Border bases for lattice ideals

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Abstract

The main ingredient to construct an $\mathcal{O}$-border basis of an ideal $I \subseteq K[x_1, \ldots, x_n]$ is the order ideal $\mathcal{O}$, which is a basis of the K-vector space $K[x_1, \ldots, x_n]/I$. In this paper we give a procedure to find all the possible order ideals associated with a lattice ideal I_M (where M is a lattice of $\mathbb{Z}^n$). The construction can be applied to ideals of any dimension (not only zero-dimensional) and shows that the possible order ideals are always in a finite number. For lattice ideals of positive dimension we also show that, although a border basis is infinite, it can be defined in finite terms. Furthermore we give an example which proves that not all border bases of a lattice ideal come from Gröbner bases. Finally, we give a complete and explicit description of all the border bases for ideals I_M in case M is a 2-dimensional lattice contained in $\mathbb{Z}^2$.

Key words: Border basis, Gröbner basis, lattice ideal, maximal clique, maximum clique, order ideal.

1. Introduction

Let I be a zero-dimensional ideal in the polynomial ring $K[x_1, \ldots, x_n]$, then a *border basis* of I is composed by a finite set $\mathcal{O}$ of monomials closed under division, which is a basis of the K-vector space $K[x_1, \ldots, x_n]/I$ and a set of polynomials $f_1, \ldots, f_m \in I$, such that $f_i = b_i - \sum_j a_{ij} t_j$, where $t_j \in \mathcal{O}$, $a_{ij} \in K$ and b_i are elements in the *border*

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of $\mathcal{O}$ (i.e. are not in $\mathcal{O}$, but are obtained multiplying an element of $\mathcal{O}$ by a variable). Border bases are a natural generalization of Gröbner bases, and indeed, given a Gröbner basis G, it is easy to construct the corresponding border basis (the set $\mathcal{O}$ is the set of irreducible monomials w.r.t. G). Border bases were introduced in [15] (see also [14]); for a discussion of their properties, see, among others, [11, 12, 13, 16, 18]. One should notice that in fact [18] deals with a more general notion of border basis.

The set we have denoted by $\mathcal{O}$ is often called an order ideal in books of commutative algebra, and we use this name throughout. But one can find in the literature at least ten other ways of naming $\mathcal{O}$: see page 6 of [13], where the alternative terminologies are linked to different branches of Mathematics.

The main difference between border bases and Gröbner bases lies in the order ideal $\mathcal{O}$ associated with them, which, for border bases, has less constraints, since it is not linked to a term order. One specific application of border bases regards the determination of the solutions of a system of polynomial equations in which the coefficients are real numbers, known with some approximation: it turns out that border bases are more stable under small perturbations of the coefficients than Gröbner bases and allow therefore to construct better values for the zeros ([3, 13, 19]). In this paper, however, we do not focus our attention on the problem of determining zeros of systems of polynomials, but we consider a different question: we want to find all the border bases of a given lattice ideal.

A *lattice ideal* is an ideal in the polynomial ring which comes from a lattice M in $\mathbb{Z}^n$. More precisely, the ideal is generated by the binomials $x^{a^+} - x^{a^-}$, where $a = (a_1, \ldots, a_n) \in M$ and $a = a^+ - a^-$ where a^+ is the n-tuple whose i-th element is a_i, if a_i is positive and 0 otherwise (a similar definition for a^-). Lattice ideals arise in many different examples: all toric ideals, for instance, are lattice ideals as well as the ideal associated with an integer programming problem. The construction of Gröbner bases for lattice ideals was studied by many authors (see [4, 6, 7, 24] and the references given there) and there are efficient symbolic computation packages which allow their computation [1, 2, 9].

In this paper, as stated, we consider an ideal I_M defined by a lattice $M \subseteq \mathbb{Z}^n$ (M can therefore equivalently be seen as a sub-module of $\mathbb{Z}^n$) and we show how to construct *all* the possible border bases of I_M. We omit a very strong condition that is usually considered for border bases, that is, we do not assume that the ideal I_M is zero-dimensional. (For another paper in which the positive dimension case for border bases is considered, see [20]). Let us remark that the main step in getting a border basis is to find an order ideal which is a K-basis for $K[x_1, \ldots, x_n]/I_M$ and this problem can be converted into the problem of determining an order ideal $\mathcal{O}$ of $\mathbb{N}^n$ (w.r.t. the partial order $\preceq$, where $u \preceq v$ if every component of u is less than or equal to the corresponding component of v) whose elements uniquely represent $\mathbb{Z}^n/M$. Section 2 therefore deals with order ideals in $K[x_1, \ldots, x_n]$ and in $\mathbb{N}^n$; an order ideal of $\mathbb{N}^n$ satisfying the above properties will be called a max-compatible order ideal (w.r.t. M); later, in section 3, we consider the problem of finding, for a given module $M \subseteq \mathbb{Z}^n$, all the possible max-compatible order ideals w.r.t. M. The construction we propose determines a finite graph whose maximal cliques allow to recover the required order ideals. In particular, in this way we see that there are only finitely many order ideals and therefore the ideal I_M has only finitely many border bases.

Section 4 deals with the case of border bases of I_M and in particular we briefly consider the case of infinite border bases: the specific shape of any order ideal $\mathcal{O}$, together with the properties of the lattice M allow us to describe both the border of $\mathcal{O}$ and the $\mathcal{O}$-border

basis in finite terms. Moreover we give some examples and in particular we show that there exist border bases for lattice ideals which cannot be obtained from any Gröbner basis of that ideal.

The final section considers the very peculiar case in which M is a module of rank 2 contained in $\mathbb{Z}^2$. We show that in this case every border basis comes from a Gröbner basis and we see that the results obtained in the previous sections allow a complete and explicit description of all the border (Gröbner) bases of I_M.

2. Order ideals

Recall that the commutative monoid $\mathbb{T}$ of the terms of $K[x_1, \ldots, x_n]$ (w.r.t. the product) is isomorphic to the additive monoid $\mathbb{N}^n$. If $t \in \mathbb{T}$ the corresponding element of $\mathbb{N}^n$ is denoted by $\lg(t)$; if $u \in \mathbb{N}^n$, the corresponding element of $\mathbb{T}$ is denoted by $\mathcal{E}(u)$ (however, if there is no risk of ambiguity, sometimes we will omit the function $\mathcal{E}$). By $e_1, \ldots, e_n$ we denote the canonical basis of $\mathbb{Z}^n$. On $\mathbb{N}^n$ we consider a partial order $\preceq$ given by $u, v \in \mathbb{N}^n$, $u \preceq v$ if every component of u is not bigger than the corresponding component of v. If $u \in \mathbb{N}^n$, let

$$D(u) = \{v \in \mathbb{N}^n \mid v \preceq u\}, \qquad C(u) = \{v \in \mathbb{N}^n \mid u \preceq v\}.$$

$D(u) \subseteq \mathbb{N}^n$ corresponds to the monomials which divide $\mathcal{E}(u)$, while $C(u)$ corresponds to the monomials which are divided by $\mathcal{E}(u)$.

If $u, v \in \mathbb{N}^n$, $\operatorname{lcm}(u, v)$ is the element $(\max(u_1, v_1), \ldots, \max(u_n, v_n))$ where u_i and v_i are the components of u and v respectively.

