi^2(X(t, \xi, \omega))$ . This can be shown applying Itô's formula similarly as in the proof of Theorem 3.3.

3) On the other hand  $X$  is obviously also a solution of (5.1), which in particular verifies (6.1). Consequently  $z^1 = \nu$ ,  $z^2 = X$  verify items i), ii), iii) of Theorem 5.1. So Theorem 5.1 implies that  $\nu \equiv X$ ; this shows that  $Y$  provides a solution to (DSNLD)( $\Phi, \mu, x_0$ ).

4) Concerning uniqueness, let  $Y^1, Y^2$  be two solutions to the (DSNLD) related to  $(\Phi, \mu, x)$ . The corresponding random fields  $X^1, X^2$  constitute the  $\mu$ -marginal laws of  $Y^1, Y^2$  respectively.

Now  $Y^i$ ,  $i = 1, 2$ , is a weak-strong solution of (DSDE)( $\gamma_i, x$ ) with  $\gamma_i(t, \xi, \omega) = \Phi(X_i(t, \xi, \omega))$ , so by Proposition 4.4  $X_i$ ,  $i = 1, 2$  fulfills (6.1). By Theorem 3.3,  $X_1$  and  $X_2$  are solutions to (1.1). By assumption 3. of the statement,  $X_1 = X_2$ . The conclusion follows by Proposition 4.1, which guarantees the uniqueness of the weak-strong solution of (DSDE)( $\gamma, x_0$ ) with  $\gamma = \gamma_1 = \gamma_2$ .  $\square$

**Remark 6.5.** *One side-effect of Theorem 6.3 is the following. Suppose  $\psi$  to be non-degenerate. Let  $X : [0, T] \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$  be a solution such that  $dP$ -a.s.*

*$\int_{[0, T] \times \mathbb{R}} X^2(s, \xi) ds d\xi < \infty$  a.s. We have the following for  $\omega$   $dP$ -a.s.*

i)  $X(t, \cdot, \omega) \geq 0$  a.e.  $\forall t \in [0, T]$ .

ii)  $E \left( \int_{\mathbb{R}} X(t, \xi) d\xi \right) = 1, \forall t \in [0, T]$  if  $e_0 = 0$ .

**Remark 6.6.** *If (1.1) has a solution, not necessarily unique, then (DSNLD) with respect to  $(\Phi, \mu, x_0)$  still admits existence.*

## 7 The degenerate case

The idea consists in proceeding similarly to [6], which treated the case  $\mu = 0$  and the case when  $x_0$  is absolutely continuous with bounded density.  $\psi$  will be assumed

to be strictly increasing after some zero  $u_c \geq 0$ , see Definition 6.1. We recall that if  $\psi$  is degenerate, then necessarily  $\Phi(0) := \lim_{x \rightarrow 0} \Phi(x) = 0$ .

The theorem below concerns existence, we do not know any uniqueness result in the degenerate case.

**Theorem 7.1.** *We suppose the following.*

1. *The functions  $e^i, 1 \leq i \leq N$  belong to  $H^1(\mathbb{R})$ .*
2. *We suppose that  $\psi : \mathbb{R} \rightarrow \mathbb{R}$  is non-decreasing, Lipschitz and strictly increasing after some zero.*
3.  *$x_0$  belongs to  $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ .*

*Then there is a weak solution to the (DSNLD)( $\Phi, \mu, x_0$ ).*

**Remark 7.2.** *If  $u_c > 0$  then  $\psi$  is necessarily degenerate and also  $\Phi$  restricted to  $[0, u_c]$  vanishes.*

**Proof** (of Theorem 7.1).

- 1) We proceed by approximation rendering  $\Phi$  non-degenerate. Let  $\kappa > 0$ . We define  $\Phi_\kappa : \mathbb{R} \rightarrow \mathbb{R}_+$  by  $\Phi_\kappa(u) = \sqrt{\Phi^2(u) + \kappa}$ ,  $\psi_\kappa(u) = \Phi_\kappa^2(u) \cdot u$ . Let  $X^\kappa$  be the solution so (1.1) with  $\psi_\kappa$  instead of  $\psi$ . According to Theorem 6.3 and Remark 6.4 4., setting

$$\tilde{\Omega}_1 = C([0, T], \mathbb{R}) \times \mathbb{R}, \quad Y(\omega_1, \omega) = \omega_1, \quad (7.1)$$

$\mathcal{H}$  the Borel  $\sigma$ -algebra of  $\tilde{\Omega}_1$ , there are families of probability kernels  $Q^\kappa$  on  $\mathcal{H} \times \Omega$ , and measurable processes  $B^\kappa$  on  $\tilde{\Omega}_0 = \tilde{\Omega}_1 \times \Omega$  such that

- i)  $B^\kappa(\cdot, \omega)$  is a  $Q^\kappa(\cdot, \omega)$ -Brownian motion;
- ii)  $Y$  is a (weak) solution, on  $(\tilde{\Omega}_1, Q^\kappa(\cdot, \omega))$ , of
$$Y_t = Y_0 + \int_0^t \Phi_\kappa(X^\kappa(s, Y_s, \omega)) dB_s^\kappa(\cdot, \omega), \quad t \in [0, T];$$
- iii)  $Y_0$  is distributed according to  $x_0 = X^\kappa(0, \cdot)$ .
- iv) The  $\mu$ -marginal weighted laws of  $Y$  under  $\mathbf{Q}^\kappa$  are  $(X^\kappa(t, \cdot))$ .

In agreement with Definition 3.1 and Definition 2.10, we need to show the existence of a suitable measurable space  $(\Omega_1, \mathcal{H})$ , a probability kernel  $Q$  on  $\mathcal{H} \times \Omega$ , a process  $B$  on  $\Omega_0 := \Omega_1 \times \Omega$  such that the following holds.

- i)  $B(\cdot, \omega)$  is a  $Q(\cdot, \omega)$ -Brownian motion.

- ii)  $Y$  is a (weak) solution on  $(\Omega_1, Q(\cdot, \omega))$  of  
 $Y_t = Y_0 + \int_0^t \Phi(X(s, Y_s, \omega)) dB_s(\cdot, \omega)$ ,  $t \in [0, T]$ , i.e. item 1) of Definition  
 2.10. Moreover items 2), 3) of the same Definition are fulfilled.
- iii)  $Y_0$  is distributed according to  $x_0$ .
- iv) For every  $t \in ]0, T]$ ,  $\varphi \in C_b(\mathbb{R})$ , if we denote  $Q^\omega = Q(\cdot, \omega)$ , we have

$$\int_{\mathbb{R}} X(t, \xi) \varphi(\xi) d\xi = E^{Q^\omega} \left( \varphi(Y_t) \mathcal{E}_t \left( \int_0^t \mu(ds, Y_s) X(s, Y_s) \right) \right).$$

- 2) We show now that  $X^\kappa$  approaches  $X$  in some sense when  $\kappa \rightarrow 0$ , where  $X$  is the solution to (1.1). This is given in the Lemma 7.3 below.

**Lemma 7.3.** *Under the assumptions of Theorem 7.1, according to Remark B.2, let  $X$  (resp.  $X^\kappa$ ) be a solution of (1.1) verifying (2.1) with  $\psi(u) = u\Phi^2(u)$  (resp.  $\psi_\kappa(u) = u(\Phi^2(u) + \kappa)$ ), for  $u > 0$ . We have the following.*

- a)  $\lim_{\kappa \rightarrow 0} \sup_{t \in [0, T]} E \left( \|X^\kappa(t, \cdot) - X(t, \cdot)\|_{H^{-1}}^2 \right) = 0$ ;
- b)  $\lim_{\kappa \rightarrow 0} E \left( \int_0^T dt \|\psi(X^\kappa(t, \cdot)) - \psi(X(t, \cdot))\|_{L^2}^2 \right) = 0$ ;
- c)  $\lim_{\kappa \rightarrow 0} \kappa E \left( \int_{[0, T] \times \mathbb{R}} dt d\xi (X^\kappa(t, \xi) - X(t, \xi))^2 \right) = 0$ .

