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# On graphs double-critical with respect to the colouring number

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The *colouring number*  $\text{col}(G)$  of a graph  $G$  is the smallest integer  $k$  for which there is an ordering of the vertices of  $G$  such that when removing the vertices of  $G$  in the specified order no vertex of degree more than  $k - 1$  in the remaining graph is removed at any step. An edge  $e$  of a graph  $G$  is said to be *double-col-critical* if the colouring number of  $G - V(e)$  is at most the colouring number of  $G$  minus 2. A connected graph  $G$  is said to be *double-col-critical* if each edge of  $G$  is double-col-critical. We characterise the double-col-critical graphs with colouring number at most 5. In addition, we prove that every 4-col-critical non-complete graph has at most half of its edges being double-col-critical, and that the extremal graphs are precisely the odd wheels on at least six vertices. We observe that for any integer  $k$  greater than 4 and any positive number  $\epsilon$ , there is a  $k$ -col-critical graph with the ratio of double-col-critical edges between  $1 - \epsilon$  and 1.

**Keywords:** graph colouring, graph characterizations, degenerate graphs, colouring number, double-critical graphs

## 1 Introduction

All graphs considered in this paper are assumed to be simple and finite.<sup>(i)</sup> The cycle on  $n$  vertices is denoted by  $C_n$ . The complete graph  $K_n$  on  $n$  vertices is referred to as an *n-clique*. Let  $G$  denote a graph. The number of vertices in a largest clique contained in  $G$  is denoted by  $\omega(G)$ . The vertex-connectivity of  $G$  is denoted by  $\kappa(G)$ . The number of vertices and edges in  $G$  is denoted by  $n(G)$  and  $m(G)$ , respectively. Given a vertex  $v$  in  $G$ ,  $N(v, G)$  denotes the set of vertices in  $G$  adjacent to  $v$ ;  $\deg(v, G)$  denotes the cardinality of  $N(v, G)$ , and it is referred to as the *degree* of  $v$  (in  $G$ ). A vertex of degree 1 is referred to as a *leaf*. The minimum degree and maximum degree of  $G$  is denoted  $\delta(G)$  and  $\Delta(G)$ , respectively. Given a subset  $S$  of the vertices of  $G$ , the subgraph of  $G$  induced by the vertices of  $S$  is denoted by  $G[S]$ , and we let  $N(S, G)$  denote the set  $\cup_{s \in S} N(s, G) \setminus S$ . The *square* of a graph  $G$ , denoted by  $G^2$ , is the graph obtained from  $G$  by adding edges between any pair of vertices of  $G$  which are at distance

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<sup>(i)</sup> The reader is referred to [1] for definitions of graph-theoretic concepts used but not explicitly defined in this paper.

2 in  $G$ . Given two graphs  $H$  and  $G$ , the *complete join* of  $G$  and  $H$ , denoted by  $G + H$ , is the graph obtained from two disjoint copies of  $H$  and  $G$  by joining each vertex of the copy of  $G$  to each vertex of the copy of  $H$ . The chromatic number of  $G$  is denoted by  $\chi(G)$ , while the list-chromatic number of  $G$  is denoted by  $\chi_\ell(G)$ . Let  $\psi$  denote some graph parameter. An edge  $e$  of  $G$  is said to be *double- $\psi$ -critical* if  $\psi(G - V(e)) \leq \psi(G) - 2$ . A connected graph  $G$  is said to be *double- $\psi$ -critical* if each edge of  $G$  is double- $\psi$ -critical. For brevity, we may also refer to double- $\chi$ -critical edges and graphs as, simply, *double-critical edges* and graphs, respectively.

The introduction of the concept of double- $\psi$ -critical graphs in [12] was inspired by a special case of the Erdős-Lovász Tihany Conjecture [2], namely the special case which states that the complete graphs are the only double-critical graphs. We refer to this special case of the Erdős-Lovász Tihany Conjecture as the *Double-Critical Graph Conjecture*. The Double-Critical Graph Conjecture is settled in the affirmative for the class of graphs with chromatic number at most 5, but remains unsettled for the class of graphs with chromatic number  $k$ , for each value of  $k \geq 6$ . [4, 11, 16, 17]. Using the computer programs SAGE [15] and geng [10], we verified the Double-Critical Graph Conjecture for all graphs on at most 12 vertices (see [13]).

In [12], it was proved that if  $G$  is a double- $\chi_\ell$ -critical graph with  $\chi_\ell(G) \leq 4$ , then  $G$  is complete. It is an open problem whether there is a non-complete double- $\chi_\ell$ -critical graph with list-chromatic number at least 5.

The double- $\kappa$ -critical graphs, which in the literature are referred to as *contraction-critical* graphs (since the vertex-connectivity drops by one after contraction of any edge), are well-understood in the case where  $\kappa$  is 4. Some structural results have been obtained for contraction-critical graphs with vertex-connectivity 5. (See [6, Sec. 4] for references on contraction-critical graphs.)

Bjarne Toft<sup>(ii)</sup> posed the problem of characterising the double-col-critical graphs. Here col denotes the colouring number which is defined in the paragraph below.

In this paper, we characterise the double-col-critical graphs with colouring number at most 5.

In the remaining part of this section, we define the colouring number and present some fundamental properties of this graph parameter.

**The colouring number of a graph.** Suppose that we are given a non-empty graph  $G$  and an ordering  $v_1, \dots, v_n$  of the vertices of  $G$ . Now we may colour the vertices of  $G$  in the order  $v_1, \dots, v_n$  such that in the  $i$ th step the vertex  $v_i$  is assigned the smallest possible positive integer which is not assigned to any neighbour of  $v_i$  among  $v_1, \dots, v_{i-1}$ . This produces a colouring of  $G$  using at most

$$\max_{i \in \{1, \dots, n\}} \deg(v_i, G[v_1, \dots, v_i]) + 1$$

colours. Taking the minimum over the set  $S_n$  of all permutations of  $\{1, \dots, n\}$ , we find that the chromatic number of  $G$  is at most

$$\min_{\pi \in S_n} \left\{ \max_{i \in \{1, \dots, n\}} \deg(v_{\pi(i)}, G[v_{\pi(1)}, \dots, v_{\pi(i)}]) \right\} + 1 \quad (1)$$

The number in (1) is called the *colouring number* of  $G$ , and it is denoted by  $\text{col}(G)$ . The colouring number of the empty graph  $K_0$  is defined to be zero. By (1),  $\text{col}(G) \leq \Delta(G) + 1$  for any graph  $G$ .

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<sup>(ii)</sup> Private communication to the second author from Bjarne Toft, Odense, August 2008.