An *order ideal* $\mathcal{O}$ (in $\mathbb{N}^n$) is a subset of $\mathbb{N}^n$ such that, if $u \in \mathcal{O}$, then $D(u) \subseteq \mathcal{O}$. The *border* of $\mathcal{O}$ is the set of elements $u \in \mathbb{N}^n$ such that $u \notin \mathcal{O}$, but there exists $i \in \{1, \ldots, n\}$ such that $u - e_i \in \mathcal{O}$. The border of $\mathcal{O}$ is denoted by $\partial\mathcal{O}$. Using the function $\mathcal{E}$, we can define order ideals in $K[x_1, \ldots, x_n]$: an *order ideal* is a subset of $\mathbb{T}$ which contains all the divisors of its elements. Analogously, the *border* of an order ideal of $K[x_1, \ldots, x_n]$ is the set of terms which are not in the order ideal, but are obtained multiplying an element of the order ideal by one of the variables. We do not require the finiteness condition of the order ideals. However, if the order ideal is finite, we do get the usual definition given, for instance, in [13]. Also the definition of the border basis of an ideal I of $K[x_1, \ldots, x_n]$ as given in [13] can immediately be extended to the case in which the order ideal is infinite (hence I is not zero dimensional). Clearly, in this case, the border basis is infinite.

Suppose $M \subseteq \mathbb{Z}^n$ is any sub-module of $\mathbb{Z}^n$ of dimension $m \leq n$ and assume it is generated by the rows of the following matrix:

$$\begin{pmatrix} d_1 & * & \ldots & * & * & \ldots & * \\ 0 & d_2 & \ldots & * & * & \ldots & * \\ \ldots & & & & & & \\ 0 & 0 & \ldots & d_m & * & \ldots & * \end{pmatrix} \tag{1}$$

where $d_1, \ldots, d_m$ are positive integers and every "$*$" above a d_j represents a non-negative integer smaller that d_j, i.e. the matrix is in Hermite Normal Form (HNF). We associate with M the following subset of $\mathbb{Z}^n$:

$$B = \{(i_1, \ldots, i_n) \mid 0 \leq i_j < d_j \text{ for } j = 1, \ldots, m,\ i_j \in \mathbb{Z} \text{ for } j = m+1, \ldots, n\}.$$

Note that if $b_1, b_2 \in B$ and $b_1 \equiv_M b_2$ (where "$\equiv_M$" means $b_1 - b_2 \in M$), then $b_1 = b_2$ and every element of $\mathbb{Z}^n$ has a unique representative, $\bmod M$, in B. Hence the elements of B are in one to one correspondence with the elements of the module $\mathbb{Z}^n/M$.

Given $b = (b_1, \ldots, b_n) \in \mathbb{Z}^n$, the construction of its representative $(\bmod M)$ $\rho(b)$ in B is easily obtained by repeated divisions as follows: suppose $b_1, \ldots, b_{r-1}$ $(r \leq m)$ are such that $0 \leq b_i < d_i$ for $i \leq r-1$ and $b_r \geq d_r$, and replace b by $b' = b - qM_r$ where M_r is the r^{th}-row of the matrix (1) and q is the quotient of b_r when divided by d_r. Then b'_r is such that $0 \leq b'_r < d_r$.

The set B is finite if and only if $m = n$. In this case B is an order ideal of $\mathbb{N}^n$ and has $d_1 d_2 \cdots d_n$ elements. Sometimes it will be convenient to label its elements with $0, 1, \ldots, d_1 d_2 \cdots d_n - 1$ in the following way: if $(a_1, \ldots, a_n) \in B$, then its label is

$$a_n + d_n a_{n-1} + d_n d_{n-1} a_{n-2} + \cdots + d_n \cdots d_2 a_1. \tag{2}$$

Consequently we can label all the elements of $\mathbb{Z}^n$, assigning the same number to equivalent elements.

Summarizing the properties of the set B when $m = n$, we have: it is an order ideal; if $b_1, b_2 \in B$ are equivalent $\bmod M$, then $b_1 = b_2$; B is maximal w.r.t. this property and every element of $\mathbb{Z}^n$ has an equivalent element in B. We capitalize on these properties in the following definition concerning any order ideal of $\mathbb{N}^n$ (finite or infinite):

Definition 2.1. Let $\mathcal{O}$ be an order ideal of $\mathbb{N}^n$. Then it is *compatible* $(\bmod M)$ if it holds: $a, b \in \mathcal{O}$ and a, b equivalent $\bmod M$, then $a = b$. The order ideal is *maximal compatible* $(\bmod M)$ if it is compatible and maximal in the set of compatible order ideals, w.r.t. inclusion. It is *max-compatible* $(\bmod M)$ if every element of $\mathbb{Z}^n$ has a representative $(\bmod M)$ in it.

Clearly, max-compatible implies maximal compatible. If $\mathcal{O}$ is finite, then max-compatible is equivalent to *maximum compatible*, i.e. compatible with the maximum number of elements.

Consider now the lattice ideal I_M associated with the module M and suppose $m = n$. Then $\mathcal{E}(B)$ is an order ideal of $K[x_1, \ldots, x_n]$ and every term $t \in \mathbb{T}$ is equivalent, modulo the ideal I_M, to an element $\mathcal{R}(t) \in \mathcal{E}(B)$ defined by $\mathcal{R}(t) = \mathcal{E}(\rho(\lg(t)))$. In particular $t - \mathcal{R}(t)$ is a binomial in I_M and the set:

$$\{u - \mathcal{R}(u) \mid \text{ for } u \in \partial(\mathcal{E}(B))\}$$

is a first example of a border basis of I_M.

Example 2.2. As a particular case, consider in $\mathbb{Z}^2$ the subgroup M generated by the rows of the matrix

$$\begin{pmatrix} 2 & 6 \\ 0 & 10 \end{pmatrix}$$

which is in HNF. The set $B \subseteq \mathbb{Z}^2$ is $\{(i,j) \mid 0 \leq i < 2,\ 0 \leq j < 10\}$, hence $\mathcal{E}(B)$ is the set of monomials $\{x^i y^j \mid 0 \leq i < 2,\ 0 \leq j < 10\}$, the border (of $\mathcal{E}(B)$) is $\{x^2 y^j \mid j = 0, \ldots, 9\} \cup \{y^{10}, xy^{10}\}$ and the corresponding border basis is:

$$\{x^2 y^i - y^{4+i} \mid i = 0, \ldots, 5\} \cup \{x^2 y^{6+j} - y^j \mid j = 0, \ldots, 3\} \cup \{x^k y^{10} - x^k \mid k = 0, 1\}.$$

3. Construction of order ideals

As usual, M denotes a sub-module of $\mathbb{Z}^n$ of rank $m \leq n$. Let $\mathcal{V}$ be the union of all compatible order ideals of $\mathbb{N}^n$ (i.e., as said, order ideals which do not contain equivalent elements mod M). If $a \in \mathbb{Z}^n$, then $\mathrm{abs}(a)$ denotes $a^+ + a^- \in \mathbb{N}^n$.

Proposition 3.1. *Let $\mathcal{A} = \{\mathrm{abs}(a) \mid a \in M \setminus \{0\}\}$ and $\mathcal{A}_1$ be the set of elements of $\mathcal{A}$ which are minimal w.r.t. the partial order $\preceq$. Then it holds:*

$$\mathcal{V} = \mathbb{N}^n \setminus \bigcup_{a \in \mathcal{A}} C(a) = \mathbb{N}^n \setminus \bigcup_{a \in \mathcal{A}_1} C(a).$$

Proof. It is clear that $\bigcup_{a\in\mathcal{A}} C(a) = \bigcup_{a\in\mathcal{A}_1} C(a)$, so it suffices to prove the first equality. Let $v \in \mathcal{V}$ and suppose there exists $\mathrm{abs}(a) \in \mathcal{A}$ such that $v \in C(\mathrm{abs}(a))$, so $\mathrm{abs}(a) \in D(v)$. Since $a = a^+ - a^- \equiv_M 0$, then $a^+ \equiv_M a^-$ and since $a^+, a^- \in D(\mathrm{abs}(a)) \subseteq D(v)$, we have that $D(v)$ is not compatible. Conversely, let $v \in \mathbb{N}^n \setminus \bigcup_{a\in\mathcal{A}} C(a)$ and suppose $D(v)$ is not compatible, hence there exist $u_1, u_2 \in D(v)$ such that $u_1 \equiv_M u_2$. If $a = u_1 - u_2 \equiv_M 0$, then $a^+ \in D(u_1)$ and $a^- \in D(u_2)$, so $\mathrm{abs}(a) = \mathrm{lcm}(a^+, a^-) \in D(v)$, which gives that $v \in C(\mathrm{abs}(a))$. □

Remark 3.2. As a consequence of the above proposition, we see that $\mathcal{E}(\mathcal{V})$ is the normal basis of the monomial ideal $J = (\mathcal{E}(\mathrm{abs}(a)) \mid a \in M \setminus \{0\})$ (i.e. $\mathcal{E}(\mathcal{V})$ is a K-basis of $K[x_1, \ldots, x_n]/J$). Moreover Dixon's lemma (see e.g. [16, page 38]) ensures that $\mathcal{A}_1$ is finite.