**Remark 7.4.** 1) a) implies of course

$$\lim_{\kappa \rightarrow 0} E \left( \int_0^T dt \|X^\kappa(t, \cdot) - X(t, \cdot)\|_{H^{-1}}^2 \right) = 0.$$

- 2) In particular Lemma 7.3 b) implies that for each sequence  $(\kappa_n) \rightarrow 0$  there is a subsequence, still denoted by the same notation, that

$$\int_{[0, T] \times \mathbb{R}} (\psi(X^{\kappa_n}(t, \xi)) - \psi(X(t, \xi)))^2 dt d\xi \xrightarrow{n \rightarrow \infty} 0 \text{ a.s.}$$

- 3) For every  $t \in [0, T]$   $X(t, \cdot) \geq 0$   $d\xi \otimes dP$  a.e. Indeed, for this it will be enough to show that a.s.

$$\int_{\mathbb{R}} d\xi \varphi(\xi) X(t, \xi) \geq 0 \text{ for every } \varphi \in \mathcal{S}(\mathbb{R}), \quad (7.2)$$

for every  $t \in [0, T]$ . Since  $X \in C([0, T]; \mathcal{S}'(\mathbb{R}))$  it will be enough to show (7.2) for almost all  $t \in [0, T]$ . This holds true since item 1) in this Remark 7.4, implies the existence of a sequence  $(\kappa_n)$  such that

$$\int_0^T dt \|X^{\kappa_n}(t, \cdot) - X(t, \cdot)\|_{H^{-1}}^2 \xrightarrow{n \rightarrow \infty} 0, \quad \text{a.s.}$$

4) Since  $\psi$  is strictly increasing after  $u_c$ , then, for  $P$  almost all  $\omega$ , for almost all  $(t, \xi) \in [0, T] \times \mathbb{R}$ , there is a sequence  $(\kappa_n)$  such that  $(X^{\kappa_n}(t, \xi) - X(t, \xi)) 1_{\{X(t, \xi) > u_c\}} \xrightarrow{n \rightarrow \infty} 0$ .

This follows from item 2) of Remark 7.4.

Since  $\Phi^2(u) = 0$  for  $0 \leq u \leq u_c$  and  $X$  is a.e. non-negative, this implies that  $dtd\xi dP$  a.e. we have

$$\Phi^2(X(t, \xi)) (X^{\kappa_n}(t, \xi) - X(t, \xi)) \xrightarrow{n \rightarrow \infty} 0. \quad (7.3)$$

*Proof (of Lemma 7.3).* By Remark B.2 3. we can write  $dP$ -a.s. the following  $H^{-1}(\mathbb{R})$ -valued equality.

$$(X^\kappa - X)(t, \cdot) = \int_0^t ds (\psi_\kappa(X^\kappa(s, \cdot)) - \psi(X(s, \cdot)))' + \sum_{i=0}^N \int_0^t (X^\kappa(s, \cdot) - X(s, \cdot)) e^i dW_s^i.$$

So

$$\begin{aligned} (I - \Delta)^{-1}(X^\kappa - X)(t, \cdot) &= - \int_0^t ds (\psi_\kappa(X^\kappa(s, \cdot)) - \psi(X(s, \cdot))) \\ &\quad + \int_0^t ds (I - \Delta)^{-1} (\psi_\kappa(X^\kappa(s, \cdot)) - \psi(X(s, \cdot))) \\ &\quad + \sum_{i=0}^N \int_0^t (I - \Delta)^{-1} (e^i (X^\kappa(s, \cdot) - X(s, \cdot))) dW_s^i. \end{aligned}$$

After regularization and application of Itô calculus with values in  $H^{-1}$ , we will be able to estimate  $g^\kappa(t) = \|(X^\kappa - X)(t, \cdot)\|_{H^{-1}}^2$ . Taking advantage of the form of  $\psi_\kappa - \psi$ , we obtain

$$\begin{aligned} g^\kappa(t) &= \sum_{i=1}^N \int_0^t \|e^i (X^\kappa - X)(s, \cdot)\|_{H^{-1}}^2 ds \quad (7.4) \\ &\quad - 2 \int_0^t \langle (X^\kappa - X)(s, \cdot), \psi_\kappa(X^\kappa(s, \cdot)) - \psi(X(s, \cdot)) \rangle_{L^2} \\ &\quad + 2 \int_0^t ds \langle (X^\kappa - X)(s, \cdot), (I - \Delta)^{-1} (\psi_\kappa(X^\kappa(s, \cdot)) - \psi(X(s, \cdot))) \rangle_{L^2} \\ &\quad + 2 \int_0^t ds \langle (X^\kappa - X)(s, \cdot), (I - \Delta)^{-1} e^0 (X^\kappa - X)(s, \cdot) \rangle_{L^2} + M_t^\kappa, \end{aligned}$$

where  $M^\kappa$  is the local martingale

$$M_t^\kappa = 2 \sum_{i=1}^N \int_0^t \langle (I - \Delta)^{-1} (X^\kappa - X)(s, \cdot), (X^\kappa - X)(s, \cdot) e^i \rangle_{L^2} dW_s^i.$$

Indeed,  $M^\kappa$  is a well-defined local martingale because, taking into account (B.1) and Remark B.2, using classical arguments, we can prove that

$$\sum_{i=1}^N \int_0^t |\langle (X^\kappa - X)(s, \cdot), (I - \Delta)^{-1} (X^\kappa - X)(s, \cdot) e^i \rangle_{L^2}|^2 ds < \infty \text{ a.s.}$$

(7.4) gives

$$\begin{aligned} g^\kappa(t) &+ 2 \int_0^t \langle (X^\kappa - X)(s, \cdot), \psi(X^\kappa(s, \cdot)) - \psi(X(s, \cdot)) \rangle_{L^2} ds \\ &+ 2\kappa \int_0^t \langle (X^\kappa - X)(s, \cdot), (X^\kappa - X)(s, \cdot) \rangle_{L^2} ds \\ &\leq -2\kappa \int_0^t ds \langle (X^\kappa - X)(s, \cdot), X(s, \cdot) \rangle_{L^2} ds + \sum_{i=1}^N \int_0^t \|e^i (X^\kappa - X)(s, \cdot)\|_{H^{-1}}^2 ds \\ &+ 2 \int_0^t ds \langle (I - \Delta)^{-1} (X^\kappa - X)(s, \cdot), \psi(X^\kappa(s, \cdot)) - \psi(X(s, \cdot)) \rangle_{L^2} \\ &+ 2\kappa \int_0^t ds \langle (I - \Delta)^{-1} (X^\kappa - X)(s, \cdot), (X^\kappa - X)(s, \cdot) \rangle_{L^2} \\ &+ 2\kappa \int_0^t ds \langle (I - \Delta)^{-1} (X^\kappa - X)(s, \cdot), X(s, \cdot) \rangle_{L^2} \\ &+ 2 \int_0^t ds \langle (I - \Delta)^{-1} (X^\kappa - X)(s, \cdot), (e^0 (X^\kappa - X)(s, \cdot)) \rangle_{L^2} + M_t^\kappa. \end{aligned}$$

We use Cauchy-Schwarz and the inequality  $2\sqrt{\kappa}b\sqrt{\kappa}c \leq \kappa b^2 + \kappa c^2$ , with first  $b = \|X^\kappa(s, \cdot) - X(s, \cdot)\|_{L^2}$ ,  $c = \|X(s, \cdot)\|_{L^2}$  and then  $b = \|X^\kappa(s, \cdot) - X(s, \cdot)\|_{H^{-2}}$ ,  $c = \|X(s, \cdot)\|_{L^2}$ . We also take into account the property of  $H^{-1}$ -multiplier for  $e^i$ ,  $0 \leq i \leq N$ . Consequently there is a constant

$\mathcal{C}(e)$  depending on  $(e^i, 0 \leq i \leq N)$  such that

$$\begin{aligned}
g^\kappa(t) &+ 2 \int_0^t \langle (X^\kappa - X)(s, \cdot), \psi(X^\kappa(s, \cdot)) - \psi(X(s, \cdot)) \rangle_{L^2} ds \\
&+ 2\kappa \int_0^t \|X^\kappa(s, \cdot) - X(s, \cdot)\|_{L^2}^2 ds \\
&\leq \kappa \int_0^t \|(X^\kappa - X)(s, \cdot)\|_{L^2}^2 ds + \kappa \int_0^t ds \|X(s, \cdot)\|_{L^2}^2 \\
&+ C(e) \int_0^t ds \|X^\kappa(s, \cdot) - X(s, \cdot)\|_{H^{-1}}^2 \\
&+ 2 \int_0^t \|(X^\kappa - X)(s, \cdot)\|_{H^{-2}} \|\psi(X^\kappa(s, \cdot)) - \psi(X(s, \cdot))\|_{L^2} \\
&+ 2\kappa \int_0^t ds g^\kappa(s) + \kappa \int_0^t ds \|(X^\kappa - X)(s, \cdot)\|_{H^{-2}}^2 + \kappa \int_0^t ds \|X(s, \cdot)\|_{L^2}^2 + M_t^\kappa.
\end{aligned} \tag{7.5}$$