The colouring number was introduced by Erdős and Hajnal [3], but equivalent concepts were introduced independently by several other authors. It can be shown (see, for instance, [19]) that the colouring number of any non-empty graph  $G$  is equal to

$$\max\{\delta(H) \mid H \text{ is an induced subgraph of } G\} + 1 \quad (2)$$

and that the colouring number can be computed in polynomial time [9]. The non-empty graphs with colouring number at most  $k + 1$  are also said to be  $k$ -degenerate [8]. Thus, a non-empty graph  $G$  is  $k$ -degenerate if and only if there is an ordering of the vertices of  $G$  such that when removing the vertices of  $G$  in the specified order no vertex of degree more than  $k$  in the remaining graph is removed at any step.

The colouring number is monotone on subgraphs, that is, if  $F$  is a subgraph of a graph  $G$  then  $\text{col}(F) \leq \text{col}(G)$ . For ease of reference, we state the following elementary facts concerning the colouring number of graphs.

**Observation 1** For any graph  $G$ ,

- (i)  $\text{col}(G) = 0$  if and only if  $G$  is the empty graph,
- (ii)  $\text{col}(G) = 1$  if and only if  $G$  contains at least one vertex but no edges,
- (iii)  $\text{col}(G) = 2$  if and only if  $G$  is forest containing at least one edge, and
- (iv)  $\text{col}(G) \geq 3$  if and only if  $G$  contains at least one cycle.

A graph  $G$  is said to be  $k$ -col-critical, or, simply, col-critical, if  $\text{col}(G) = k$  and  $\text{col}(F) < k$  for every proper subgraph  $F$  of  $G$ . Similarly, a graph  $G$  is said to be  $k$ -col-vertex-critical, or, simply, col-vertex-critical, if  $\text{col}(G) = k$  and  $\text{col}(F) < k$  for every induced proper subgraph  $F$  of  $G$ . It is easy to see that every connected  $r$ -regular graph is  $(r + 1)$ -col-critical.

**Observation 2** For any col-vertex-critical graph  $G$ ,

- (i)  $\text{col}(G) = 0$  if and only if  $G \simeq K_0$ ,
- (ii)  $\text{col}(G) = 1$  if and only if  $G \simeq K_1$ ,
- (iii)  $\text{col}(G) = 2$  if and only if  $G \simeq K_2$ , and
- (iv)  $\text{col}(G) = 3$  if and only if  $G$  is a cycle.

**Observation 3** For any graph  $G$  and any element  $x \in E(G) \cup V(G)$ , if  $\text{col}(G - x) < \text{col}(G)$  then  $\text{col}(G - x) = \text{col}(G) - 1$ .

**Observation 4** A graph  $G$  is col-vertex-critical if and only if  $\text{col}(G - v) < \text{col}(G)$  for every vertex  $v$  in  $G$ .

**Observation 5** Given any graph  $G$ , there is a col-critical subgraph  $F$  of  $G$  with  $\text{col}(G) = \text{col}(F) = \delta(F) + 1$ . In particular, if  $G$  is col-critical then  $\text{col}(G) = \delta(G) + 1$ .

**Proof:** Recall that  $\text{col}(G) = \max\{\delta(H) \mid H \subseteq G\} + 1$ . Among the subgraphs  $H$  of  $G$  with  $\text{col}(G) = \delta(H) + 1$ , let  $F$  denote a minimal one, that is,  $\delta(F') < \delta(F)$  for every proper subgraph  $F'$  of  $F$ . (This minimum exists since  $G$  is finite.) Then  $F$  is col-critical with  $\text{col}(F) = \delta(F) + 1 = \text{col}(G)$ .  $\square$

**Observation 6** *Given any graph  $G$ , there is a col-vertex-critical induced subgraph  $F$  of  $G$  with  $\text{col}(G) = \text{col}(F) = \delta(F) + 1$ . In particular, if  $G$  is col-vertex-critical then  $\text{col}(G) = \delta(G) + 1$ .*

**Proof:** Let  $F$  denote a minimal induced subgraph of  $G$  with  $\text{col}(F) = \text{col}(G)$ . This implies  $\text{col}(F') < \text{col}(F)$  for any induced proper subgraph  $F'$  of  $F$ , in particular,  $F$  is a col-vertex-critical graph. Suppose  $\text{col}(F) > \delta(F) + 1$ . Then there is some proper induced subgraph  $F'$  of  $F$  with  $\delta(F') + 1 = \text{col}(F)$ , and so  $\text{col}(F') \geq \text{col}(F)$ , a contradiction. Hence  $\text{col}(F) = \delta(F) + 1$ . If  $G$  is col-vertex-critical, then  $F = G$ , and the desired result follows.  $\square$

The two following results may be of interest in their own right.

**Proposition 1 (Pedersen [12])** *For any two non-empty disjoint graphs  $G_1$  and  $G_2$ , the colouring number of the complete join  $G_1 + G_2$  is at most*

$$\min\{\text{col}(G_1) + n(G_2), \text{col}(G_2) + n(G_1)\} \quad (3)$$

and at least

$$\min\{\text{col}(G_1) + n(J_2), \text{col}(G_2) + n(J_1)\} \quad (4)$$

where, for each  $i \in \{1, 2\}$ ,  $J_i$  is any subgraph of  $G_i$  with minimum degree equal to  $\text{col}(G_i) - 1$ .

If, in addition,  $\text{col}(G_i) = \delta(G_i) + 1$  for each  $i \in \{1, 2\}$  (in particular, if both  $G_1$  and  $G_2$  are col-vertex-critical), then the colouring number of the complete join  $G_1 + G_2$  is equal to the minimum in (3).

A graph  $G$  is said to be *decomposable* if there is a partition of  $V(G)$  into two (non-empty) sets  $V_1$  and  $V_2$  such that, in  $G$ , every vertex of  $V_1$  is adjacent to every vertex of  $V_2$ . Given any graph  $G$ , we let  $V_\delta(G)$  denote the set of vertices of  $G$  of minimum degree in  $G$ . Clearly,  $V_\delta(G)$  is non-empty for any non-empty graph.

**Proposition 2 (Pedersen [12])** *Let  $G$  denote a decomposable graph. Then  $G$  is col-critical if and only if the vertex set of  $G$  can be partitioned into two sets  $V_1$  and  $V_2$  such that  $G = G_1 + G_2$ , where  $G_i := G[V_i]$  for  $i \in \{1, 2\}$ ,  $G_1$  is regular, and*

- (i)  $V(G_2) \setminus V_\delta(G_2)$  is an independent set of  $G_2$ , and

$$\delta(G_1) + n(G_2) = \delta(G_2) + n(G_1)$$

or

- (ii)  $G_2$  is an edgeless graph, and

$$n(G_1) - \delta(G_1) - n(Q) < n(G_2) < n(G_1) - \delta(G_1)$$

where  $Q$  denotes a smallest component of  $G_1$  (in terms of the number of vertices).