Proposition 3.3. *It holds:* $\mathrm{rank}\,(M) = n$ *if and only if the set $\mathcal{V}$ is finite.*

Proof. If $\mathrm{rank}\,(M) = n$, then for every $i \in \{1, \ldots, n\}$ we can find an element $t_i e_i \in M$ (where $t_i \in \mathbb{N}$ and e_i is an element of the canonical basis of $\mathbb{Z}^n$). If $\mathcal{O}$ is a compatible order ideal, then necessarily $\mathcal{O} \subseteq D(t_1, \ldots, t_n)$. Therefore $\mathcal{V} \subseteq D(t_1, \ldots, t_n)$ and is a finite set. If $\mathrm{rank}\,(M) < n$, then there exists $i \in \{1, \ldots, n\}$ such that $te_i \notin M$ for all $t \in \mathbb{N}$, hence $\mathcal{O} = \{te_i \mid t \in \mathbb{N}\}$ is an infinite compatible order ideal, hence $\mathcal{V}$ is infinite. □

Example 3.4. We consider again example 2.2, i.e. the module in $\mathbb{Z}^2$ generated by $(2, 6)$ and $(0, 10)$. Since the rank of M is 2, the set $\mathcal{V}$ is finite. The set $\mathcal{A}$ is given by the blue

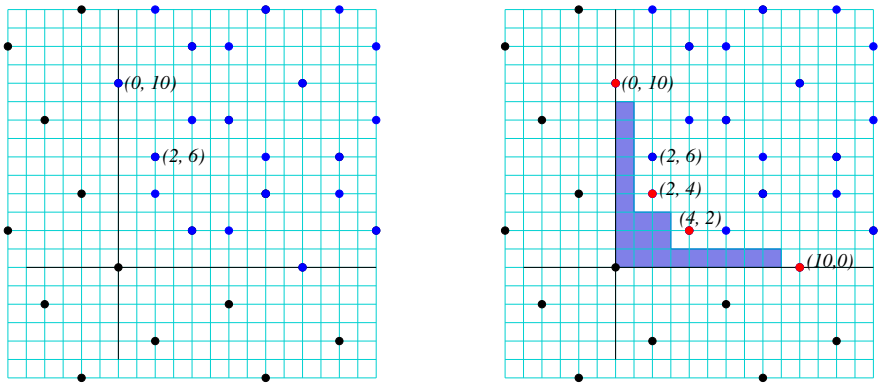

Fig. 1. On the left: the lattice generated by $(2,6)$ and $(0,10)$ and the set $\mathcal{A}$ (in blue); on the right: the elements of $\mathcal{A}_1$ (in red) and the set $\mathcal{V}$ (the shadow region).

dots in figure 1, left, the set $\mathcal{A}_1$ is the set $\{(10,0),\ (4,2),\ (2,4),\ (0,10)\}$ and the set $\mathcal{V}$ is the shadow region in figure 1, right.

Let $G_{\mathcal{V}}$ be the graph whose vertexes are the elements of $\mathcal{V}$ and two vertexes u and v are connected by an edge if in the set $D(u) \cup D(v)$ there are no equivalent elements $(\mathrm{mod}\, M)$. Suppose first that M has rank $m = n$. Then in this case $G_{\mathcal{V}}$ is a finite graph and a maximal compatible order ideal (see definition 2.1) corresponds to a *maximal clique* of $G_{\mathcal{V}}$ and a maximum compatible order ideal (i.e. max-compatible) corresponds to a *maximum clique* of $G_{\mathcal{V}}$. Therefore the problem of finding maximal and maximum compatible order ideals is reduced, at least in the case of rank n, to the problem of finding maximal and maximum cliques of a (finite) graph, which can be done by the Bron-Kerbosch algorithm (see [8], [10]) (an implementation of the Bron-Kerbosh algorithm can be found, for instance, in [22]). If the rank of M is less than n, the maximal cliques again give the maximal compatible order ideals, but $G_{\mathcal{V}}$ is an infinite graph. We shall now see how to overcome this problem.

Set $P = \{(k_1, \ldots, k_n) \in \mathbb{N}^n \mid \forall j\ k_j \neq 0\}$ and $-P = \{-k \mid k \in P\}$. We consider the following set:

$$\mathcal{X} = \left\{(c^+, c^-) \mid c \in M \setminus (P \cup -P \cup \{0\})\right\} \subseteq \mathbb{N}^n \times \mathbb{N}^n.$$

(Note that $\mathcal{X}$ could also be defined as the set of (c^-, c^+) since, if $c \in M \setminus (P \cup -P \cup \{0\})$, then also $-c \in M \setminus (P \cup -P \cup \{0\})$). On $\mathcal{X}$ we define a partial order $\sqsubseteq$ given by:

$$\text{if } a, b \in \mathcal{X},\ \ a \sqsubseteq b \ \text{ if } \ a_0 \preceq b_0 \text{ and } a_1 \preceq b_1, \text{ or } a_0 \preceq b_1 \text{ and } a_1 \preceq b_0$$

(where a_0 and a_1 denote the two coordinates of $a \in \mathcal{X}$). By $\mathcal{X}_1$ we denote the set of the minimal elements of $\mathcal{X}$ w.r.t. $\sqsubseteq$.

Proposition 3.5. *The partial order $\sqsubseteq$ is a well-founded order and the set $\mathcal{X}_1$ is finite.*

Proof. The partial order $\sqsubseteq$ is well-founded since $\preceq$ is well-founded. To see that $\mathcal{X}_1$ is finite, let $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ be such that each $\epsilon_i \in \{-1, 1\}$ (it is convenient to consider each ϵ as an identifier of an orthant of $\mathbb{Z}^n$) and let $M_\epsilon = \{\epsilon x \mid x \in M\}$, where $\epsilon x = (\epsilon_1 x_1, \ldots, \epsilon_n x_n)$.

The module M_ϵ is constructed in such a way that the part of M_ϵ contained in the positive orthant corresponds to the part of M contained in the orthant in which the signs of the coordinates are given by the vector ϵ. Let B_ϵ be the Hilbert basis of $(M_\epsilon \setminus \{0\}) \cap \mathbb{N}^n$ (w.r.t. the partial order $\preceq$) (see e.g. [13, 6.1.B]). In particular, B_ϵ is a finite set and for every element u of $(M_\epsilon \setminus \{0\}) \cap \mathbb{N}^n$ there exists an element $b \in B_\epsilon$ such that $b \preceq u$. Let $c \in M \setminus (P \cup -P \cup \{0\})$ and assume the orthant of c $(= c^+ - c^-)$ is given by the vector ϵ. Then there exists $b \in B_\epsilon$ such that $b \preceq \epsilon c = c^+ + c^-$, hence $(\epsilon b)^+ \preceq c^+$ and $(\epsilon b)^- \preceq c^-$. From this it follows that $(c^+, c^-) \in \mathcal{X}_1 \Rightarrow c \in \cup_\epsilon \epsilon B_\epsilon$ and hence $\mathcal{X}_1$ is finite. □

Proposition 3.6. *Let $u, v \in G_\mathcal{V}$. The following are equivalent:*
(1) u and v are not connected;
(2) there exists $a \in \mathcal{X}$ such that $a_0 \in D(u)$ and $a_1 \in D(v)$;
(3) there exists $a \in \mathcal{X}_1$ such that $a_0 \in D(u)$ and $a_1 \in D(v)$.