Since  $\psi$  is Lipschitz, it follows  $(\psi(r) - \psi(r_1))(r - r_1) \geq \alpha(\psi(r) - \psi(r_1))^2$ , for any  $r, r_1 \geq 0$ , for some  $\alpha > 0$ . Consequently, the inequality  $2bc \leq b^2\alpha + \frac{c^2}{\alpha}$ , with  $b, c \in \mathbb{R}$  and the fact that  $\|\cdot\|_{H^{-2}} \leq \|\cdot\|_{H^{-1}}$  give

$$\begin{aligned}
&2 \int_0^t ds \|(X^\kappa - X)(s, \cdot)\|_{H^{-2}} \|\psi(X^\kappa(s, \cdot)) - \psi(X(s, \cdot))\|_{L^2} \\
&\leq \int_0^t ds \alpha g^\kappa(s, \cdot) + \int_0^t ds \langle \psi(X^\kappa(s, \cdot)) - \psi(X(s, \cdot)), X^\kappa(s, \cdot) - X(s, \cdot) \rangle_{L^2}.
\end{aligned}$$

So (7.5) yields

$$\begin{aligned}
g^\kappa(t) &+ \int_0^t \langle X^\kappa(s, \cdot) - X(s, \cdot), \psi(X^\kappa(s, \cdot)) - \psi(X(s, \cdot)) \rangle_{L^2} ds \\
&+ \kappa \int_0^t ds \|X^\kappa(s, \cdot) - X(s, \cdot)\|_{L^2}^2 ds \\
&\leq 2\kappa \int_0^t ds \|X(s, \cdot)\|_{L^2}^2 + M_t^\kappa + (C(e) + \alpha + 3\kappa) \int_0^t g^\kappa(s) ds.
\end{aligned} \tag{7.6}$$

Taking the expectation we get

$$E(g^\kappa(t)) \leq (C(e) + \alpha + 3\kappa) \int_0^t E(g^\kappa(s)) ds + 2\kappa \int_0^t E(\|X(s, \cdot)\|_{L^2}^2) ds,$$

for every  $t \in [0, T]$ . By Gronwall lemma we get

$$E(g^\kappa(t)) \leq 2\kappa E \left\{ \int_0^T ds \|X(s, \cdot)\|_{L^2}^2 \right\} e^{(C(e)+\alpha+3\kappa)T}, \quad \forall t \in [0, T]. \quad (7.7)$$

Taking the supremum and letting  $\kappa \rightarrow 0$ , item a) of Lemma 7.3 is now established.

We go on with item b). Since  $\psi$  is Lipschitz, (7.6) implies that, for  $t \in [0, T]$ ,

$$\begin{aligned} & \int_0^t ds \|\psi(X^\kappa(s, \cdot)) - \psi(X(s, \cdot))\|_{L^2}^2 \\ & \leq \frac{1}{\alpha} ds \left\langle \psi(X^\kappa(s, \cdot)) - \psi(X(s, \cdot)), X^\kappa(s, \cdot) - X(s, \cdot) \right\rangle_{L^2} \\ & \leq \frac{\kappa}{2\alpha} \int_0^t ds \|X(s, \cdot)\|_{L^2}^2 + C(e, \alpha) \int_0^t g^\kappa(s) ds + M_t^\kappa, \end{aligned}$$

where  $C(e, \alpha)$  is a constant depending on  $e^i, 0 \leq i \leq N$  and  $\alpha$ . Taking the expectation for  $t = T$ , we get

$$E \left( \int_0^T ds \|\psi(X^\kappa(s, \cdot)) - \psi(X(s, \cdot))\|_{L^2}^2 \right) \leq \frac{\kappa}{2\alpha} E \left( \int_0^T ds \|X(s, \cdot)\|_{L^2}^2 \right) + C(e, \alpha) \int_0^T E(g^\kappa(s)) ds.$$

Taking  $\kappa \rightarrow 0$ , (2.1) and (7.7) provide the conclusion of item b) of Lemma 7.3.

c) Coming back to (7.6), and  $t = T$ , we have

$$\kappa \int_0^T ds \|X^\kappa(s, \cdot) - X(s, \cdot)\|_{L^2}^2 \leq 2\kappa \int_0^T ds \|X(s, \cdot)\|_{L^2}^2 + M_T^\kappa + (C(e) + \alpha + 3\kappa) \int_0^T ds g^\kappa(s).$$

Taking the expectation we have

$$\kappa E \left( \int_0^T ds \|X^\kappa(s, \cdot) - X(s, \cdot)\|_{L^2}^2 \right) \leq 2\kappa E \left( \int_0^T ds \|X(s, \cdot)\|_{L^2}^2 \right) + (C(e) + \alpha + 3\kappa) E \left( \int_0^T g^\kappa(s) ds \right).$$

Using item a) and the fact that  $E \left( \int_{[0, T] \times \mathbb{R}} X^2(s, \xi) ds d\xi \right) < \infty$ , the result follows.

Lemma 7.3 is finally completely established.

□

We need now another intermediate lemma concerning the paths of a solution to (1.1).

**Lemma 7.5.** *For almost all  $\omega \in \Omega$ , almost all  $t \in [0, T]$ ,*

- 1)  $\xi \mapsto \psi(X(t, \xi, \omega)) \in H^1(\mathbb{R})$ ,
- 2)  $\xi \mapsto \Phi(X(t, \xi, \omega))$  is continuous.

*Proof.* Item 1) is established in [8], see Definition 3.2 and Theorem 3.4. 1) implies that  $\xi \mapsto \psi(X(t, \xi, \omega))$  is continuous. See also Remark B.2 1. By the same arguments as in Proposition 4.22 in [6], we can deduce item 2). □

- 3) We go on with the proof of Theorem 7.1. We keep in mind i), ii), iii), iv) at the beginning of item 1) of the proof. Since  $\Phi$  is bounded, for  $P$ -almost all  $\omega$ , using Burkholder-Davis-Gundy inequality one obtains

$$E^{Q^\kappa(\cdot, \omega)} (Y_t - Y_s)^4 \leq \text{const}(t - s)^2, \quad (7.8)$$

where const does not depend on  $\omega$ . On the other hand, for all  $Q^\kappa(\cdot, \omega)$ ,  $Y_0$  is distributed according to  $x_0$ .

At this point, we need a version of Kolmogorov-Centsov theorem for the stable convergence. Let  $\tilde{\Omega}_0 = \tilde{\Omega}_1 \times \Omega$  as at the beginning of the proof of Theorem 7.1. We recall that  $\tilde{\Omega}_1 = C([0, T]) \times \mathbb{R}$ ,  $Y(\omega_1, \omega) = \omega_1$ ,  $\mathcal{H}$  is the Borel  $\sigma$ -field on  $\tilde{\Omega}_1$ .

**Lemma 7.6.** *Let be a sequence  $Q^\kappa(\cdot, \omega)$  of random kernel on  $\mathcal{H} \times \Omega$ . Let us denote by  $\mathbf{Q}^\kappa$  the sequence of marginal laws of the probabilities on  $(\tilde{\Omega}_0, \mathcal{H} \otimes \mathcal{F})$  given by  $Q^\kappa(\cdot, \omega)P(d\omega)$ . Suppose the following.*

- The sequences of marginal laws of the probabilities  $\mathbf{Q}^\kappa$  at zero are tight.
- There are  $\alpha, \beta > 0$  such that

$$E^{Q^\kappa(\cdot, \omega)} |Y_t - Y_s|^\alpha \leq C(\omega)(t - s)^{1+\beta}, \quad 0 \leq s \leq t \leq T,$$

for some positive  $P$ -integrable random constant  $C$ .