Moreover,  $\text{col}(G) = \delta(G_1) + n(G_2) + 1$  in both (i) and (ii).

## 2 Double-col-critical graphs

The analogue of the Double-Critical Graph Conjecture with  $\chi$  replaced by  $\text{col}$  does not hold. For instance, the non-complete graph  $C_6^2$  is 4-regular, 5-col-critical, and double-col-critical. Since  $C_6^2$  is planar, it also follows that it is not even true that every double-col-critical graph with colouring number 5 contains a  $K_5$  minor. (In [5], it was proved that every double-critical graph  $G$  with  $\chi(G) \leq 7$  at least contains a  $K_{\chi(G)}$  minor.) It is easy to see that the square of any cycle of length at least 5 is a double-col-critical graph with colouring number 5.

**Observation 7** *Any double-col-critical graph is col-vertex-critical.*

**Proof:** Let  $G$  denote a double-col-critical graph. If there are no vertices in  $G$ , then we are done. Let  $v$  denote an arbitrary but fixed vertex of  $G$ . If there is no vertex in  $G$  adjacent to  $v$ , then we are done, since then, by the connectedness of  $G$ ,  $G$  is just the singleton  $K_1$ . Let  $u$  denote a neighbour of  $v$ . By Observation 4, we need to show  $\text{col}(G - v) < \text{col}(G)$ . The fact that  $G$  is double-col-critical implies  $\text{col}(G - u - v) \leq \text{col}(G) - 2$ . Suppose  $\text{col}(G - v) \geq \text{col}(G)$ . Then

$$\text{col}((G - v) - u) \leq \text{col}(G) - 2 = \text{col}(G - v) - 2$$

which contradicts Observation 3. This shows  $\text{col}(G - v)$  is strictly less than  $\text{col}(G)$ , as desired.  $\square$

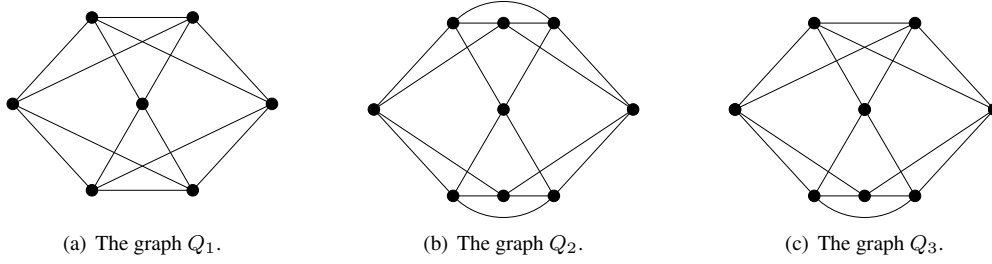
**Observation 8** *For each integer  $k \in \{0, 1, 2, 3, 4\}$ , the only double-col-critical graph with colouring number  $k$  is the  $k$ -clique.*

**Proof:** Let  $G$  denote a double-col-critical graph, and define  $k := \text{col}(G)$ . Then, by Observation 7,  $G$  is also col-vertex-critical, and so, by Observation 6,  $\delta(G) = k - 1$ . If  $k \leq 3$ , then the desired result follows immediately from Observation 2. Suppose  $k = 4$ . Then, for any edge  $e \in E(G)$ ,  $\text{col}(G - V(e)) \leq 2$  and so, by Observation 1,  $G - V(e)$  is a forest. Fix an edge  $xy \in E(G)$ . If  $G - x - y$  contains no edges, then  $G$  is 2-degenerate and so  $\text{col}(G) \leq 3$ , a contradiction. Let  $T$  denote a component of  $G - x - y$  with at least one edge, and let  $u$  and  $v$  denote two leafs of  $T$ . Since, as noted above,  $\delta(G) = 3$ , it follows that both  $u$  and  $v$  are adjacent to both  $x$  and  $y$ . If  $u$  and  $v$  are adjacent in  $T$ , then  $G[\{u, v, x, y\}] \simeq K_4$ , and so, since  $G$  is also col-vertex-critical,  $G \simeq K_4$ . Hence we may assume that  $u$  has a neighbour  $t$  in  $T - v$ . Now  $G[\{x, y, v\}]$  is a 3-clique in  $G - t - u$  and so  $\text{col}(G - t - u) \geq \text{col}(G[\{x, y, v\}]) = 3$ , a contradiction. This completes the proof.  $\square$

It is easy to verify that the graphs  $Q_1$ ,  $Q_2$ , and  $Q_3$  in Figure 1 are double-col-critical and have colouring number 5. None of the graphs  $Q_1$ ,  $Q_2$ , and  $Q_3$  are squares of a cycle. We shall see that  $Q_1$ ,  $Q_2$ ,  $Q_3$ , and the squares of the cycles of length at least 5 are all the double-col-critical graphs with colouring number 5. First a few preliminary observations.

**Observation 9** *If  $G$  is a double-col-critical graph, then  $\delta(G) = \text{col}(G) - 1$  and every pair of adjacent vertices of  $G$  has a common neighbour of degree  $\delta(G)$  in  $G$ .*

**Proof:** Let  $G$  denote a double-col-critical graph. Then, by Observation 7,  $G$  is also col-vertex-critical, and so, by Observation 6,  $\delta(G) = \text{col}(G) - 1$ . Let  $xy$  denote an arbitrary edge of  $G$ . Now, by the definition of the colouring number,  $G - x - y$  has minimum degree at most  $(\text{col}(G) - 2) - 1$  which is equal to  $\delta(G) - 2$ . This means that some vertex of  $V(G) \setminus \{x, y\}$ , say  $z$ , which has degree at least  $\delta(G)$  in  $G$  has



**Fig. 1:** The graphs  $Q_1$ ,  $Q_2$ , and  $Q_3$ , depicted above, are the only double-col-critical graphs with colouring number 5 which are not squares of cycles.

degree at most  $\delta(G) - 2$  in  $G - x - y$ . The only way this can happen is if  $z$  has degree  $\delta(G)$  in  $G$  and is adjacent to both  $x$  and  $y$  in  $G$ . This completes the argument.  $\square$

**Observation 10** *If  $G$  is a non-complete double-col-critical graph, then  $G$  does not contain a clique of order  $\text{col}(G) - 1$ .*