Proof. Suppose u and v are not connected, hence there exist $u_1 \in D(u)$ and $v_1 \in D(v)$ such that $u_1 \equiv_M v_1$. Then $c = u_1 - v_1$ is an element of M such that $c^+ \in D(u)$ and $c^- \in D(v)$. Moreover, $c^+ \neq 0$, and $c^- \neq 0$ (if, for instance, $c^+ = 0$, then $c^- \equiv_M 0$ and this is a contradiction, since $v \in \mathcal{V}$), therefore $a = (c^+, c^-) \in \mathcal{X}$. If 2. holds, let $b \in \mathcal{X}$ be such that $b \sqsubseteq a$. Then either $b_0 \preceq a_0$ and $b_1 \preceq a_1$ (hence $b_0 \in D(u)$ and $b_1 \in D(v)$), or $b_0 \preceq a_1$ and $b_1 \preceq a_0$ and in this case it is enough to consider $\beta = -b$. Then $\beta \sqsubseteq a$ and $\beta_0 \in D(u)$ and $\beta_1 \in D(v)$. From this 3. follows. Finally, if 3 holds, then $a_0 \equiv_M a_1$ and u and v are not connected. □

We now define a partition on $\mathcal{V}$ (hence on the vertexes of $G_\mathcal{V}$) as follows: if $u \in \mathcal{V}$, then we set

$$R_u = \{v \in \mathcal{V} \mid \text{for all } a \in \mathcal{X}_1,\ a_0 \in D(v) \text{ iff } a_0 \in D(u)\}. \tag{3}$$

Again, let us remark that R_u can also be defined by:

$$R_u = \{v \in \mathcal{V} \mid \text{for all } a \in \mathcal{X}_1,\ a_1 \in D(v) \text{ iff } a_1 \in D(u)\}$$

since $(a_0, a_1) \in \mathcal{X}_1$ if and only if $(a_1, a_0) \in \mathcal{X}_1$.
We have

Proposition 3.7. *If $u_1, u_2 \in R_u$, then u_1 and u_2 are connected. Moreover, $u, v \in G_\mathcal{V}$ are connected if and only if every element of R_u is connected to every element of R_v.*

Proof. Suppose u_1 and u_2 are not connected. Then, by proposition 3.6, there exists $a \in \mathcal{X}$ (minimal) such that $a_0 \in D(u_1)$, $a_1 \in D(u_2)$, so $a_0 \in D(u)$ and, analogously, $a_1 \in D(u)$, but this gives a contradiction, since $u \in \mathcal{V}$ and $a_0 \equiv_M a_1$. Suppose now that u and v are not connected and let $u_1 \in R_u$ and $v_1 \in R_v$. Hence there exists $a \in \mathcal{X}$ such that $a_0 \in D(u)$ and $a_1 \in D(v)$, so $a_0 \in D(u_1)$ and $a_1 \in D(v_1)$, hence u_1 and v_1 are not connected. If $u_1 \in R_u$ and $v_1 \in R_v$ are not connected, a similar argument shows that u and v are not connected. □

Consider the set $\mathcal{Y} = \{a_0 \mid a \in \mathcal{X}_1\} \cup \{0\}$ $(= \{a_1 \mid a \in \mathcal{X}_1\} \cup \{0\})$, and let $c_1, \ldots, c_l \in \mathbb{N}^n$ be the elements of $\mathcal{Y}$. With every element $u \in \mathcal{V}$ we associate the l-tuple $s(u) = (\chi(c_i, u) \mid i = 1, \ldots, l)$, where

$$\chi(c_i, u) = \begin{cases} 0 \text{ if } c_i \notin D(u) \\ 1 \text{ if } c_i \in D(u) \end{cases}.$$

Proposition 3.8. *For each $u \in \mathcal{V}$, we have:*

$$R_u = \{v \in \mathcal{V} \mid s(u) = s(v)\}.$$

Proof. Immediate. □

Corollary 3.9. *The set $\{R_u \mid u \in \mathcal{V}\}$ is a finite set.*

Proof. The l-tuples $s(u)$ can only assume a finite number of values. □

We recall that a hyper-rectangle of $\mathbb{N}^n$ is a set of points $(a_1, \ldots, a_n) \in \mathbb{N}^n$, such that each coordinate a_i is subject to a condition of the form $l_i \leq a_i < L_i$ where $l_i \in \mathbb{N}$ and $L_i \in \mathbb{N} \cup \{+\infty\}$.

Proposition 3.10. *Each set R_u is a finite union of hyper-rectangles.*

Proof. Let $v \in R_u$ and $c_i \in \mathcal{Y}$. If $c_i \in D(u)$, then $c_i \in D(v)$, so each coordinate v_j of v is such that $c_{ij} \leq v_j$ (where c_{ij} is the j-th coordinate of c_i). If $c_i \notin D(u)$, then $c_i \notin D(v)$, so there exists a k such that $v_k < c_{ik}$. Moreover, $v \in \mathcal{V}$ and proposition 3.1 give some further upper bounds for the coordinates of v. Considering all these bounds we see that the elements of R_u are subject to a finite number of conditions each of which defines a hyper-rectangle. □

Starting from the partition $R_u, u \in \mathcal{V}$ of the vertexes of $G_{\mathcal{V}}$, we can construct the quotient graph $\widetilde{G}_{\mathcal{V}}$ whose vertexes are the elements of the partition and two vertexes R_u and R_v of $\widetilde{G}_{\mathcal{V}}$ are connected if and only if u and v are connected in $G_{\mathcal{V}}$. According to proposition 3.7, the connection is well-defined; moreover, by corollary 3.9, $\widetilde{G}_{\mathcal{V}}$ is a finite graph.

A clique of $G_{\mathcal{V}}$ is a compatible order ideal. A maximal clique of $G_{\mathcal{V}}$ is a maximal, compatible order ideal and gives a maximal clique of $\widetilde{G}_{\mathcal{V}}$. Conversely, from a maximal clique of $\widetilde{G}_{\mathcal{V}}$, taking the union of its vertexes (considered as sets), we get a maximal clique of $G_{\mathcal{V}}$ which is a maximal, compatible order ideal. Hence, the above constructions allow us to obtain all the maximal compatible order ideals associated with a module M.