Then there is a random kernel  $Q^\infty$  on  $\mathcal{H} \times \Omega$  and a subsequence  $(\kappa_n)$  such that for every bounded continuous functional  $G : \tilde{\Omega}_1 \rightarrow \mathbb{R}$ , for every bounded  $\mathcal{F}$ -measurable r.v.  $F : \Omega \rightarrow \mathbb{R}$ , we have

$$\int_{\Omega} F(\omega) dP(\omega) \int_{\tilde{\Omega}_1} G(Y(\omega_1)) Q^{\kappa_n}(d\omega_1, \omega) \xrightarrow{n \rightarrow \infty} \int_{\Omega} F(\omega) dP(\omega) \int_{\tilde{\Omega}_1} G(Y(\omega_1)) Q^\infty(d\omega_1, \omega). \quad (7.9)$$

*Proof.* Taking the expectation with respect to  $P$  we obtain

$$E^{\mathbf{Q}^\kappa} (Y_t - Y_s)^\alpha \leq C_0(t-s)^{1+\beta}, \quad 0 \leq s \leq t,$$

where  $C_0$  is the expectation of  $C$ . First, by usual arguments as Chebyshev inequality, one can show the following:

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} \sup_{\kappa} \mathbf{Q}^\kappa \{(\omega_1, \omega) \mid |(W^1, \dots, W^N)(\omega)(0)| > \lambda; |\omega_1(0)| > \lambda\} &= 0, \\ \lim_{\delta \rightarrow 0} \sup_{\kappa} \mathbf{Q}^\kappa \{(\omega_1, \omega) \mid m((W^1, \dots, W^N, \omega_1); \delta) > \varepsilon\} &= 0, \forall \varepsilon > 0, \end{aligned}$$

where  $m$  denotes the modulus of continuity. By Theorem 4.10 of [18], the sequences of probabilities  $\mathbf{Q}^\kappa, \kappa > 0$ , on  $\tilde{\Omega}_1 \times \Omega$  are tight. Let  $\mathbf{Q}^{\kappa_n}$  be a sequence converging weakly to a probability  $\mathbf{Q}^\infty$  on  $\mathcal{H} \otimes \mathcal{F}$ . Since  $\mathcal{F}$  is separable and  $C([0, T])^N$ , which is space value of process  $W$ , is a Polish space equipped with its Borel  $\sigma$ -algebra, according to [23], it is possible to desintegrate  $\mathbf{Q}^\infty$ , i.e. there is random kernel  $Q^\infty(\cdot, \omega)$  such that for every bounded continuous functional  $G : \tilde{\Omega}_1 \rightarrow \mathbb{R}$ , for every bounded continuous  $\tilde{F} : C([0, T])^N \rightarrow \mathbb{R}$  such that (7.9) holds for every  $F = \tilde{F}(W)$ , where  $W = (W^1, \dots, W^N)$ . Since continuous bounded functionals  $\tilde{F}$  are dense in  $L^2(C([0, T])^N)$  equipped with Wiener measure, (7.9) holds also for any  $F$  bounded  $\mathcal{F}$ -measurable r.v. with  $\mathbf{Q}^\infty(d\omega_1, d\omega) = Q^\infty(d\omega_1, \omega)P(d\omega)$ .  $\square$

By (7.8), we apply Lemma 7.6 with  $\alpha = 2, \beta = 1$  and we consider the corresponding  $Q^{\kappa_n}(\cdot, \omega)$  and the limit random kernel  $Q(\cdot, \omega) := Q^\infty(\cdot, \omega)$ . We define also the probability  $\mathbf{Q} := \mathbf{Q}^\infty$  on  $\tilde{\Omega}_0 = \tilde{\Omega}_1 \times \Omega$  according to the conventions introduced before Definition 2.3. In the sequel we denote again by  $d\mathbf{Q}^\kappa(\omega_1, \omega) := dP(\omega)Q^\kappa(d\omega_1, \omega)$  and also  $Q^{\omega, \kappa} := Q^\kappa(\cdot, \omega), Q^\omega := Q(\cdot, \omega)$ .

From Lemma 7.6 derives the following.

**Corollary 7.7.** *For any bounded random element  $F : \tilde{\Omega}_1 \times \Omega \rightarrow \mathbb{R}$  such that for almost all  $\omega \in \Omega, F(\cdot, \omega) \in C(\tilde{\Omega}_1)$ . Then*

$$\int_{\Omega} dP(\omega) \int_{\tilde{\Omega}_1} (dQ^{\omega, \kappa_n}(\omega_1) - dQ^\omega(\omega_1)) F(Y, \omega) \text{ converges to zero.}$$

*Proof.* See Appendix A.  $\square$

We need here a technical lemma.

**Lemma 7.8.** *Let  $t \in [0, T], p \in \mathbb{R}$ .*

1. *There is  $C(p) > 0$  such that*

$$E^{\mathbf{Q}^\kappa} \left( \mathcal{E}_t \left( \int_0^t \mu(ds, Y_s) \right)^p \right) \leq C(p), \quad \forall \kappa > 0.$$

2. For almost all  $\omega \in \Omega$ , and every  $p \in \mathbb{R}$  there is a random constant  $C(p, \omega)$  such that the random variables

$$E^{Q^{\omega, \kappa}} \left( \mathcal{E}_t \left( \int_0^t \mu(ds, Y_s) \right)^p \right) \leq C(p, \omega), \quad \forall \kappa > 0. \quad (7.10)$$

*Proof.* Without restriction of generality we can of course suppose  $e^0 = 0$ .

1. We can write

$$\begin{aligned} \mathcal{E}_t \left( \int_0^t \mu(ds, Y_s) \right)^p &= \mathcal{E}_t \left( p \int_0^t \mu(ds, Y_s) \right) \exp \left( \frac{p^2 - p}{2} \sum_{i=1}^N \left( \int_0^t e^i(Y_s)^2 ds \right) \right) \\ &\leq \mathcal{E}_t \left( p \int_0^t \mu(ds, Y_s) \right) \exp \left( T \frac{p^2 - p}{2} \sum_{i=1}^N \|e^i\|_\infty^2 \right). \end{aligned}$$

Since  $p \int_0^t \mu(ds, Y_s)$  is a  $(\mathcal{G}_t)$ - $Q^\kappa$ -martingale, the result follows.

2. Let  $\omega \in \Omega$  excepted on a  $P$ -null set. The integrand of the expectation in (7.10) equals  $\exp(J_1(n) + J_2(n))$ , where

$$J_1(n) := p \sum_{i=1}^N \left( W_t^i e^i(Y_t) - \frac{1}{2} \int_0^t e^i(Y_s)^2 ds - \frac{1}{2} \int_0^t W_s^i (e^i)''(Y_s) \Phi^2(X(s, Y_s, \omega)) ds \right)$$

and  $J_2(n) = -p \sum_{i=1}^N \int_0^t W_s^i (e^i)'(Y_s) dY_s$ . For each  $\omega$ ,  $\exp(J_1(n))$  is bounded, so it remains to prove the existence of a random constant  $C(p, \omega)$  such that for every  $0 \leq i \leq N$

$$E^{Q^{\omega, \kappa}} \left( \exp \left( -p \int_0^t W_s^i (e^i)'(Y_s) dY_s \right) \right) \leq C(p, \omega). \quad (7.11)$$

Since  $-p \int_0^t W_s^i (e^i)'(Y_s) dY_s$  is a  $Q^{\omega, \kappa}$ -martingale,

$$\mathcal{E}_t^\kappa := \exp \left( -p \int_0^t W_s^i (e^i)'(Y_s) dY_s - \frac{p^2}{2} \int_0^t (W_s^i)^2 (e^i)'^2(Y_s) \Phi_\kappa^2(X^\kappa(s, Y_s, \omega)) ds \right)$$

is an (exponential) martingale, with respect to  $Q^{\omega, \kappa}$ . Consequently the left-

hand side of (7.11) is bounded by

$$\begin{aligned} & E^{Q^{\omega, \kappa}} \left( \mathcal{E}_t^\kappa \exp \left( \frac{p^2}{2} \int_0^t (W_s^i)^2 ((e^i)')^2(Y_s) \Phi_\kappa^2(X^\kappa(s, Y_s, \omega)) ds \right) \right) \\ & \leq C(p, \cdot) := \exp \left( \frac{p^2}{2} \|(e^i)'\|_\infty^2 \left( \|\Phi\|_\infty^2 + 1 \right) \int_0^T (W_s^i)^2 ds \right). \end{aligned}$$

This concludes the proof of Lemma 7.8.  $\square$

**Lemma 7.9.** *We fix  $\omega \in \Omega$  excepted on some  $P$ -null set. Let  $\varphi : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  continuous with compact support. The random variables*

$$E^{Q^{\omega, \kappa}} \left( \int_0^T |\Phi_\kappa(X^\kappa(r, Y_r, \omega)) - \Phi(X(r, Y_r, \omega))| \varphi(r, Y_r) dr \right)$$

converge to zero a.s. and in  $L^p(\Omega, P)$  for every  $p \geq 1$ , when  $\kappa \rightarrow 0$ .