**Proof:** Let  $G$  denote a non-complete double-col-critical graph. By Observation 7,  $G$  is col-vertex-critical, and so, since  $G$  is also non-complete,  $G$  cannot contain a clique of order more than  $\text{col}(G) - 1$ . Also, by Observation 7,  $\delta(G) = \text{col}(G) - 1$ . Suppose that  $G$  contains a clique  $K$  of order  $\text{col}(G) - 1$ . Clearly,  $G - V(K)$  is not empty. If  $G - V(K)$  contains an edge  $xy$ , then  $\text{col}(G) - 1 = \text{col}(K) \leq \text{col}(G - x - y) \leq \text{col}(G) - 2$ , a contradiction. Hence  $G - V(K)$  is edgeless, and so, since  $\delta(G) = \text{col}(G) - 1 = n(K)$ , it follows that each vertex of  $V(G) \setminus V(K)$  is adjacent to every vertex of  $V(K)$ , in particular,  $G$  contains a clique of order  $\text{col}(G)$ , a contradiction.  $\square$

**Proposition 3** *Every double-col-critical graph with colouring number at least 3 is 2-connected.*

**Proof:** Let  $G$  denote a double-col-critical graph with  $\text{col}(G) \geq 3$ . Since  $G$  is col-vertex-critical, it is connected with  $\text{col}(G) = \delta(G) + 1$ . Suppose  $G$  is not 2-connected, and let  $x$  denote a cutvertex of  $G$ . Each component of  $G - x$  has minimum degree at least  $\delta(G) - 1$ . Let  $C$  denote a component of  $G - x$ , and let  $e$  denote some edge of  $G - V(C)$ . Then, using the fact that  $G$  is double-col-critical, col is monotone, and  $C$  is a subgraph of  $G - V(e)$ , we obtain

$$\text{col}(G) - 1 = (\delta(G) + 1) - 1 \leq \text{col}(C) \leq \text{col}(G - V(e)) = \text{col}(G) - 2$$

a contradiction. This shows that  $G$  must be 2-connected.  $\square$

## 2.1 Double-col-critical graphs with colouring number 5.

We shall say that a  $k$ -neighbour of a vertex  $x$  is a neighbour of  $x$  of degree  $k$ .

**Observation 11** *If  $G$  is a double-col-critical graph with colouring number 5 and  $ab \in E(G)$ , then  $a$  or  $b$  has degree 4 in  $G$ .*

**Proof:** Let  $G$  denote a double-col-critical graph with colouring number 5. Then  $\delta(G) = 4$ . Let  $ab$  denote an edge of  $G$ . Suppose that both  $a$  and  $b$  have degree greater than 4 in  $G$ . By Observation 9, there is a common 4-neighbour  $c$  of  $a$  and  $b$ . We shall make repeated use of Observation 9 and Observation 10. The latter observation implies that  $G$  contains no 4-clique. There is a common 4-neighbour  $d$  of  $a$  and  $c$ . Since  $\omega(G) \leq 3$ ,  $d$  is not adjacent to  $b$ . This implies that there is a common 4-neighbour  $e$  of  $b$  and  $c$  and  $e$  is not identical to  $d$ . The vertex  $e$  is not adjacent to  $a$ . The vertex  $a$  has degree at least 5, and so the common 4-neighbour of  $c$  and  $d$  must be  $e$ . We note that  $\{a, b, c, d, e\}$  induces a subgraph of  $G$  of minimum degree 3. Hence  $G - \{a, b, c, d, e\}$  contains no edges. Moreover,  $\text{col}(G - \{c, d, e\}) \leq \text{col}(G) - 2$ , and so  $G - \{c, d, e\}$  contains a vertex  $f$  of degree at most 2. Since  $a$  and  $b$  both have degree at least 3 in  $G - \{c, d, e\}$  and  $N(c, G) = \{a, b, d, e\}$ , it follows that this vertex  $f$  must be in the set  $V(G) \setminus \{a, b, c, d, e\}$  and that  $f$  is adjacent both  $d$  and  $e$ . The vertex  $f$  has degree 4 in  $G$ . It also follows from the fact that  $G - \{a, b, c, d, e\}$  contains no edges that  $f$  must be adjacent to both  $a$  and  $b$ . Since  $a$  has degree at least 5 in  $G$ , it follows that  $a$  must be adjacent to some vertex  $g \in V(G) \setminus \{a, b, c, d, e, f\}$ . Then  $\text{col}(G - a - g) \leq \text{col}(G) - 2 = 3$ . On the other hand,  $\{b, c, d, e, f\}$  induce a subgraph of  $G - a - g$  of minimum degree 3, a contradiction. This contradiction implies that  $G$  contains no two adjacent vertices both of which have degree greater than 4.  $\square$  Recently, the first author obtained a characterisation of what he called *minimal critical graphs*

with minimum degree 4. It turns out that our double-col-critical graphs with colouring number 5 are such graphs, and so – using the characterisation of minimal critical graphs of minimum degree 4 – we obtain a characterisation of the double-col-critical graphs with colouring number 5.

In the following result, which is the main result of this paper, we let  $Q_1$ ,  $Q_2$ , and  $Q_3$  denote the graphs depicted in Figure 1.

**Theorem 1** *A graph is double-col-critical with colouring number 5 if and only if it is isomorphic to  $Q_1$ ,  $Q_2$ ,  $Q_3$ , or the square of a cycle of length at least 5.*

The graph  $Q_2$  is the dual of the Herschel graph which is the smallest nonhamiltonian polyhedral graph.