In a maximal compatible order ideal, by definition, all the elements are different mod M, but it is not true, in general, that a maximal compatible order ideal contains a representative of all the elements of $\mathbb{Z}^n/M$, in other words, not all maximal compatible order ideals are also max-compatible (as defined in section 2). However among the maximal compatible order ideals there are all the max-compatible ones. The finiteness of $\widetilde{G}_{\mathcal{V}}$ then yields:

Proposition 3.11. *Given a module $M \subseteq \mathbb{Z}^n$, there are only finitely many max-compatible order ideals associated with M.*

To conclude this section, we sketch here an algorithm which allows to compute all the maximal compatible order ideals of a given module M. It can be summarized as follows:

Algorithm 1 (Computation of maximal compatible order ideals)**.**
INPUT: A module (lattice) $M \subseteq \mathbb{Z}^n$ given by a finite set of generators.
OUTPUT: All the maximal compatible order ideals w.r.t. M.
Step 1. Compute the set $\mathcal{A}_1$ of minimal elements of $\mathcal{A}$ w.r.t. the partial order $\preceq$;
Step 2. Let $\mathcal{V} = \mathbb{N}^n \setminus \bigcup_{a \in \mathcal{A}_1} C(a)$;
Step 3. Construct the set $\mathcal{X}_1$ of the minimal elements of $M \setminus (P \cup -P \cup \{0\})$ w.r.t. $\sqsubseteq$;
Step 4. Define the partition on $\mathcal{V}$ as in (3) and 3.8;
Step 5. Construct the graph $\widetilde{G}_{\mathcal{V}}$ whose vertexes are the elements of the above partition and two vertexes R_u and R_v are not connected if and only if there exists $a \in \mathcal{X}_1$ such that $a_0 \in D(u)$ and $a_1 \in D(v)$ (see proposition 3.6);
Step 6. Compute the maximal cliques of $\widetilde{G}_{\mathcal{V}}$;
Step 7. Recover the maximal cliques of $G_{\mathcal{V}}$: if $R_{u_1}, \ldots, R_{u_k}$ is a maximal clique of $\widetilde{G}_{\mathcal{V}}$, then the corresponding maximal clique of $G_{\mathcal{V}}$ is $R_{u_1} \cup \cdots \cup R_{u_k}$.
Step 8. Return all the maximal cliques computed in Step 7.

Remark 3.12. The computation of $\mathcal{A}_1$ in step 1 can be done in a finite number of steps (according to remark 3.2, the problem is equivalent to the problem of finding a minimal set of generators of a monomial ideal; one way to proceed, is suggested in the proof of proposition 3.5); the set $\mathcal{V}$ of step 2 can be infinite, but is described in finite terms; a possible construction of $\mathcal{X}_1$ is given again in the proof of proposition 3.5; to get the partition of $\mathcal{V}$ in step 4 it is enough to find elements $u \in \mathcal{V}$ such that the l-tuples $s(u)$ assume all the possible (finite) values.

Example 3.13. We consider again the module of example 2.2. Since in M we have the elements $(-2, 4)$ and $(6, -2)$, in $\mathcal{X}$ we have, among others, the four elements $((2,0),(0,4))$, $((6,0),(0,2))$ and $((0,4),(2,0))$, $((0,2),(6,0))$. It is easy to see that these four elements are all the elements of $\mathcal{X}_1$.
Starting from $\mathcal{X}_1$ we can divide the set $\mathcal{V}$ (showed in figure 1, right) into six regions, according to (3). The six regions are $R_{(0,0)}$, $R_{(0,2)}$, $R_{(0,4)}$, $R_{(2,0)}$, $R_{(2,2)}$, $R_{(6,0)}$ and in figure 2 are labeled, respectively, by A, B, C, D, E and F. Hence $\widetilde{G}_{\mathcal{V}}$ has 6 vertexes and the edges that are not connected (according to proposition 3.6) are: BF, CD, CE, CF, EF. The maximal cliques of $\widetilde{G}_{\mathcal{V}}$ are (A, B, C), (A, B, D, E) and (A, D, F). They correspond to the maximal cliques of $G_{\mathcal{V}}$ which are $(A \cup B \cup C)$, $(A \cup B \cup D \cup E)$ and $(A \cup D \cup F)$. These sets (each of 20 elements) are all the maximal compatible order ideals (w.r.t. M) and are all maximum (note that $\widetilde{G}_{\mathcal{V}}$ has only one maximum clique, which is (A, B, D, E), hence it is evident that maximum cliques of $G_{\mathcal{V}}$ in general do not correspond to maximum cliques of $\widetilde{G}_{\mathcal{V}}$).

Example 3.14. Let $M = \langle (2,1,4), (0,3,-3) \rangle \subseteq \mathbb{Z}^3$. M is a module of rank 2 in $\mathbb{Z}^3$. The set $\mathcal{A}_1$ is $\{(0,3,3), (2,1,4), (2,4,1), (6,0,15), (6,15,0)\}$. The set $\mathcal{V}$ is therefore the complement (in $\mathbb{N}^3$) of the cones $C(a)$ where $a \in \mathcal{A}_1$. The set $\mathcal{X}_1$ is:

$$((4,11,0),(0,0,1)),\ ((0,3,0),(0,0,3)),\ ((4,0,11),(0,1,0)),$$
$$((2,0,7),(0,2,0)),\ ((2,7,0),(0,0,2))$$

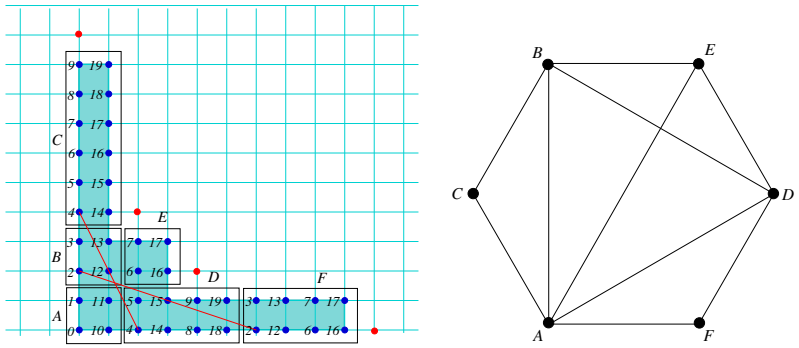

Fig. 2. On the left: the lattice of example 2.2 and the set $\mathcal{V}$ (in light blue), whose elements are labeled according to (2) and divided into the regions A, B, ..., F. The two red segments represent the elements of $\mathcal{X}_1$. On the right we have the graph $\widetilde{G}_{\mathcal{V}}$, whose vertexes are the regions $A, B, \ldots, F$.

and 5 other couples obtained inverting the above couples. Therefore the set $\mathcal{Y}$ is:

$$(0,0,0),(4,11,0),(0,0,1),(0,3,0),(0,0,3),(4,0,11),$$
$$(0,1,0),(2,0,7),(0,2,0),(2,7,0),(0,0,2)$$

and we get a partition of $\mathcal{V}$ into 19 classes R_u, where u is one of the following points of $\mathbb{Z}^3$:

$$(0,0,0),(0,0,1),(0,0,2),(0,0,3),(0,1,0),(0,1,1),(0,1,2),$$
$$(0,1,3),(0,2,0),(0,2,1),(0,2,2),(0,2,3),(0,3,0),$$
$$(0,3,1),(0,3,2),(2,0,7),(2,7,0),(4,0,11),(4,11,0).$$

For instance, $R_{(0,0,0)}$ is the hyper-rectangle $\{(i,0,0) \mid i \in \mathbb{N}\}$, while $R_{(4,0,11)}$ is given by the union of two hyper-rectangles:

$$R_{(4,0,11)} = \{(i,0,h) \mid i \geq 4,\ 11 \leq h \leq 14\} \cup \{(i,0,h) \mid 4 \leq i \leq 5,\ h \geq 15\}\,.$$

The graph $\widetilde{G}_{\mathcal{V}}$ has 19 vertexes; the computation of the maximal cliques gives 6 elements, hence the module M has 6 maximal order ideals. An example of a maximal clique is given by: $R_{(0,0,0)}$, $R_{(0,0,1)}$, $R_{(0,0,2)}$, $R_{(0,0,3)}$, $R_{(2,0,7)}$, $R_{(4,0,11)}$ and the union of these sets gives the corresponding maximal order ideal $\mathcal{O} = H_1 \cup H_2$, where

$$H_1 = \{(i,0,j) \mid i \geq 0,\ 0 \leq j \leq 14\}, \quad H_2 = \{(i,0,j) \mid 0 \leq i \leq 5,\ j \geq 15\}.$$

In general it is not true that maximal compatible order ideals are max-compatible. Here is an example in the case of a rank 3 module in $\mathbb{Z}^3$: let $M = \langle (1,1,2),(0,3,1),(0,0,4)\rangle \subseteq \mathbb{Z}^3$; the above algorithm gives 23 maximal compatible order ideals, 19 of 12 elements (maximum), 2 of 9 elements and 2 of 8 elements. In particular: $D(2,0,0) \cup D(0,2,0) \cup D(0,0,3)$ is a compatible order ideal with 8 elements which is maximal but not maximum.