*Proof.* Let  $\omega \in \Omega$ . Since  $\varphi$  has compact support, by Cauchy-Schwarz with respect to the measure  $\varphi(r, Y(r))dr$  on  $[0, T]$ , it is enough to prove that

$$E^{Q^{\omega, \kappa}} \left( \int_0^T (\Phi_\kappa(X^\kappa(r, Y_r, \omega)) - \Phi(X(r, Y_r, \omega)))^2 \varphi(r, Y_r) dr \right) \quad (7.12)$$

converges to zero. Since  $\Phi$  is bounded it is enough to prove the convergence to zero for almost all  $\omega \in \Omega$ . In order not to overcharge the notation, in this proof we will omit the argument of  $\omega$  of  $Y$ . By Fubini's theorem the left-hand side of (7.12) equals

$$\int_0^T dr E^{Q^{\omega, \kappa}} \left( (\Phi_\kappa(X^\kappa(r, Y_r, \omega)) - \Phi(X(r, Y_r, \omega)))^2 \varphi(r, Y_r) \right).$$

Using also Lebesgue dominated convergence theorem, given a sequence  $(\kappa_n)$ , when  $n \rightarrow \infty$ , it is enough to find a subsequence  $(\kappa_{n_\ell})$  such that for all  $r \in [0, T]$  outside a possible Lebesgue null set

$$E^{Q^{\omega, \kappa_{n_\ell}}} \left\{ \left( \Phi_{\kappa_{n_\ell}}(X^{\kappa_{n_\ell}}(r, Y_r, \omega)) - \Phi(X(r, Y_r, \omega)) \right)^2 \varphi(r, Y_r) \right\} \rightarrow_{\ell \rightarrow \infty} 0.$$

We set  $Z_r(\omega_1, \omega) = \mathcal{E}_r \left( \int_0^{\cdot} \mu(\omega)(ds, Y_s(\omega_1)) \right)$ . We will substitute from now on  $(n_\ell)$  with  $n$ .

Taking into account Lemma 7.8 and Cauchy-Schwarz with respect to the finite measure  $Z_r(\omega_1, \omega)Q^{\omega, \kappa_n}(d\omega_1)$ , it is enough to prove that for  $r$  a.e.

$$E^{Q^{\omega, \kappa_n}} \left\{ (\Phi_{\kappa_n}(X^{\kappa_n}(r, Y_r, \omega)) - \Phi(X(r, Y_r, \omega)))^2 \varphi(r, Y_r) Z_r(\cdot, \omega) \right\}$$

converges to zero when  $n$  goes to infinity.

Since  $X^\kappa$  constitutes the family of  $\mu$ -marginal weighted laws of  $Y$  under  $Q^{\omega, \kappa}$ , previous expression gives

$$\begin{aligned} & \int_{\mathbb{R}} |\varphi|(r, y) (\Phi_{\kappa_n}(X^{\kappa_n}(r, y, \omega)) - \Phi(X(r, y, \omega)))^2 X^{\kappa_n}(r, y, \omega) dy \\ & \leq I_{11}(\kappa_n, r) + I_{12}(\kappa_n, r) + I_{13}(\kappa_n, r) + I_{14}(\kappa_n, r), \end{aligned} \quad (7.13)$$

where we have developed the square in the first line of (7.13) using the definition of  $\psi$  and  $\Phi_\kappa$ . Indeed we get

$$\begin{aligned} I_{11}(\kappa, r) &= \int_{\mathbb{R}} dy |\varphi|(r, y) |\psi(X^\kappa(r, y, \omega)) - \psi(X(r, y, \omega))|, \\ I_{12}(\kappa, r) &= \int_{\mathbb{R}} dy |\varphi|(r, y) |\Phi^2(X^\kappa(r, y, \omega)) - \Phi^2(X(r, y, \omega))| |X - X^\kappa|(r, y, \omega), \\ I_{13}(\kappa, r) &= \int_{\mathbb{R}} dy |\varphi|(r, y) |\kappa| |X^{\kappa_n} - X|(r, y, \omega), \\ I_{14}(\kappa, r) &= \int_0^T dr \int_{\mathbb{R}} dy \kappa |X(r, y, \omega)| |\varphi(r, y)|. \end{aligned}$$

We denote  $I_{1j}(\kappa) := \int_0^T I_{1j}(\kappa, r) dr$ ,  $j = 1, 2, 3, 4$ . It is of course enough to prove that, up to a subsequence  $I_{1j}(\kappa_n) \rightarrow 0$ ,  $j = 1, 2, 3, 4$ , where  $n \rightarrow \infty$ . By Cauchy-Schwarz,  $I_{11}^2(\kappa)$  is bounded by

$$\|\varphi\|_{L^2([0, T] \times \mathbb{R})}^2 \int_0^T dr \int_{\mathbb{R}} (\psi(X^\kappa(r, y, \omega)) - \psi(X(r, y, \omega)))^2 dy.$$

This converges to zero according to Remark 7.4 2), after extracting a subsequence  $\kappa_n$  (not depending on  $\omega$ ). The square of the expectation of  $I_{12}(\kappa)$  is bounded by

$$\|\varphi\|_{L^2([0, T] \times \mathbb{R})}^2 \int_{[0, T] \times \mathbb{R}} dr dy \Phi^4(X(r, y, \omega)) |X^\kappa - X|^2(r, y, \omega).$$

The expectation of previous expression is indeed uniformly bounded in  $\kappa$  because of (7.6) and (7.7). So the family of r.v.

$\Phi^2(X^{\kappa_n}(r, y, \omega)) |(X - X^\kappa)(r, y, \omega)|$  is uniformly integrable with respect to the finite measure  $dP(\omega)|\varphi|(t, y)dtdy$ . Consequently  $I_{12}(\kappa)$  goes to zero because of (7.3) in Remark 7.4 4).

$I_{13}^2(\kappa)$  is bounded by  $\kappa \|\varphi\|_{L^2([0, T] \times \mathbb{R})}^2 \int_{[0, T] \times \mathbb{R}} drdy |X^\kappa - X|^2(r, y, \omega)$ . After extracting a subsequence  $\kappa_n$ , previous expression converges to zero because of Lemma 7.3 c). Finally  $I_{14}(\kappa) \xrightarrow{n \rightarrow \infty} 0$  by Cauchy-Schwarz and the fact that

$\int_{[0, T] \times \mathbb{R}} drdy X^2(r, y, \omega) < \infty$   $dP$ -a.s. This establishes the proof of Lemma 7.9.  $\square$

Let  $(\kappa_n)$  be the sequence introduced by the statement of Lemma 7.6. Previous Corollary 7.7 and Lemma 7.9 have the following consequences. Let  $Q(d\omega_1, \omega)$  be the random kernel introduced in Lemma 7.5 and the related probability  $\mathbf{Q}(d\omega_1, d\omega) = dP(\omega)Q(d\omega_1, \omega)$ .

**Corollary 7.10.** *Let  $R : \Omega \rightarrow \mathbb{R}$  be a bounded measurable r.v. Let  $\varphi : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  be a function with compact support. The sequence*

$$\int_{\Omega} R(\omega) dP(\omega) \int_{\tilde{\Omega}_1} dQ^{\omega, \kappa_n}(\omega_1) \int_0^T \varphi(r, Y_r) \Phi_{\kappa_n}^2(X^{\kappa_n}(r, Y_r, \omega)) dr \quad (7.14)$$

converges, when  $n \rightarrow \infty$ , to

$$\int_{\Omega} R(\omega) dP(\omega) \int_{\tilde{\Omega}_1} dQ(\omega_1, \omega) \int_0^T \varphi(r, Y_r) \Phi^2(X(r, Y_r, \omega)) dr. \quad (7.15)$$

*Proof.* We split the difference between (7.14) and (7.15) which gives  $I_1(n) + I_2(n)$  where

$$I_1(n) = \int_{\Omega} R(\omega) dP(\omega) \int_{\tilde{\Omega}_1} dQ^{\omega, \kappa_n}(\omega_1) \left( \int_0^T \varphi(r, Y_r) (\Phi_{\kappa_n}^2(X^{\kappa_n}(r, Y_r, \omega)) dr - \Phi^2(X(r, Y_r, \omega)) dr) \right),$$

and

$$I_2(n) = \int_{\Omega} R(\omega) dP(\omega) \int_{\tilde{\Omega}_1} (Q^{\omega, \kappa_n}(d\omega_1) - Q(d\omega_1, \omega)) \left( \int_0^T \varphi(r, Y_r) \Phi^2(X(r, Y_r, \omega)) dr \right).$$