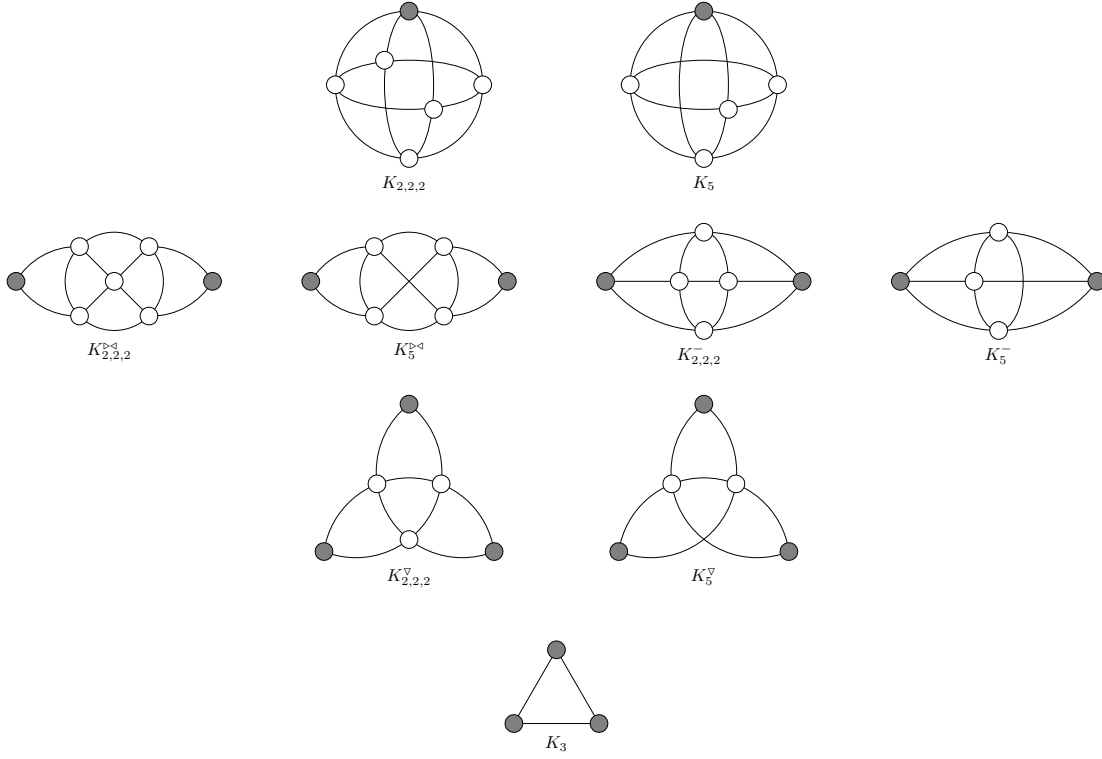
In order to prove Theorem 1, we first need to introduce a bit of notation and state the above-mentioned characterisation of minimal critical graphs with minimum degree 4.

For the remaining part of this section we shall be using the following notation. We shall let  $\mathcal{C}$  denote the set of simple connected graphs of minimum degree at least 4. An edge  $e$  of a graph  $G$  in  $\mathcal{C}$  is *essential* if the graph  $G - e$  obtained from  $G$  by deleting  $e$  is not in  $\mathcal{C}$ , and let us call  $e$  *critical* if the graph  $G/e$  obtained by contracting  $e$  and simplifying is not in  $\mathcal{C}$ . An edge  $e$  is essential if and only if  $e$  is a bridge or at least one of its endvertices has degree 4; and  $e$  is critical if and only if the endvertices of  $e$  have a common 4-neighbour or  $N(V(e), G)$  consists of three common neighbours of the endvertices of  $e$ . We are now interested in the *minimal critical* graphs in  $\mathcal{C}$ , that is, graphs  $G \in \mathcal{C}$  with the property that each edge of  $G$  is both essential and critical.

For the description of the minimal critical graphs in  $\mathcal{C}$ , we shall consider a number of *bricks*, that is, any graph isomorphic to one of the following nine graphs:  $K_5$ ,  $K_{2,2,2}$ ,  $K_5^-$ ,  $K_{2,2,2}^-$ ,  $K_5^\nabla$ ,  $K_{2,2,2}^\nabla$ ,  $K_5^{\triangleright\triangleleft}$ ,  $K_{2,2,2}^{\triangleright\triangleleft}$ , or  $K_3$  which are depicted in Figure 2. Each brick comes together with its *vertices of attachment*: For  $K_5$  and  $K_{2,2,2}$ , this is an arbitrary single vertex, for the other seven bricks these are its vertices of degree less than 4. The remaining vertices of the brick are its *internal vertices*, and the edges connecting two inner vertices are called its *internal edges*. Observe that every brick  $B$  has one, two, or three vertices of attachment, and that they are pairwise nonadjacent unless  $B$  is the triangle, that is,  $K_3$ .

It turns out that the minimal critical graphs from  $\mathcal{C}$  are either squares of cycles of length at least 5, or they are the edge disjoint union of bricks, following certain rules. This is made precise in the following





**Fig. 2:** The nine bricks. Vertices of attachment are displayed solid.

theorem.

**Theorem 2 (Kriesell [7])** *A graph is a minimal critical graph in  $\mathcal{C}$  if and only if it is the square of a cycle of length at least 5 or arises from a connected multihypergraph  $H$  of minimum degree at least 2 with at least one hyperedge and  $|V(e)| \in \{1, 2, 3\}$  for all hyperedges  $e$  by replacing each hyperedge  $e$  by a brick  $B_e$  (see Figure 2) such that the vertices of attachment of  $B_e$  are those in  $V(e)$  and at the same time the only objects of  $B_e$  contained in more than one brick, and*

(TB) *the brick  $B_e$  is triangular only if each vertex  $x \in V(e)$  is incident with precisely one hyperedge  $f_x$  different from  $e$  and the corresponding brick  $B_{f_x}$  is neither  $K_5$ ,  $K_5^{\bar{-}}$ ,  $K_{2,2,2}$ , nor  $K_{2,2,2}^{\bar{-}}$ , and, for any other vertex  $y \in V(e) \setminus \{x\}$  and hyperedge  $f_y$  containing  $y$  but distinct from  $e$ , we have*

- (i)  $V(f_x) \cap V(f_y) \neq \emptyset$  only if not both of  $B_{f_x}$  and  $B_{f_y}$  are triangular, and
- (ii)  $f_x = f_y$  only if  $B_{f_x}$  is  $K_5^{\triangleright\triangleleft}$  or  $K_{2,2,2}^{\triangleright\triangleleft}$ .

**Proof of Theorem 1.:** In order to prove the desired result, we prove the following equivalent statement.

A graph is double-col-critical with colouring number 5 if and only if it is the square of a cycle of length at least 5 or one of the three graphs obtained by taking the union of two graphs  $G_1$  and  $G_2$  such that  $G_i \simeq K_5^\nabla$  or  $G_i \simeq K_{2,2,2}^\nabla$  for  $i \in \{1, 2\}$  and  $x \in V(G_1) \cap V(G_2)$  if and only if  $x$  has degree 2 in  $G_1$  and degree 2 in  $G_2$ .

The ‘if’-part of the statement above is straightforward to verify and it is left to the reader.

Let  $G$  denote an arbitrary double-col-critical graph with colouring number 5. It follows from the definition of double-col-critical graphs, Observation 6, Observation 7, Observation 9, and Observation 11 that  $G$  has the following properties.