When rank $M = n$, the max-compatible order ideals, as shown in the above example, can easily be selected counting their elements. When the rank of M is less than n, it is necessary to have another criterion to select, from the maximal order ideals, the max-compatible ones.

One possible way to proceed is to check if $\rho(\mathcal{O}) = B$, where the map ρ is described in section 2 (and is obtained by successive divisions by the rows of M). The order ideal $\mathcal{O}$ is a finite union of R_u's, hence $\mathcal{O}$ is a finite union of hyper-rectangles by proposition 3.10. It is possible to show that, using the pivot elements of the matrix M, the image under ρ of a hyper-rectangle can be decomposed into a finite union of sets of points of the form $(F_1(i_1, \ldots, i_r), \ldots, F_n(i_1, \ldots, i_r))$, where $F_1, \ldots, F_n$ are linear functions and $i_1, \ldots, i_r$ are integer numbers bounded by suitable inequalities. From this it follows that the image under the map ρ of a hyper-rectangle can be described in finite terms and the check $\rho(\mathcal{O}) = B$ can be done algorithmically. We note, moreover, that the sketched construction allows to also obtain the map $\sigma : B \longrightarrow \mathcal{O}$ which is the inverse of ρ.

An example can clarify this construction: take the order ideal $\mathcal{O}$ considered in example 3.14, which is the union of the hyper-rectangles H_1 and H_2 and let us see how to compute $\rho(H_1)$ and $\rho(H_2)$. Since the pivot element of the first row of M is 2, in order to compute the reduction of H_1 by the matrix M, we decompose H_1 into two disjoint parts: $H_1 = \{(2h, 0, j)\} \cup \{(2h + 1, 0, j)\}$, $(h \geq 0)$. The elements of the form $(2h, 0, j)$ are reduced w.r.t. the first row of M to $(0, -h, j - 4h)$ (while $(2h + 1, 0, j)$ reduces to $(1, -h, j - 4h)$). The pivot element of the second row of M is 3, so we consider three cases: $h = 3l$, $h = 3l + 1$, $h = 3l + 2$ $(l \geq 0)$. For instance, in case $h = 3l$, the elements $(0, -h, j - 4h)$ become $(0, -3l, j - 12l)$ which reduce to $(0, 0, j - 15l)$. In this way we can see that

$$\rho(H_1) = \{(i, 0, j) \mid j \leq 14\} \cup \{(i, 1, j) \mid j \leq 3\} \cup \{(i, 2, j) \mid j \leq 7\}$$

and, similarly,

$$\rho(H_2) = \{(i, 0, j) \mid j \geq 15\} \cup \{(i, 1, j) \mid j \geq 4\} \cup \{(i, 2, j) \mid j \geq 8\}$$

where $0 \leq i \leq 1$, and it is clear that $\rho(H_1) \cup \rho(H_2) = B$, since $B = \{(i, j, h) \mid 0 \leq i \leq 1,\ 0 \leq j \leq 2,\ h \in \mathbb{Z}\}$.
Repeating these constructions for all the order ideals obtained in example 3.14, it is possible to verify that all these order ideals are indeed max-compatible.

Proposition 3.15. *If $\mathcal{O}$ is a max-compatible order ideal and $z \in \mathbb{Z}^n$, then it is possible to compute an element $b \in \mathcal{O}$ such that $b \equiv_M z$.*

Proof. According to *Step 7* of the previous algorithm, the order ideal $\mathcal{O}$ is a union of sets of the form R_u, where each R_u is a finite union of hyper-rectangles (proposition 3.10), hence it is enough to solve the following problem: given a hyper-rectangle R and an element $z \in \mathbb{Z}^n$, check if there exists $b \in R$ such that $b \equiv_M z$. It is easy to see that this problem can be converted into the problem of solving a set of linear Diophantine equations. □

4. Border bases

If I_M is a lattice ideal given by the lattice $M \subseteq \mathbb{Z}^n$ and if $\mathcal{O}$ is a max-compatible $(\mathrm{mod}\, M)$ order ideal of $\mathbb{N}^n$, then the set $\mathcal{E}(\mathcal{O})$ is an order ideal of $K[x_1, \ldots, x_n]$ and is a basis of $K[x_1, \ldots, x_n]/I_M$ as a K-vector space and we can define the $\mathcal{E}(\mathcal{O})$-border basis of I_M, which is given by:

$$G_{\mathcal{O}} = \{\mathcal{E}(b) - \mathcal{E}(\bar{b}) \mid \text{for } b \in \partial(\mathcal{O})\} \tag{4}$$

where $\bar{b}$ is the representative of b in $\mathcal{O}$. Since a border basis is constructed starting from a max-compatible order ideal, as a consequence of proposition 3.11 we have:

Proposition 4.1. *The number of border bases of any lattice ideal I_M (where* $\operatorname{rank} M \leq n$*) is finite.*

Notice that the situation here parallels that of Gröbner bases, where the number of all possible reduced Gröbner bases of an ideal is finite (see [17]) and can be read off from the construction of the Gröbner fan (see also [5, 23]).

Regarding the computation of (all) the border bases of a lattice ideal I_M, we have that if $\operatorname{rank} M = n$, then any max-compatible order ideal is finite, hence (4) gives the border basis $G_{\mathcal{O}}$ as a finite set. If $\operatorname{rank} M < n$, then the border of a max-compatible order ideal $\mathcal{O}$ is infinite. As a consequence of section 3, $\mathcal{O}$ is a finite union of hyper-rectangles contained in $\mathbb{N}^n$, hence its border is contained in the borders of hyper-rectangles.

Assume $R = \{(a_1, \ldots, a_n) \mid l_i \leq a_i < L_i\}$ is one of the hyper-rectangles of the decomposition of $\mathcal{O}$ (moreover, it is not restrictive to assume $l_i = 0$, since $\mathcal{O}$ is an order ideal). For each j such that $L_j \neq +\infty$ we consider the elements:

$$\{(a_1, \ldots, L_j, \ldots, a_n) \mid l_i \leq a_i < L_i,\ i \neq j\}.$$

These hyper-rectangles give the border of R. In this way we eventually describe the border of $\mathcal{O}$ (as a finite union of hyper-rectangles). To obtain the representatives of the elements of $\partial\mathcal{O}$ in $\mathcal{O}$ (and hence to obtain the $\mathcal{O}$-border basis), it is enough to compute $\sigma(\rho(b))$ for each $b \in \partial\mathcal{O}$ (or to use proposition 3.15). Although $\partial\mathcal{O}$ is infinite, its description in terms of finite hyper-rectangles allows to describe the $\mathcal{O}$-border basis in finite terms. Here we explain this construction with an example.