We have

$$|I_1(n)| \leq 2 \|\Phi\|_{\infty} \|R\|_{\infty} \int_{\Omega} dP(\omega) \int_{\tilde{\Omega}_1} dQ^{\omega, \kappa_n}(\omega_1) \left( \int_0^T |\varphi(r, Y_r)| |\Phi_{\kappa_n}(X^{\kappa_n}(r, Y_r, \omega)) - \Phi(X(r, Y_r, \omega))| dr \right).$$

$I_1(n)$  converges to zero by Lemma 7.9. Concerning  $I_2(n)$ , by Fubini's theorem, we first observe that

$$I_2(n) = \int_0^T dr \int_{\Omega} dP(\omega) \left( \int_{\tilde{\Omega}_1} (Q^{\omega, \kappa_n}(d\omega_1) - Q(d\omega_1, \omega)) \varphi(r, Y_r) \Phi^2(X(r, Y_r, \omega)) R(\omega) \right).$$

We apply now Corollary 7.7, setting for fixed  $r$ ,  $F(\omega_1, \omega) = R(\omega)\varphi(r, \omega_1(r))\Phi^2(X(r, \omega_1(r), \omega))$  and the result follows.  $\square$

5) We go on with the proof of Theorem 7.1.

We want now to prove that  $Y(\cdot, \omega)$  is a (weak) solution of

$$Y_t = Y_0 + \int_0^t \Phi(X(s, Y_s, \cdot)) d\beta_s^\omega,$$

for some Brownian motion  $\beta^\omega$ . This is related to item 1) of Definition 2.10 with  $\gamma(t, \xi, \omega) = \Phi(X(t, \xi, \omega))$ . According to Remark 2.11 c), for this it is enough to show that for  $dP$ -a.s.  $\omega$   $Y(\cdot, \omega)$  is a solution of the following (local) martingale problem. For every  $f \in C^{1,2}([0, T] \times \mathbb{R})$  with compact support, the process

$$Z_t^f := f(t, Y_t) - f(0, Y_0) - \frac{1}{2} \int_0^t \partial_{xx}^2 f(r, Y_r) \Phi^2(X(r, Y, \omega)) dr - \int_0^t \partial_r f(r, Y_r) dr,$$

is a (local) martingale under  $Q^\omega$ .

This will be a consequence of the lemma below.

**Lemma 7.11.** *Let  $F$  be a bounded  $\mathcal{F}_s$ -measurable, let  $A : C([0, s]) \rightarrow \mathbb{R}$  bounded continuous functional. Let  $G = A(Y_r, r \leq s)$ . Then, for  $0 \leq s \leq t \leq T$  we have*

$$E(FE^{Q^\omega}(GZ_t^f)) = E(FE^{Q^\omega}(GZ_s^f)). \quad (7.16)$$

*Proof.* We set

$$Z_t^{\kappa, f} = f(t, Y_t) - f(0, Y_0) - \frac{1}{2} \int_0^t \partial_{xx}^2 f(r, Y_r) \Phi_\kappa^2(X^\kappa(r, Y, \omega)) dr - \int_0^t \partial_r f(r, Y_r) dr.$$

Let  $(\kappa_n)$  be the sequence introduced by Lemma 7.6. The difference of the right and left-hand side of (7.16) is the sum  $(I_1 + I_2 + I_3)(\kappa_n)$  where

$$\begin{aligned} I_1(\kappa) &= E\left(F(E^{Q^\omega}(GZ_t^f) - E^{Q^{\omega, \kappa}}(GZ_t^{\kappa, f}))\right) \\ I_2(\kappa) &= E\left(FE^{Q^{\omega, \kappa}}(G(Z_t^{\kappa, f} - Z_s^{\kappa, f}))\right) \\ I_3(\kappa) &= E\left(F(E^{Q^{\omega, \kappa}}(GZ_s^{\kappa, f}) - E^{Q^\omega}(GZ_s^f))\right). \end{aligned}$$

$I_1(\kappa_n) + I_3(\kappa_n)$  converges to zero by Lemma 7.6, Corollary 7.10 and Lemma 7.9.  $I_2(\kappa_n) = 0$  since  $Z^{\kappa, f}$  is a  $Q^{\kappa, \omega}$ -martingale.  $\square$

- 6) After previous intermediary step 5) we need to show that  $Y$  defined in (7.1) is a weak-strong solution of DSDE( $\gamma, x_0$ ) with  $\gamma(s, \xi, \omega) = \Phi(X(s, \xi, \omega))$  and  $X$  is a solution of (1.1). We recall that the kernel  $Q(\cdot, \omega)$  has been introduced through Lemma 7.6 on  $(\tilde{\Omega}_1 \times \Omega, \mathcal{H} \otimes \mathcal{F})$ . So, according to step 5), under  $Q^\omega := Q(\cdot, \omega)$ ,  $Y$  is a martingale with  $[Y]_t = \int_0^t \Phi^2(X(s, Y_s, \omega)) ds$ . To conclude the proof of item 1) in Definition 2.10, it remains to construct the suitable required process  $B$ . For this, we need to enlarge the probability space  $\tilde{\Omega}_1$  as follows. We set  $\Omega_1 = \tilde{\Omega}_1 \times C([0, T]; \mathbb{R})$ ; the second component allows to define a Brownian motion. By an abuse of notation, we set again  $Y_t(\omega_1, \omega) = \omega_1^0(t)$ , this time with  $\omega_1 = (\omega_1^0, \omega_1^1)$ . In spite of adding the component  $\omega_1^1$ , in step 5) we have already shown  $Q^\omega := Q(\cdot, \omega)$ , is by construction the law of  $Y(\cdot, \omega)$ . We need to construct a process  $B$  on  $\Omega \times \Omega_1$ , such that for almost all  $\omega$ ,  $B(\cdot, \omega)$  is a  $Q^\omega$ -Brownian motion and (2.7) holds for  $\gamma(t, \cdot, \omega) = \Phi(X(t, \cdot, \omega))$ .

On  $\Omega_1$  we set  $\beta_t(\omega_1) = \omega_1^1(t)$ . We equip  $C([0, T]; \mathbb{R})$  in  $\Omega_1$  with the Wiener measure  $\mathcal{W}$  so that  $\beta$  is a standard Brownian motion on  $\Omega_1$ .  $\beta$  can also be considered to be a Brownian motion on  $\Omega_0 = \Omega_1 \times \Omega$  which is  $Q^\omega$ -independent of  $Y$  for  $P$ -almost all  $\omega \in \Omega$ . Of course  $\beta$  is also independent of  $Y$  on the probability space  $(\Omega_1 \times \Omega, \mathcal{B}(\Omega_1) \times \mathcal{F}, d\mathbf{Q}(\omega_1, \omega) := Q^\omega(d\omega_1)dP(\omega))$ .  $\beta$  is also independent of  $(\mathcal{F}_t)$ .

We set now

$$B_t(\cdot, \omega) = \int_0^t dY_s(\cdot, \omega) 1_{\{\gamma(s, \xi, \omega) \neq 0\}} \frac{1}{\gamma(s, \xi, \omega)} + \int_0^t 1_{\{\gamma(s, \xi, \omega) = 0\}} d\beta_s.$$

Now for  $Q^\omega$ -a.s. the quadratic variation of the  $Q^\omega$ -martingale  $B(\cdot, \omega)$  is  $t$ , so that, by Lévy characterization theorem,  $B(\cdot, \omega)$  is a Brownian motion under  $Q^\omega$ .

It remains to show items 2) and 3) of the definition of weak-strong solution. Let  $(\mathcal{Y}_t)$  be the canonical filtration of the process  $Y(\cdot, \omega)$ . Item 3) follows because of item 1) and because  $\gamma(t, \cdot, \omega) = \Phi(X(t, \cdot, \omega))$  is progressively measurable. Concerning item 2) we see that under  $\mathbf{Q}$  defined by  $P$  and the kernel  $Q(\cdot, \omega)$ ,  $W^1, \dots, W^N$  are  $\mathbf{Q}$ -martingales with  $(\mathcal{G}_t)$  as defined in Definition 2.10. Indeed let  $F$  be a bounded  $\mathcal{F}_s$ -measurable random variable and  $G$  be a bounded  $\mathcal{Y}_s$ -measurable r.v. Let  $1 \leq i \leq N$ . By item 3)  $E^{Q^\omega}(G)$  is  $\mathcal{F}_s$ -measurable, so

$$E^{\mathbf{Q}}((W_t^i - W_s^i)FG) = E((W_t^i - W_s^i)FE^{Q^\omega}(G)) = 0,$$

since  $W^i$  is an  $\mathcal{F}_s$ -martingale.