- (a)  $G$  has minimum degree 4;
- (b) if  $x$  and  $y$  are adjacent vertices, then at least one of them has degree 4;
- (c) if  $x$  and  $y$  are adjacent vertices, then they have a common 4-neighbour; and
- (d) if  $x$  and  $y$  are adjacent in  $G$ , then  $G - x - y$  has no induced subgraph of minimum degree at least 3.

By (a),  $G$  is in  $\mathcal{C}$ . By (b), every edge of  $G$  is essential, and, by (c), every edge of  $G$  is critical. Thus,  $G$  is minimal critical in  $\mathcal{C}$ , and so Theorem 2 applies. Suppose that  $G$  is not the square of a cycle. Then  $G$  has a representation by a multihypergraph  $H$  as described in Theorem 2.

If  $e$  is a 1-hyperedge then the unique attachment vertex of the corresponding brick  $B_e$  in  $G$  is a cutvertex of  $G$ , a contradiction to Proposition 3.

Suppose that there exists a 2-hyperedge  $e$  with  $V(e) = \{u, v\}$ . If  $B_e$  is  $K_5^{\triangleright\triangleleft}$  or  $K_{2,2,2}^{\triangleright\triangleleft}$  then  $B_e - V(e)$  is an induced subgraph of  $G$  of minimum degree 3, and since the vertex  $u \in V(e)$  has only two neighbours in that subgraph, there is an edge in  $G - (V(B_e) \setminus V(e))$ , contradicting (d). If, otherwise,  $B_e$  is  $K_5^-$  or  $K_{2,2,2}^-$  then, by Theorem 2,  $B_e$  is an induced subgraph of  $G$  of minimum degree 3. By (a), there is a vertex in  $V(G) \setminus V(B_e)$ , and it has a neighbour in  $V(G) \setminus V(B_e)$ . This contradicts (d). Hence there are only 3-hyperedges in  $H$ .

Suppose that  $H$  contains a 3-hyperedge  $e$  for which the corresponding brick  $B_e$  is triangular. It follows from (a) and (d) that some vertex  $q \in V(G) \setminus V(B_e)$  is adjacent to at least two vertices in  $V(B_e)$ . The vertex  $q$  is not adjacent to all three vertices of  $V(B_e)$ , since otherwise  $G[V(B_e) \cup \{q\}]$  would induce a 4-clique in  $G$  which contradicts Observation 10. Let  $x$  and  $y$  denote the neighbours of  $q$  in  $V(B_e)$ . By Theorem 2 (TB),  $x$  is incident to exactly one hyperedge  $f_x$  different from  $e$ . Similarly,  $y$  is incident to exactly one hyperedge  $f_y$  different from  $e$ . If  $f_x = f_y$ , then, by Theorem 2 (TB.ii),  $B_{f_x}$  is  $K_5^{\triangleright\triangleleft}$  or  $K_{2,2,2}^{\triangleright\triangleleft}$ , in particular,  $f_x$  is a 2-hyperedge, a contradiction. Hence  $f_x \neq f_y$  and so, since  $q \in V(f_x) \cap V(f_y)$ , it follows from Theorem 2 (TB.i) that not both  $B_{f_x}$  and  $B_{f_y}$  are triangular bricks. The fact that  $f_x$  and  $f_y$  are distinct and  $q \in V(f_x) \cap V(f_y)$  implies that  $q$  is an attachment vertex of both  $B_{f_x}$  and  $B_{f_y}$ . Since  $q$  is adjacent to  $x$  and both  $q$  and  $x$  are attachment vertices, it follows that  $B_{f_x}$  must be triangular. Similarly,  $B_{f_y}$  must be triangular, and so we have obtained a contradiction. This shows that each hyperedge in  $H$  is of the type  $K_5^\nabla$  or  $K_{2,2,2}^\nabla$ .

Let  $e$  denote an arbitrary 3-hyperedge of  $H$ . If there are two vertices  $x, y \in V(e)$  of degree exceeding 4 in  $G$  then  $G - (V(B_e) \setminus \{x, y\})$  has minimum degree at least 3, contradicting (d) applied to any internal edge of  $B_e$ . Therefore, if there is a vertex  $x \in V(e)$  of degree exceeding 4 in  $G$ , then the two vertices  $y, z \in V(e) \setminus \{x\}$  are incident with precisely one further 3-hyperedge  $f_y$  and  $f_z$ , respectively, both distinct

from  $e$ . If  $f_y \neq f_z$  then one may argue as above that  $G - V(B_e - x)$  has minimum degree at least 3, contradicting (d) applied to any internal edge of  $B_e$ . Hence  $f_y = f_z =: f$ . Let  $w$  be the vertex in  $V(f) \setminus \{y, z\}$ . If  $w \neq x$  then  $\{w, x\}$  forms a 2-separator, and otherwise  $w = x$  is a cutvertex as  $x$  has degree exceeding 4. In either case,  $G - (V(B_e - x) \cup V(B_f))$  has minimum degree at least 3, again contradicting (d).

Hence all vertices of attachment have degree 4 in  $G$ . Let  $e$  denote a 3-hyperedge in  $H$ , and let  $x, y$ , and  $z$  denote the vertices of  $V(e)$ . Again let  $f_x, f_y, f_z$  denote the unique 3-hyperedge distinct from  $e$  incident with  $x, y, z$ , respectively. If they are pairwise distinct then  $G - V(B_e)$  has minimum degree at least 3, contradiction to (d). If  $f := f_y = f_z \neq f_x$  then let  $w$  be the vertex in  $V(f)$  distinct from  $y, z$ . As  $f_x \neq f$ , we have  $w \neq x$ , so that  $G - (V(e) \cup V(f))$  has minimum degree at least 3, contradicting (d), unless there is a vertex in  $V(f_x)$  adjacent to both  $x$  and  $w$ ; in this latter case, the unique 3-hyperedge distinct from  $f$  incident with  $w$  must be  $f_x$ , and so the vertex  $u$  in  $V(f_x) \setminus \{w, x\}$  is a cutvertex of  $G$ , a contradiction to Proposition 3. Hence  $f_x = f_y = f_z$ , and the desired statement follows.  $\square$  Given our

success in characterising the double-col-critical graphs with colouring number 5, we venture to ask for a characterisation of the double-col-critical graphs with colouring number 6. If  $G$  is a double-col-critical graph, then  $G + K_k$  is a double-col-critical graph with  $\text{col}(G + K_k) = \text{col}(G) + k$  (see Proposition 4). This implies that the graphs  $Q_1 + K_1, Q_2 + K_1, Q_3 + K_1$ , and  $C + K_1$ , where  $C$  is the square of any cycle of length at least 5, are all double-col-critical graphs with colouring number 6. These are not the only double-col-critical graphs with colouring number 6; the icosahedral graph is yet another double-col-critical graph with colouring number 6. This latter fact was also observed by Stiebitz [18, p. 323], although in a somewhat different setting. Using the computer programs SAGE [15] and geng [10], we determined all the double-col-critical graphs with colouring number 6 and at most 10 vertices; there are 116 such graphs and they are available at the second authors homepage [14].