Example 4.2. Let us take again the module M considered in example 3.14 and in particular the order ideal $\mathcal{O} \subseteq \mathbb{N}^n$ defined in there. The corresponding lattice ideal I_M is $(y - x^4z^{11}, x^6z^{15} - 1)$. The border of $\mathcal{O}$ is the union of the following sets: $B_1 = \{(6+p, 0, 15) \mid p \geq 0\}$, $B_2 = \{(6, 0, 16+p) \mid p \geq 0\}$, $B_3 = \{(p, 1, 15+q) \mid 0 \leq p \leq 5,\ q \geq 0\}$ and $B_4 = \{(p, 1, q) \mid p \geq 0,\ 0 \leq q \leq 14\}$. Finally, each element of $\partial\mathcal{O}$ has a representative in $\mathcal{O}$ according to the following scheme:

$b \in \partial\mathcal{O}$	$\bar{b} \in \mathcal{O}$	$b \in \partial\mathcal{O}$	$\bar{b} \in \mathcal{O}$
$(6+i, 0, 15)$	$(i, 0, 0)$	$(6, 0, 16+i)$	$(0, 0, 1+i)$
$(j, 1, 15+i)$	$(4+j, 0, 26+i)$	$(2+h, 1, 15+i)$	$(h, 0, 11+i)$
$(i, 1, h)$	$(4+i, 0, 11+h)$	$(j, 1, 4+k)$	$(4+j, 0, 15+j)$
$(2+i, 1, 4+k)$	$(i, 0, k)$		

where $i \geq 0$, $0 \leq j \leq 1$, $0 \leq h \leq 3$ and $0 \leq k \leq 10$.

Note, however, that if I_M is a lattice ideal and if $\mathcal{E}(\mathcal{O})$ is an order ideal (assume $\mathcal{O}$ max-compatible), to express any $f \in K[x_1, \ldots, x_n]$ as a linear combination of monomials in $\mathcal{E}(\mathcal{O})$, it is not necessary to have the $\mathcal{E}(\mathcal{O})$-border basis and then use a reduction as is usually done in the general case (see [13], proposition 6.4.11). As a consequence of proposition 3.15, any monomial of f can directly be reduced to a monomial of $\mathcal{E}(\mathcal{O})$, avoiding the construction of the border basis.

It is well known that any Gröbner basis of an ideal I gives an order ideal in $K[x_1, \ldots, x_n]$ which is a K-basis of $K[x_1, \ldots, x_n]/I$ and, consequently, a border basis of I (for instance, the order ideal considered in example 4.2 comes from the Gröbner basis of I_M computed w.r.t. the lex term-order in which $x < z < y$). It is also well known that in general there exist border bases which do not come from any Gröbner bases, hence it is interesting to see what can be said regarding the border bases and the Gröbner bases of a lattice ideal I_M. Although in many examples border bases are indeed Gröbner bases (see next section), there are several cases of border bases of lattice ideals which cannot come from Gröbner bases. Here we give an example.

Let us consider the module $M \subseteq \mathbb{Z}^3$ generated by the vectors $(1, -2, -1)$, $(1, -1, 2)$ and $(-2, -1, 1)$. The corresponding HNF for M is:

$$M_H = \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{pmatrix}.$$

The set $\mathcal{V}$ is given by $\mathbb{N}^3 \setminus \cup_{a \in \mathcal{A}_1} C(a)$, where $\mathcal{A}_1 = \{(0,0,14), (1,0,5), (0,14,0), (0,4,2), (14,0,0), (2,1,1), (0,5,1), (1,2,1), (0,1,3), (1,3,0), (4,2,0), (3,0,1), (1,1,2), (2,0,4), (5,1,0)\}$, hence can be described by:

$$\mathcal{V} = \bigcup_{i=1}^{10} D(P_i)$$

where $P_1 = (13,0,0)$, $P_2 = (0,13,0)$, $P_3 = (0,0,13)$, $P_4 = (4,1,0)$, $P_5 = (3,2,0)$, $P_6 = (2,0,3)$, $P_7 = (1,0,4)$, $P_8 = (0,3,2)$, $P_9 = (0,4,1)$, $P_{10} = (1,1,1)$.

The binomial ideal I_M associated with M is:

$$I_M = \left(xz^2 - y, y^2z - x, y^3 - x^2z, xy^2 - z^3, x^2y - z, x^3 - yz^2, z^4 - x^2, yz^3 - 1\right).$$

The computation of the Gröbner fan of I_M (one possibility is to use the implementation presented in Sage, see [22]) shows that I_M has 33 reduced Gröbner bases, while the computation of all the possible max-compatible order ideals of I_M obtained with the techniques developed in section 3 gives 35 order ideals. Hence there must be two max-compatible order ideals (in $\mathbb{N}^3$) which do not come from Gröbner bases. They are the following:

$$\mathcal{O}_1 = D(1,2,0) \cup D(2,0,1) \cup D(0,1,2) \cup D(1,1,1)$$

and

$$\mathcal{O}_2 = D(3,0,0) \cup D(0,3,0) \cup D(0,0,3) \cup D(1,1,1).$$

They both have 14 elements (according either to the determinant of the matrix M_H or to the dimension of $K[x,y,z]/I_M$ as a K vector space). The border basis corresponding to $\mathcal{E}(\mathcal{O}_1)$ contains the following three binomials:

$$x^3 - yz^2, \quad y^3 - x^2z, \quad z^3 - xy^2$$

(where x^3, y^3 and z^3 are in the border of $\mathcal{E}(\mathcal{O}_1)$, while yz^2, x^2z and xy^2 are in $\mathcal{E}(\mathcal{O}_1)$). If $\mathcal{E}(\mathcal{O}_1)$ were an order ideal coming from a Gröbner basis corresponding to a term order $<_\sigma$, then we would have: $x^3 >_\sigma yz^2,\ y^3 >_\sigma x^2z,\ z^3 >_\sigma xy^2$ and these conditions are not compatible.
A similar contradiction can be found with the order ideal $\mathcal{O}_2$.
It is possible to verify that the matrix M_H of this example is minimal, in the sense that any other matrix of the form

$$\begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & c \end{pmatrix}$$

where $a \leq 5$, $b \leq 3$ and $c \leq 14$, $(a,b,c) \neq (5,3,14)$, corresponds to an ideal in which all border bases come from Gröbner bases.

5. The case $m = n = 2$

The case of a two dimensional module M in $\mathbb{Z}^2$ is particularly simple. In this section we show the main points. We consider the following two matrices obtained from the generators of M:

$$\begin{pmatrix} a_1 & a_2 \\ 0 & a_3 \end{pmatrix}, \qquad \begin{pmatrix} b_1 & b_2 \\ b_3 & 0 \end{pmatrix}. \tag{5}$$

The first is the usual matrix in HNF (hence $a_1 > 0$, $0 \leq a_2 < a_3$), the second is in HNF with respect to the second and first column, hence we have $b_2 > 0$, $0 \leq b_1 < b_3$; moreover, the relations between the a's and b's are: $b_2 = \gcd(a_2, a_3)$, $b_3 = a_1a_3/b_2$ and $b_1 = a_1 \cdot \min\{\lambda \in \mathbb{N} \mid \exists \mu \in \mathbb{Z} : b_2 = \lambda a_2 + \mu a_3\}$.

Let $\mathcal{B}_1$ be the set of minimal elements of $M \cap \mathbb{N}^2 \setminus \{0\}$ w.r.t. $\preceq$ and $\mathcal{B}_2$ be the set of minimal elements of $\{(p,q) \in \mathbb{N}^2 \mid (p,-q) \in M \setminus \{0\}\}$ again w.r.t. $\preceq$. Both $\mathcal{B}_1$ and $\mathcal{B}_2$ contain the elements $(b_3, 0)$ and $(0, a_3)$. The set $\mathcal{A}_1$ of proposition 3.1 is contained in $\mathcal{B}_1 \cup \mathcal{B}_2$ hence (by proposition 3.1), we have:

$$\mathcal{V} = \mathbb{N}^2 \setminus \left(\bigcup_{P_1 \in \mathcal{B}_1} C(P_1) \cup \bigcup_{P_2 \in \mathcal{B}_2} C(P_2) \right).$$

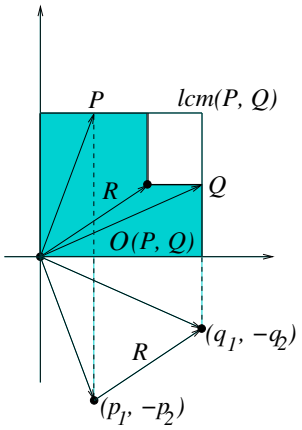

Fig. 3. The construction of the order ideal $\mathcal{O}(P,Q)$ of proposition 5.1.