- 7) The final step consists in proving that  $X$  is the family of  $\mu$ -marginal weighted laws of  $Y$ . We need to show that for almost all  $\omega$ , for every  $t \in [0, T]$ ,  $\varphi \in \mathcal{S}(\mathbb{R})$ , that

$$\int_{\mathbb{R}} d\xi \varphi(\xi) X(t, \xi, \omega) = E^{Q^\omega} \left( \varphi(Y_t) \mathcal{E}_t \left( \int_0^t \mu(ds, Y_s) \right) \right).$$

Since both sides of previous equality are  $\mathcal{F}_t$ -measurable, given a bounded  $\mathcal{F}_t$ -measurable random variable  $R$  it will be enough to show that

$$\int_{\Omega} dP(\omega) R(\omega) \int_{\mathbb{R}} d\xi \varphi(\xi) X(t, \xi, \omega) = \int_{\Omega} dP(\omega) R(\omega) E^{Q^\omega} \left( \varphi(Y_t) \mathcal{E}_t \left( \int_0^t \mu(ds, Y_s) \right) \right). \quad (7.17)$$

Let  $\omega \in \Omega$  outside some  $P$ -null set.

By step 1) of the proof of this Theorem 7.1, we know that  $X^\kappa$  fulfills, for almost all  $\omega$ ,

$$\int_{\mathbb{R}} d\xi X^\kappa(t, \xi) \varphi(\xi) = E^{Q^\kappa(\cdot, \omega)} \left( \varphi(Y_t) \mathcal{E}_t \left( \int_0^t \mu(ds, Y_s) \right) \right),$$

for all  $\varphi \in \mathcal{S}(\mathbb{R})$ . Consequently if  $(\kappa_n)$  is the sequence obtained via Lemma 7.5, we have

$$\int_{\Omega} dP(\omega) R(\omega) \int_{\mathbb{R}} d\xi X^{\kappa_n}(t, \xi) \varphi(\xi) = \int_{\Omega} dP(\omega) R(\omega) E^{Q^{\omega, \kappa_n}} \left( \varphi(Y_t) \mathcal{E}_t \left( \int_0^t \mu(ds, Y) \right) \right), \quad (7.18)$$

for every  $\varphi \in \mathcal{S}(\mathbb{R})$ .

Since  $t \mapsto X(t, \cdot)$  is continuous from  $[0, T]$  to  $\mathcal{S}'(\mathbb{R})$  and the right-hand side of (7.17) is continuous on  $[0, T]$  for fixed  $\varphi \in \mathcal{S}(\mathbb{R})$ , it is enough to show (7.17) for almost all  $t \in [0, T]$ .

Now for almost all  $t$ , the left-hand side of (7.17) is approached by the left-hand side of (7.18). Let us fix  $t \in [0, T]$ . It remains to show that the right-hand side of (7.17) is the limit of the right-hand side of (7.18). We fix  $\omega \in \Omega$  outside a null set. We set  $\mathcal{E}_t := \mathcal{E}_t \left( \int_0^t \mu(ds, Y_s) \right)$ ,  $t \in [0, T]$ . By Theorem 2 of [25] and uniform integrability arguments, similarly as after (2.5), we have

$$\mathcal{E}_t = \exp(\psi_\omega(Y)),$$

where  $\psi_\omega : \tilde{\Omega}_1 \rightarrow \mathbb{R}$  is a continuous modification of

$$\omega \mapsto \left( \eta \mapsto \int_0^t e^i(\eta_s) dW_s^i - \frac{1}{2} \int_0^t e^i(\eta_s)^2 ds \right).$$

Indeed, previous random field, indexed by  $\eta \in \tilde{\Omega}_1$ , admits a continuous modification; to prove this we make use of Kolmogorov-Centsov theorem and Doob's inequality, which says that for any  $0 \leq i \leq N$ , there is a constant  $\text{const} = \text{const}((e^i)')$  with

$$E \left( \left| \int_0^t (e^i(\eta_s^1) - e^i(\eta_s^2)) dW_s^i \right|^4 \right) \leq \text{const} \sup_{s \in [0, T]} |\eta^1 - \eta^2|^2(s), \quad \eta^1, \eta^2 \in \tilde{\Omega}_1.$$

At this point we fix  $M > 0$ . We decompose the difference of the right-hand sides of (7.18) and (7.17) as

$$J_1(n, M) + J_2(n, M) + J_3(n, M), \quad (7.19)$$

where

$$\begin{aligned} J_1(n, M) &= \int_{\Omega} dP(\omega) R(\omega) E^{Q^{\omega, \kappa_n}} (\varphi(Y_t) \mathcal{E}_t - \varphi(Y_t)(\mathcal{E}_t \wedge M)), \\ J_2(n, M) &= \int_{\Omega} dP(\omega) R(\omega) (E^{Q^{\omega, \kappa_n}} - E^{Q^\omega}) (\varphi(Y_t)(\mathcal{E}_t \wedge M)), \\ J_3(n, M) &= \int_{\Omega} dP(\omega) R(\omega) E^{Q^\omega} (\varphi(Y_t)(\mathcal{E}_t \wedge M) - \varphi(Y_t) \mathcal{E}_t). \end{aligned}$$

Setting  $\mathbf{Q}^{\kappa_n}(d\omega, d\omega_1) = dP(\omega) Q^{\omega, \kappa_n}(d\omega_1)$ , by Cauchy-Schwarz and Chebyshev inequalities, for every  $p > 1$ , we have

$$|J_1(n, M)| = \left| \int_{\Omega_1 \times \Omega} d\mathbf{Q}^{\kappa_n} \varphi(Y_t) \mathcal{E}_t 1_{\{\mathcal{E}_t > M\}} \right| \leq \|\varphi\|_{\infty} \frac{E^{Q^{\kappa_n}}(\mathcal{E}_t^p)}{M^{p-1}}.$$

By Lemma 7.8, we get  $\sup_n |J_1(n, M)| \rightarrow 0$  if  $M \rightarrow \infty$ . By a similar reasoning, replacing  $Q^{\kappa_n}(d\omega, d\omega_1)$  with  $Q(d\omega, d\omega_1) = dP(\omega) Q^\omega(d\omega_1)$ , we can prove that  $\sup_n |J_3(n, M)| \rightarrow 0$ . Let  $\varepsilon > 0$ . Let  $M$  such that  $\sup_n |J_1(n, M) + J_3(n, M)| \leq \varepsilon$ . On the other hand we have

$$J_2(n, M) = \int_{\Omega} dP(\omega) R(\omega) (E^{Q^{\omega, \kappa_n}} - E^{Q^\omega}) (\varphi(Y_t)(\psi_\omega(Y) \wedge M)).$$

Since for almost all  $\omega$ ,  $F(\eta, \omega) := R(\omega) \varphi(\eta(t)) \psi_\omega(\eta)$  is bounded and continuous, Corollary 7.7 implies that  $J_2(n, M)$  goes to zero when  $n \rightarrow \infty$ .