The standard 6-regular toroidal graphs obtained from the toroidal grids by adding all diagonals in the same direction have colouring number 7 and are double-col-critical.

**Complete joins of double-col-critical graphs.** In [5], it was observed that if  $G$  is the complete join  $G_1 + G_2$ , then  $G$  is double-critical if and only if both  $G_1$  and  $G_2$  are double-critical. Next we prove that the ‘if’-part of the analogous statement for double-col-critical graphs is true. The ‘only if’-part is not true, as follows from considering the double-col-critical graph  $C_6^2$ : We have  $C_6^2 \simeq C_4 + \overline{K_2}$  but neither  $C_4$  nor  $\overline{K_2}$  is double-col-critical.

**Proposition 4** *If  $G_1$  and  $G_2$  are two disjoint double-col-critical graphs, then the complete join  $G_1 + G_2$  is also double-col-critical with*

$$\text{col}(G_1 + G_2) = \min\{\text{col}(G_1) + n(G_2), \text{col}(G_2) + n(G_1)\}$$

**Proof:** Let  $G_1$  and  $G_2$  denote two disjoint double-col-critical graphs. Then, by Observation 7, both  $G_1$  and  $G_2$  are col-vertex-critical, and so, by Proposition 1,

$$\text{col}(G_1 + G_2) = \min\{\text{col}(G_1) + n(G_2), \text{col}(G_2) + n(G_1)\}$$

We need to prove that  $\text{col}((G_1 + G_2) - x - y) \leq \text{col}(G_1 + G_2) - 2$  for every edge  $e = xy \in E(G)$ ; by symmetry, it suffices to consider (1)  $x, y \in V(G_1)$  and (2)  $x \in V(G_1)$  and  $y \in V(G_2)$ . Suppose

$x, y \in V(G_1)$ . Then

$$\begin{aligned} \text{col}((G_1 + G_2) - x - y) &= \text{col}((G_1 - x - y) + G_2) \\ &\leq \min\{\text{col}(G_1 - x - y) + n(G_2), \text{col}(G_2) + n(G_1 - x - y)\} \\ &\leq \min\{\text{col}(G_1) - 2 + n(G_2), \text{col}(G_2) + n(G_1) - 2\} \\ &= \min\{\text{col}(G_1) + n(G_2), \text{col}(G_2) + n(G_1)\} - 2 \end{aligned}$$

where we applied Proposition 1 and the fact that  $G_1$  is double-col-critical. A similar argument applies in case (2). We omit the details.  $\square$  Proposition 4 and the fact that both  $C_6^2$  and  $K_t$  are double-col-critical

immediately implies the following result, which, in particular, shows that, for each integer  $k \geq 6$ , there is a non-regular double-col-critical graph with colouring number  $k$ .

**Corollary 1** *For any positive integer  $t$ , the graph  $G_t := C_6^2 + K_t$  is a double-col-critical graph with  $\text{col}(G_t) = t + 5$ ,  $\delta(G_t) = n(G_t) - 2$  and  $\Delta(G_t) = n(G_t) - 1$ .*

### 3 Double-col-critical edges

In [5], Kawarabayashi, the second author, and Toft initiated the study of the number of double-critical edges in graphs. In this section, we study the number of double-col-critical edges in graphs. Kawarabayashi, the second author, and Toft proved the following theorem for which we shall prove an analogue for the colouring number.

The complete join  $C_n + K_1$  of a cycle  $C_n$  and a single vertex is referred to as a *wheel*, and it is denoted  $W_n$ . If  $n$  is odd, we refer to  $W_n$  as an *odd wheel*.

**Theorem 3 (Kawarabayashi, Pedersen & Toft [5])** *If  $G$  denotes a 4-critical non-complete graph, then  $G$  contains at most  $m(G)/2$  double-critical edges. Moreover,  $G$  contains precisely  $m(G)/2$  double-critical edges if and only if  $G$  is an odd wheel of order at least 6.*

The following result is just a slight reformulation of Theorem 3.

**Corollary 2** *If  $G$  denotes a 4-chromatic graph with no 4-clique, then  $G$  contains at most  $m(G)/2$  double-critical edges. Moreover,  $G$  contains precisely  $m(G)/2$  double-critical edges if and only if  $G$  is an odd wheel of order at least 6.*

**Proof:** Let  $G$  denote a 4-chromatic graph with no 4-clique. If  $e = xy$  is a double-critical edge in  $G$ , then  $e$  is a critical edge of  $G$  and  $x$  is a critical vertex of  $G$ . We remove non-critical elements from  $G$  until we are left with a 4-critical subgraph  $G'$ . At no point did we remove an endvertex of a double-critical edge. Thus, the number of double-critical edges in  $G$  is equal to the number of double-critical edges in  $G'$ . Clearly,  $G'$  is a non-complete graph, and so, by Theorem 3, the number of double-critical edges in  $G'$  is at most  $m(G')/2$  which is at most  $m(G)/2$ . The second part of the corollary now follows easily.  $\square$

The following result — which is an analogue of Theorem 3 with the chromatic number replaced by the colouring number — extends Observation 8.

**Proposition 5** *If  $G$  denotes a 4-col-critical non-complete graph, then  $G$  contains at most  $m(G)/2$  double-col-critical edges. Moreover,  $G$  contains precisely  $m(G)/2$  double-col-critical edges if and only if  $G$  is a wheel of order at least 6.*

**Lemma 1** *If  $e$  and  $f$  are two double-col-critical edges in a 4-col-critical non-complete graph, then  $e$  and  $f$  are incident.*

**Proof:** Let  $G$  denote a col-critical graph with  $\text{col}(G) = 4$ . Then, by Observation 5,  $\delta(G) = 3$ . We must have  $\omega(G) \leq 3$ , since  $G$  is a 4-col-critical non-complete graph.