We call $P, Q \in \mathcal{B}_2$ *consecutive* if

$$\mathcal{B}_2 \cap D(\mathrm{lcm}(P,Q)) = \{P,\ Q\}.$$

Proposition 5.1. *Let $P = (p_1, p_2)$ and $Q = (q_1, q_2)$ be two consecutive elements of $\mathcal{B}_2$, let $R \in M$ be such that $R = (p_1 - q_1, q_2 - p_2)$ (if $p_1 \geq q_1$ and $q_2 \geq p_2$) or $R = (q_1 - p_1, p_2 - q_2)$ (if $q_1 \geq p_1$ and $p_2 \geq q_2$) and let $\mathcal{O}(P,Q) = D(\mathrm{lcm}(P,Q)) \setminus C(R)$. Then we have:*

(1) $\mathcal{O}(P,Q)$ is a compatible order ideal;
(2) $\mathcal{O}(P,Q)$ has $a_1 a_3$ elements (hence is max-compatible);
(3) If $\mathcal{O}$ is any compatible order ideal, then there exist $P, Q \in \mathcal{B}_2$ consecutive, such that $\mathcal{O} \subseteq \mathcal{O}(P,Q)$.

Proof. Clearly $\mathcal{O}(P,Q)$ is an order ideal. If $A, B \in \mathcal{O}(P,Q)$ are equivalent, we can assume that $A = (\alpha, 0)$ and $B = (0, \beta)$. The vector $(\alpha, -\beta)$ is an element of M and it is easy to see that (α, β) is minimal w.r.t. $\preceq$, so $(\alpha, \beta) \in \mathcal{B}_2$, in contradiction with the consecutivity of P and Q. Again, since P and Q are consecutive, in the parallelogram whose vertexes are O, $(p_1, -p_2)$, $(q_1, -q_2)$ and $(p_1, -p_2) + (q_1, -q_2)$ there are no other points of M, hence M is generated by $(p_1, -p_2)$, $(q_1, -q_2)$ and therefore the determinant of the matrix:

$$\begin{pmatrix} p_1 & -p_2 \\ q_1 & -q_2 \end{pmatrix}$$

(which is $p_2 q_1 - p_1 q_2$) must be equal (in absolute value) to $a_1 a_3$. A direct computation of the number of elements with integer coordinates contained in $\mathcal{O}(P,Q)$ gives $|p_2 q_1 - p_1 q_2|$, hence $\mathcal{O}(P,Q)$ has $a_1 a_3$ elements. In particular $\mathcal{O}(P,Q)$ is maximum. Finally, take any compatible order ideal $\mathcal{O}$. Let $P = (p_1, p_2), Q = (q_1, q_2) \in \mathcal{B}_2$ be consecutive, such that $(p_1, 0) \in \mathcal{O}$ but $(q_1, 0) \notin \mathcal{O}$ (P and Q can always be found, since $\mathcal{B}_2$ contains $(b_3, 0)$ and $(0, a_3)$). Then clearly $\mathcal{O} \subseteq \mathcal{O}(P,Q)$. □

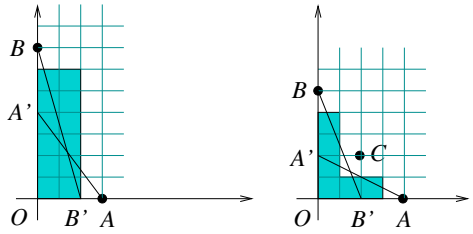

Fig. 4. The possible shapes of an order ideal for two dimensional lattices.

As a consequence of the above proposition, we see that all the possible maximum compatible order ideals of M are of the form $\mathcal{O}(P,Q)$, where $P,Q \in \mathcal{B}_2$ are consecutive. It is easy to verify that $(a_1, a_3 - a_2)$ and $(0, a_3)$ are elements of $\mathcal{B}_2$ and are consecutive, analogously $(b_3 - b_1, b_2)$ and $(b_3, 0)$ are also in $\mathcal{B}_2$ and are consecutive. From them we get the following two maximum, compatible order ideals: $\{(\alpha,\beta) \mid 0 \leq \alpha < a_1,\ 0 \leq \beta < a_3\}$ and $\{(\alpha,\beta) \mid 0 \leq \alpha < b_3,\ 0 \leq \beta < b_2\}$ (which are two rectangles). If now $P,Q \in \mathcal{B}_2$ are any consecutive elements (different from the two couples considered above), neither P nor Q lies on one of the two coordinate axes, hence the order ideal $\mathcal{O}(P,Q)$ is given by the difference of two rectangles, as in figure 3. In conclusion we have:

Proposition 5.2. *Let $M \subseteq \mathbb{Z}^2$ be a two dimensional sub-module generated by the rows of the first and hence also the second matrix in (5). Then the maximal compatible order ideals of M are:*

(1) $\{(\alpha,\beta) \mid 0 \leq \alpha < a_1,\ 0 \leq \beta < a_3\}$*;*

(2) $\{(\alpha,\beta) \mid 0 \leq \alpha < b_3,\ 0 \leq \beta < b_2\}$ *(where $b_2 = \gcd(a_2, a_3)$ and $b_3 = a_1 a_3 / b_2$);*

(3) a difference of two rectangles: $D(\mathrm{lcm}(P,Q)) \setminus C(R)$, where $P,Q \in \mathcal{B}_2$ are consecutive, with no zero coordinates and R is as defined in proposition 5.1.

If $\mathcal{O}$ is one of the order ideals described by the above proposition, then the *corners* of $\mathcal{E}(\mathcal{O})$ (as defined in [13], page 428) are either two elements (in case (1) and (2)) or three (in case (3)), as shown by the points A and B and A, B and C in figure 4. If A, B are corners, let A' and B' denote their representatives in the order ideal (the representative of C is necessarily O, since C is an element of M). It is clear that A' and B' have one coordinate 0 (if not, we could construct two equivalent elements in the order ideal) and (recalling the characterizations of term-orders given in [21]), any line through O which has a slope between the slope of the line BB' and the slope of AA' gives rise to a term order $<_\sigma$ in $\mathbb{N}^2$ (and hence in $K[x,y]$) in which $A' <_\sigma A$, $B' <_\sigma B$ (and $0 <_\sigma C$). As a consequence of the above considerations and of [13], proposition 6.4.18, we have:

Proposition 5.3. *Let $M \subseteq \mathbb{Z}^2$ be as above and let I_M be the corresponding lattice ideal. Any maximal compatible order ideal w.r.t. M corresponds to the lattice ideal constructed from a Gröbner bases of I_M (and conversely). I_M has two reduced Gröbner bases of two elements which are $\{x^{a_1} - 1,\ x^{a_3} - 1\}$ and $\{x^{b_3} - 1,\ x^{b_2} - 1\}$ and all the other reduced Gröbner bases have three elements of the form $x^\alpha - y^{\alpha'}$, $y^\beta - y^{\beta'}$, $x^{\gamma_1} y^{\gamma_2} - 1$, where $A = (\alpha, 0)$, $A' = (0, \alpha')$, $B = (0,\beta)$, $B' = (\beta', 0)$, $C = (\gamma_1, \gamma_2)$, A, B, C are corners of an order ideal, A', B' are the representative of A and B in the order ideal.*

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