Taking the limsup in (7.19) we get

$$\limsup_{n \rightarrow \infty} |J_1(n, M) + J_2(n, M) + J_3(n, M)| \leq \varepsilon.$$

Since  $\varepsilon$  is arbitrarily small, we get  $\lim_{n \rightarrow \infty} |J_1(n, M) + J_2(n, M) + J_3(n, M)| = 0$  and the result follows.  $\square$

## A Technicalities

**Proposition A.1.** *Let  $Y_0$  be distributed according to  $x_0$ . Let  $a : [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$  be a Borel function such there are  $0 < c < C$  with  $c \leq a(s, \xi) \leq C$ ,  $\forall (s, \xi) \in [0, T] \times \mathbb{R}$ . We fix  $0 \leq r \leq t \leq T$ . We set  $a_n(t, x) = \int_{\mathbb{R}} \rho_n(x - y) a(t, y) dy$  where  $(\rho_n)$  is the usual sequence of mollifiers converging to the Dirac delta. The unique solutions  $S^n$  to  $S_t^n = Y_0 + \int_r^t a_n(s, S_s^n) dB_s$ ,  $B$  being a classical Wiener process, converges in law to the (weak unique solution) of  $S_t = Y_0 + \int_r^t a(s, S_s) dB_s$ .*

*Proof.* The proof follows by standard arguments, see Stroock-Varadhan ([27], Problem 7.3.3), tightness and Kolmogorov-Centsov type arguments. For a detailed proof, the reader may consult [7].  $\square$

**Proof** (of Corollary 7.7).

By (7.8), the family  $(Q^{\kappa_\ell}, \ell \in \mathbb{N}, \omega \in \Omega)$  is tight. So, for every positive integer  $n$  there exists a compact subset  $K_n$  of  $\tilde{\Omega}_1$  such that

$$Q^{\kappa_\ell}(K_n^c, \omega) < \frac{1}{n}, \forall \ell \in \mathbb{N}, \omega \in \Omega. \quad (1.1)$$

Since each  $C(K_n) := C(K_n; \mathbb{R})$  is separable with respect to the sup-norm  $\|\cdot\|_\infty$  then  $C(K_n), \|\cdot\|_\infty$  is a separable Banach space. So we apply Appendix 1, Lemma A.1.4 in [21], to the map  $\Omega \ni \omega \mapsto F(\cdot|_{K_n}, \omega) \in C(K_n)$ , where  $F(\cdot|_{K_n}, \omega)$  denotes the map  $K_n \ni \eta \mapsto F(\eta, \omega)$ . Therefore we can find a sequence  $\tilde{F}_{n,k} : \Omega \rightarrow C(K_n)$ ,  $\omega \mapsto \tilde{F}_{n,k}(\cdot, \omega) \in K_n$  such that for  $\|F\|_\infty := \sup_{\eta \in \tilde{\Omega}_1, \omega \in \Omega} |F(\eta, \omega)|$ , we have

$$\|\tilde{F}_{n,k}\|_\infty \leq 1 + \|F\|_\infty, \quad \tilde{F}_{n,k}(\Omega) \subset \{\tilde{g}_{n,k}^{(1)}, \dots, \tilde{g}_{n,k}^{(N_{n,k})}\} \subset C(K_n),$$

where  $\tilde{g}_{n,k}^{(i)} \neq \tilde{g}_{n,k}^{(j)}$  if  $i \neq j$ , and for all  $\omega \in \Omega$

$$\sup_{\eta \in K_n} |F(\eta, \omega) - F_{n,k}(\eta, \omega)| \rightarrow 0, \quad (1.2)$$

as  $k \rightarrow \infty$ . Clearly, for all  $\omega \in \Omega$ ,  $\tilde{F}_{n,k}(\cdot, \omega) = \sum_{j=1}^{N_{n,k}} \tilde{g}_{n,k}^{(j)} 1_{\{\tilde{g}_{n,k}^{(j)}\}} \circ \tilde{F}_{n,k}(\cdot, \omega)$ .

By Tietze's extension theorem there exist extensions  $g_{n,k}^{(1)}, \dots, g_{n,k}^{(N_{n,k})} \in C(\tilde{\Omega}_1)$  of  $\tilde{g}_{n,k}^{(1)}, \dots, \tilde{g}_{n,k}^{(N_{n,k})}$  such that for all  $1 \leq j \leq N_{n,k}$ ,  $\sup_{\eta \in \tilde{\Omega}_1} |g_{n,k}^{(j)}(\eta)| \leq \sup_{\eta \in \tilde{K}_n} |\tilde{g}_{n,k}^{(j)}(\eta)|$ . Now we define  $F_{n,k} : \Omega \rightarrow C(\tilde{\Omega}_1)$ ,  $\omega \mapsto F_{n,k}(\cdot, \omega)$  by

$$F_{n,k}(\cdot, \omega) = \sum_{j=1}^{N_{n,k}} g_{n,k}^{(j)} 1_{\{\tilde{g}_{n,k}^{(j)}\}} \circ \tilde{F}_{n,k}(\cdot, \omega).$$

Clearly, still

$$\|F_{n,k}\|_\infty \leq 1 + \|F\|_\infty. \quad (1.3)$$

Note that for all  $\eta \in \tilde{\Omega}_1$

$$\tilde{F}_{n,k}(\eta, \omega) = \sum_{j=1}^{N_{n,k}} g_{n,k}^{(j)}(\eta) 1_{\{\tilde{g}_{n,k}^{(j)}\}} \circ \tilde{F}_{n,k}(\eta, \omega),$$

hence of the form that Lemma 7.6 applies. Therefore using the standard notation  $\mu(f) := \int f d\mu$ , for a measure  $\mu$  and a function  $f$ , we can argue as follows. Fix  $n \in \mathbb{N}$ . Then for all  $\ell, k \in \mathbb{N}$

$$\begin{aligned} & \left| \int Q^{\kappa_\ell}(F(\cdot, \omega), \omega) P(d\omega) - \int Q(F(\cdot, \omega), \omega) P(d\omega) \right| \\ & \leq \left| \int Q^{\kappa_\ell}(F(\cdot, \omega) 1_{K_n}, \omega) P(d\omega) - \int Q(F(\cdot, \omega) 1_{K_n}, \omega) P(d\omega) \right| + \frac{2}{n} \|F\|_\infty \\ & \leq \int \underbrace{Q^{\kappa_\ell}(|F(\cdot, \omega) - F_{n,k}(\cdot, \omega)| 1_{K_n}, \omega)}_{\leq \sup_{\eta \in K_n} |F(\eta, \omega) - F_{n,k}(\eta, \omega)|} P(d\omega) \\ & + \left| \int Q^{\kappa_\ell}(F_{n,k}(\cdot, \omega), \omega) P(d\omega) - \int Q(F_{n,k}(\cdot, \omega), \omega) P(d\omega) \right| \\ & + \frac{2}{n} (1 + \|F\|_\infty) + \int Q(|F(\cdot, \omega) - F_{n,k}(\cdot, \omega)|, \omega) 1_{K_n} P(d\omega) + \frac{2}{n} \|F\|_\infty. \end{aligned}$$

The first inequality is a consequence of (1.1), the second one of (1.1) and (1.3). Now, letting first  $\ell \rightarrow \infty$  (using Lemma 7.6), then  $k \rightarrow \infty$  (using (1.2)) and finally  $n \rightarrow \infty$ , the assertion follows.  $\square$

## B Uniqueness for the porous media equation with noise

We state here a general uniqueness lemma which only holds under even weaker hypotheses than Assumption 1.1 i.e.  $\psi : \mathbb{R} \rightarrow \mathbb{R}$  is Lipschitz and that the functions belong to  $W^{1,\infty}$ .

**Theorem B.1.** *Let  $x_0 \in \mathcal{S}'(\mathbb{R}^d)$  and suppose  $\psi : \mathbb{R} \rightarrow \mathbb{R}$  to be Lipschitz. Then equation (1.1) admits at most one solution among the random fields  $X : ]0, T] \times \mathbb{R} \times \Omega \rightarrow \mathbb{R}$  such that*

$$\int_{[0, T] \times \mathbb{R}} X^2(s, \xi) ds d\xi < \infty \quad a.s. \quad (\text{B.1})$$

**Remark B.2.** 1. *Suppose moreover that  $e^i, 0 \leq i \leq N$ , belong to  $H^1$ . If  $x_0 \in L^2$  or  $\psi$  is non-degenerate then Theorem 3.4 of [8] provides an existence theorem for (1.1). It states the existence of a random field  $X$  such that*

$$E \left( \int_{[0, T] \times \mathbb{R}} X^2(s, \xi) ds d\xi \right) < \infty,$$

such that  $t \mapsto X(t, \cdot)$  belongs to  $C([0, T]; H^{-1}(\mathbb{R}))$  and  $t \mapsto \int_0^t \psi(X(s, \cdot)) ds \in C([0, T]; H^1(\mathbb{R}))$  a.s.

2. So, under the assumption of item 1., the solution  $X$  is unique among those fulfilling (B.1).

3.  $X$  of point ii) fulfills the equation, for almost all  $\omega$ , in  $H^{-1}$

$$X(t, \cdot) = x_0 + \int_0^t \Delta(\psi(X(s, \cdot))) ds + \int_0^t \mu(ds, \cdot) X(s, \cdot), \quad t \in [0, T]. \quad (2.1)$$

The proof of Theorem B.1 is a consequence of the result stated in Theorem B.1 of [24], see also [7].

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