Suppose  $e$  is an arbitrary double-col-critical edge in  $G$ . Then  $\text{col}(G - V(e)) = 2$  which, by Observation 1 (iii), means that  $G - V(e)$  is a forest containing at least one edge and, since  $\delta(G) = 3$ ,  $\delta(G - V(e)) \geq 1$  and each leaf in  $G - V(e)$  is adjacent to both endvertices of  $e$  in  $G$ . Let  $u$  and  $v$  denote two leafs of  $G - V(e)$ . Now, if  $G$  contains some double-col-critical edge  $f$  which is not incident to  $e$ , then  $G - V(f)$  contains no cycles, since  $\text{col}(G - V(f)) = 2$ , and so  $f$  is incident to both  $u$  and  $v$ , which implies  $G[\{u, v\} \cup V(e)] \simeq K_4$ , a contradiction. This means that any two double-col-critical edges of  $G$  are incident, and the proof is complete.  $\square$

**Proof of Proposition 5:** Let  $G$  denote a 4-col-critical non-complete graph. Then  $n(G) \geq 5$  and, by Observation 5,  $\delta(G) = 3$  which implies  $m(G) \geq \lceil n(G) \cdot \delta(G)/2 \rceil \geq 8$ . By Lemma 1, we only have to consider two cases: (i)  $G$  contains three incident double-col-critical edges  $xy$ ,  $yz$ , and  $xz$ , or (ii) there is a vertex  $v \in V(G)$  such that every double-col-critical edge of  $G$  is incident to  $v$ . If (i) holds, then, since  $m(G) \geq 8$ , the desired statement follows. Suppose (ii) holds. Then the number of double-col-critical edges in  $G$  is at most  $\deg(v, G)$ . We may assume that there is at least one double-col-critical edge, say,  $vw$  in  $G$ . Suppose  $G - v$  is disconnected. Then, since  $\delta(G) = 3$ , each component of  $G - v$  has minimum degree at least 2, and so, in particular, some component of  $G - v - w$  has minimum degree at least 2. This, however, contradicts the fact that  $G - v - w$  is a forest. Hence  $G - v$  is connected. By Observation 3,  $\text{col}(G - v) \geq 3$  and so, by Observation 1 (iii),  $G - v$  contains a cycle. Hence  $G - v$  is a connected graph with at least one cycle, and so  $m(G - v) \geq n(G - v)$ . Thus,

$$m(G) = \deg(v, G) + m(G - v) \geq \deg(v, G) + (n(G) - 1) \geq 2 \deg(v, G)$$

which implies that the number of double-col-critical edges is at most  $m(G)/2$  and that the number of double-col-critical edges is equal to  $m(G)/2$  only if  $\deg(v, G) = n(G) - 1$  and  $G - v$  is a cycle.

Conversely, if  $G$  is a wheel on at least five vertices, then it is easy to see that exactly  $m(G)/2$  edges of  $G$  are double-col-critical. This completes the proof.  $\square$  Let  $k$  denote some integer greater

than 3. Let  $D_k$  denote the  $2k$ -cycle with vertices labelled cyclically  $v_0 v_1 \dots v_k u_{k-1} u_{k-2} \dots u_1$ . Let  $F_k$  denote the graph

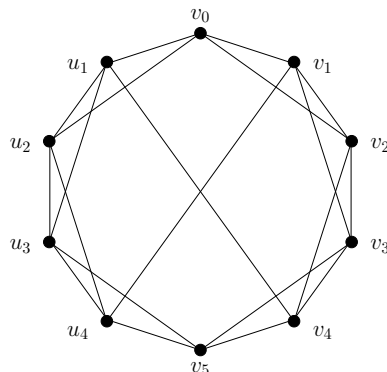
$$D_k^2 - u_1 v_1 - u_{k-1} v_{k-1} + v_1 v_{k-1} + u_1 u_{k-1}$$

Figure 3 depicts a drawing of  $F_5$ .

**Observation 12** *For every integer  $k$  greater than 3, the graph  $F_k$ , as defined as above, is a 5-col-critical graph with colouring number 5 in which all edges except  $v_1 v_{k-1}$  and  $u_1 u_{k-1}$  are double-col-critical.*

**Proposition 6** *For each integer  $p$  greater than 4 and positive real number  $\epsilon$ , there is a  $p$ -col-critical graph  $G$  with the ratio of double-col-critical edges between  $1 - \epsilon$  and 1.*

**Proof:** If  $p = 5$ , the desired result follows directly from Observation 12 by letting  $k$  tend to infinity. Let  $p$  denote an integer greater than 5 and  $\epsilon$  a positive real number. Let  $k$  denote an integer a lot greater than  $p$ , and let  $G$  denote the graph obtained by taking the complete join of  $F_k$  and  $\overline{K_{p-5}}$ . Then, by



**Fig. 3:** The graph  $F_5$  has colouring number 5 and all edges of  $F_5$ , except  $u_1v_4$  and  $v_1u_4$ , are double-col-critical.

Proposition 2 (ii),  $G$  is  $p$ -col-critical, and, since  $k$  is a lot greater than  $p$ , all but the edges  $v_1v_{k-1}$  and  $u_1u_{k-1}$  are double-col-critical in  $G$ . By letting  $k$  tend to infinity the ratio of double-col-critical edges in  $G$  will from a certain point onwards be between  $1 - \epsilon$  and 1. This completes the argument.  $\square$

Proposition 6 means that there is no result corresponding to Proposition 5 for colouring numbers greater than 4.

## 4 Concluding remarks

By a theorem of Mozhan [11] and, independently, Stiebitz [16],  $K_5$  is the only double-critical graph with chromatic number 5, that is,  $K_5$  is the only 5-chromatic graph with 100% double-critical edges, but we do not know whether there are non-complete 5-chromatic graphs with the percentage of double-critical edges arbitrarily close to a 100. In [5], Kawarabayashi, the second author, and Toft conjectured that if  $G$  is a 5-critical non-complete graph, then  $G$  contains at most  $(2 + \frac{1}{3n(G)-5}) \frac{m(G)}{3}$  double-critical edges.

As we have seen, the story is a bit different for the colouring number. By Proposition 6 for  $p = 5$ , there are non-complete graphs with colouring number 5 with the percentage of double-col-critical edges arbitrarily close to a 100. This only makes Theorem 1 all the more interesting. By Theorem 1, we are able to distinguish between graphs with colouring number 5 having 99.99% double-col-critical edges and those that have a 100% double-col-critical edges.

The problem of obtaining a concise structural description of the double-col-critical graphs with colouring number  $k \geq 6$  remains open. Given any graph  $G$ , it can be decided in polynomial time whether or not  $G$  is double-col-critical, since the colouring number itself can be computed in polynomial time. Nevertheless, given the structural complexity of the double-col-critical graphs with colouring number 6 mentioned on page 58, it seems likely that even the problem of obtaining a concise structural description of the double-col-critical graphs with colouring number 6 is non-trivial.

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